COMPARISON OF GENERALIZED AND SIMPLE LAGUERRE FUNCTIONS FOR TIME-DELAY SYSTEM APPROXIMATION

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Abstract: The simple and generalized Laguerre functions for time invariant system approximations are discussed. The expressions needed for these approximations are presented. For the purpose of getting the best results, the choice of the free parameters is included for both the simple and the generalized Laguerre functions. In addition, the approximation of first and second order systems is shown and the results are discussed. According to this, the conditions in which the simple and the generalized Laguerre functions yield better results are presented.

Keywords: Laguerre functions, system approximation, time-delay system

1 INTRODUCTION

Approximation of systems with orthogonal functions is an often discussed topic in many research fields. The Laguerre functions represent one of the posibble choices for this purpose. There are many research papers written on the uses of the simple Laguerre functions. However, the generalized Laguerre functions appear in the literature less frequently.

The aim of this paper is to show the differences between the use of the simple and the generalized Laguerre functions for system approximations. The approximations are shown for different values of time delay. According to this, the results are evaluated, for which the quadratic error criterion is used. Furthermore, the conditions in which the generalized or simple Laguerre functions give better results are stated.

2 GENERALIZED AND SIMPLE LAGUERRE FUNCTIONS

The generalized Laguerre functions are given by the equation [1]:

$$\lambda_m^{(\alpha)}(\sigma;t) = \sqrt{\frac{\sigma m!}{\Gamma(m+\alpha+1)}} \sum_{n=0}^{m} (-1)^n \binom{m+\alpha}{m-n} \frac{(\sigma t)^{n+\frac{\alpha}{2}}}{n!} e^{-\sigma\frac{t}{2}}, \tag{1}$$

where m is the function's order $(m = 0, 1, ...)$ and t represents time $(t > 0)$. α and σ are the free parameters of the generalized Laguerre functions, α is the order of generalization and σ is the timescale. If $\alpha = 0$ is substituted to equation (1), the simple Laguerre functions are obtained.

The generalized Laguerre functions are orthonormal in $L_2(\mathbb{R}^+)$ [1]. This means, that any function $f(t) \in L^2$ can be expressed with it's Laguerre series. This is given by [1], [2]

$$f(t) = \sum_{m=0}^{\infty} C_m \lambda_m^{(\alpha)}(\sigma;t), \tag{2}$$

where

$$C_m = \langle f(t), \lambda_m^{(\alpha)}(\sigma;t)\rangle. \tag{3}$$

In the previous expression $\langle f(t), \lambda_m^{(\alpha)}(\sigma;t)\rangle$ represents the inner product of functions $f(t)$ and $\lambda_m^{(\alpha)}(\sigma;t)$. For numerical computation this equation will be used in the form of

$$f_{apr}(t) = \sum_{m=0}^{M} C_m \lambda_m^{(\alpha)}(\sigma;t), \tag{4}$$

where the upper limit of the sum was changed from infinity to M. Here M is an integer, which represents the highest order of the approximation.

3 CHOOSING THE VALUES OF THE FREE PARAMETERS

To be able to use equation (4), the values of the free parameters α and σ for the generalized Laguerre functions, and the value of σ for the simple Laguerre functions are need to be chosen correctly.

For the generalized Laguerre functions the equations [3]

$$\alpha = \frac{2m_0}{m_{-1}}\sqrt{\frac{m_{-1}\mu_1}{m_{-1}m_1 - m_0^2}}, \tag{5}$$

$$\sigma = 2\sqrt{\frac{m_{-1}\mu_1}{m_{-1}m_1 - m_0^2}}, \tag{6}$$

will be used to determine appropriate values for the free parameters. The variables m_i and μ_i represent signal measurements or "moments". They are defined as [3]:

$$m_i = \langle f(t), g_i(t)\rangle, \quad g_i(t) = t^i f(t), \tag{7}$$

$$\mu_i = \langle f'(t), \tilde{g}_i(t)\rangle, \quad \tilde{g}_i(t) = t^i f'(t). \tag{8}$$

In equations (7) and (8) $f(t)$ is the approximated function.

The signal measurements m_i and μ_i are positive numbers for $t \in (0,\infty)$. This is applicable for this paper, because the Laguerre functions are defined for $(t > 0)$.

The simple Laguerre functions have only one free parameter, the timescale σ. This means that other resources need to be considered to get appropriate values for this parameter. For this purpose the following expression will be used [4]:

$$\sigma = 2\sqrt{\frac{M_2}{M_1}}. \tag{9}$$

The variables M_1 and M_2 represent signal measurements similarly as m_i and μ_i in the previous equations. They are defined in literature [4] as:

$$M_1 = \frac{\int_0^\infty t f^2(t)dt}{\int_0^\infty f^2(t)dt}, \quad M_2 = \frac{\int_0^\infty t(f'(t))^2 dt}{\int_0^\infty f^2(t)dt}. \tag{10}$$

With the help of expressions (10) we can simplify equation (9) as

$$\sigma = 2\sqrt{\frac{M_2}{M_1}} = 2\sqrt{\frac{\mu_1}{m_1}}. \tag{11}$$

4 SYSTEM APPROXIMATIONS WITH TIME DELAY

4.1 Evaluation of the results

For the purpose of numerical evaluation of the results the quadratic criterion will be used in the form of [4]

$$J = \frac{\int_0^\infty \left(f(t) - f_{apr}(t)\right)^2 dt}{\int_0^\infty f^2(t)dt} = 1 - \frac{\sum_{m=0}^{M} C_m^2}{\sum_{m=0}^{\infty} C_m^2}, \tag{12}$$

where $f(t)$ is the original, $f_{apr}(t)$ is the approximated function. Expression (12) was further simplified using equations (2) and (4). With the help of the orthonormality property of the Laguerre functions the formula on the right side of the equation was obtained. This way only the C_m coefficients are needed for calculating the quadratic error.

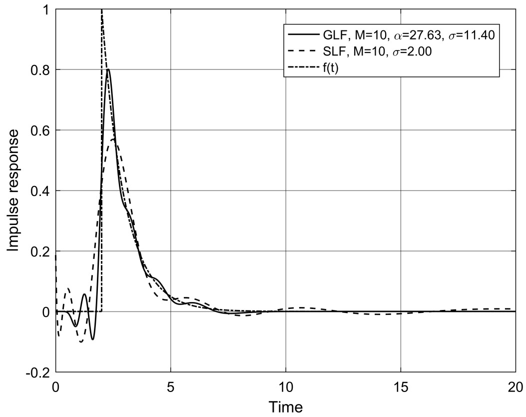
4.2 First order system

First, the approximation of first order systems will be shown. The transfer function of these systems is given by

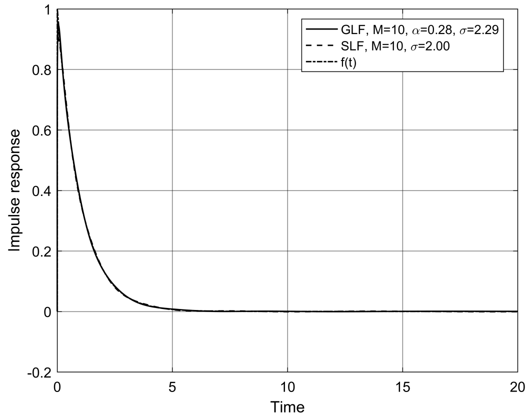
$$F(s) = \frac{K}{T_1 s + 1} e^{-Ts}, \tag{13}$$

where K is the system's gain, T_1 is the system's time constant and T is the time delay. The values of the free parameters for the generalized and simple Laguerre functions are numerically computed with the help of equations (5), (6) and (11) from section 3.

The first order system constants' values are chosen as follows: $K = 1$, $T_1 = 1$. The approximations are shown on figure 1 for different time delays. On the left figure it can be seen that the generalized Laguerrre functions give better results for order $M = 10$. On the right figure the time delay is negligibly small. In this case both approximations give better results, they overlap with the system's impulse response.



(a) Impulse response of first order system ($T = 2$)

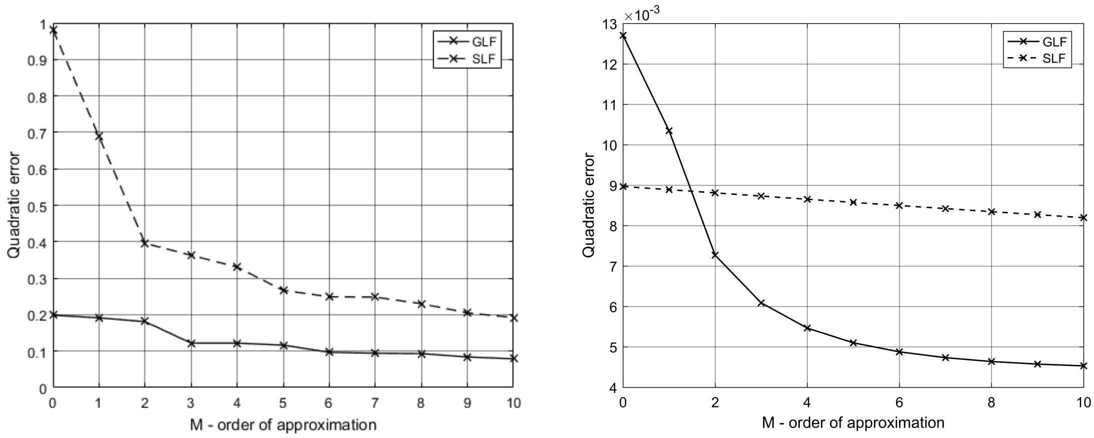


(b) Impulse response of first order system ($T = 0.005$)

Figure 1: Impulse response of first order system with different time delays

Figure 2 shows the values of the quadratic error (given by equation (12)) for different orders of approximation. It can be seen from both plots that the error is decreasing with increasing order. The left figure clearly shows that the generalized Laguerre functions give better results for all values.

However, from the right plot it is clear that the simple Laguerre functions are better for orders $M = 0$ and $M = 1$. This is caused by the time delay's small value (in comparison to the system's time constant).



(a) First order system with time delay $T = 2$

(b) First order system with time delay $T = 0.005$

Figure 2: Comparison of quadratic errors for different time delays - first order system

The fact that the simple Laguerre functions give better results for first order systems when the time delay approaches zero can be proven analytically. The impulse response of first order systems without time delay can be written as

$$g(t) = \frac{K}{T_1} e^{-\frac{t}{T_1}}. \tag{14}$$

The zeroth order simple Laguerre function is given by

$$\lambda_0(\sigma;t) = \sqrt{\sigma} e^{-\sigma\frac{t}{2}}. \tag{15}$$

The previously given impulse response for first order systems can be approximated only with this function. For the approximation the next equation is obtained with the help of (4) for $m = 0$:

$$f_{apr}(t) = C_0 \sqrt{\sigma} e^{-\sigma\frac{t}{2}}. \tag{16}$$

After comparing (14) and (16) it is clear, that the value of σ should be chosen as:

$$\sigma = \frac{2}{T_1}. \tag{17}$$

For C_0 the following expression is obtained with the help of equation (3):

$$C_0 = \frac{2K\sqrt{\sigma}}{\sigma T_1 + 2} = \frac{K\sqrt{\sigma}}{2}. \tag{18}$$

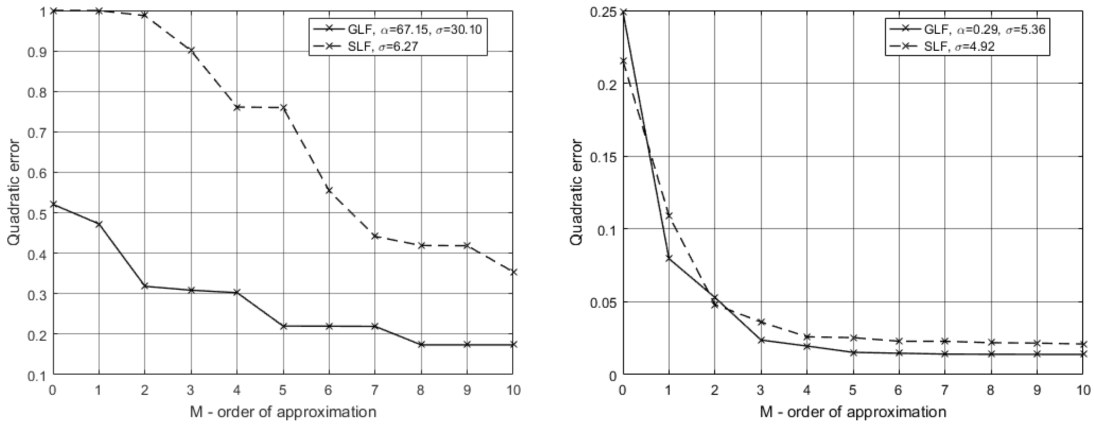
After using expresions (17) and (18) in (16) we get $f_{apr}(t) = g(t)$. This shows that any first order system without time delay can be replaced with a zeroth order simple Laguerre function. This explains the results obtained from figure 2b.

4.3 SECOND ORDER SYSTEM

The approximation of second order systems will be evaluated to show that the Laguerre functions can also be used for higher order systems. The transfer function of the approximated system is given by

$$F(s) = \frac{10s}{2s^2 + 6s + 3} e^{-Ts}. \tag{19}$$

As it can be seen from figure 3a, the generalized Laguerre functions yield once again better results for higher values of time delay. On the other hand, for really small delays, the approximation with simple Laguerre functions can be more accurate for lower orders. This can be seen from figure 3b.



(a) Second order system with time delay $T = 2$ **(b)** Second order system with time delay $T = 0.005$

Figure 3: Comparison of quadratic errors for different time delays - second order system

5 CONCLUSION

This paper compared the simple and the generalized Laguerre functions for time-delay system approximations. It was shown that in most cases the generalized Laguerre functions give better results. However, if certain conditions are met this must not be the case. These conditions are the system's time delay approaching zero and the low order of the approximation. This can be seen from the quadratic error in figures 2 and 3. It was also shown, that for first order systems the impulse response of the system can be replaced with just the zeroth order simple Laguerre function if the time delay is not present. This further explains why are the results given by the simple Laguerre functions better for negligibly small time delays.

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