ON THE STABILITY OF SOME PROPERTIES OF PARTIAL ALGEBRAS UNDER POWERS

N. CHAISANSUK* — S. LEERATANAVALLEE* — J. ŠLAPAL**

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ABSTRACT. In the paper, we study eight properties of partial algebras most of which are related to diagonality. For each of the properties, we give sufficient conditions under which this property is preserved by powers of partial algebras.

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1. Introduction

In his pioneering paper [1], G. Birkhoff introduced the operation of a cardinal power of partially ordered sets and showed the validity of the first exponential law for the operation, i.e., the law $(A^B)^C \cong A^{B \times C}$. In [6], J. Šlapal extended the concept of a power from partially ordered sets onto arbitrary n -ary relational systems and studied its behavior. The same author introduced and studied powers of n -ary algebras in [7] and also powers of partial algebras in [8]. He showed the importance of idempotency, diagonality (in the sense of [5]), mediality (in the sense of [3]), and commutativity (in the sense of [4]) of algebras and partial algebras when studying their powers. Moreover, in [9], J. Šlapal defined and studied the following eight properties of partial algebras: idempotency, diagonality, commutativity, strong diagonality, weak diagonality, local diagonality, local antidiagonality, and weak local antidiagonality. The aim of this paper

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is to give sufficient conditions for universal partial algebras under which these properties are preserved by exponentiating the algebras and under which the exponentiation satisfies the first exponential law.

2. Partial algebras

For the basic concepts used concerning partial algebras we refer to [2]. Let Ω be a nonempty set. A family $\tau = (K_\lambda; \lambda \in \Omega)$ of sets will be called a *type*. By a *partial algebra* of type τ we understand a pair $\mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega) \rangle$ where G is a nonempty set, the so-called *underlying set* of $\mathbf{G}$, and p_λ is a K_λ -ary partial operation on G , i.e., a map $p_\lambda: D_{p_\lambda} \rightarrow G$ where $D_{p_\lambda} \subseteq G^{K_\lambda}$, for each $\lambda \in \Omega$. The set D_{p_λ} is called the *domain* of p_λ . If K_λ is finite, then p_λ is said to be *finitary*. Throughout the paper, the type τ of any partial algebra $\langle G, (p_\lambda; \lambda \in \Omega) \rangle$ is understood to be the family denoted by $(K_\lambda; \lambda \in \Omega)$. The underlying set of a partial algebra $\mathbf{G}$ will be denoted by $|\mathbf{G}|$. Of course, given a partial algebra $\langle G, (p_\lambda; \lambda \in \Omega) \rangle$, when writing $p_\lambda(x_i; i \in K_\lambda) = x$ (for some $x \in G$), we automatically mean that $p_\lambda(x_i; i \in K_\lambda)$ is defined, i.e., that $(x_i; i \in K_\lambda) \in D_{p_\lambda}$.

Let $\mathbf{H} = \langle H, (q_\lambda; \lambda \in \Omega) \rangle$, $\mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega) \rangle$ be a pair of partial algebras of type τ . Then $\mathbf{G}$ is called a *partial subalgebra* of $\mathbf{H}$ provided that $G \subseteq H$ and, for each $\lambda \in \Omega$, whenever $(x_i; i \in K_\lambda) \in G^{K_\lambda}$ and $x \in H$,

$$p_\lambda(x_i; i \in K_\lambda) = x \iff q_\lambda(x_i; i \in K_\lambda) = x$$

(note that $D_{p_\lambda} = D_{q_\lambda} \cap G^{K_\lambda}$). A map $f: H \rightarrow G$ is called a *homomorphism* of $\mathbf{H}$ into $\mathbf{G}$ if, for each $\lambda \in \Omega$, $q_\lambda(x_i; i \in K_\lambda) = x$ implies $p_\lambda(f(x_i); i \in K_\lambda) = f(x)$. The set of all homomorphisms of $\mathbf{H}$ into $\mathbf{G}$ will be denoted by $\text{Hom}(\mathbf{H}, \mathbf{G})$.

The *direct product* of a family $\mathbf{G}_i = \langle G_i, (p_{i_\lambda}; \lambda \in \Omega) \rangle$, $i \in I$ of partial algebras of type τ is the partial algebra $\prod_{i \in I} \mathbf{G}_i = \langle \prod_{i \in I} G_i, (q_\lambda; \lambda \in \Omega) \rangle$ where $\prod_{i \in I} G_i$ denotes the cartesian product of sets and, for any $\lambda \in \Omega$, any $(f_k; k \in K_\lambda) \in (\prod_{i \in I} G_i)^{K_\lambda}$ and any $f \in \prod_{i \in I} G_i$, $q_\lambda(f_k; k \in K_\lambda) = f$ if and only if $p_{i_\lambda}(f_k(i); k \in K_\lambda) = f(i)$ for each $i \in I$. If $\mathbf{G}_i = \mathbf{G}$ for every $i \in I$, then we write $\mathbf{G}^I$ instead of $\prod_{i \in I} \mathbf{G}_i$.

Given a partial algebra $\langle G, (p_\lambda; \lambda \in \Omega) \rangle$ and $\lambda \in \Omega$, an element $x \in G$ is said to be a p_λ -*idempotent* if $p_\lambda(x_i; i \in K_\lambda) = x$ whenever $x_i = x$ for each $i \in K_\lambda$.

Let G, K, L be sets. By a $K \times L$ -matrix M over G we understand any map $M: K \times L \rightarrow G$, i.e., $M = (x_{ij}; i \in K, j \in L)$ where $x_{ij} \in G$ whenever $i \in K$

and $j \in L$. We denote by M^T the transposed matrix to M , i.e., the matrix $M^T = (x_{ji}; j \in L, i \in K)$ over G .

Let $x \in G$ and let $M = (x_{ij}; i \in K, j \in L)$ be a $K \times L$ -matrix M over G . Then M is said to be *x-constant* provided that $x_{ij} = x$ whenever $i \in K$ and $j \in L$, and it is said to have *x-constant diagonal* if $K = L$ and $x_{ii} = x$ for all $i \in K$. Let q be an L -ary partial operation on G . Then M is said to be *q-operational* provided that $(x_{ij}; j \in L) \in D_q$ for each $i \in K$. Let p be a K -ary partial operation on G . Then M is said to be *pq-operational* provided that it is *q-operational* and $(q(x_{ij}; j \in L); i \in K) \in D_p$; we put $pq(M) = p(q(x_{ij}; j \in L); i \in K)$. Finally, M is said to be *diagonally p-operational* provided that $K = L$ and $(x_{ii}; i \in K) \in D_p$; we put $\Delta_p(M) = p(x_{ii}; i \in K)$.

DEFINITION 2.1. ([9]) A partial algebra $\langle G, (p_\lambda; \lambda \in \Omega) \rangle$ of type τ is called

- (1) *idempotent* if, for each $\lambda \in \Omega$, every element $x \in G$ is a p_λ -idempotent.
- (2) *commutative* if, for any $\lambda, \mu \in \Omega$ and any $p_\lambda p_\mu$ -operational $K_\lambda \times K_\mu$ -matrix M over G such that M^T is p_λ -operational, M^T is $p_\mu p_\lambda$ -operational with $p_\lambda p_\mu(M) = p_\mu p_\lambda(M^T)$,
- (3) *diagonal* if, for any $\lambda \in \Omega$, every $p_\lambda p_\lambda$ -operational $K_\lambda \times K_\lambda$ matrix M over G is diagonally p_λ -operational with $p_\lambda p_\lambda(M) = \Delta_{p_\lambda}(M)$,
- (4) *strongly diagonal* if, for any $\lambda \in \Omega$ and any $x \in G$, any p_λ -operational $K_\lambda \times K_\lambda$ matrix M over G is $p_\lambda p_\lambda$ -operational with $p_\lambda p_\lambda(M) = x$ if and only if it is diagonally p_λ -operational with $\Delta_{p_\lambda}(M) = x$,
- (5) *weakly diagonal* if, for any $\lambda \in \Omega$ and any $p_\lambda p_\lambda$ -operational $K_\lambda \times K_\lambda$ matrix M over G , whenever M^T is $p_\lambda p_\lambda$ -operational with $p_\lambda p_\lambda(M) = p_\lambda p_\lambda(M^T)$, M is diagonally p_λ -operational with $p_\lambda p_\lambda(M) = \Delta_{p_\lambda}(M)$,
- (6) *locally diagonal* if, for any $\lambda \in \Omega$ and any $x \in G$, every p_λ -operational $K_\lambda \times K_\lambda$ matrix M over G with x -constant diagonal is $p_\lambda p_\lambda$ -operational with $p_\lambda p_\lambda(M) = x$,
- (7) *locally antidiagonal* if, for any $\lambda \in \Omega$, any $x \in G$ and any $p_\lambda p_\lambda$ -operational $K_\lambda \times K_\lambda$ matrix M over G with x -constant diagonal, $p_\lambda p_\lambda(M) = x$ implies that M is x -constant,
- (8) *weakly locally antidiagonal* if, for any $\lambda \in \Omega$, any $x \in G$ and any $p_\lambda p_\lambda$ -operational $K_\lambda \times K_\lambda$ matrix M over G with x -constant diagonal such that M^T is $p_\lambda p_\lambda$ -operational too, $p_\lambda p_\lambda(M) = p_\lambda p_\lambda(M^T) = x$ implies that M is x -constant.

Clearly, strong diagonality implies diagonality, diagonality implies weak diagonality and local antidiagonality implies weak local antidiagonality. If $\mathbf{G}$ is idempotent, then strong diagonality implies local diagonality.

Example 2.2. Let G be a set and ρ be a binary relation on G . Let p be the binary partial operation on G given by

$$D_p = \rho \quad \text{and} \quad p(a, b) = a \quad \text{whenever} \quad (a, b) \in D_p.$$

Let q be the binary partial operation on G dual to p , i.e., given by

$$D_q = \rho \quad \text{and} \quad q(c, d) = d \quad \text{whenever} \quad (c, d) \in D_q.$$

Then $\langle G, p, q \rangle$ is a partial algebra of type $(2, 2)$ and we have:

- (a) $\langle G, p, q \rangle$ is commutative,
- (b) ρ is reflexive if and only if $\langle G, p, q \rangle$ is idempotent,
- (c) ρ is transitive if and only if $\langle G, p, q \rangle$ is diagonal,
- (d) $\langle G, p, q \rangle$ is locally diagonal if ρ is symmetric,
- (e) $\langle G, p, q \rangle$ is strongly diagonal if ρ is transitive and symmetric,
- (f) ρ is antisymmetric if and only if $\langle G, p, q \rangle$ is weakly locally antidiagonal.

Example 2.3. Let (G, p) be a partial rectangular band, i.e., a partial algebra of type (2) such that there are sets X, Y with $G = X \times Y$ and p is the binary partial operation on G given by

$$D_p = \{((a_1, b_1), (a_2, b_2)) \in (X \times Y)^2 : b_1 = a_2\}$$

and

$$p((a_1, b_1), (a_2, b_2)) = (a_1, b_2) \quad \text{whenever} \quad ((a_1, b_1), (a_2, b_2)) \in D_p.$$

Then (G, p) is a weakly diagonal partial algebra.

Example 2.4. Let X be a set and let p be a binary partial operation on the power set $\mathcal{P}(X)$ of X given by

$$(A, B) \in D_p \iff [A = B \text{ or } (A \neq \emptyset \ \& \ B \neq \emptyset \ \& \ A \cap B = \emptyset)]$$

$$\text{and then} \quad p(A, B) = A \cap B.$$

Then $(\mathcal{P}(X), p)$ is a locally antidiagonal partial algebra.

Let $\mathbf{H} = \langle H, (q_\lambda; \lambda \in \Omega) \rangle$ and $\mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega) \rangle$ be partial algebras of the same type, let $\lambda \in \Omega$ and let $(f_i; i \in K_\lambda) \in (\text{Hom}(\mathbf{H}, \mathbf{G}))^{K_\lambda}$. Let $f \in \text{Hom}(\mathbf{H}, \mathbf{G})$ be a homomorphism such that, for all $(x_i; i \in K_\lambda) \in H^{K_\lambda}$ and all $x \in H$, $q_\lambda(x_i; i \in K_\lambda) = x$ implies $p_\lambda(f_i(x_i); i \in K_\lambda) = f(x)$. Then f need not be a unique homomorphism of $\mathbf{H}$ into $\mathbf{G}$ having this property — see the following example.

Example 2.5. Let $\mathbf{H} = (H, q)$, $\mathbf{G} = (G, p)$ be partial algebras of type (2) where $H = \{1, 2\}$, $G = \{a, b\}$ and p, q are given by the tables

q	1	2
1	2	2
2	2	2

and

p	a	b
a	a	a
b	a	a

Let $f: H \rightarrow G$ be the map given by $f(1) = a$, $f(2) = a$ and $g: H \rightarrow G$ be the map given by $g(1) = b$, $g(2) = a$. Then $f, g \in \text{Hom}(\mathbf{H}, \mathbf{G})$ and, for any $x, y, z \in \{1, 2\}$, $q(x, y) = z$ implies $p(f(x), f(y)) = f(z)$ and $p(f(x), f(y)) = g(z)$. But $f \neq g$.

LEMMA 2.6. *Let $\mathbf{H} = \langle H, (q_\lambda; \lambda \in \Omega) \rangle$, $\mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega) \rangle$ be partial algebras of the same type, $\mathbf{H}$ idempotent, and let $\lambda \in \Omega$ and $(f_i; i \in K_\lambda) \in (\text{Hom}(\mathbf{H}, \mathbf{G}))^{K_\lambda}$. If $f \in \text{Hom}(\mathbf{H}, \mathbf{G})$ is a homomorphism such that, for all $(x_i; i \in K_\lambda) \in H^{K_\lambda}$ and all $x \in H$,*

$$q_\lambda(x_i; i \in K_\lambda) = x \implies p_\lambda(f_i(x_i); i \in K_\lambda) = f(x),$$

then f is uniquely determined.

Proof. Let $f, g \in \text{Hom}(\mathbf{H}, \mathbf{G})$ be homomorphisms such that, for all $(x_i; i \in K_\lambda) \in H^{K_\lambda}$ and all $x \in H$, $q_\lambda(x_i; i \in K_\lambda) = x$ implies both $p_\lambda(f_i(x_i); i \in K_\lambda) = f(x)$ and $p_\lambda(g_i(x_i); i \in K_\lambda) = g(x)$. Let $x \in H$. As $\mathbf{H}$ is idempotent, putting $x_i = x$ for each $i \in K_\lambda$, we get $q_\lambda(x_i; i \in K_\lambda) = x$, $p_\lambda(f_i(x); i \in K_\lambda) = f(x)$ and $p_\lambda(g_i(x); i \in K_\lambda) = g(x)$. Hence, $f = g$. $\square$

3. Stability of properties of partial algebras under powers

DEFINITION 3.1. Let $\mathbf{H} = \langle H, (q_\lambda; \lambda \in \Omega) \rangle$ and $\mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega) \rangle$ be partial algebras of the same type τ . The *power* of $\mathbf{G}$ and $\mathbf{H}$ is the partial algebra $\mathbf{G}^{\mathbf{H}} = \langle \text{Hom}(\mathbf{H}, \mathbf{G}), (r_\lambda; \lambda \in \Omega) \rangle$ of type τ where, for any $\lambda \in \Omega$, any $(f_i; i \in K_\lambda) \in (\text{Hom}(\mathbf{H}, \mathbf{G}))^{K_\lambda}$ and any $f \in \text{Hom}(\mathbf{H}, \mathbf{G})$, $r_\lambda(f_i; i \in K_\lambda) = f$ if and only if f is a unique homomorphism of $\mathbf{H}$ into $\mathbf{G}$ with the property that, for all $(x_i; i \in K_\lambda) \in H^{K_\lambda}$ and all $x \in H$, $q_\lambda(x_i; i \in K_\lambda) = x$ implies $p_\lambda(f_i(x_i); i \in K_\lambda) = f(x)$.

Lemma 2.6 and Definition 3.1 immediately result in

COROLLARY 3.2. *Let $\mathbf{G}, \mathbf{H}$ be partial algebras of the same type. If $\mathbf{H}$ is idempotent, then the power $\mathbf{G}^{\mathbf{H}}$ is idempotent, too.*

LEMMA 3.3. *Let $\mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega) \rangle$, $\mathbf{H} = \langle H, (q_\lambda; \lambda \in \Omega) \rangle$ be partial algebras of the same type, $\mathbf{H}$ idempotent, and let $\mathbf{G}^{\mathbf{H}} = \langle \text{Hom}(\mathbf{H}, \mathbf{G}), (r_\lambda; \lambda \in \Omega) \rangle$. Let $\lambda, \mu \in \Omega$ and let $M = (f_{ij}; i \in K_\lambda, j \in K_\mu)$ be a matrix over $\text{Hom}(\mathbf{H}, \mathbf{G})$. If M is r_μ -operational with $r_\mu(f_{ij}; j \in K_\mu) = f_i$ for each $i \in K_\lambda$, $r_\lambda r_\mu$ -operational with $r_\lambda r_\mu(M) = f$ or diagonally r_λ -operational with $\Delta_{r_\lambda}(M) = f$, respectively, then, whenever $(y_j; j \in K_\mu) \in D_{q_\mu}$ and $q_\mu(y_j; j \in K_\mu) = y$, the matrix $M^* = (f_{ij}(y_j); i \in K_\lambda, j \in K_\mu)$ over G is p_μ -operational with $p_\mu(f_{ij}(y_j); j \in K_\mu) = f_i(y)$ for each $i \in K_\lambda$, $p_\lambda p_\mu$ -operational with $p_\lambda p_\mu(M^*) = f(y)$ or diagonally p_λ -operational with $\Delta_{p_\lambda}(M^*) = f(y)$, respectively.*

Proof.

(1) Let M be r_μ -operational with $r_\mu(f_{ij}; j \in K_\mu) = f_i$ for each $i \in K_\lambda$ and let $(y_j; j \in K_\mu) \in D_{q_\mu}$, $q_\mu(y_j; j \in K_\mu) = y$. Then it follows immediately from Definition 3.1 that M^* is p_μ -operational with $p_\mu(f_{ij}(y_j); j \in K_\mu) = f_i(y)$ for each $i \in K_\lambda$.

(2) Let M be $r_\lambda r_\mu$ -operational with $r_\lambda r_\mu(M) = f$ and put $r_\mu(f_{ij}; j \in K_\mu) = f_i$ for each $i \in K_\lambda$. Let $(y_j; j \in K_\mu) \in D_{q_\mu}$, $q_\mu(y_j; j \in K_\mu) = y$. By (1), M^* is p_μ -operational with $p_\mu(f_{ij}(y_j); j \in K_\mu) = f_i(y)$ for each $i \in K_\lambda$. Thus,

$$p_\lambda p_\mu(M^*) = p_\lambda(p_\mu(f_{ij}(y_j)); j \in K_\mu, i \in K_\lambda) = p_\lambda(f_i(y); i \in K_\lambda).$$

As $\mathbf{H}$ is idempotent, we get $p_\lambda(f_i(y); i \in K_\lambda) = f(y)$. Consequently, M^* is $p_\lambda p_\mu$ -operational with $p_\lambda p_\mu(M^*) = f(y)$.

(3) Let M be diagonally r_λ -operational with $\Delta_{r_\lambda}(M) = f$, i.e., let $r_\lambda(f_{ii}; i \in K_\lambda) = f$. Let $(y_j; j \in K_\mu) \in D_{q_\mu}$, $q_\mu(y_j; j \in K_\mu) = y$. Then

$$\Delta_{p_\lambda}(M^*) = p_\lambda(f_{ii}(y_i); i \in K_\lambda) = f(y).$$

Hence, M^* is diagonally p_λ -operational with $\Delta_{p_\lambda}(M^*) = f(y)$. $\square$

Using Lemma 3.3, one can easily prove the following theorem.

THEOREM 3.4. *Let $\mathbf{G}, \mathbf{H}$ be partial algebras of the same type, $\mathbf{H}$ idempotent. If $\mathbf{G}$ is commutative, diagonal, strongly diagonal or weakly diagonal, respectively, then so is $\mathbf{G}^{\mathbf{H}}$.*

LEMMA 3.5. *Let $\mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega) \rangle$ be an idempotent, locally diagonal partial algebra and let $\lambda \in \Omega$. If $(x_i; i \in K_\lambda), (y_i; i \in K_\lambda) \in G^{K_\lambda}$, $p_\lambda(y_i; i \in K_\lambda) = y$ and there exists $i_0 \in K_\lambda$ such that $x_{i_0} = y$ and $x_i = y_{i_0}$ for each $i \in K_\lambda - \{i_0\}$, then $p_\lambda(x_i; i \in K_\lambda) = y_{i_0}$.*

Proof. Let $(x_i; i \in K_\lambda), (y_i; i \in K_\lambda) \in G^{K_\lambda}$, $p_\lambda(y_i; i \in K_\lambda) = y$ and let $i_0 \in K_\lambda$ be an element such that $x_{i_0} = y$ and $x_i = y_{i_0}$ for each $i \in K_\lambda - \{i_0\}$. Let $M = (a_{ij}; i \in K_\lambda, j \in K_\lambda)$ be a $K_\lambda \times K_\lambda$ -matrix over $\mathbf{G}$ such that $a_{ij} = y_{i_0}$

for each $i \in K_\lambda - \{i_0\}$ and $a_{i_0j} = y_j$ for each $j \in K_\lambda$. Since $\mathbf{G}$ is idempotent, M is p_λ -operational with y_{i_0} -constant diagonal. As $\mathbf{G}$ is locally diagonal, we have

$$p_\lambda p_\lambda(M) = p_\lambda(p_\lambda(a_{ij}; j \in K_\lambda); i \in K_\lambda) = y_{i_0}.$$

Thus, $p_\lambda(x_i; i \in K_\lambda) = y_{i_0}$ because $x_i = p_\lambda(a_{ij}; j \in K_\lambda)$ for every $i \in K_\lambda$. $\square$

THEOREM 3.6. *Let $\mathbf{G}, \mathbf{H}$ be partial algebras of the same type, $\mathbf{H}$ idempotent and locally diagonal. If $\mathbf{G}$ is locally diagonal, then so is $\mathbf{G}^{\mathbf{H}}$.*

Proof. Let $\mathbf{H} = \langle H, (q_\lambda; \lambda \in \Omega) \rangle$, $\mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega) \rangle$, $\mathbf{G}^{\mathbf{H}} = \langle \text{Hom}(\mathbf{H}, \mathbf{G}), (r_\lambda; \lambda \in \Omega) \rangle$ and let $\mathbf{G}$ be locally diagonal. Let $\lambda \in \Omega$ and $f \in \text{Hom}(\mathbf{H}, \mathbf{G})$. Let $M = (f_{ij}; i \in K_\lambda, j \in K_\lambda)$ be an r_λ -operational matrix over $\text{Hom}(\mathbf{H}, \mathbf{G})$ with f -constant diagonal. Put $r_\lambda(f_{ij}; j \in K_\lambda) = f_i$ for each $i \in K_\lambda$. Let $(y_i; i \in K_\lambda) \in D_{q_\lambda}$, $q_\lambda(y_i; i \in K_\lambda) = y$ and, for every $i \in K_\lambda$, put $a_{ii} = y$ and $a_{ij} = y_i$ for each $j \in K_\lambda - \{i\}$. As $\mathbf{H}$ is idempotent and locally diagonal, we have $q_\lambda(a_{ij}; j \in K_\lambda) = y_i$ for every $i \in K_\lambda$ by Lemma 3.5. Thus, $p_\lambda(f_{ij}(a_{ij}); j \in K_\lambda) = f_i(y_i)$ for every $i \in K_\lambda$. Put $M^* = (f_{ij}(a_{ij}); i \in K_\lambda, j \in K_\lambda)$. Then M^* is a p_λ -operational $K_\lambda \times K_\lambda$ -matrix over $\mathbf{G}$ with $f(y)$ -constant diagonal. Since $\mathbf{G}$ is locally diagonal, we have $p_\lambda p_\lambda(M^*) = f(y)$. Therefore,

$$p_\lambda(f_i(y_i); i \in K_\lambda) = f(y).$$

Since $\mathbf{H}$ is idempotent, f is uniquely determined. Hence, $r_\lambda(f_i; i \in K_\lambda) = r_\lambda r_\lambda(M) = f$. $\square$

THEOREM 3.7. *Let $\mathbf{G}, \mathbf{H}$ be partial algebras of the same type. $\mathbf{H}$ idempotent. If $\mathbf{G}$ is locally antidiagonal or weakly locally antidiagonal, respectively, then so is $\mathbf{G}^{\mathbf{H}}$.*

Proof. Let $\mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega) \rangle$, $\mathbf{H} = \langle H, (q_\lambda; \lambda \in \Omega) \rangle$ be partial algebras of the same type, $\mathbf{H}$ idempotent, and let $\mathbf{G}^{\mathbf{H}} = \langle \text{Hom}(\mathbf{H}, \mathbf{G}), (r_\lambda; \lambda \in \Omega) \rangle$. Let $\lambda \in \Omega$ and $f \in \text{Hom}(\mathbf{H}, \mathbf{G})$. Let $M = (f_{ij}; i \in K_\lambda, j \in K_\lambda)$ be an $r_\lambda r_\lambda$ -operational matrix over $\text{Hom}(\mathbf{H}, \mathbf{G})$ with f -constant diagonal and let $y \in H$. Since $\mathbf{H}$ is idempotent, $M^* = (f_{ij}(x); i \in K_\lambda, j \in K_\lambda)$ is a $p_\lambda p_\lambda$ -operational matrix over G by Lemma 3.3. It is also evident that M^* has $f(y)$ -constant diagonal. Using these facts, we easily get the statement. $\square$

THEOREM 3.8. *Let $\mathbf{G}, \mathbf{H}, \mathbf{K}$ be partial algebras of the same type. If $\mathbf{H}, \mathbf{K}$ are idempotent, then*

$$(\mathbf{G}^{\mathbf{H}})^{\mathbf{K}} \cong \mathbf{G}^{\mathbf{H} \times \mathbf{K}}.$$

Proof. Let

$$\begin{aligned} \mathbf{H} &= \langle H, (q_\lambda; \lambda \in \Omega) \rangle, & \mathbf{G} &= \langle G, (p_\lambda; \lambda \in \Omega) \rangle, \\ \mathbf{K} &= \langle K, (s_\lambda; \lambda \in \Omega) \rangle, & \mathbf{G}^{\mathbf{H}} &= \langle \text{Hom}(\mathbf{H}, \mathbf{G}), (R_\lambda; \lambda \in \Omega) \rangle, \end{aligned}$$

$\mathbf{H} \times \mathbf{K} = \langle H \times K, (V_\lambda; \lambda \in \Omega) \rangle$, $(\mathbf{G}^{\mathbf{H}})^{\mathbf{K}} = \langle \text{Hom}(\mathbf{K}, \text{Hom}(\mathbf{H}, \mathbf{G})), (T_\lambda; \lambda \in \Omega) \rangle$,
 $\mathbf{G}^{\mathbf{H} \times \mathbf{K}} = \langle \text{Hom}(\mathbf{H} \times \mathbf{K}, \mathbf{G}), (U_\lambda; \lambda \in \Omega) \rangle$
 and let $\mathbf{H}, \mathbf{K}$ be idempotent.

Let $\lambda \in \Omega$ and define a map $\varphi: \text{Hom}(\mathbf{K}, \text{Hom}(\mathbf{H}, \mathbf{G})) \rightarrow \text{Hom}(\mathbf{H} \times \mathbf{K}, \mathbf{G})$ by $\varphi(g)(y, z) = g(z)(y)$ whenever $g \in \text{Hom}(\mathbf{K}, \text{Hom}(\mathbf{H}, \mathbf{G}))$, $z \in K$ and $y \in H$. We will show first that the map φ is well defined, i.e., that $\varphi(g) \in \text{Hom}(\mathbf{H} \times \mathbf{K}, \mathbf{G})$ whenever $g \in \text{Hom}(\mathbf{K}, \text{Hom}(\mathbf{H}, \mathbf{G}))$. For this account, let $g \in (\text{Hom}(\mathbf{K}, \text{Hom}(\mathbf{H}, \mathbf{G})))$ and let $((y_i, z_i); i \in K_\lambda) \in D_{V_\lambda}$, $V_\lambda((y_i, z_i); i \in K_\lambda) = (y, z)$. Then $q_\lambda(y_i; i \in K_\lambda) = y$ and $s_\lambda(z_i; i \in K_\lambda) = z$ and, since $g \in (\text{Hom}(\mathbf{K}, \text{Hom}(\mathbf{H}, \mathbf{G})))$, we get $R_\lambda(g(z_i); i \in K_\lambda) = g(z)$. This implies

$$p_\lambda(g(z_i)(y_i); i \in K_\lambda) = g(z)(y)$$

and, therefore,

$$p_\lambda(\varphi(g)(y_i, z_i); i \in K_\lambda) = p_\lambda(g(z_i)(y_i); i \in K_\lambda) = g(z)(y) = \varphi(g)(y, z).$$

Hence, $\varphi(g) \in \text{Hom}(\mathbf{H} \times \mathbf{K}, \mathbf{G})$.

Further, define a map $\alpha: \text{Hom}(\mathbf{H} \times \mathbf{K}, \mathbf{G}) \rightarrow \text{Hom}(\mathbf{K}, \text{Hom}(\mathbf{H}, \mathbf{G}))$ by $\alpha(h)(z)(y) = h(y, z)$ whenever $h \in \text{Hom}(\mathbf{H} \times \mathbf{K}, \mathbf{G})$, $z \in K$ and $y \in H$. We will show that α is well defined, too.

Let $h \in \text{Hom}(\mathbf{H} \times \mathbf{K}, \mathbf{G})$. To show that $\alpha(h)(z) \in \text{Hom}(\mathbf{H}, \mathbf{G})$ for each $z \in K$, let $z \in K$ and $(y_i; i \in K_\lambda) \in D_{q_\lambda}$, $q_\lambda(y_i; i \in K_\lambda) = y$. Since $\mathbf{K}$ is idempotent, we have $V_\lambda((y_i, z); i \in K_\lambda) = (y, z)$. From $h \in \text{Hom}(\mathbf{H} \times \mathbf{K}, \mathbf{G})$ it follows that $p_\lambda(h(y_i, z); i \in K_\lambda) = h(y, z)$. Thus,

$$p_\lambda(\alpha(h)(z)(y_i); i \in K_\lambda) = p_\lambda(h(y_i, z); i \in K_\lambda) = h(y, z) = \alpha(h)(z)(y),$$

so that $\alpha(h)(z) \in \text{Hom}(\mathbf{H}, \mathbf{G})$.

To show that $\alpha(h) \in \text{Hom}(\mathbf{K}, \text{Hom}(\mathbf{H}, \mathbf{G}))$, let $(z_i; i \in K_\lambda) \in D_{s_\lambda}$, $s_\lambda(z_i; i \in K_\lambda) = z$ and let $(y_i; i \in K_\lambda) \in D_{q_\lambda}$, $q_\lambda(y_i; i \in K_\lambda) = y$. Then $V_\lambda((y_i, z_i); i \in K_\lambda) = (y, z)$. Since $h \in \text{Hom}(\mathbf{H} \times \mathbf{K}, \mathbf{G})$, we get $p_\lambda(h(y_i, z_i); i \in K_\lambda) = h(y, z)$. So,

$$p_\lambda(\alpha(h)(z_i)(y_i); i \in K_\lambda) = p_\lambda(h(y_i, z_i); i \in K_\lambda) = h(y, z) = \alpha(h)(z)(y).$$

Since $\mathbf{H}$ is idempotent, $\alpha(h)(z)$ is uniquely determined. Consequently,

$$R_\lambda(\alpha(h)(z_i); i \in K_\lambda) = \alpha(h)(z),$$

so that $\alpha(h) \in \text{Hom}(\mathbf{K}, \text{Hom}(\mathbf{H}, \mathbf{G}))$.

Further, we have

$$\varphi(\alpha(h))(y, z) = \alpha(h)(z)(y) = h(y, z) \quad \text{and} \quad \alpha(\varphi(g))(z)(y) = \varphi(g)(y, z) = g(z)(y)$$

for all $h \in \text{Hom}(\mathbf{H} \times \mathbf{K}, \mathbf{G})$, $g \in \text{Hom}(\mathbf{K}, \text{Hom}(\mathbf{H}, \mathbf{G}))$, $y \in H$ and $z \in K$. Consequently, φ and α are bijections inverse to each other.

To show that φ is a homomorphism, let $(g_i; i \in K_\lambda) \in D_{T_\lambda}$, $T_\lambda(g_i; i \in K_\lambda) = g$ and $((y_i, z_i); i \in K_\lambda) \in D_{V_\lambda}$, $V_\lambda((y_i, z_i); i \in K_\lambda) = (y, z)$. Then $q_\lambda(y_i; i \in K_\lambda) = y$ and $s_\lambda(z_i; i \in K_\lambda) = z$. Hence, $R_\lambda(g_i(z_i); i \in K_\lambda) = g(z)$ and, consequently, $p_\lambda(g_i(z_i)(y_i); i \in K_\lambda) = g(z)(y)$ and $g(z)$ is uniquely determined because $\mathbf{H} \times \mathbf{K}$ is idempotent. Therefore,

$$p_\lambda(\varphi(g_i)(y_i, z_i); i \in K_\lambda) = p_\lambda(g_i(z_i)(y_i); i \in K_\lambda) = g(z)(y) = \varphi(g)(y, z)$$

and $\varphi(g)$ is uniquely determined because $\mathbf{H} \times \mathbf{K}$ is idempotent. We have $U_\lambda(\varphi(g_i); i \in K_\lambda) = \varphi(g)$. Thus, φ is a homomorphism.

To show that α is a homomorphism, let $(h_i; i \in K_\lambda) \in D_{U_\lambda}$, $U_\lambda(h_i; i \in K_\lambda) = h$, $(z_i; i \in K_\lambda) \in D_{s_\lambda}$, $s_\lambda(z_i; i \in K_\lambda) = z$ and $(y_i; i \in K_\lambda) \in D_{q_\lambda}$, $q_\lambda(y_i; i \in K_\lambda) = y$. Then we have $V_\lambda((y_i, z_i); i \in K_\lambda) = (y, z)$ and, consequently, $p_\lambda(h_i(y_i, z_i); i \in K_\lambda) = h(y, z)$. Therefore,

$$p_\lambda(\alpha(h_i)(z_i)(y_i); i \in K_\lambda) = p_\lambda(h_i(y_i, z_i); i \in K_\lambda) = h(y, z) = \alpha(h)(z)(y)$$

and $\alpha(h)(z)$ is uniquely determined because $\mathbf{H}$ is idempotent. Consequently, $R_\lambda(\alpha(h_i)(z_i); i \in K_\lambda) = \alpha(h)(z)$. The idempotency of $\mathbf{K}$ implies the uniqueness of $\alpha(h)$, hence $T_\lambda(\alpha(h_i); i \in K_\lambda) = \alpha(h)$. Therefore, α is a homomorphism.

We have shown that $\varphi: \text{Hom}(\mathbf{K}, \text{Hom}(\mathbf{H}, \mathbf{G})) \rightarrow \text{Hom}(\mathbf{H} \times \mathbf{K}, \mathbf{G})$ is an isomorphism (with $\varphi^{-1} = \alpha$), i.e., $(\mathbf{G}^{\mathbf{H}})^{\mathbf{K}} \cong \mathbf{G}^{\mathbf{H} \times \mathbf{K}}$. $\square$

THEOREM 3.9. *Let $\mathbf{G}, \mathbf{H}$ be partial algebras of the same type. If $\mathbf{G}$ is commutative, then there exists a partial subalgebra $\mathbf{K}$ of the direct product $\mathbf{G}^{|\mathbf{H}|}$ such that $|\mathbf{K}| = \text{Hom}(\mathbf{H}, \mathbf{G})$.*

Proof. Let $\mathbf{H} = \langle H, (q_\lambda; \lambda \in \Omega) \rangle$, $\mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega) \rangle$, $\mathbf{G}^{|\mathbf{H}|} = \langle G^H, (R_\lambda; \lambda \in \Omega) \rangle$ and let $\mathbf{G}$ be commutative. Let $\lambda \in \Omega$, let $(f_i; i \in K_\lambda) \in D_{R_\lambda} \cap (\text{Hom}(\mathbf{H}, \mathbf{G}))^{K_\lambda}$, $R_\lambda(f_i; i \in K_\lambda) = f$. Let $\mu \in \Omega$ and let $(y_j; j \in K_\mu) \in D_{q_\mu}$, $q_\mu(y_j; j \in K_\mu) = y$. Consider the $K_\lambda \times K_\mu$ -matrix $M = (f_i(y_j); i \in K_\lambda, j \in K_\mu)$ over $\mathbf{G}$. Then, for each $i \in K_\lambda$, $p_\lambda(f_i(y_j); j \in K_\mu) = f_i(y)$ because $f_i \in \text{Hom}(\mathbf{H}, \mathbf{G})$ for each $i \in K_\lambda$. Further, we have $p_\lambda(f_i(y); i \in K_\lambda) = f(y)$ and $p_\lambda(f_i(y_j); i \in K_\lambda) = f(y_j)$ for each $j \in K_\mu$ (by the definition of the direct product of partial algebras). Thus, M is $p_\lambda p_\mu$ -operational with $p_\lambda p_\mu(M) = f(y)$ and M^T is p_λ -operational. Since $\mathbf{G}$ is commutative, we have $p_\mu p_\lambda(M^T) = f(y)$. Thus, $p_\lambda(f(y_j); j \in K_\mu) = f(y)$, which yields $f \in \text{Hom}(\mathbf{H}, \mathbf{G})$. $\square$

COROLLARY 3.10. *Let $\mathbf{G}, \mathbf{H}$ be partial algebras of the same type. If $\mathbf{G}$ is both diagonal and commutative and $\mathbf{H}$ is idempotent, then the power $\mathbf{G}^{\mathbf{H}}$ is a partial subalgebra of the direct product $\mathbf{G}^{|\mathbf{H}|}$.*

Proof. Let $\mathbf{H} = \langle H, (q_\lambda; \lambda \in \Omega) \rangle$, $\mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega) \rangle$, $\mathbf{G}^{\mathbf{H}} = \langle \text{Hom}(\mathbf{H}, \mathbf{G}), (r_\lambda; \lambda \in \Omega) \rangle$, $\mathbf{G}^{\mathbf{H}} = \langle G^{\mathbf{H}}, (R_\lambda; \lambda \in \Omega) \rangle$ and let $\mathbf{G}$ be both diagonal and commutative and $\mathbf{H}$ be idempotent. Let $\lambda \in \Omega$, $(f_i; i \in K_\lambda) \in D_{R_\lambda}$, $R_\lambda(f_i; i \in K_\lambda) = f$ and let $(y_i; i \in K_\lambda) \in D_{q_\lambda}$, $q_\lambda(y_i; i \in K_\lambda) = y$. Consider the $K_\lambda \times K_\lambda$ -matrix $M = (f_{ij}(y_i); i \in K_\lambda, j \in K_\lambda)$ over G . As $p_\lambda(f_j(y_i); j \in K_\lambda) = f(y_i)$ for each $i \in K_\lambda$, M is p_λ -operational. By Theorem 3.9, $f \in \text{Hom}(\mathbf{H}, \mathbf{G})$, which yields $p_\lambda(f(y_i); i \in K_\lambda) = f(y)$. Thus, M is $p_\lambda p_\lambda$ -operational with $p_\lambda p_\lambda(M) = f(y)$. Since G is diagonal, M is p_λ -operational with $\Delta p_\lambda(M) = f(y)$. Hence, $p_\lambda(f_i(y_i); i \in K_\lambda) = f(y)$. The idempotency of $\mathbf{H}$ implies the uniqueness of f . Therefore, $r_\lambda(f_i; i \in K_\lambda) = f$.

Conversely, let $(f_i; i \in K_\lambda) \in D_{r_\lambda}$, $r_\lambda(f_i; i \in K_\lambda) = f$. Then $f \in \text{Hom}(\mathbf{H}, \mathbf{G})$ and, since $\mathbf{H}$ is idempotent, we have $p_\lambda(f_i(x); i \in K_\lambda) = f(x)$ for each $x \in K_\lambda$. Hence, $R_\lambda(f_i; i \in K_\lambda) = f$. $\square$

A partial algebra $\langle G, (p_\lambda; \lambda \in \Omega) \rangle$ is said to be a (total) *algebra* provided that the domain D_{p_λ} equals G for every $\lambda \in \Omega$. If $\mathbf{G}$ and $\mathbf{H}$ are algebras of the same type, then the power $\mathbf{G}^{\mathbf{H}}$ need not be an algebra even if $\mathbf{H}$ is idempotent — see the following example:

Example 3.11. Let $\mathbf{H} = (H, q)$, $\mathbf{G} = (G, p)$ be algebras of type (2) where $H = \{1, 2\}$, $G = \{a, b\}$ and the binary operations q and p are given by the tables

q	1	2
1	1	2
2	2	2

and

p	a	b
a	a	a
b	a	b

Clearly, $\mathbf{H}$ is idempotent. Let $f: H \rightarrow G$ be the map given by $f(1) = a$, $f(2) = a$, $g: H \rightarrow G$ be the map given by $g(1) = b$, $g(2) = b$ and $h: H \rightarrow G$ be the map given by $h(1) = b$, $h(2) = a$. Then $\text{Hom}(\mathbf{H}, \mathbf{G}) = \{f, g, h\}$. Let $\mathbf{G}^{\mathbf{H}} = (\text{Hom}(\mathbf{H}, \mathbf{G}), r)$. Then we have $r(g, h) \neq g$, $r(g, h) \neq h$ and $r(g, h) \neq f$, so $(g, h) \notin D_r$. Thus, $D_r \neq (\text{Hom}(\mathbf{H}, \mathbf{G}))^2$, which implies that $\mathbf{G}^{\mathbf{H}}$ is not a (total) algebra.

It is evident that, for algebras, the diagonality and strong diagonality coincide and, for commutative algebras, also the diagonality and weak diagonality coincide. By Corollary 3.10, given algebras $\mathbf{G}$ and $\mathbf{H}$ of the same type, $\mathbf{G}^{\mathbf{H}}$ is an algebra whenever $\mathbf{G}$ is commutative and diagonal and $\mathbf{H}$ is idempotent (because the direct product of a family of algebras of the same type is an algebra, and a partial subalgebra of an algebra is an algebra too). Thus, by Theorem 3.4, we get:

COROLLARY 3.12. *Let $\mathbf{G}, \mathbf{H}$ be algebras of the same type, $\mathbf{H}$ idempotent. If $\mathbf{G}$ is both diagonal and commutative, then so is the algebra $\mathbf{G}^{\mathbf{H}}$.*

Further, as a consequence of Theorem 3.11, we get:

THEOREM 3.13. *Let $\mathbf{G}, \mathbf{H}, \mathbf{K}$ be algebras of the same type. If $\mathbf{G}$ is both diagonal and commutative and $\mathbf{H}, \mathbf{K}$ are idempotent, then $(\mathbf{G}^{\mathbf{H}})^{\mathbf{K}}$ and $\mathbf{G}^{\mathbf{H} \times \mathbf{K}}$ are isomorphic algebras.*

Remark 3.14.

(a) Clearly, for algebras with just one operation, diagonality implies commutativity. The idempotent and diagonal algebras with just one finitary operation were studied by J. Plonka in [5]. He showed that these algebras are, up to isomorphisms, the algebras $(X_1 \times \cdots \times X_n, p)$ where n a positive integer, $X_1, \dots, X_n$ are sets and the operation $p: (X_1 \times \cdots \times X_n)^n \rightarrow X_1 \times \cdots \times X_n$ is given by $p((x_1^1, \dots, x_n^1), \dots, (x_1^n, \dots, x_n^n)) = (x_1^1, x_2^2, \dots, x_n^n)$.

(b) Commutative groupoids are usually called *medial* — see [3] where these groupoids are studied. Clearly, a groupoid $\mathbf{G} = (G, \cdot)$ is diagonal if and only if it is a semigroup with $xyz = xz$ whenever $x, y, z \in G$. Idempotent and diagonal groupoids are called *rectangular bands*.

(c) Recall that a groupoid with a neutral element is called a *loop* provided that it satisfies both the left and right unique division laws. By [4], a loop $\mathbf{G}$ is medial if and only if it has the so-called *Hamiltonian property* (i.e., each subloop of $\mathbf{G}$ is a block of a congruence on $\mathbf{G}$).

(d) It may easily be seen that a diagonal algebra is locally diagonal if and only if it is idempotent. It is also evident that, for diagonal algebras, local antidiagonality is equivalent to weak local antidiagonality. Since every diagonal algebra $\langle G, (p_\lambda; \lambda \in \Omega) \rangle$ has a p_λ -idempotent for any $\lambda \in \Omega$, namely, an arbitrary element $p_\lambda(x_i; i \in K_\lambda)$ with $x_i = x$ ($x \in G$ an element) for all $i \in K_\lambda$, a diagonal algebra $\langle G, (p_\lambda; \lambda \in \Omega) \rangle$ is locally antidiagonal if and only if $\text{card } G = 1$ or $\text{card } K_\lambda = 1$ for every $\lambda \in \Omega$. Therefore, no interesting results may be obtained concerning the stability of the “local” properties of algebras under their powers.

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**Department of Mathematics*

Chiang Mai University

Faculty of Science

Chiang Mai

THAILAND

E-mail: nitimachaisansuk@gmail.com

sorasak.l@cmu.ac.th

***Institute of Mathematics*

Brno University of Technology

CZ-616 69 Brno

CZECH REPUBLIC

E-mail: slapal@fme.vutbr.cz