BRNO UNIVERSITY OF TECHNOLOGY

VYSOKÉ UČENÍ TECHNICKÉ V BRNĚ

FACULTY OF MECHANICAL ENGINEERING

FAKULTA STROJNÍHO INŽENÝRSTVÍ

INSTITUTE OF MATHEMATICS

ÚSTAV MATEMATIKY

DISCRETE REGULAR VARIATION AND DIFFERENCE EQUATIONS

DISKRÉTNÍ REGULÁRNÍ VARIACE A DIFERENČNÍ ROVNICE

MASTER'S THESIS

DIPLOMOVÁ PRÁCE

AUTHOR Bc. Daniel Čaputa

AUTOR PRÁCE

SUPERVISOR prof. Mgr. Pavel Řehák, Ph.D.

VEDOUCÍ PRÁCE

BRNO 2022

BRNO UNIVERSITY OF TECHNOLOGY
FACULTY OF MECHANICAL ENGINEERING

Assignment Master's Thesis

Institut:	Institute of Mathematics
Student:	**Bc. Daniel Čaputa**
Degree programm:	Applied Sciences in Engineering
Branch:	Mathematical Engineering
Supervisor:	**prof. Mgr. Pavel Řehák, Ph.D.**
Academic year:	2021/22

As provided for by the Act No. 111/98 Coll. on higher education institutions and the BUT Study and Examination Regulations, the director of the Institute hereby assigns the following topic of Master's Thesis:

Discrete Regular Variation and Difference Equations

Brief Description:

The concept of discrete regular variation is practically as old as regularly varying functions, but its development is not generally close to, and sometimes far from, a simple imitation of arguments for regularly varying functions. Only recently it has been shown that the theory of Karamata sequences finds remarkable applications in the study of qualitative properties of difference equations.

Master's Thesis goals:

1. To gather and complete basic information on Karamata sequences and related objects that are spread in the literature (and are useful in difference equations).
2. To demonstrate usefulness of Karamata sequences in asymptotic theory of difference equations.

Recommended bibliography:

BINGHAM, N. H., GOLDIE, C. M., TEUGELS, J. L. Regular Variation, Encyclopedia of Mathematics and its Applications. Vol. 27, Cambridge Univ. Press, 1987.

BOJANIC, R., SENETA, E. A Unified Theory of Regularly Varying Sequences. Math. Z. 134, 91-106, 1973.

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ŘEHÁK, P. Asymptotic Formulae for Solutions of Linear Second-Order Difference Equations. J. Difference Equ. Appl. 22, 107-139, 2016.

Deadline for submission Master's Thesis is given by the Schedule of the Academic year 2021/22

In Brno,

L. S.

doc. Mgr. Petr Vašík, Ph.D. Director of the Institute	doc. Ing. Jaroslav Katolický, Ph.D. FME dean

Abstrakt

Táto práca sa zaoberá asymptotickou analýzou lineárnej diferenčnej rovnice druhého rádu s využitím teórie Karamatovských postupností. Sú zhromaždené vlastnosti regulárne sa meniacich postupností, ktoré sú užitočné v asymtotickej teórii. Pomocou transformácie diferenčnej rovnice na dynamickú rovnicu na vhodnú časovú škálu a dokázaním všeobecného výsledku pre dynamickú rovnicu je odvodená podmienka, ktorá zaručí regulárnu variáciu priestoru riešení diferenčnej rovnice. Kombináciou rôznych techník sú odvodené asymptotické formule a riešenia diferenčnej rovnice sú klasifikované do istých asymptotických tried.

Summary

This thesis deals with the asymptotic analysis of a linear second-order difference equation using the theory of Karamata sequences. Properties of regularly varying sequences that are useful in asymptotic theory are gathered. Using a transformation of a difference equation into the dynamic equation on the appropriate time scale and proving a general result for the dynamic equation, the condition that guarantees a regular variation of the solution space of a difference equation is obtained. By the combination of the variety of techniques, asymptotic formulae are established and the solutions of the difference equation are classified into certain asymptotic classes.

Klíčová slova

diferenčná rovnica, pozitívne riešenie, regulárne sa meniaca postupnosť, časová škála, dynamická rovnica, asymptotická formula

Keywords

difference equation, positive solution, regularly varying sequence, time scale, dynamic equation, asymptotic formula

ČAPUTA, D. *Discrete Regular Variation and Difference Equations.* Master's Thesis. Brno University of Technology, Faculty of Mechanical Engineering, 2022. 57s. Supervisor prof. Mgr. Pavel Řehák, Ph.D.

Rozšírený abstrakt

Táto práca sa zaoberá asymptotickou analýzou lineárnej diferenčnej rovnice druhého rádu

$$\Delta(r_k \Delta y_k) + p_k y_{k+1} = 0,$$

kde $r > 0$, s rôznymi znamienkovými podmienkami pre p. Táto rovnica je častým objektom záujmu, nakoľko sa často vyskytuje v rôznych aplikáciách.

V práci sú najprv popísané základné informácie o tejto diferenčnej rovnici a pridružené pojmy — oscilácia, Riccatiho rovnica a istá algebraická rovnica, ktorá sa vyskytuje pri analýze tejto rovnice. Sú definované asymptotické triedy riešení vzhľadom na ich správanie v nekonečne a je ukázaná základná klasifikácia riešení do týchto tried. Sú predstavené pojmy časová škála a kalkulus na časových škálach, čo je zovšeobecnením (nielen) klasického diferenciálneho kalkulu a diskrétneho kalkulu.

Ďalej sú v práci zhromaždené vlastnosti regulárne sa meniacich postupností, ktoré sú užitočné v asymptotickej teórii diferenčných rovníc. Jednou z najdôležitejších viet je Karamatova veta, ktorá ukáže, že pomaly sa meniace postupnosti násobené mocninnými postupnosťami a integrované, sa asymptoticky správajú ako konštanty.

Je diskutovaná otázka, za akých predpokladov sú všetky eventuálne pozitívne riešenia danej rovnice regulárne sa meniace postupnosti. Sú odvodené podmienky, na základe ktorých má všeobecnejšia dynamická rovnica na časovej škále regulárne sa meniace riešenia. Využitím tohto výsledku a transformácie diferenčnej rovnice na vhodnú časovú škálu získame nutnú a postačujúcu podmienku pre to, aby všetky eventuálne pozitívne riešenia boli regulárne sa meniace. Tým, že dokážeme, že všetky riešenia sú regulárne sa meniace, získame, skrz vlastnosti regulárne sa meniacich postupností, netriviálne informácie o týchto riešeniach.

V práci sú odvodené asymtotické formule pre riešenia danej diferenčnej rovnice, ktoré majú veľký význam skrz fakt, že všeobecne nie je táto rovnica analyticky riešiteľná. Riešenia sú klasifikované podľa ich správania v nekonečne do asymptotických tried. Sú diskutované rôzne poznámky o výsledkoch, o možnom ďalšom smerovaní a je naznačené, ako sa dá inak pozerať na výsledky v tejto práci skrz transformáciu na rekurentnú rovnicu.

Sú demonštrované rôzne techniky a postupy, ktoré sa používajú v (nielen) asymptotickej teórii. Transformáciami závislej, resp. nezávislej premennej prevedieme „zložitú“ diferenčnú rovnicu na (v kontexte aktuálneho skúmania rovnice) „jednoduchšiu“ diferenčnú alebo dynamickú rovnicu. Medzi použité transformácie napríklad patrí lineárna transformácia, princíp reciprocity alebo transformácia z jednej časovej škály na inú časovú škálu. Ďalej je predvedená Riccatiho technika, ktorá je v kontexte neoscilatívnych riešení veľmi silným nástrojom na analýzu kvalitatívnych vlastností rovníc.

Je ukázaná využiteľnosť teórie regulárne sa meniach postupností ich vlastností pre (nielen) asymptotickú teóriu.

Jedným z najpodstatnejších prínosov tejto práce je časť, kde sa dokazuje regulárna variácia všetkých eventuálne pozitívnych riešení diferenčnej rovnice. Sú odvodené podmienky, za ktorých má všeobecnejšia dynamická rovnica na diskrétnej časovej škále

$$x^{\Delta\Delta}(t) + p(t)y^{\sigma}(t) = 0,$$

kde $r > 0$ a p ľubovoľné, regulárne sa meniaci priestor riešení, čo je nový výsledok, ktorý zovšeobecňuje známe výsledky z hľadiska neprítomnosti znamienkovej podmienky $p < 0$ a z hľadiska všeobecnejšieho definičného oboru. Ďalším novým výsledkom je dokázanie pod akými predpokladmi má diferenčná rovnica priestor riešení tvorený z regulárne sa meniacich postupností. Tento fakt je dokázaný pomocou transformácie diferenčnej rovnice

na dynamickú rovnicu a dáva návod, ako sa dá vysporiadať s rôznymi situáciami, keď je (nielen) diferenčná rovnica v „zložitom“ tvare. Tento výsledok je nový z hľadiska $r \not\equiv 1$, resp. z hľadiska neprítomnosti znamienkovej podmienky $p < 0$.

Ďalším prínosom tejto práce je doplnenie asymptotických formulí pre $p > 0$ a ich unifikácia so známymi výsledkami. Okrem asymptotických formulí je pre prípad $p > 0$ doplnená klasifikácia riešení diferenčnej rovnice na základe ich asymptotického správania.

Prínosom je ukážka využiteľnosti teórie regulárne sa meniach postupností a taktiež spomínaná demonštrácia rôznych techník a postupov, ktoré sú často používané.

I hereby declare that I have written my Master's Thesis Discrete Regular Variation and Difference Equations independently under the supervision of prof. Mgr. Pavel Řehák, Ph.D. using literature listed in the bibliography section.

Bc. Daniel Čaputa

At this place I would like to express my huge gratitude to my supervisor, doc. Mgr. Pavel Řehák, Ph.D., for his willingness, consultations, professional guidance and valuable comments.

Bc. Daniel Čaputa

CONTENTS

INTRODUCTION

The linear second-order difference equation

$$\Delta(r_k \Delta y_k) + p_k y_{k+1} = 0$$

is a frequent object of interest. It arises out, for example, from a discretization of a differential equation, as an Euler-Lagrange equation of a certain quadratic functional or directly as a discrete model.

In general, this equation is not analytically solvable. Because of that, it is of interest to study this difference equation from the qualitative point of view. Asymptotic theory, which is treated in this thesis, forms a large part of qualitative theory.

The Karamata theory of regular variation has been proved to be a very useful tool in the asymptotic analysis of differential equations, for example, in [8]. The concept of discrete regular variation has also found applications in the study of qualitative properties of difference equations. One can mention a paper [10] or works [7] and [6], where difference equations in relation to theory of regularly varying sequences are studied using different techniques.

The aim of this thesis is to demonstrate the usefulness of the Karamata sequences in asymptotic theory of difference equations and to show how a combination of various techniques including regular variation enables us to make a precise description of the solutions of the difference equation.

In the first chapter we give basic information about the linear second-order difference equation and oscillation, we present a basic classification of nonoscillatory solutions and we introduce a concept of a time scale. In the next chapter we recall a concept of discrete regular variation on $\mathbb{Z}$ and a concept of regular variation on time scales. Chapter 3 is concerned with the existence of regularly varying solutions and related considerations. The fact that, under certain assumptions, the solution space of the difference equation consists of regularly varying sequences is proved via transformation of a difference equation into the dynamic equation on a suitable time scale. In the last chapter we establish asymptotic formulae, we discuss the classification of the solutions of the difference equation and lastly, we present several remarks concerning our results and provide some directions for a future research.

CHAPTER 1

SECOND-ORDER LINEAR DIFFERENCE EQUATIONS

We consider the linear second-order difference equation

$$\Delta(r_k \Delta y_k) + p_k y_{k+1} = 0 \tag{1.1}$$

on $[m,\infty)_{\mathbb{Z}}$ where r is a positive sequence and p is eventually of one sign. By Δ we mean the usual difference operator

$$\Delta y_k = y_{k+1} - y_k$$

by $\Delta^2 = \Delta \circ \Delta$. Further, we denote $[a,\infty)_{\mathbb{Z}} = \{a, a+1, \cdots\}$ and $[a,b]_{\mathbb{Z}} = \{a, a+1, \cdots, b\}$ where $a, b \in \mathbb{Z}$. Fundamentals about difference equations can be found in [5].

The equation (1.1) arises out in several contexts. It is the Euler-Lagrange equation of the quadratic functional

$$\sum_{k=m}^{n} \left(r_k (\Delta \xi_k)^2 - p_k \xi_{k+1}^2 \right),$$

thus it is the Jacobi equation of a general discrete functional. It can serve directly as a discrete model, e.g. the Fibonacci reccurence relation. It can be understood as the discretization of the linear second-order differential equation

$$(r(t)y'(t))' + p(t)y(t) = 0. \tag{1.2}$$

The discretization goes on as follows. Let us consider (1.2), where r, p are continuous on $[a,b]$. For small $h = (b-a)/n$, $n \in \mathbb{N}$, we have

$$y'(t) \approx \frac{y(t) - y(t-h)}{h}$$

and

$$(r(t)y'(t))' \approx \frac{1}{h}\left(\frac{r(t+h)(y(t+h) - y(t))}{h} - \frac{r(t)(y(t) - y(t-h))}{h} \right).$$

Let $t = a + kh$, where $k \in [0,n]_{\mathbb{Z}}$. If y is solution of (1.2) on $[a,b]$, then

$$r(a+(k+1)h)\left[y(a+(k+1)h) - y(a+kh)\right] - r(a+kh)\left[y(a+kh) - y(a+(k-1)h)\right] \\ + h^2 p(a+kh) y(a+kh) \approx 0.$$

Denote $y_k = y(a+(k-1)h)$, $r_k = r(a+kh)$ and $p_k = h^2 p(a+kh)$. Hence,

$$r_{k+1}(y_{k+2} - y_{k+1}) - r_k(y_{k+1} - y_k) + p_k y_{k+1} \approx 0,$$

and so

$$\Delta(r_k\Delta y_k) + p_k y_{k+1} \approx 0$$

for $k \in [0, n-2]_{\mathbb{Z}}$.

Oscillation

We will work only with nonoscillatory solutions that is solutions which are eventually of one sign. Since p is eventually monotone, all nonoscillatory solutions are eventually of one sign. By the discrete Sturm separation theorem, if one solution is nonoscillatory, then all solutions are nonoscillatory. Therefore we can talk about (non)oscillation of an equation. For $p_k < 0$ for large k, nonoscillation of the equation (1.1) follows from the Sturm comparison theorem. For $p_k > 0$ for large k, equation (1.1) can be both oscillatory and nonoscillatory. There exist criteria for determining, whether (1.1) is oscillatory or not. But, as a matter of fact, nonoscillation of this equation arises out as a by-product of our considerations.

Basic classification of the solution space

Without loss of generality we can examine only eventually positive solutions. Denote

$$\mathcal{S} = \{y : y \text{ is a positive solution of (1.1) for large } k\}.$$

Since y is eventually monotone, we can divide S into two disjoint classes

$$\mathcal{IS} = \{y \in \mathcal{S} : \Delta y_k > 0 \text{ for large } k\}$$

and

$$\mathcal{DS} = \{y \in \mathcal{S} : \Delta y_k < 0 \text{ for large } k\}.$$

Denote

$$\mathcal{IS}_\infty = \{y \in \mathcal{IS} : \lim_{k\to\infty} y_k = \infty\} \text{ and } \mathcal{IS}_B = \{y \in \mathcal{IS} : \lim_{k\to\infty} y_k \in (0,\infty)\}$$

and

$$\mathcal{DS}_0 = \{y \in \mathcal{DS} : \lim_{k\to\infty} y_k = 0\} \text{ and } \mathcal{DS}_B = \{y \in \mathcal{DS} : \lim_{k\to\infty} y_k = \vartheta \in (0,\infty)\}.$$

We can further divide these classes into subclasses based on asymptotic behaviour of the quasidifference $r_k\Delta y_k$. Denote

$$\mathcal{IS}_{u,v} = \{y \in \mathcal{IS} : \lim_{k\to\infty} y_k = u, \lim_{k\to\infty} r_k\Delta y_k = v\}$$

and

$$\mathcal{DS}_{u,v} = \{y \in \mathcal{DS} : \lim_{k\to\infty} y_k = u, \lim_{k\to\infty} r_k\Delta y_k = v\},$$

where we will write $u = B$ or $v = B$ when the value of the limit is a real nonzero number.

Further, denote

$$\mathcal{S}_{\mathcal{SV}} = \mathcal{S} \cap \mathcal{SV}$$

and

$$\mathcal{S}_{\mathcal{RV}}(\vartheta) = \mathcal{S} \cap \mathcal{RV}(\vartheta).$$

Basic classification when $p_k > 0$ for large k

Let $p_k > 0$ for large k. Then quasiderivative $r\Delta y$ eventually decreases. If $y \in \mathcal{IS}$, then $r\Delta y$ is positive and if $y \in \mathcal{DS}$, then $r\Delta y$ is negative. Therefore, only following subclasses make sense:

$$\mathcal{IS}_{\infty,B},\ \mathcal{IS}_{\infty,0},\ \mathcal{IS}_{B,B},\ \mathcal{IS}_{B,0},\ \mathcal{DS}_{B,B},\ \mathcal{DS}_{B,\infty},\ \mathcal{DS}_{0,B},\ \mathcal{DS}_{0,\infty}.$$

Lemma 1.1. *Let* $\sum_{j=k}^{\infty} 1/r_j = \infty$. *Then* $\mathcal{S} = \mathcal{IS}_{B,0} \cup \mathcal{IS}_{\infty,0} \cup \mathcal{IS}_{\infty,B}$.

Proof. Take $y \in \mathcal{DS}$. Then there exists a constant $-M \in (-\infty, 0)$ such that

$$r_k \Delta y_k \le -M$$

and by dividing by r_k and summing from n to $k-1$

$$y_k \le y_n - M \sum_{j=n}^{k-1} \frac{1}{r_j} \to -\infty \text{ as } k \to \infty,$$

that contradicts $y_k > 0$.

Take $y \in \mathcal{IS}_{B,B}$. Then $r_k \Delta y_k$ has the limit $M \in (0, \infty)$ and

$$r_k \Delta y_k \ge M$$

and by dividing by r_k and summing from n to $k-1$

$$y_k \ge y_n + M \sum_{j=n}^{k-1} \frac{1}{r_j} \to \infty \text{ as } k \to \infty,$$

that contradicts the finitness of the limit M. □

Lemma 1.2. *Let* $\sum_{j=k}^{\infty} p_j = \infty$. *Then* $\mathcal{S} = \mathcal{DS}_{0,B} \cup \mathcal{DS}_{0,\infty} \cup \mathcal{DS}_{B,\infty}$.

Proof. Take $y \in \mathcal{IS}$. Then, from (1.1) by summation from n to $k-1$,

$$0 < r_k \Delta y_k = r_n \Delta y_n - \sum_{j=n}^{k-1} p_j y_{j+1} \le r_n \Delta y_n - y_n \sum_{j=n}^{k-1} p_j \to -\infty \text{ as } k \to \infty,$$

that contradicts $r_k \Delta y_k > 0$.

Taky $y \in \mathcal{DS}_{B,B}$. It holds that

$$r_k \Delta y_k = r_n \Delta y_n - \sum_{j=n}^{k-1} p_j y_{j+1} \to -\infty \text{ as } k \to \infty,$$

that contradicts the finitness of the limit $\lim\limits_{k\to\infty} r_k \Delta y_k$. □

Corollary 1.3. *Let* $\sum_{j=k}^{\infty} p_j = \infty = \sum_{j=k}^{\infty} 1/r_j$. *Then* $\mathcal{S} = \emptyset$, *i.e. the equation* (1.1) *is oscillatory.*

Basic classification when $p_k < 0$ for large k

Let $p < 0$ for large k. Then quasiderivative $r\Delta y$ eventually increases. If $y \in \mathcal{IS}$, then $r\Delta y$ is positive and if $y \in \mathcal{DS}$, then $r\Delta y$ is negative. Therefore only following subclasses make sense:

$$\mathcal{IS}_{\infty,B},\ \mathcal{IS}_{\infty,\infty},\ \mathcal{IS}_{B,B},\ \mathcal{IS}_{B,\infty},\ \mathcal{DS}_{B,B},\ \mathcal{DS}_{B,0},\ \mathcal{DS}_{0,B},\ \mathcal{DS}_{0,0}.$$

Riccati equation

The transformation of (1.1) into the Riccati equation will be one of the tools frequently used in our proofs. It goes as follows.

Lemma 1.4. *Let y be a nonoscillatory solution of* (1.1). *Set $w_k = (r_k \Delta y_k)/y_k$. Then w satisfies Riccati equation*

$$\Delta w_k + p_k + \frac{w_k^2}{r_k + w_k} = 0 \tag{1.3}$$

and $r_k + w_k > 0$ for large k.

Associated algebraic equation and its properties

Analysis of regular variation of the solution space of (1.1) will lead us to the algebraic equation

$$\vartheta^2 - (1-\gamma)\vartheta + A = 0, \tag{1.4}$$

where $A < \left(\frac{1-\gamma}{2}\right)^2$, i.e. (1.4) has two distinct real roots.

As it will turn out, the solution space of (1.1) will consist of regularly varying sequences of indices corresponding to the roots of this equation. Hence, it is of interest to analyse properties of this roots.

Lemma 1.5. *The next observations about the roots $\vartheta_1 < \vartheta_2$ of* (1.4) *hold:*

i) $\vartheta_1 + \vartheta_2 = 1 - \gamma$.

ii) Let $\gamma < 1$. Then

- $0 < \vartheta_1 < \dfrac{1-\gamma}{2} < \vartheta_2 < 1 - \gamma$ *provided that* $A > 0$,
- $\vartheta_1 < 0 < 1 - \gamma < \vartheta_2$ *provided that* $A < 0$,
- $\vartheta_1 = 0, \vartheta_2 = 1 - \gamma$ *provided that* $A = 0$.

iii) Let $\gamma > 1$. Then

- $\vartheta_1 < \dfrac{1-\gamma}{2} < \vartheta_2 < 1 - \gamma < 0$ *provided that* $A > 0$,
- $\vartheta_1 < 1 - \gamma < 0 < \vartheta_2$ *provided that* $A < 0$,
- $\vartheta_1 = 1 - \gamma, \vartheta_2 = 0$ *provided that* $A = 0$.

iv) $1 - \gamma - 2\vartheta_1 = \sqrt{(1-\gamma)^2 - 4A} > 0$.

Time scales and dynamic equations on time scales

In this section, we will introduce the concept of a time scale. Information concerning the time scales are drawn from [3]. One of the most important features of time scale calculus is the unification of differential calculus and calculus of finite differences. It allows us to study properties of a dynamic equation on a general set — time scale that includes both real numbers $\mathbb{R}$, integer numbers $\mathbb{Z}$ or any other nonempty, closed subset of real numbers. Then, by choosing a specific time scale, for example $\mathbb{R}$ or $\mathbb{Z}$, the result for a general dynamic equation yields a result for a differential equation or a difference equation respectively.

Moreover, the time scale calculus can explain the similarities or the differences between the results in continuous and discrete settings.

Further, the time scale calculus allows more variability when discretizing in the sense that it allows the step size to be varied in time.

Another example of the usefulness of time scale calculus is the idea of transformation of a "difficult" problem on one time scale into the "simpler" problem on another time scale, which in fact a major use case of the time scale calculus in our theory.

Definition 1.6. A nonempty, closed subset $\mathbb{T}$ of the real numbers is called a *time scale*. On time scale $\mathbb{T}$ we define:

- *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}, \sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$,
- *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}, \rho(t) = \sup\{s \in \mathbb{T} : s < t\}$,
- *graininess function* $\mu : \mathbb{T} \to [0, \infty), \mu(t) = \sigma(t) - t$,
- the set $\mathbb{T}^\kappa$ as follows. If for maximum m of $\mathbb{T}$ it holds that $\rho(m) < m$ (m is left scattered), then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$. Else $\mathbb{T}^\kappa = \mathbb{T}$,
- *righ-dense point* $t \in \mathbb{T}$ such that $t = \sigma(t)$,
- *left-dense point* $t \in \mathbb{T}$ such that $t = \rho(t)$,
- the time scale interval $\mathbb{BC}_\mathbb{T}[a, b] = [a, b] \cap \mathbb{T}$ and $\mathbb{BC}_\mathbb{T}[a, \infty) = [a, \infty) \cap \mathbb{T}$,
- by f^σ we mean $f^\sigma = f \circ \sigma$.

Definition 1.7. Let $f : \mathbb{T} \to \mathbb{R}$ be a function. For $t \in \mathbb{T}^\kappa$ we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given $\varepsilon > 0$, there is a neighborhood U of t such that

$$\left|f(\sigma(t)) - f(t) - f^\Delta(t)(\sigma(t) - t)\right| \le \varepsilon \left|\sigma(t) - t\right|$$

for all $s \in U$. We call $f^\Delta(t)$ the *delta (or Hilger) derivative.*

Theorem 1.8. *Let f, g be delta differentiable at $t \in \mathbb{T}^\kappa$. Then*

a) $(f + g)^\Delta(t) = f^\Delta(t) + g^\Delta(t)$

b) $(\alpha f)^\Delta(t) = \alpha f^\Delta(t)$ *for every* $\alpha \in \mathbb{R}$

c) $(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t))$

d) $\left(\frac{1}{f}\right)^\Delta(t) = \frac{-f^\Delta(t)}{f(t)f(\sigma(t))}$, *provided by* $f(t)f(\sigma(t)) \neq 0$

e) $\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g(\sigma(t))}$, *provided by* $g(t)g(\sigma(t)) \neq 0$

Definition 1.9. A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided its right-sided limits exist at all right-dense points in $\mathbb{T}$ and its left-sided limits exist at all left-dense points in $\mathbb{T}$.

Denote the set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ as C_{rd} and the set of functions $f : \mathbb{T} \to \mathbb{R}$ that are delta differentiable and whose derivative is rd-continuous as C^1_{rd}.

Next, the definition of integral on time scale follows. A time scale integral can be made under more general setting, but the presented version is sufficient for our purposes.

Definition 1.10. A function $F : \mathbb{T} \to \mathbb{R}$ is called an *antiderivative* of $f : \mathbb{T} \to \mathbb{R}$ provided

$$F^{\Delta}(t) = f(t) \text{ for all } t \in \mathbb{T}^{\kappa}.$$

Definition 1.11. Define the *delta integral* on a time scale $\mathbb{T}$ for $f \in C_{rd}$ by

$$\int_r^s f(t)\,\Delta t = F(s) - F(r) \text{ for all } r, s \in \mathbb{T}.$$

Theorem 1.12. *Let $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$ and $f, g \in C_{rd}$. Then*

a) $\int_a^b f(s) + g(s)\,\Delta s = \int_a^b f(s)\,\Delta s + \int_a^b g(s)\,\Delta s$

b) $\int_a^b \alpha f(s)\,\Delta s = \alpha \int_a^b f(s)\,\Delta s$

c) $\int_a^b f(s)\,\Delta s = \int_a^c f(s)\,\Delta s + \int_c^b f(s)\,\Delta s$

d) $\int_a^b f(s) g^{\Delta}(s)\,\Delta s = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(s) g(\sigma(s))\,\Delta s$

Theorem 1.13. *Let $f \in C_{rd}$ and $t \in \mathbb{T}^{\kappa}$. Then*

$$\int_t^{\sigma(t)} f(s)\,\Delta s = \mu(t) f(t)$$

Example 1.14. Let $\mathbb{T} = \mathbb{R}$. Then $\sigma(t) = t$, $\mu(t) \equiv 0$,

$$f^{\Delta}(t) = \lim_{t \to s} \frac{f(t) - f(s)}{t - s} = f'(t). \tag{1.5}$$

and

$$\int_a^b f(s)\,\Delta s = \int_a^b f(s)\,ds \tag{1.6}$$

Example 1.15. Let $\mathbb{T} = \mathbb{Z}$. Then $\sigma(t) = t + 1$, $\mu(t) \equiv 1$,

$$f^{\Delta}(t) = f(t+1) - f(t) = \Delta f_t \tag{1.7}$$

and

$$\int_a^b f(s)\,\Delta s = \sum_{j=a}^{b-1} f_j \tag{1.8}$$

Definition 1.16. A time scale $\mathbb{T}$ is said to be *discrete* if for every $t \in \mathbb{T}$ $\rho(t) < t < \sigma(t)$.

Theorem 1.17. *Let $\mathbb{T}$ be a discrete time scale. Then for every $f : \mathbb{T} \to \mathbb{R}$*

$$f^{\Delta}(t) = \frac{f^{\sigma}(t) - f(t)}{\mu(t)}. \tag{1.9}$$

Theorem 1.18. *Let $\mathbb{T}$ be a discrete time scale, $a, b \in \mathbb{T}$, $a < b$. Then*

$$\int_a^b f(s)\,\Delta s = \sum_{s \in [a,b)_{\mathbb{T}}} \mu(s) f(s).$$

Definition 1.19. Let $\mathbb{T}$ be a time scale and p function satisfying $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^\kappa$. A solution to the initial value problem

$$y^\Delta(t) = p(t)y(t), \quad y(t_0) = 1$$

on $\mathbb{T}$ is called a *generalized exponential function* $e_p(t, t_0)$.

Consider the dynamic self-adjoint equation of a second order

$$(r(t)x^\Delta(t))^\Delta + p(t)x^\sigma(t) = 0 \tag{1.10}$$

on time scale $\mathbb{T}$, where $r > 0$ and $1/r, p \in C_{rd}$. A solution x of (1.10) is called *nonoscillative*, if $x(t)x^\sigma(t) > 0$ for all $t \in \mathbb{T}$.

Theorem 1.20. *Let x be a nonoscillatory solution of* (1.10)*. Then* $z(t) = \dfrac{r(t)x^\Delta(t)}{x(t)}$ *satisfies the dynamic Riccati equation*

$$z^\Delta(t) + p(t) + \frac{z^2(t)}{r(t) + \mu(t)z(t)} = 0 \tag{1.11}$$

with $r(t) + \mu(t)z(t) > 0$ for all $t \in \mathbb{T}^\kappa$.

CHAPTER 2

DISCRETE KARAMATA THEORY

Before proceeding to discrete Karamata theory of regularly varying sequences, let us present asymptotic notation.

Definition 2.1. Sequences y and x are said to be *asymptotically equivalent*, denoted by $x \sim y$, if

$$\lim_{k\to\infty} \frac{y_k}{x_k} = 1. \tag{2.1}$$

By

$$x_k = o(y_k) \text{ as } k \to \infty,$$

we mean

$$\lim_{k\to\infty} \frac{x_k}{y_k} = 0.$$

By

$$x_k \sum_{j=m}^{k-1} (1 + o(1)) y_j \text{ as } k \to \infty$$

we mean that there exists sequence ε such that $\lim_{k\to\infty} \varepsilon_k = 0$ and

$$x_k = \sum_{j=m}^{k-1} (1 + \varepsilon_j) y_j. \tag{2.2}$$

2.1 Karamata theory on $\mathbb{Z}$

In this section we will deal with basics of Karamata theory of regular variation on $\mathbb{Z}$. In particular, we will provide several theorems about Karamata sequences that will be useful in analysing asymptotic properties of the solutions of difference equations. Information in this section are drawn from [2], [4] and [1].

Definition 2.2. A positive sequence $y = \{y_k\}_{k=m}^{\infty}$, is said to be *regularly varying of index* ϑ, if there is a positive sequence α satisfying

$$\lim_{k\to\infty} \frac{y_k}{\alpha_k} = C, \ \lim_{k\to\infty} \frac{k\Delta\alpha_k}{\alpha_k} = \vartheta, \tag{2.3}$$

where $C > 0$ is a constant. If $\vartheta = 0$, y is said to be *slowly varying sequence.*

Definition 2.3. A positive sequence y, is said to be *normalized regularly varying of index* ϑ, if it satisfies

$$\lim_{k\to\infty} \frac{k\Delta y_k}{y_k} = \vartheta. \tag{2.4}$$

If $\vartheta = 0$, y is said to be normalized slowly varying sequence.

Definition 2.4. Denote $(N)\mathcal{RV}(\vartheta)$ a set of all (normalized) regularly varying sequences of index ϑ and $(N)\mathcal{SV}$ a set of all (normalized) slowly varying sequences.

Example 2.5. It holds that $\ln^\gamma k \in \mathcal{NSV}$ and $k^\vartheta \ln k \in \mathcal{NRV}(\vartheta)$ for every $\vartheta, \gamma \in \mathbb{R}$ and $1 + (-1^k)/k \in \mathcal{SV}/\mathcal{NSV}$. A sequence 2^k is not regularly varying.

Definition 2.6. A "falling factorial power" sequence $k^{(\vartheta)}$ is defined as,

$$k^{(\vartheta)} = \frac{\Gamma(k+1)}{\Gamma(k-\vartheta+1)},$$

where Γ is usual Gamma function.

Remark 2.7. It holds that $\Gamma(k+1) = k\Gamma(k)$ and $\Gamma(k+\vartheta) \sim \Gamma(k)k^\vartheta$. Hence,

$$k^\vartheta \sim k^{(\vartheta)} \text{ as } k \to \infty$$

and $k^{(\vartheta)} \in \mathcal{NRV}(\vartheta)$.

Theorem 2.8. *The following statements are equivalent:*

a) $y \in \mathcal{RV}(\vartheta)$,

b)

$$y_k = k^\vartheta L_k, \tag{2.5}$$

where $L \in \mathcal{NSV}$,

c)

$$y_k = k^\vartheta \varphi_k \exp\left\{\sum_{j=m}^{k-1} \frac{\psi_j}{j}\right\}, \tag{2.6}$$

$k \geq m$, *where* $\varphi_k \to C \in (0, \infty)$ *and* $\psi_k \to 0$ *as* $k \to \infty$,

d)

$$y_k = \varphi_k \exp\left\{\sum_{j=m}^{k-1} \frac{\delta_j}{j}\right\}, \tag{2.7}$$

$k \geq m$, *where* $\varphi_k \to C \in (0, \infty)$ *and* $\delta_k \to \vartheta$ *as* $k \to \infty$,

e)

$$y_k = k^\vartheta \varphi_k \prod_{j=m}^{k-1} \left(1 + \frac{\psi_j}{j}\right), \tag{2.8}$$

$k \geq m$, *where* $\varphi_k \to C \in (0, \infty)$ *and* $\psi_k \to 0$ *as* $k \to \infty$, *f)*

$$y_k = \varphi_k \prod_{j=m}^{k-1} \left(1 + \frac{\delta_j}{j}\right), \tag{2.9}$$

$k \geq m$, where $\varphi_k \to C \in (0, \infty)$ and $\delta_k \to \vartheta$ as $k \to \infty$.

g)

$$y_k = k^{(\vartheta)} L_k, \tag{2.10}$$

where $L \in \mathcal{SV}$ and $k^{(\vartheta)}$ is "falling factorial power" sequence defined in Definition 2.6.

Theorem 2.9. *A sequence y_k belongs to $\mathcal{NRV}(\vartheta)$ if and only if*

$$y_k = k^{\vartheta} L_k, \tag{2.11}$$

or

$$y_k = k^{(\vartheta)} L_k, \tag{2.12}$$

where $L \in \mathcal{NSV}$. Moreover, $y \in \mathcal{NRV}(\vartheta)$ if and only if y satisfies any of (2.6), (2.7), (2.8) *or* (2.9) *with $\varphi_k \equiv C$, where C is a positive constant.*

In view of Theorem 2.8 b), we can denote a "slowly varying component" $L^{<y>}$ of a regularly varying sequence y as

$$L_k^{<y>} = \frac{y_k}{k^{\vartheta}}. \tag{2.13}$$

Theorem 2.10. *Let $x \in \mathcal{RV}(\vartheta_1)$, $y \in \mathcal{RV}(\vartheta_2)$. Then $x + y \in \mathcal{RV}(\max\{\vartheta_1, \vartheta_2\})$, $xy \in \mathcal{RV}(\vartheta_1 + \vartheta_2)$ and $1/x \in \mathcal{RV}(-\vartheta_1)$. The same holds if RV is replaced by NRV.*

Theorem 2.11. *Let $y \in \mathcal{RV}(\vartheta)$. Then $y_{k+1}/y_k \sim 1$ as $k \to \infty$.*

Theorem 2.12. *Let $y \in \mathcal{RV}(\vartheta)$. Then $\Delta \ln y_k \sim \Delta y_k / y_k$ as $k \to \infty$.*

The next theorem plays a very important role in our theory.

Theorem 2.13 (Karamata type theorem)**.** *Let $L \in \mathcal{SV}$. Then*

$$\sum_{j=m}^{k-1} j^{\vartheta} L_j \sim \frac{1}{\vartheta + 1} k^{\vartheta+1} L_k \tag{2.14}$$

as $k \to \infty$, provided that $\vartheta > -1$, and

$$\sum_{j=k}^{\infty} j^{\vartheta} L_j \sim \frac{1}{-\vartheta - 1} k^{\vartheta+1} L_k \tag{2.15}$$

as $k \to \infty$, provided that $\vartheta < -1$.

2.2 Karamata theory on $\mathbb{T}$

In this section, we will outline Karamata theory on time scales and provide several definitions and theorems that will be used to get condition that guarantees the regular variation of the solution space of (1.10). We omit many details as Kamarata theory on time scales is not the main subject of this thesis. Information in this section are drawn from [12].

Definition 2.14. Let $\mu(t)/t \to 0$ as $t \to \infty$. A $f \in C^1_{rd}$ is said to be *normalized regularly varying of index* $\vartheta \in \mathbb{R}$, if

$$\lim_{t\to\infty} \frac{tf^{\Delta}(t)}{f(t)} = \vartheta$$

and we write $f \in \mathcal{NRV}_{\mathbb{T}}(\vartheta)$. If $\vartheta = 0$, then f is said to be *normalized slowly varying function* and we write $f \in \mathrm{NSV}_{\mathbb{T}}$.

Theorem 2.15. *Let $\mu(t)/t \to 0$ as $t \to \infty$. Then $f \in C^1_{rd}$ belongs to $\mathcal{NRV}_{\mathbb{T}}(\vartheta)$ if and only if*

$$f(t) = e_{\eta}(t, a),$$

where $e_{\eta}(t, a)$ is a generalized exponential function and $t\eta(t) \to \vartheta$ as $t \to \infty$.

Theorem 2.16. *Function f belongs to $\mathcal{NRV}_{\mathbb{T}}(\vartheta)$ if and only if*

$$f(t) = t^{\vartheta} L(t),$$

where $L \in \mathrm{NSV}_{\mathbb{T}}$.

Theorem 2.17. *Let $\mu(t)/t \to 0$ as $t \to \infty$ and $f \in \mathcal{NRV}_{\mathbb{T}}(\vartheta)$. Then $f(t) \sim f^{\sigma}(t)$ as $t \to \infty$.*

Theorem 2.18 (Karamata integration theorem on time scales)**.** *Let $\mu(t)/t \to 0$ as $t \to \infty$ and $L \in \mathrm{NSV}_{\mathbb{T}}$. Then*

$$\int_t^{\infty} s^{\vartheta} L(s)\,\Delta s \sim \frac{1}{-\vartheta - 1} t^{\vartheta+1} L(t)$$

as $t \to \infty$, provided by $\vartheta < -1$, and

$$\int_a^{t} s^{\vartheta} L(s)\,\Delta s \sim \frac{1}{\vartheta + 1} t^{\vartheta+1} L(t)$$

as $t \to \infty$, provided by $\vartheta > -1$.

CHAPTER 3

REGULAR VARIATION OF THE SOLUTION SPACE

We want to establish the condition, under which the solution space of (1.1) consists of regularly varying sequences. In order to have the difference equation (1.1) in a sufficiently general setting, we use the idea of transformation of a difference equation into a dynamic equation on a time scale and prove that this dynamic equation has, under certain assumptions, the solution space formed by regularly varying functions (on time scales). The regular variation of all solutions of (1.1) will follow from the inverse transformation and properties of regularly varying sequences. This chapter contains new results and improvements over existing results.

Consider the equation

$$x^{\Delta\Delta} + p(t)x^{\sigma} = 0 \tag{3.1}$$

on a discrete time scale $\mathbb{T}$, where $p(t)$ is an arbitrary function. We want to show that, under certain assumptions, this equation has only regularly varying solutions. Let us emphatize that there is no sign condition on p.

The regular variation of solutions of

$$\Delta^2 y_k + p_k y_{k+1} = 0,$$

where p is an arbitrary sequence, is discussed in [9]. The next theorem generalizes this result to the dynamic equation (3.1).

Theorem 3.1. *Let $\mathbb{T}$ be a discrete time scale satisfying $\mu(t)/t \to 0$ and $\mu(t) \sim \mu^{\sigma}(t)$ as $t \to \infty$. Then there exists a fundamental system of solutions of* (3.1) *$y \in \mathcal{NRV}_{\mathbb{T}}(\vartheta_1), x \in \mathcal{NRV}_{\mathbb{T}}(\vartheta_2)$ if and only if*

$$t \int_t^{\infty} p(s)\, \Delta s \to C \in (-\infty, 1/4) \text{ for } t \to \infty,$$

where $\vartheta_1 < \vartheta_2$ are the real roots of the algebraic equation

$$\vartheta^2 - \vartheta + C = 0. \tag{3.2}$$

Moreover, every eventually positive solution z of (3.1) *is normalized regularly varying, with $z \in \mathcal{NRV}_{\mathbb{T}}(\vartheta_1) \cup \mathcal{NRV}_{\mathbb{T}}(\vartheta_2)$.*

Proof. „$\Rightarrow$": Let $y \in \mathcal{NRV}_{\mathbb{T}}(\vartheta_1)$ be a solution of the equation (3.1) and set $w = y^\Delta/y$. Then w satisfies the dynamic Riccati equation

$$w^\Delta(t) + p(t) + \frac{w^2(t)}{1 + \mu(t)w(t)} = 0 \tag{3.3}$$

and in view of $tw(t) \to \vartheta_1$ as $t \to \infty$ and $\mu(t)w(t) \to 0$ as $t \to \infty$, we get that $w(t) \to 0$ as $t \to \infty$ and $1 + \mu(t)w(t) > 0$.

We want to integrate the equation (3.3) from t to ∞. We need to show that the integrals exist. It holds that $1 - \mu(t)w(t) \geq 1 - \varepsilon$ for $\varepsilon \in (0, 1)$ and $|w(t)| \leq M/t$ for sufficiently large t. Also, in view of $\dfrac{t^\sigma}{t} = \dfrac{t + \mu(t)}{t} \sim 1$ as $t \to \infty$ we have $\dfrac{t^\sigma}{t} \leq N$ for sufficiently large t. Further, since $\left(\dfrac{-1}{t}\right)^\Delta = \dfrac{1}{t\sigma(t)}$, we obtain

$$\int_a^\infty \frac{1}{s^2}\,\Delta s = \int_a^\infty \frac{1}{s\sigma(s)}\frac{\sigma(s)}{s}\,\Delta s \leq N\left[\frac{-1}{t}\right]_a^\infty < \infty.$$

Then

$$\int_a^\infty \frac{w^2(s)}{1 + \mu(s)w(s)}\,\Delta s \leq \frac{1}{1-\varepsilon}\int_a^\infty w^2(s)\,\Delta s \leq \frac{M^2}{1-\varepsilon}\int_a^\infty \frac{1}{s^2}\,\Delta s < \infty.$$

Integrating (3.3) from t to ∞ and multiplying by t we get

$$t\int_t^\infty p(s)\,\Delta s = tw(t) - t\int_t^\infty \frac{w^2(s)}{1 + \mu(s)w(s)}\,\Delta s. \tag{3.4}$$

Time scale analogue of the L'Hospital rule yields

$$\lim_{t\to\infty} \frac{\int_t^\infty \frac{w^2(s)}{1+\mu(s)w(s)}\,\Delta s}{\frac{1}{t}} = \lim_{t\to\infty} \frac{\frac{-w^2(t)}{1+\mu(s)w(t)}}{-t\sigma(t)} = \lim_{t\to\infty} \frac{t^2w^2(t)}{1 + \mu(t)w(t)} = \vartheta_1^2.$$

Hence,

$$\lim_{t\to\infty} t\int_t^\infty p(s)\,\Delta s = \lim_{t\to\infty} tw(t) - \lim_{t\to\infty}\int_t^\infty \frac{w^2(s)}{1 + \mu(s)w(s)}\,\Delta s = \vartheta_1 - \vartheta_1^2 = C. \tag{3.5}$$

"$\Leftarrow$": Set $\psi(t) = t\int_t^\infty p(s)\,\Delta s - C$. We search for a solution of (3.1) in the form

$$y(t) = e_u(t, a), \text{ where } u(t) = \frac{\vartheta_1 + \psi(t) + w(t)}{t} \text{ and } a \in \mathbb{T}. \tag{3.6}$$

In order that y if a solution of (3.1), we need to determine $w(t)$ such that $u(t)$ is a solution of the dynamic Riccati equation

$$u^\Delta(t) + p(t) + \frac{u^2(t)}{1 + \mu(t)u(t)} = 0 \tag{3.7}$$

and $u(t)$ satisfies $1 + \mu(t)u(t) > 0$ for large t. If, moreover, $w(t) \to 0$ as $t \to \infty$, then $y \in \mathcal{NRV}_{\mathbb{T}}(\vartheta_1)$ by Theorem 2.15. Because of the definition of u

$$\psi^\Delta(t) = \int_t^\infty p(s)\,\Delta s - \sigma(t)p(t)$$

and

$$u^{\Delta}(t) = -p(t) + \frac{tw(t)^{\Delta} - \vartheta_1^2 - w(t)}{t\sigma(t)}$$

we can write (3.7) in terms of w as

$$w^{\Delta}(t) + \frac{-\vartheta_1^2 - w(t)}{t} + \frac{\sigma(t)(\vartheta_1 + \psi(t) + w(t))^2}{t^2 + t\mu(t)(\vartheta_1 + \psi(t) + w(t))} = 0, \tag{3.8}$$

that is

$$w^{\Delta}(t) + \frac{2\vartheta_1 - 1 + 2\psi(t)}{t} + \frac{w^2(t) + \psi^2(t) + 2\vartheta_1\psi(t)}{t} + G[w](t) = 0, \tag{3.9}$$

where

$$\begin{aligned} G[w](t) &= \frac{\sigma(t)(\vartheta_1 + \psi(t) + w(t))^2}{t^2 + t\mu(t)(\vartheta_1 + \psi(t) + w(t))} - \frac{(\vartheta_1 + \psi(t) + w(t))^2}{t} \\ &= \frac{(\vartheta_1 + \psi(t) + w(t))^2 - (\vartheta_1 + \psi(t) + w(t))^3}{\frac{t^2}{\mu(t)} + t(\vartheta_1 + \psi(t) + w(t))}. \end{aligned}$$

Set $h(t) = e_v(t, a)$, where $v(t) = (2\vartheta_1 - 1 + 2\psi(t))/t$. From Theorem 2.15 we get $h \in \mathcal{NRV}_{\mathbb{T}}(2\vartheta_1 - 1)$ and h is decreasing towards zero, as $2\vartheta_1 - 1 < 0$ by Theorem 1.5 iii). Multiply (3.9) by h to obtain

$$h(t)w^{\Delta}(t) + h^{\Delta}w(t) + h(t)\frac{w^2(t) + \psi^2(t) + 2\vartheta_1\psi(t)}{t} + h(t)G[w](t) = 0, \tag{3.10}$$

what we can rewrite using the identity $(hw)^{\Delta}(t) = h^{\Delta}(t)w(t) + h(t)w^{\Delta}(t) + \mu(t)h^{\Delta}(t)w^{\Delta}(t)$ as

$$(hw)^{\Delta}(t) + \frac{h(t)}{t}(w^2(t) + \psi^2(t) + 2\vartheta_1\psi(t)) + h(t)G[w] - \mu(t)h^{\Delta}(t)w^{\Delta}(t) = 0. \tag{3.11}$$

If $h(t)w(t) \to 0$ for $t \to \infty$, then integration from t to ∞ of (3.11) yields

$$\begin{aligned} w(t) = \frac{1}{h(t)}\int_t^{\infty} \frac{h(s)}{s}(w^2(s) + \psi^2(s) + 2\vartheta_1\psi(s))\,\Delta s + \frac{1}{h(t)}\int_t^{\infty} h(s)G[w](s)\,\Delta s \\ - \frac{1}{h(t)}\int_t^{\infty} \mu(t)h^{\Delta}(s)w^{\Delta}(s)\,\Delta s. \end{aligned} \tag{3.12}$$

We want to apply the contraction mapping theorem on the equation (3.12). We will work in the Banach space $\mathbb{BC}_{\mathbb{T}}[a, \infty)$ — bounded, continuous functions on $[a, \infty)_{\mathbb{T}}$, endowed with the supremum norm. Introduce the set Ω as

$$\Omega = \{w \in \mathbb{BC}_{\mathbb{T}}[a, \infty) : |w(t)| \le \delta \text{ for } t \ge a\},$$

where the values of δ, a will be determined later and define the operator $\tau : \Omega \to \mathbb{BC}_{\mathbb{T}}[a, \infty)$ by

$$\begin{aligned} \tau[w](t) = \frac{1}{h(t)}\int_t^{\infty} \frac{h(s)}{s}(w^2(s) + \psi^2(s) + 2\vartheta_1\psi(s))\,\Delta s + \frac{1}{h(t)}\int_t^{\infty} h(s)G[w](s)\,\Delta s \\ - \frac{1}{h(t)}\int_t^{\infty} \mu(s)h^{\Delta}(s)w^{\Delta}(s)\,\Delta s. \end{aligned}$$

We need to show that $\tau[w] \in \Omega$ for $w \in \Omega$ and $\|\tau[w] - \tau[v]\| \leq \gamma \|w - v\|$, where $\gamma < 1, w, v \in \Omega$. Before proceeding to proof, let us show some properties of h that will be used in the proof. It holds that

$$\lim_{t\to\infty} \frac{1}{h(t)} \int_t^\infty \frac{h(s)}{s} \Delta s = \lim_{t\to\infty} \frac{\frac{-h(t)}{t}}{h(t)\frac{2\vartheta_1 - 1 + 2\psi(t)}{t}} = \frac{1}{1 - 2\vartheta_1} > 0 \tag{3.13}$$

and for sequence α, $\lim_{t\to\infty} \alpha(t) = 0$,

$$\lim_{t\to\infty} \frac{1}{h(t)} \int_t^\infty \frac{h(s)}{s} \alpha(s)\, \Delta s = \lim_{t\to\infty} \frac{\frac{-h(t)}{t}\alpha(t)}{\frac{h(t)}{t}\left(2\vartheta_1 - 1 + 2\psi(t)\right)} = \lim_{t\to\infty} \frac{\alpha(t)}{2\vartheta_1 - 1} = 0, \tag{3.14}$$

where the time scale analogue of the L'Hospital rule was used. The next property is a consequence of the time scale analogue of the L'Hospital rule, the assumption $\mu(t) \sim \mu^\sigma(t)$ as $t \to \infty$, $h^\Delta \in \mathcal{NRV}_\mathbb{T}(2\vartheta_1 - 2)$ and of Theorem 2.17:

$$\begin{aligned}
\lim_{t\to\infty} \frac{1}{h(t)} \int_t^\infty (\mu(s)h^\Delta(s))^\Delta\, \Delta s &= \lim_{t\to\infty} \frac{-(\mu(t)h^\Delta(t))^\Delta}{h^\Delta(t)} \\
&= \lim_{t\to\infty} \frac{\frac{1}{\mu(t)}(\mu^\sigma(t)h^{\Delta\sigma}(t) - \mu(t)h^\Delta(t))}{h^\Delta(t)} \\
&= \lim_{t\to\infty} \frac{\frac{\mu^\sigma(t)}{\mu(t)}h^{\Delta\sigma}(t) - h^\Delta(t)}{h^\Delta(t)} \\
&= \lim_{t\to\infty} \frac{\mu^\sigma(t)}{\mu(t)} \frac{h^{\Delta\sigma}(t)}{h^\Delta(t)} - 1 = 1 - 1 = 0.
\end{aligned} \tag{3.15}$$

Denote

$$\widetilde{\psi}(t) = \sup_{s\geq t} |\psi(s)|$$

Since $\mu(t)/t \to 0$ as $t \to \infty$ and conditions (3.13), (3.15) holds, we can choose $\delta > 0$ and $a \in \mathbb{R}$ such that the following inequalities are satisfied:

$$\sup_{t\geq a} \frac{1}{h(t)} \int_t^\infty \frac{h(s)}{s} \Delta s \leq \frac{2}{1 - 2\vartheta_1}, \tag{3.16}$$

$$\frac{12\delta}{1 - 2\vartheta_1} \leq 1, \tag{3.17}$$

$$\widetilde{\psi}^2(a) + 2\,|\vartheta_1|\,\psi(a) \leq \delta^2, \tag{3.18}$$

$$\frac{(|\vartheta_1| + \widetilde{\psi}(a) + \delta)^2 + (|\vartheta_1| + \widetilde{\psi}(a) + \delta)^3}{\sup_{t\geq a} \frac{t}{\mu(t)} - (|\vartheta_1| + \widetilde{\psi}(a) + \delta)} \leq \frac{\delta(1 - 2\vartheta_1)}{6}, \tag{3.19}$$

$$\sup_{t\geq a} \frac{\mu(t)}{t}(1 - 2\vartheta_1 + 2\widetilde{\psi}(a)) \leq \frac{1}{6}, \tag{3.20}$$

$$\sup_{t\geq a} \frac{1}{h(t)} \int_t^\infty \left|\left(\mu(s)h^\Delta(s)\right)^\Delta\right| \Delta s \leq \frac{1}{6}, \tag{3.21}$$

$$\begin{aligned}
\gamma = \frac{4\delta}{1 - 2\vartheta_1} &+ \frac{2}{1 - 2\vartheta_1} \sup_{t\geq a} \frac{(1 + |\vartheta_1| + \widetilde{\psi}(a) + \delta)^2 \left(\frac{t}{\mu(t)} + |\vartheta_1| + \widetilde{\psi}(a) + \delta\right)}{\left(\frac{t}{\mu(t)} - (|\vartheta_1| + \widetilde{\psi}(a) + \delta)\right)^2} \\
&+ \sup_{t\geq a} \frac{\mu(t)}{t}(1 - 2\vartheta_1 + 2\psi) + \sup_{t\geq a} \frac{1}{h(t)} \int_t^\infty \left|\left(\mu(s)h^\Delta(s)\right)^\Delta\right| \Delta s < 1.
\end{aligned} \tag{3.22}$$

Let us show that $\tau[w] \in \Omega$ holds for every $w \in \Omega$. Let $w \in \Omega$. Then

$$|\tau[w](t)| \le K_1(t) + K_2(t) + K_3(t) \tag{3.23}$$

for $t \in \mathbb{T}, t > a$, where

$$\begin{aligned} K_1(t) &= \left| \frac{1}{h(t)} \int_t^\infty \frac{h(s)}{s} (w^2(s) + \psi^2(s) + 2\vartheta_1 \psi(s)) \, \Delta s \right| \\ &\le \frac{1}{h(t)} \int_t^\infty \frac{h(s)}{s} (\delta^2 + \widetilde{\psi}^2(a) + 2 |\vartheta_1| \, \widetilde{\psi}(a)) \, \Delta s \overset{(3.16),(3.18)}{\le} \frac{2}{1-2\vartheta_1} 2\delta^2 \overset{(3.16)}{\le} \frac{\delta}{3}, \end{aligned} \tag{3.24}$$

$$\begin{aligned} K_2(t) &= \left| \frac{1}{h(t)} \int_t^\infty h(s) \frac{(\vartheta_1 + \psi(s) + w(s))^2 - (\vartheta_1 + \psi(s) + w(s))^3}{\frac{s}{\mu(s)} + t(\vartheta_1 + \psi(s) + w(s))} \, \Delta s \right| \\ &\le \frac{1}{h(t)} \int_t^\infty \frac{h(s)}{s} \frac{(|\vartheta_1| + \widetilde{\psi}(a) + \delta)^2 + (|\vartheta_1| + \widetilde{\psi}(a) + \delta)^3}{\frac{s}{\mu(s)} - (|\vartheta_1| + \widetilde{\psi}(a) + \delta)} \, \Delta s \\ &\overset{(3.16),(3.19)}{\le} \frac{2}{1-2\vartheta_1} \frac{\delta(1-2\vartheta_1)}{6} = \frac{\delta}{3} \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} K_3(t) &= \left| \frac{1}{h(t)} \int_t^\infty \mu(s) h^\Delta(s) w^\Delta(s) \, \Delta s \right| \\ &\overset{\text{Thm 1.12}-d)}{\le} \left| \frac{1}{h(t)} \lim_{\xi \to \infty} \left[w(s)\mu(s)h^\Delta(s) \right]_t^\xi \right| + \left| \frac{1}{h(t)} \int_t^\infty (\mu(s)h^\Delta(s))^\Delta w^\sigma(s) \, \Delta s \right| \\ &\overset{(3.15)}{=} \left| 0 - \frac{1}{h(t)} \frac{\mu(t)}{t} w(t) h(t) (1 - 2\vartheta_1 + 2\psi(t)) \right| + \left| \frac{1}{h(t)} \int_t^\infty (\mu(s)h^\Delta(s))^\Delta w^\sigma(s) \, \Delta s \right| \\ &\le \frac{\mu(t)}{t} \delta(1 - 2\vartheta_1 + 2\widetilde{\psi}(a)) + \frac{\delta}{h(t)} \int_t^\infty \left| (\mu(s)h^\Delta(s))^\Delta w^\sigma(s) \right| \, \Delta s \overset{(3.20),(3.21)}{\le} \frac{\delta}{6} + \frac{\delta}{6} = \frac{\delta}{3}. \end{aligned} \tag{3.26}$$

Overall, $|\tau[w](t)| \le \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta$. It remains to prove the fact that $\tau[w](t) \to 0$ as $t \to \infty$. It is the consequence of $w(t) \to 0$, $\psi(t) \to 0$, $\mu(t)/t \to 0$ as $t \to \infty$, (3.14) and (3.15). Hence, $\tau[w] \in \Omega$ for every $w \in \Omega$. Next, we need to show that $\|\tau w - \tau v\| \le \gamma \|w - v\|$ for $w, v \in \Omega$, where $\gamma < 1$. Let $w, v \in \Omega$. Then

$$|\tau[w](t) - \tau[v](t)| \le H_1(t) + H_2(t) + H_3(t), \tag{3.27}$$

where

$$H_1(t) = \left| \frac{1}{h(t)} \int_t^\infty \frac{h(s)}{s} (w^2(s) - v^2(s)) \, \Delta s \right|,$$

$$H_2(t) = \left| \frac{1}{h(t)} \int_t^\infty h(s) (G[w](s) - G[v](s)) \, \Delta s \right|$$

and

$$H_3(t) = \left| \frac{1}{h(t)} \int_t^\infty \mu(s) h^\Delta(s) (w(s) - v(s))^\Delta \, \Delta s \right|.$$

It holds that

$$H_1(t) \le \|w - v\| \frac{1}{h(t)} \int_t^\infty \frac{h(s)}{s} 2\delta \, \Delta s \le \frac{4\delta}{1 - 2\vartheta_1}. \tag{3.28}$$

Before we examine $H_2(t)$, let us observe that

$$\frac{\partial G[w]}{\partial w}(t) = \frac{\frac{2t^2}{\mu(t)}(\vartheta_1 + \psi(t) + w(t)) + t(\vartheta_1 + \psi(t) + w(t))^2}{\left[\frac{t^2}{\mu(t)} + t(\vartheta_1 + \psi(t) + w(t))\right]^2} + \frac{-3\frac{t^2}{\mu(t)}(\vartheta_1 + \psi(t) + w(t))^2 - 2t(\vartheta_1 + \psi(t) + w(t))^3}{\left[\frac{t^2}{\mu(t)} + t(\vartheta_1 + \psi(t) + w(t))\right]^2} \tag{3.29}$$

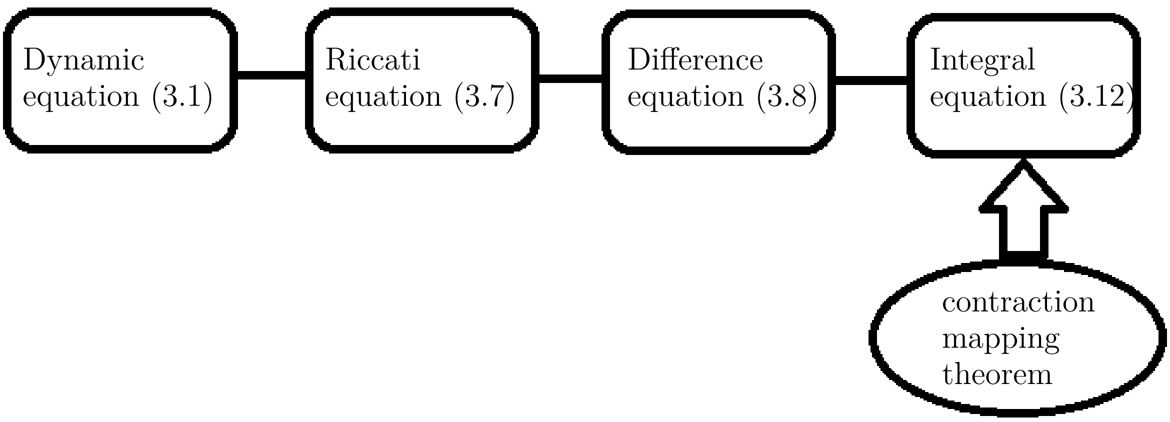
Therefore, the mean value theorem yields

$$\begin{aligned} H_2(t) &\leq \frac{1}{h(t)} \int_t^\infty \frac{h(s)}{s} s \left| \sup \frac{\partial G}{\partial w}(\zeta) \right| |w(s) - v(s)| \,\Delta s \\ &\leq \|w - v\| \sup_{t \geq a} \frac{(1 + |\vartheta_1| + \widetilde{\psi}(a) + \delta)^2 (\frac{t}{\mu(t)} + |\vartheta_1| + \widetilde{\psi}(a) + \delta)}{\frac{t}{\mu(t)} - (|\vartheta_1| + \widetilde{\psi}(a) + \delta)} \frac{1}{h(t)} \int_t^\infty \frac{h(s)}{s} \,\Delta s, \end{aligned} \tag{3.30}$$

where $\min\{w(t), v(t)\} \leq \zeta \leq \max\{w(t), v(t)\}$ for $t \geq a$ and lastly, Theorem 1.12 yields

$$\begin{aligned} H_3(t) &\leq \left| \frac{1}{h(t)} \lim_{\xi \to \infty} \left[(w(s) - v(s)) \mu(s) h^\Delta(s) \right]_t^\xi \right| \\ &\quad + \left| \frac{1}{h(t)} \int_t^\infty (\mu(s) h^\Delta(s))^\Delta (w^\sigma(s) - v^\sigma(s)) \,\Delta s \right| \\ &\overset{(3.15)}{\leq} \|w - v\| \frac{\mu(t)}{t} (1 - 2\vartheta_1 + 2\widetilde{\psi}(a)) + \|w - v\| \left| \frac{1}{h(t)} \int_t^\infty (\mu(s) h^\Delta(s))^\Delta \,\Delta s \right|. \end{aligned} \tag{3.31}$$

Overall, $\|\tau[w] - \tau[v]\| \leq \gamma \|w - v\|$ for $w, v \in \Omega$, where γ satisfies $\gamma < 1$ by the virtue of (3.22). So, τ is a contraction and the assumptions of the contraction mapping theorem are satisfied. Therefore, there exists an unique function $w \in \Omega$ that is a solution of (3.12) and also of (3.8) with w positive, decreasing towards zero. Then u defined in (3.6) is a solution of Riccati equation (3.7) and $1 + \mu(t)u(t) > 0$ for large enough t. Thus, y defined in (3.6) is a nonoscillatory solution of (3.1) and $y \in \mathcal{NRV}_{\mathbb{T}}(\vartheta_1)$ by Theorem 2.15.



Next, we find a linearly independent solution x. We achieve that by using the reduction of order formula

$$x(t) = y(t) \int_a^t \frac{1}{y(s)y^\sigma(s)}\, \Delta s. \tag{3.32}$$

Denote $z(t) = 1/y^2(t)$. Then $z \in \mathcal{NRV}_{\mathbb{T}}(-2\vartheta_1)$ and $z(t) \sim 1/(y(t)y^\sigma(t))$ as $t \to \infty$. Hence, by the discrete L'Hospital rule,

$$\begin{aligned}
\lim_{t\to\infty} \frac{\frac{t}{y(t)}}{x(t)} &= \lim_{t\to\infty} \frac{tz(t)}{\int_a^t \frac{1}{y(s)y^\sigma(s)}\, \Delta s} = \lim_{t\to\infty} \frac{z(t)^\sigma + tz^\Delta(t)}{\frac{1}{y(t)y^\sigma(t)}} = \lim_{t\to\infty} \frac{z(t) + tz^\Delta(t)}{z(t)} \\
&= \lim_{t\to\infty} \left(1 + \frac{tz^\Delta(t)}{z(t)}\right) = 1 - 2\vartheta_1,
\end{aligned}$$

and so

$$x(t)(1-2\vartheta_1) \sim \frac{t}{y(t)} = \frac{t^{1-\vartheta_1}}{L(t)} \text{ as } t \to \infty, \tag{3.33}$$

that is

$$x(t) \sim t^{1-\vartheta_1} \widetilde{L}(t) \text{ as } t \to \infty,$$

where

$$\widetilde{L} = \frac{1}{(1-2\vartheta_1)L} \tag{3.34}$$

and $x \in \mathcal{RV}_{\mathbb{T}}(1-\vartheta_1) = \mathcal{RV}_{\mathbb{T}}(\vartheta_2)$. Let us show that $x \in \mathcal{NRV}_{\mathbb{T}}(\vartheta_2)$. Indeed,

$$\begin{aligned}
\lim_{t\to\infty} \frac{tx^\Delta(t)}{x(t)} &= \lim_{t\to\infty} \frac{ty^\Delta(t) \int_a^t y(s)y^\sigma(s)\, \Delta s + \frac{t}{y(t)}}{x(t)} \\
&= \lim_{t\to\infty} \frac{ty^\Delta(t)}{y(t)} + \frac{t}{x(t)y(t)} \overset{(3.33)}{=} \vartheta_1 + 1 - 2\vartheta_1 = 1 - \vartheta_1 = \vartheta_2,
\end{aligned} \tag{3.35}$$

and so $x \in \mathcal{NRV}_{\mathbb{T}}(\vartheta_2)$.

It remains to prove that every eventually positive solution z of (3.1) is normalized regularly varying. Since (3.1) is linear, we can write z in form $z(t) = c_1 y(t) + c_2 x(t)$. If $c_1 = 0$ or $c_2 = 0$, then (as z is eventually positive) $c_2 > 0$ or $c_1 > 0$ and $z \in \mathcal{NRV}_{\mathbb{T}}(\vartheta_2)$ or $z \in \mathcal{NRV}_{\mathbb{T}}(\vartheta_1)$ respectively. Since $y(t)/x(t) \to \infty$ (as $\vartheta_1 < \vartheta_2$) for $t \to \infty$, it holds that

$$\frac{tz^\Delta(t)}{z(t)} = \frac{c_1 ty^\Delta(t) + c_2 tx^\Delta(t)}{c_1 y(t) + c_2 x(t)} = \frac{c_1 t \frac{y^\Delta(t)}{y(t)} \frac{y(t)}{x(t)} + c_2 \frac{tx^\Delta(t)}{x(t)}}{c_1 \frac{y(t)}{x(t)} + c_2} \sim \frac{tx^\Delta(t)}{x(t)}$$

as $t \to \infty$, hence z is normalized regularly varying function. □

Let us return to the difference equation (1.1). Next, we establish conditions that guarantee the fact that solution space of this equation consists only of regularly varying sequences. Theorem 3.2 and Theorem 3.3 generalizes result of [9] in case $r_k \not\equiv 1$.

Theorem 3.2. *Let $r \in \mathcal{NRV}_{\mathbb{Z}}(\gamma)$, $\gamma < 1$. Then equation* (1.1) *has a fundamental system of solutions $y^{[1]} \in \mathcal{NRV}_{\mathbb{Z}}(\vartheta_1(1-\gamma))$, $y^{[1]} \in \mathcal{NRV}_{\mathbb{Z}}(\vartheta_2(1-\gamma))$ if and only if*

$$\lim_{k\to\infty} \sum_{j=m}^{k-1} \frac{1}{r_j} \sum_{j=k}^{\infty} p_j = C < \frac{1}{4}, \tag{3.36}$$

where $\vartheta_1 < \vartheta_2$ are the real roots of the algebraic equation (3.2).

Proof. We want to transform the equation (1.1) to the equation on time scale and then apply Theorem 3.1. Let y be a solution of (1.1). Set

$$u(t) = u(\tau(k)) = u(\tau_k) = y_k,$$

where $\tau_k > 0$ and $\Delta\tau_k > 0$. Then

$$\mu(t) = \sigma(t) - t = \sigma(\tau(k)) - \tau(k) = \tau_{k+1} - \tau_k = \Delta\tau_k, \tag{3.37}$$

$$u^{\Delta}(t) = \frac{u^{\sigma}(t) - u(t)}{\mu(t)} = \frac{y_{k+1} - y_k}{\Delta\tau_k} = \frac{\Delta y_k}{\Delta\tau_k} \tag{3.38}$$

and

$$\Delta(r_k \Delta y_k) = \mu(t)\left(r\left(\tau^{-1}(t)\right)\mu(t)u^{\Delta}(t)\right)^{\Delta}, \tag{3.39}$$

and so u satisfies

$$(\widetilde{r}(t)u^{\Delta}(t))^{\Delta} + \widetilde{p}(t)u^{\sigma}(t) = 0, \tag{3.40}$$

where

$$\widetilde{r}(t) = r\left(\tau^{-1}(t)\right)\mu(t) \tag{3.41}$$

and

$$\widetilde{p}(t) = \frac{p\left(\tau^{-1}(t)\right)}{\mu(t)}. \tag{3.42}$$

Set $\tau_k = \sum_{j=m}^{k-1} 1/r_j$. Since $\gamma < 1$, we have $\sum_{j=k}^{\infty} 1/r_j = \infty$, and so $\tau_k \to \infty$ as $k \to \infty$, i.e. $\mathbb{T}$ is unbounded. Further, $\mu(t) = \Delta\tau_k > 0$ and $\mathbb{T}$ is discrete. We have $\Delta\tau_k = 1/r_k$ and in view of $\mu(t) = \Delta\tau_k$ we get

$$\widetilde{r}(t) = \frac{r\left(\tau^{-1}(t)\right)}{r\left(\tau^{-1}(t)\right)} \equiv 1. \tag{3.43}$$

It holds that

$$\frac{\mu(t)}{t} = \frac{\Delta\tau_k}{\tau_k} = \frac{1}{r_k \sum_{j=m}^{k-1} \frac{1}{r_j}} \overset{\text{Thm 2.18}}{\sim} \frac{1}{r_k \frac{k}{r_k}} \to 0, \tag{3.44}$$

$$\mu(t) = \Delta\tau_k = \frac{1}{r_k} \overset{\text{Thm 2.17}}{\sim} \frac{1}{r_{k+1}} = \Delta\tau_{k+1} = \mu^{\sigma}(t) \tag{3.45}$$

as $t \to \infty$. Theorem (1.18) yields

$$t\int_t^{\infty} \widetilde{p}(s)\,\Delta s = \tau(k)\int_{\tau(k)}^{\infty} \frac{p\left(\tau^{-1}(s)\right)}{\mu(s)}\,\Delta s = \tau_k \sum_{j=k}^{\infty} p_j = \sum_{j=m}^{k-1} \frac{1}{r_j} \sum_{j=k}^{\infty} p_j = C < \frac{1}{4}. \tag{3.46}$$

Hence, the equation (3.40) satisfies the assumptions of Theorem 3.1, and so (3.40) has a fundamental system of solutions $u_i \in \mathcal{NRV}_{\mathbb{T}}(\vartheta_i)$, $i = 1, 2$. It holds that

$$\vartheta_1 \leftarrow \frac{t u_1^{\Delta}(t)}{u_1(t)} = \frac{\tau(k) u_1^{\Delta}(\tau_k)}{u_1(\tau(k))} = \frac{\tau_k}{\Delta\tau_k} \frac{\Delta y_k^{[1]}}{y_k^{[1]}} = \frac{\tau_k}{k\Delta\tau_k} \frac{k\Delta y_k^{[1]}}{y_k^{[1]}} \tag{3.47}$$

as $t \to \infty$ (i.e. $k \to \infty$) and since $r \in \mathcal{NRV}_{\mathbb{Z}}(\gamma)$,

$$\tau_k = \sum_{j=m}^{k-1} \frac{1}{r_j} \sim \frac{1}{1-\gamma} \frac{k}{r_k} \text{ as } k \to \infty, \tag{3.48}$$

therefore

$$\frac{k\Delta\tau_k}{\tau_k} = \frac{\frac{k}{r_k}}{\tau_k} \sim \frac{\frac{k}{r_k}}{\frac{1}{1-\gamma}\frac{k}{r_k}} = 1-\gamma \text{ as } k \to \infty. \tag{3.49}$$

and $\tau_k \in \mathcal{NRV}_{\mathbb{Z}}(1-\gamma)$. From (3.47) and (3.49) we get, as $k \to \infty$,

$$\frac{k\Delta y_k^{[1]}}{y_k^{[1]}} \to \vartheta_1(1-\gamma),$$

from what we get $y^{[1]} \in \mathcal{NRV}_{\mathbb{Z}}(\vartheta_1(1-\gamma))$. Similarly, we get that $y^{[2]} \in \mathcal{NRV}_{\mathbb{Z}}(\vartheta_2(1-\gamma))$. □

Remark 3.3. The relation (3.46) is, in fact, an improvement of [11] in case $0 < C < 1/4$.

Theorem 3.4. *Let $r \in \mathcal{NRV}_{\mathbb{Z}}(\gamma)$, $\gamma > 1$. Then equation* (1.1) *has a fundamental system of solutions $y^{[1]} \in \mathcal{NRV}_{\mathbb{Z}}(\eta_1(1-\gamma))$, $y^{[2]} \in \mathcal{NRV}_{\mathbb{Z}}(\eta_2(1-\gamma))$ if and only if*

$$\lim_{k\to\infty} \frac{1}{R_k} \sum_{j=k}^{\infty} p_j R_{j+1}^2 = C < \frac{1}{4}, \tag{3.50}$$

where

$$R_k = \sum_{j=k}^{\infty} \frac{1}{r_j}$$

and η_i are the roots of algebraic equation

$$\eta^2 + \eta + C = 0. \tag{3.51}$$

Proof. First note that thanks to $\gamma > 1$, we have $\sum_{j=k}^{\infty} 1/r_j < \infty$, and so R_k is well-defined. Let y be a solution of (1.1). Set $y = hz$, $h \neq 0$. Then z satisfies the difference equation

$$\Delta(\widetilde{r}_k \Delta z_k) + \widetilde{p}_k z_{k+1} = 0, \tag{3.52}$$

where

$$\widetilde{r}_k = r_k h_k h_{k+1} \tag{3.53}$$

and

$$\widetilde{p}_k = h_{k+1}\left[\Delta(r_k \Delta h_k) + p_k h_{k+1}\right]. \tag{3.54}$$

Set

$$h_k = \sum_{j=k}^{\infty} \frac{1}{r_j} = R_k. \tag{3.55}$$

The Karamata theorem for R_k yields

$$\frac{k\Delta R_k}{R_k} \sim \frac{k\frac{-1}{r_k}}{\frac{1}{\gamma-1}\frac{k}{r_k}} = 1-\gamma \text{ as } k \to \infty, \tag{3.56}$$

and so $R_k \in \mathcal{NRV}_{\mathbb{Z}}(1-\gamma)$. Hence,

$$\widetilde{r}_k = r_k R_k R_{k+1} \in \mathcal{NRV}_{\mathbb{Z}}(\gamma + 1 - \gamma + 1 - \gamma) = \mathcal{NRV}_{\mathbb{Z}}(2-\gamma) \tag{3.57}$$

and

$$\widetilde{p}_k = R_k[\Delta(r_k \Delta R_k) + p_k R_{k+1}] = p_k R_k R_{k+1}. \tag{3.58}$$

It holds that
$$\sum_{j=m}^{k-1}\frac{1}{\widetilde{r}_j}=\frac{1}{R_k}-\frac{1}{R_m}\sim\frac{1}{R_k}\text{ as }k\to\infty,$$
therefore
$$\lim_{k\to\infty}\sum_{j=m}^{k-1}\frac{1}{\widetilde{r}_j}\sum_{j=k}^{\infty}\widetilde{p}_j=\lim_{k\to\infty}\frac{1}{R_k}\sum_{j=k}^{\infty}p_jR_{j+1}^2=C<\frac{1}{4}\tag{3.59}$$
and since $2-\gamma<1$, we can apply Theorem 1.2 to obtain $z^{[i]}\in\mathcal{NRV}_{\mathbb{Z}}(\vartheta_i(1-\gamma))$, $i=1,2$, where ϑ_i are solutions of (3.2). Combining this with $h=R\in\mathcal{NRV}_{\mathbb{Z}}(1-\gamma)$, $\eta_2=-\vartheta_1$ and $\vartheta_1+1=1-\eta_2=\eta_1$ we get
$$y^{[1]}=hz^{[1]}=R_kz_k^{[1]}\in\mathcal{NRV}_{\mathbb{Z}}(\vartheta_1(1-\gamma)+(1-\gamma))=\mathcal{NRV}_{\mathbb{Z}}(\eta_1(1-\gamma)).\tag{3.60}$$
Similarly, we get that $y^{[2]}\in\mathcal{NRV}_{\mathbb{Z}}(\eta_2(1-\gamma))$. □

The next corollary is a direct consequence of the previous theorem with a special setting that will be used in the next chapters.

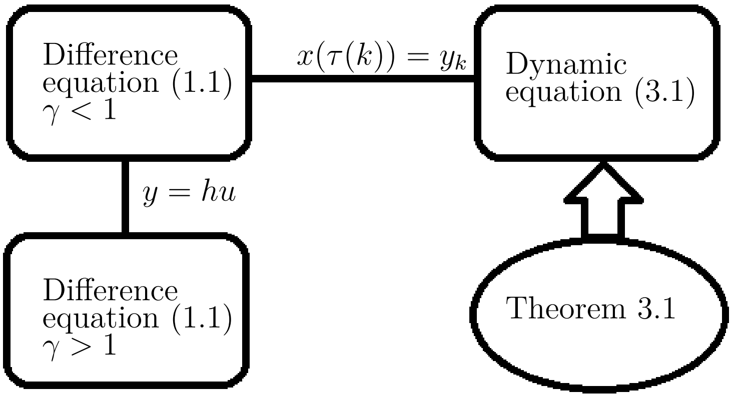
Corollary 3.5. *Let $r_k\in\mathcal{NRV}_{\mathbb{Z}}(\gamma)$ and $p_k\in\mathcal{NRV}_{\mathbb{Z}}(\gamma-2)$ with $\gamma\neq1$. Then* (1.1) *has a fundamental set of solutions $\{y_1,y_2\}$, where $y_i\in\mathcal{NRV}_{\mathbb{Z}}(\vartheta_i)$ if and only if*
$$\lim_{k\to\infty}\frac{k^2p_k}{r_k}=A<\left(\frac{\gamma-1}{2}\right)^2,\tag{3.61}$$
where $\vartheta_1\neq\vartheta_2$ are the reals roots of the algebraic equation
$$\vartheta^2-(1-\gamma)\vartheta+A=0.\tag{3.62}$$
Moreover, all eventually positive solutions are regularly varying sequences.

Proof. Let $\gamma<1$. The Karamata theorem yields
$$\sum_{j=m}^{k-1}\frac{1}{r_j}\sum_{j=k}^{\infty}p_j\sim\frac{k}{r_k(-\gamma+1)}\cdot\frac{kp_k}{-\gamma+1}=\frac{k^2p_k}{r_k}\cdot\frac{1}{(1-\gamma)^2}\text{ as }k\to\infty.\tag{3.63}$$
Let $\gamma>1$. Using the Karamata theorem, we get
$$\frac{1}{R_k}\sum_{j=k}^{\infty}p_jR_{j+1}^2\sim\frac{r_k(\gamma-1)}{k}\cdot\frac{1}{(\gamma-1)^2}\sum_{j=k}^{\infty}\frac{j^2p_j}{r_j^2}\sim\frac{r_k}{k(\gamma-1)}\cdot\frac{k^3p_k}{r_k^2(\gamma-1)}=\frac{k^2p_k}{r_k}\cdot\frac{1}{(1-\gamma)^2}\tag{3.64}$$
as $k\to\infty$.

The relationship between equations (3.2) and (3.62) is given by following linear transformation. Let ζ the root of (3.2). Then $\vartheta=\zeta(1-\gamma)$ is the root of (3.62) and it holds that $A=C(1-\gamma)^2$.

Hence, (3.61) ensures that (3.36) and (3.50) holds and the assumptions of Theorem 3.2 or Theorem 3.4, respectively, are satisfied and (1.1) has a fundamental set of solutions formed by regularly varying sequences. The indices of regular variation of y_i follow from those theorems.

In general, the existence of the (finite) limit (3.61) implies the existence of the (finite) limit of (3.36) or (3.50), but the converse does not hold. However, under conditions $p\in\mathcal{RV}(\delta),r\in\mathcal{RV}(\delta+2),\delta\neq-1$, the sufficient condition (3.61) becomes necessary as (3.61) is equivalent with (3.36) or (3.50) based on $\delta\lesseqgtr-1$ respectively. □



We have proved that all elements of the solution space of (1.1) are regularly varying sequences via transforming this equation to the equation (3.1) on a time scale. For this dynamic equation, we established conditions, under which all of its solutions are regularly varying functions. Both results, regular variation of solutions of (1.1) and regular variation of solutions of (3.1), are new and an improvement over the known results.

Alternatively, instead of transforming (1.1) into a dynamic equation, we could proceed "more directly" and using Riccati equation

$$\Delta w_k + p_k + \frac{w_k^2}{r_k + w_k} = 0 \tag{3.65}$$

to construct a regularly varying solution. But such an approach would require more strict assumptions. The next theorem shows the neccesity part of this alternative approach.

Theorem 3.6. *Let $r \in \mathcal{NRV}_{\mathbb{Z}}(\gamma)$ and $y \in S \cap \left(\bigcup_{\vartheta\in \mathrm{R}} \mathcal{NRV}_{\mathbb{Z}}(\vartheta)\right) \neq \emptyset$. Then*

i)

$$\lim_{k\to\infty} \frac{k}{r_k} \sum_{j=k}^{\infty} p_j = A \in \left(-\infty, \frac{1-\gamma}{4}\right),$$

where A satisfies

$$\vartheta - A - \frac{\vartheta^2}{1-\gamma} = 0$$

in case $\gamma < 1$.

ii)

$$\lim_{k\to\infty} \frac{k}{r_k} \sum_{j=m}^{k-1} p_j = A \in \left(-\infty, \frac{\gamma-1}{4}\right),$$

where A satisfies

$$\vartheta + A - \frac{\vartheta^2}{1-\gamma} = 0$$

in case $\gamma > 1$.

Proof. Let $\gamma < 1$ and $y \in \mathcal{NRV}_{\mathbb{Z}}(\vartheta)$ be a solution of (1.1). Set

$$w_k = \frac{r_k \Delta y_k}{y_k}.$$

Then w_k satisfies

$$\Delta w_k + p_k + \frac{w_k^2}{r_k + w_k} = 0. \tag{3.66}$$

It holds that $(kw_k)/r_k \to \vartheta$, $w_k/r_k \to 0$, $w_k \to 0$ as $k \to \infty$ and $r_k + w_k > 0$. Summing (3.66) from k to ∞ and multiplying by k/r_k we get

$$\frac{kw_k}{r_k} - \frac{k}{r_k}\sum_{j=k}^{\infty} p_j - \frac{k}{r_k}\sum_{j=k}^{\infty}\frac{w_j^2}{r_j + w_j} = 0.$$

The discrete L'Hospital rule yields

$$\begin{aligned}\lim_{k\to\infty}\frac{\sum_{j=k}^{\infty}\frac{w_j^2}{r_j+w_j}}{\frac{r_k}{k}} &= \lim_{k\to\infty}\frac{\frac{-w_k^2}{r_k+w_k}}{\frac{r_{k+1}}{k+1}-\frac{r_k}{k}} = \lim_{k\to\infty}\frac{(k+1)kw_k^2}{r_k(r_k+w_k)(1-\frac{k\Delta r_k}{r_k})}\\ &= \lim_{k\to\infty}\frac{(k+1)kw_k^2}{r_k^2} = \lim_{k\to\infty}\frac{1}{(1+\frac{w_k}{r_k})(1-\frac{k\Delta r_k}{r_k})} = \frac{\vartheta^2}{1-\gamma}.\end{aligned}$$

Hence,

$$\frac{k}{r_k}\sum_{j=k}^{\infty} p_j \to \vartheta - \frac{\vartheta^2}{1-\gamma} = A \text{ as } k\to\infty.$$

Let $\gamma > 1$ and y, w_k be as in the previous part. By summing (3.66) from n to $k-1$ and multiplying by k/r_k we get

$$\frac{kw_k}{r_k} - \frac{kw_n}{r_n} + \frac{k}{r_k}\sum_{j=m}^{k-1} p_j + \frac{k}{r_k}\sum_{j=m}^{k-1}\frac{w_j^2}{r_j + w_j} = 0.$$

It holds that

$$\begin{aligned}\lim_{k\to\infty}\frac{k}{r_k}\sum_{j=m}^{k-1}\frac{w_j^2}{r_j+w_j} &= \lim_{k\to\infty}\frac{\frac{w_k^2}{r_k+w_k}-\frac{w_n^2}{w_n+r_n}}{\frac{k\Delta r_k - r_k}{k(k+1)}}\\ &= \lim_{k\to\infty}\left[\frac{\frac{k(k+1)w_k^2}{r_k+w_k}}{(\frac{k\Delta r_k}{r_k}-1)r_k} - \frac{\frac{k(k+1)w_n^2}{r_n+w_n}}{(\frac{k\Delta r_k}{r_k}-1)r_k}\right] = \frac{-\vartheta^2}{1-\gamma}.\end{aligned}$$

Hence,

$$\lim_{k\to\infty}\frac{k}{r_k}\sum_{j=m}^{k-1} p_j \to -\vartheta - \frac{\vartheta^2}{1-\gamma} = A \text{ as } k\to\infty.$$

$\square$

CHAPTER 4

DISCRETE KARAMATA THEORY AND DIFFERENCE EQUATIONS

4.1 Asymptotic formulae for $\mathcal{SV}$ solutions

The following conditions play an important role in the investigation of the asymptotic behauvior of the solutions of (1.1):

$$p \in \mathcal{RV}(\delta), \quad r \in \mathcal{RV}(\delta+2) \tag{4.1}$$

and

$$\lim_{k\to\infty} \frac{k^2 p_k}{r_k} = A \in \left(-\infty, \left(\frac{\delta+1}{2}\right)^2\right). \tag{4.2}$$

Denote

$$G_k = \frac{kp_k}{r_k}$$

and note that $G \in \mathcal{RV}(-1)$ and in general, $\sum_{j=k}^{\infty} G_j$ may converge on diverge.

Lemma 4.1 ([10])**.** *Let $p < 0$, $|p| \in \mathcal{RV}(\delta)$ and $r \in \mathcal{RV}(\delta+2)$. Then $\mathcal{S}_{\mathcal{SV}} \subseteq \mathcal{DS}$ provided that $\delta < -1$ and $\mathcal{S}_{\mathcal{SV}} \subseteq \mathcal{IS}$ provided that $\delta > -1$.*

Lemma 4.2. *Let $|p| \in \mathcal{RV}(\delta)$, $r \in \mathcal{RV}(\delta+2)$ with $\delta \neq 1$. If $y \in \mathcal{S}_{\mathcal{SV}}$, then*

$$r_k \Delta y_k \sim \frac{kp_k y_k}{-\delta-1} \text{ as } k \to \infty. \tag{4.3}$$

Proof. Let $p > 0$ and $\delta < -1$. Then $\mathcal{S} = \mathcal{IS}_{B,0} \cup \mathcal{IS}_{\infty,0} \cup \mathcal{IS}_{\infty,B}$. Suppose $r_k \Delta y_k \to B \in (0,\infty)$ as $k \to \infty$. Then, since $r\Delta y$ is decreasing, for $k \geq m$ it holds that

$$r_k \Delta y_k \geq B$$

and the division by r_k and the summation from n to $k-1$ yields

$$y_k \geq y_n + \sum_{j=n}^{k-1} \frac{B}{r_j} \geq \sum_{j=n}^{k-1} \frac{B}{r_j} \in \mathcal{RV}(-\delta-1).$$

Since $-\delta-1>0$, y_k can not be slowly varying sequence and so $\lim\limits_{k\to\infty} r_k\Delta y_k=0$.

Take $y\in\mathcal{S}_{\mathcal{SV}}$. By summing (1.1) from k to ∞ and applying the Karamata theorem we get

$$-r_k\Delta y_k=-\sum_{j=k}^{\infty}p_jy_{j+1}\sim-\sum_{j=k}^{\infty}p_jy_{j+1}\sim\frac{-kp_ky_k}{-\delta-1}\text{ as }k\to\infty. \tag{4.4}$$

Let $p>0$ and $\delta>-1$. Then $\mathcal{S}=\mathcal{DS}_{0,B}\cup\mathcal{DS}_{0,\infty}\cup\mathcal{DS}_{B,\infty}$. Take $y\in\mathcal{DS}_{B,\infty}$. Then $r_k\Delta y_k$ is decreasing with limit $-B\in(-\infty,0)$, so

$$r_k\Delta y_k\geq -B$$

and by dividing by r_k and by the summation from k to ∞ we get

$$-y_k\geq\sum_{j=k}^{\infty}\frac{-B}{r_j}.$$

Hence,

$$y_k\leq B\sum_{j=n}^{k-1}\frac{1}{r_j}\in\mathcal{RV}(-\delta-1).$$

Since $-\delta-1<0$, y_k can not be slowly varying sequence and so $\lim\limits_{k\to\infty} r_k\Delta y_k=\infty$.

Take $y\in\mathcal{S}_{\mathcal{SV}}$. By summing (1.1) from n to $k-1$ and applying the Karamata theorem we get

$$r_k\Delta y_k=r_n\Delta y_n-\sum_{j=n}^{k-1}p_jy_{j+1}\sim-\sum_{j=n}^{k-1}p_jy_{j+1}\sim\frac{-kp_ky_k}{\delta+1}\text{ as }k\to\infty. \tag{4.5}$$

Let $p<0$ and $\delta<-1$ and take $y\in\mathcal{S}_{\mathcal{SV}}$. Then $\sum_{j=k}^{\infty}1/r_k=\infty$, $r_\Delta y_k$ is negative increasing and $y\in\mathcal{DS}$ by Lemma 4.1. Suppose $r_k\Delta y_k$ has limit $-B\in(-\infty,0)$. Then

$$r_k\Delta y_k\leq -B$$

and by dividing by r_k and by summing from n to $k-1$ we obtain

$$y_k\leq y_n-M\sum_{j=n}^{k-1}\frac{1}{r_j}\to-\infty\text{ as }k\to\infty,$$

that contradicts $y_k>0$, and so $\lim\limits_{k\to\infty} r_k\Delta y_k=0$.

Take $y\in\mathcal{S}_{\mathcal{SV}}$. By summing (1.1) from t to ∞ and proceeding similarly as in case $p>0$, $\delta<-1$ we get (4.4).

Let $p<0$ and $\delta>-1$ and take $y\in\mathcal{S}_{\mathcal{SV}}$. Then $\sum_{j=k}^{\infty}p_j=\infty$ and $y\in\mathcal{IS}$ by Lemma 4.1. Summation of (1.1) from n to ∞ yields

$$r_k\Delta y_k=r_n\Delta y_n-\sum_{j=n}^{k-1}p_jy_{j+1}\geq-\sum_{j=n}^{k-1}p_jy_{j+1}\to\infty\text{ as }k\to\infty$$

and so, $\lim\limits_{k\to\infty} r_k\Delta y_k=\infty$. By proceeding similarly as in case $p>0$, $\delta>-1$ we get (4.5). $\square$

Theorem 4.3. *Let* $|p| \in \mathcal{RV}(\delta)$, $r \in \mathcal{RV}(\delta+2)$, $\delta \neq -1$ *and* $\lim_{k\to\infty} \frac{k^2 p_k}{r_k} = 0$. *Then*

$$y_k = \exp\left\{-\sum_{j=m}^{k-1}(1+o(1))\frac{G_j}{\delta+1}\right\} \text{ as } k \to \infty \tag{4.6}$$

provided that $\sum_{j=k}^{\infty} G_j = \infty$, *and*

$$y_k = A\exp\left\{\sum_{j=k}^{\infty}(1+o(1))\frac{G_j}{\delta+1}\right\} \text{ as } k \to \infty, \tag{4.7}$$

where $A = A(y) = \lim_{k\to\infty} y_k$, *provided that* $\sum_{j=k}^{\infty} G_j < \infty$.

Proof. From Corollary 3.5 and Lemma 1.5 (with $\gamma = \delta+2$) we have $S = \mathcal{S}_{\mathcal{SV}} \cup \mathcal{S}_{\mathcal{RV}}(-\delta-1)$. Take $y \in \mathcal{S}_{\mathcal{SV}}$. Then by Lemma 4.2 we have

$$r_k \Delta y_k \sim \frac{k p_k y_k}{-\delta-1} \text{ as } k \to \infty$$

and Theorem 2.12 yields

$$\Delta \ln y_k \sim \frac{\Delta y_k}{y_k} \sim \frac{k p_k}{(-\delta-1) r_k} \text{ as } k \to \infty. \tag{4.8}$$

Let $\sum_{j=k}^{\infty} G_j = \infty$. By summing (4.8) from n to $k-1$ we get

$$\ln \frac{y_k}{y_n} \sim \sum_{j=n}^{k-1} \frac{G_j}{-\delta-1} \text{ as } k \to \infty \tag{4.9}$$

and

$$y_k = y_n \exp\left\{\sum_{j=n}^{k-1}(1+o(1))\frac{G_j}{-\delta-1}\right\} \text{ as } k \to \infty \tag{4.10}$$

and since $\sum_{j=k}^{\infty} G_j = \infty$,

$$y_k = \exp\left\{-\sum_{j=m}^{k-1}(1+o(1))\frac{G_j}{\delta+1}\right\} \text{ as } k \to \infty. \tag{4.11}$$

Let $\sum_{j=k}^{\infty} G_j < \infty$. By summing (4.8) from k to ∞ we get

$$\ln \frac{A}{y_k} \sim \sum_{j=k}^{\infty} \frac{G_j}{-\delta-1} \text{ as } k \to \infty \tag{4.12}$$

and

$$y_k = A\exp\left\{\sum_{j=m}^{k-1}(1+o(1))\frac{G_j}{\delta+1}\right\} \text{ as } k \to \infty, \tag{4.13}$$

where $A = \lim_{k\to\infty} y_k$. $\square$

From Corollary 3.5 and Lemma 1.5 we know that, under the assummptions of Theorem 4.3, the solution space of (1.1) is formed by slowly varying sequences and regularly varying sequences with index $-\delta-1$. The next remark shows a possible way to obtain asympotic formulae for the other half of the solution space using so-called reciprocity principle, which allow us to use the previous theorem.

Remark 4.4. Let the assumptions of Theorem 4.3 hold. Take $y \in \mathcal{S}_{\mathcal{RV}}(-\delta-1)$. Set $u_k = r_k \Delta y_k$. Then u satisfies

$$\Delta(\widetilde{r}_k \Delta u_k) + \widetilde{p}_k u_{k+1} = 0, \tag{4.14}$$

where $\widetilde{r}_k = 1/p_k$ and $\widetilde{p}_k = 1/r_{k+1}$. If $p \in \mathcal{RV}(\delta), r \in \mathcal{RV}(\delta+2)$, then $\widetilde{p} \in \mathcal{RV}(\widetilde{\delta}), \widetilde{r} \in \mathcal{RV}(\widetilde{\delta}+2)$ with $\widetilde{\delta} = -\delta-2$. It holds that

$$\frac{k^2 \widetilde{p}_k}{\widetilde{r}_k} = \frac{k^2(1/r_{k+1})}{1/p_k} \sim \frac{k^2 p_k}{r_k} \to 0 \text{ as } k \to \infty. \tag{4.15}$$

Since $y \in \mathcal{RV}(-\delta-1)$, $\Delta y_k \in \mathcal{RV}(-\delta-2)$ and $r \in \mathcal{RV}(\delta+2)$, we get $u \in \mathcal{SV}$. Denote $\widetilde{G}_k = k\widetilde{p}_k/\widetilde{r}_k$. It holds that

$$\widetilde{G}_k = k\widetilde{p}_k/\widetilde{r}_k = \frac{k(1/r_{k+1})}{1/p_k} \sim \frac{kp_k}{r_k} \text{ as } k \to \infty.$$

Hence, if $\sum_{j=k}^{\infty} G_j = \infty$, then also $\sum_{j=k}^{\infty} \widetilde{G}_j = \infty$ and we can apply Theorem 4.3 to obtain

$$r_k \Delta y_k = u_k = \exp\left\{-\sum_{j=n}^{k-1}(1+o(1))\frac{\widetilde{G}_j}{\widetilde{\delta}+1}\right\} = \exp\left\{(1+o(1))\sum_{j=n}^{k-1}\frac{G_j}{\delta+1}\right\} \text{ as } k \to \infty \tag{4.16}$$

and a formula for y follows.

If $\sum_{j=k}^{\infty} G_j < \infty$, then also $\sum_{j=k}^{\infty} \widetilde{G}_j < \infty$ and we can apply Theorem 4.3 to obtain

$$\begin{aligned} r_k \Delta y_k = u_k &= M \exp\left\{\sum_{j=n}^{k-1}(1+o(1))\frac{\widetilde{G}_j}{\widetilde{\delta}+1}\right\} \\ &= M \exp\left\{-\sum_{j=n}^{k-1}(1+o(1))\frac{G_j}{\delta+1}\right\} \text{ as } k \to \infty \end{aligned} \tag{4.17}$$

and a formula for y follows.

4.2 Asymptotic formulae for RV solutions

Theorem 4.5. *Let $r \in \mathcal{NRV}(\gamma), \gamma \neq 1$,*

$$\lim_{k\to\infty} \frac{k^2 p_k}{r_k} = A \tag{4.18}$$

and

$$L_k^{[i]} = \frac{k^2 p_k}{r_k} + \vartheta_i \frac{k \Delta r_k}{r_k} + \vartheta_i(\vartheta_i - 1) \tag{4.19}$$

with $\left|L^{[i]}\right| \in \mathcal{SV}$ *and* $\vartheta_i \neq 0$ *the real root of* (1.4), $i = 1, 2$. *Then*

$$y_k^{[1]} \sim k^{(\vartheta_i)} \exp\left\{\sum_{j=m}^{k-1}(1 + o(1))\frac{L_j^{[i]}}{(1 - \gamma - 2\vartheta_i)j}\right\} \text{ as } k \to \infty \tag{4.20}$$

provided that $\sum_{j=k}^{\infty} L_j^{[i]} = \infty$ *and*

$$y_k^{[i]} \sim Dk^{(\vartheta_i)} \exp\left\{\sum_{j=k}^{\infty}(1 + o(1))\frac{L_j^{[i]}}{(\gamma + 2\vartheta_i - 1)j}\right\} \text{ as } k \to \infty, \tag{4.21}$$

where $D = \lim_{k\to\infty} y_k/k^{(\vartheta_i)}$, *provided that* $\sum_{j=k}^{\infty} L_j^{[i]} < \infty$, $i = 1, 2$.

Proof. Let $0 \neq \vartheta_1 < \vartheta_2 \neq 0$ be the roots of (1.4). From Theorem 3.3, Theorem 3.5 and the relationship between (1.4) and (3.2) we have $S \subseteq \mathcal{NRV}(\vartheta_1) \cup \mathcal{NRV}(\vartheta_2)$. Take $y \in \mathcal{S}_{\mathcal{RV}}(\vartheta_i)$, $i = 1, 2$. Set $y = hu$, where $h_k = k^{(\vartheta_i)}$, $i = 1, 2$. Then $u^{[i]}$ satisfies (3.52) with (3.53) and (3.54) and

$$\widetilde{r}_k = r_k^{[i]} = r_k k^{(\vartheta_i)}(k + 1)^{(\vartheta_i)}$$

$$\Delta(r_k\Delta h_k) = \Delta(r_k\vartheta_i k^{(\vartheta_i-1)}) = \Delta r_k\vartheta_i k^{(\vartheta_i-1)} + r_{k+1}\vartheta_i(\vartheta_i - 1)k^{(\vartheta_i-2)}.$$

From the properties of the Gamma function we have

$$\frac{k^{(\vartheta_i-1)}}{k^{(\vartheta_i-2)}} = k + 2 - \vartheta_i$$

and

$$\frac{(k + 1)^{(\vartheta_i)}}{k^{(\vartheta_i-2)}} = (k + 1)(k + 2 - \vartheta_i).$$

Further,

$$\begin{aligned}
\widetilde{p}_k = p_k^{[i]} &= (k + 1)^{(\vartheta_i)}[\Delta r_k\vartheta_i k^{(\vartheta_i-1)} + r_{k+1}\vartheta_i(\vartheta_i - 1)k^{(\vartheta_i-2)} + p_k(k + 1)^{(\vartheta_i)}] \\
&= (k + 1)^{(\vartheta_i)}k^{(\vartheta_i-2)}r_{k+1}\left[\frac{k^{(\vartheta_i-1)}\vartheta_i}{k^{(\vartheta_i-2)}} \cdot \frac{\Delta r_k}{r_{k+1}} + \vartheta_i(\vartheta_i - 1) + \frac{p_k}{r_{k+1}} \cdot \frac{(k + 1)^{(\vartheta_i)}}{k^{(\vartheta_i-2)}}\right] \\
&= (k + 1)^{(\vartheta_i)}k^{(\vartheta_i-2)}r_{k+1}\widehat{L}_k,
\end{aligned}$$

where

$$\widehat{L}_k^{[i]} = (k + 2 - \vartheta_i)\frac{\vartheta_i\Delta r_k}{r_{k+1}} + \vartheta_i(\vartheta_i - 1) + \frac{p_k}{r_{k+1}}(k + 1)(k + 2 - \vartheta_i).$$

We want to show that $\widehat{L}_k^{[i]} \sim L_k^{[i]}$ as $k \to \infty$. It holds that

$$\begin{aligned}
\widehat{L}_k &= L_k^{[i]} - \frac{k^2 p_k}{r_k} + \frac{p_k}{r_{k+1}}(k + 1)(k + 2 - \vartheta_i) - \vartheta_i\frac{k\Delta r_k}{r_k} + (k + 2 - \vartheta_i)\frac{\vartheta_i\Delta r_k}{r_{k+1}} \\
&= L_k^{[i]} + A_k^{[i]} + B_k^{[i]},
\end{aligned}$$

where

$$\begin{aligned}
A_k^{[i]} &= \vartheta_i\left[-\frac{k\Delta r_k}{r_k} + (k + 2 - \vartheta_i)\frac{\Delta r_k}{r_{k+1}}\right] \\
&= \vartheta_i\frac{k\Delta r_k}{r_k}\left(\frac{k + 2 - \vartheta_i}{r_k} \cdot \frac{r_k}{r_{k+1}} - 1\right) \to 0 \text{ as } k \to \infty
\end{aligned}$$

and

$$\begin{aligned} B_k^{[i]} &= \frac{p_k}{r_{k+1}}(k+1)(k+2-\vartheta_i) - \frac{k^2 p_k}{r_k} = p_k\left(\frac{(k+1)(k+2-\vartheta_i)}{r_{k+1}} - \frac{k^2}{r_k}\right) \\ &= \frac{k^2 p_k}{r_k}\left(\frac{(k+1)(k+2-\vartheta_i)}{k^2}\cdot\frac{r_k}{r_{k+1}} - 1\right) \to 0 \text{ as } k\to\infty. \end{aligned}$$

Hence

$$\widehat{L}_k^{[i]} \sim L_k^{[i]} \text{ as } k\to\infty$$

and $\left|\widehat{L}^{[i]}\right| \in \mathcal{SV}$. Further, we have $|\widetilde{p}| \in \mathcal{RV}(\widetilde{\delta})$, $\widetilde{r} \in \mathcal{RV}(\widetilde{\delta}+2)$ with $\widetilde{\delta} = \gamma + 2\vartheta_1 - 2$. From Lemma 1.5 we obtain $\widetilde{\delta} \neq -1$. Denote $G_k^{[i]} = \frac{k\widetilde{p}_k^{[i]}}{\widetilde{r}_k^{[i]}}$. Then $G_k^{[i]} \sim L_k^{[i]}/k$ as $k\to\infty$.

Let $\sum_{j=m}^{k-1} L_j^{[i]}/j = \infty$. By applying Theorem 4.3 we obtain the asymptotic formula for $u^{[i]}$

$$u_k^{[i]} = \exp\left\{\sum_{j=m}^{k-1}(1+o(1))\frac{G_j^{[i]}}{-\widetilde{\delta}-1}\right\} \text{ as } k\to\infty$$

and the asymptotic formula for $y^{[i]}$ follows

$$y_k^{[i]} = k^{(\vartheta_i)}\exp\left\{\sum_{j=m}^{k-1}(1+o(1))\frac{L_j^{[i]}}{(1-\gamma-2\vartheta_i)j}\right\} \text{ as } k\to\infty. \tag{4.22}$$

Let $\sum_{j=m}^{k-1} L_j^{[i]}/j < \infty$. By applying Theorem 4.3 we obtain the asymptotic formula for $u^{[i]}$

$$u_k^{[i]} = D^{[i]}\exp\left\{\sum_{j=k}^{\infty}(1+o(1))\frac{G_j^{[i]}}{\widetilde{\delta}+1}\right\} \text{ as } k\to\infty,$$

where $D^{[i]} = \lim\limits_{k\to\infty} y_k^{[i]}/k^{(\vartheta_1)}$ and the asymptotic formula for $y^{[i]}$ follows

$$y_k^{[i]} = Dk^{(\vartheta_i)}\exp\left\{\sum_{j=k}^{\infty}(1+o(1))\frac{L_j^{[i]}}{(\gamma+2\vartheta_i-1)j}\right\} \text{ as } k\to\infty. \tag{4.23}$$

□

Example 4.6.

$$r_k = k^\gamma\left(1-\frac{1}{\ln^\zeta k}\right),$$

$$p_k = Ak^{\gamma-2}\left(1+\frac{1}{\ln^\eta k}\right),$$

$\zeta,\eta \in (0,\infty)$, $\gamma\in\mathbb{R}$ and $A \in \left(-\infty, \left(\frac{\gamma-1}{2}\right)^2\right)$. Then

$$\frac{k^2 p_k}{r_k} = A\frac{1+1/(\ln^\eta k)}{1-1/(\ln^\zeta k} \to A \text{ as } k\to\infty.$$

Further,

$$\widehat{L}_k = \frac{k^2 p_k}{r_k} - A = A(1+o(1))\left(\frac{1}{\ln^\eta k}+\frac{1}{\ln^\zeta k}\right)$$

and $\widehat{L} \in \mathcal{SV}$. It holds that

$$\widehat{L}_k^{[i]} = (1 + o(1))\widehat{L}_k,$$

$i = 1, 2$ with $\mathrm{sgn}\widehat{L} = \mathrm{sgn}A$. The integral criterion yields that

$$\sum_{j=m}^{\infty} \frac{\left|\widehat{L}_k^{[i]}\right|}{j} < \infty, i = 1, 2 \text{ if and only if } \min \eta, \zeta > 1.$$

4.3 Classification

In this section we establish asymtotic formulae for regularly varying solutions of (1.1) under certain conditions. We complete the results of [10] in case $p > 0$ and we unify these results. Aside from that, asymptotic formulae established in Theorem 4.7 and Theorem 4.8 can be used to complete classification in [10].

Theorem 4.7. *Let $p \in \mathcal{RV}(\delta)$, $r \in \mathcal{RV}(\delta + 2)$, $\delta \neq -1$, and $\lim_{k\to\infty} \frac{k^2 p_k}{r_k} = 0$.*

(i) If $\delta < -1$ and $\sum_{j=k}^{\infty} G_j = \infty$, then

$$\mathcal{S} = \mathcal{IS}_{\infty,0} = \mathcal{S}_{\mathcal{SV}} \cup \mathcal{S}_{\mathcal{RV}}(-\delta - 1).$$

(ii) If $\delta < -1$ and $\sum_{j=k}^{\infty} G_j < \infty$, then

$$\mathcal{S} = \mathcal{S}_{\mathcal{SV}} \cup \mathcal{S}_{\mathcal{RV}}(-\delta - 1),$$

$$\mathcal{S}_{\mathcal{SV}} = \mathcal{IS}_{B,0},$$

$$\mathcal{S}_{\mathcal{RV}}(-\delta - 1) = \mathcal{IS}_{\infty,B}.$$

(iii) If $\delta > -1$ and $\sum_{j=k}^{\infty} G_j = \infty$, then

$$\mathcal{S} = \mathcal{DS}_{0,\infty} = \mathcal{S}_{\mathcal{SV}} \cup \mathcal{S}_{\mathcal{RV}}(-\delta - 1).$$

(iv) If $\delta > -1$ and $\sum_{j=k}^{\infty} G_j < \infty$, then

$$\mathcal{S} = \mathcal{S}_{\mathcal{SV}} \cup \mathcal{S}_{\mathcal{RV}}(-\delta - 1),$$

$$\mathcal{S}_{\mathcal{SV}} = \mathcal{DS}_{B,\infty},$$

$$\mathcal{S}_{\mathcal{RV}}(-\delta - 1) = \mathcal{DS}_{0,B}.$$

Proof. Let $\delta < -1$. Then $\sum_{j=k}^{\infty} 1/r_j = \infty$ and

$$\mathcal{S} = \mathcal{IS}_{B,0} \cup \mathcal{IS}_{\infty,0} \cup \mathcal{IS}_{\infty,B}$$

by Lemma 1.1. Take $y \in \mathcal{S}_{\mathcal{SV}}$. From proof of Theorem 4.3 we know that

$$\lim_{k\to\infty} r_k \Delta y_k = 0$$

Let $\sum_{j=k}^{\infty} G_j = \infty$. Then y satisfies (4.6) by Theorem 4.3 and since $G > 0$ and $\delta + 1 < 0$, we get

$$\lim_{k\to\infty} y_k = \infty.$$

Overall, we have $y_k \in \mathcal{IS}_{\infty,0}$.

Let $\sum_{j=k}^{\infty} G_j < \infty$. Then y satisfies (4.7) by Theorem 4.3 and since $G > 0$ and $\delta + 1 < 0$, we get

$$\lim_{k\to\infty} y_k \in (0, \infty),$$

thus $y \in \mathcal{IS}_{B,0}$.

Take $y \in \mathcal{S}_{\mathcal{RV}}(-\delta - 1)$. Since $-\delta - 1 > 0$, $\lim_{k\to\infty} y_k = \infty$. Let $\sum_{j=k}^{\infty} G_j = \infty$. Then, Remark 4.4 yields that $r\Delta y$ satisfies (4.16) and

$$\lim_{k\to\infty} r_k \Delta y_k = 0,$$

thus $y \in \mathcal{IS}_{\infty,0}$.

Let $\sum_{j=k}^{\infty} G_j < \infty$. Then, by Remark 4.4, $r\Delta y$ satisfies (4.17) and

$$\lim_{k\to\infty} r_k \Delta y_k = \in (0, \infty).$$

Hence, $y \in \mathcal{IS}_{\infty,0}$.

Let $\delta > -1$. Then $\sum_{j=k}^{\infty} p_j = \infty$ and

$$\mathcal{S} = \mathcal{DS}_{0,B} \cup \mathcal{DS}_{0,\infty} \cup \mathcal{DS}_{B,\infty}$$

by Lemma 1.2. Take $y \in \mathcal{S}_{\mathcal{SV}}$. From proof of Theorem 4.3 we know that

$$\lim_{k\to\infty} r_k \Delta y_k = -\infty.$$

Let $\sum_{j=k}^{\infty} G_j = \infty$. Then y satisfies (4.6) by Theorem 4.3 and since $G > 0$ and $\delta + 1 > 0$, we get

$$\lim_{k\to\infty} y_k = 0.$$

Overall, $y \in \mathcal{DS}_{0,\infty}$.

Let $\sum_{j=k}^{\infty} G_j < \infty$. Then y satisfies (4.7) by Theorem 4.3 and since $G > 0$ and $\delta + 1 > 0$, we get

$$\lim_{k\to\infty} y_k = \in (0, \infty)$$

and we have $y \in \mathcal{DS}_{B,\infty}$.

Take $y \in \mathcal{S}_{\mathcal{RV}}(-\delta - 1)$. Let $\sum_{j=k}^{\infty} G_j = \infty$. Then, Remark 4.4 yields that $r\Delta y$ satisfies (4.16) and

$$\lim_{k\to\infty} r_k \Delta y_k = -\infty.$$

From $-\delta - 1 < 0$ we get that

$$\lim_{k\to\infty} y_k = 0,$$

and $y \in \mathcal{DS}_{0,\infty}$.

Let $\sum_{j=k}^{\infty} G_j < \infty$. Then, by Remark 4.4, $r\Delta y$ satisfies (4.17) and

$$\lim_{k\to\infty} r_k \Delta y_k = \in (-\infty, 0).$$

Since $-\delta - 1 < 0$, we get

$$\lim_{k\to\infty} y_k = 0$$

and $y \in \mathcal{DS}_{0,B}$.

Take case (i). We have proved that $\mathcal{S}_{\mathcal{SV}} \cup \mathcal{S}_{\mathcal{RV}}(() - \delta - 1) \subseteq \mathcal{IS}_{\infty,B}$. The relation $\mathcal{IS}_{\infty,B} \subseteq \mathcal{S}_{\mathcal{SV}} \cup \mathcal{S}_{\mathcal{RV}}(() - \delta - 1)$ results from the fact that we are, by Corrolary 3.5, dealing with all regularly varying solutions. Thus there cannot be a regularly varying solution that belongs to another class. This observation hold for all other cases and also for Theorem 4.8, Theorem 4.9 and Theorem 4.10. □

Theorem 4.8 ([10]). *Let* $-p \in \mathcal{RV}(\delta)$, $r \in \mathcal{RV}(\delta+2)$, $\delta \neq -1$, *and* $\lim_{k\to\infty} \dfrac{k^2 p_k}{r_k} = 0$.

(i) If $\delta < -1$ *and* $\sum_{j=k}^{\infty} G_j = \infty$, *then*

$$\mathcal{S} = \mathcal{S}_{\mathcal{SV}} \cup \mathcal{S}_{\mathcal{RV}}(-\delta-1),$$

$$\mathcal{S}_{\mathcal{SV}} = \mathcal{DS} = \mathcal{DS}_{0,0},$$
$$\mathcal{S}_{\mathcal{RV}}(-\delta-1) = \mathcal{IS} = \mathcal{IS}_{\infty,\infty}.$$

(ii) If $\delta < -1$ *and* $\sum_{j=k}^{\infty} G_j < \infty$, *then*

$$\mathcal{S} = \mathcal{S}_{\mathcal{SV}} \cup \mathcal{S}_{\mathcal{RV}}(-\delta-1),$$

$$\mathcal{S}_{\mathcal{SV}} = \mathcal{DS} = \mathcal{DS}_{B,0},$$
$$\mathcal{S}_{\mathcal{RV}}(-\delta-1) = \mathcal{IS} = \mathcal{IS}_{\infty,B}.$$

(iii) If $\delta > -1$ *and* $\sum_{j=k}^{\infty} G_j = \infty$, *then*

$$\mathcal{S} = \mathcal{S}_{\mathcal{SV}} \cup \mathcal{S}_{\mathcal{RV}}(-\delta-1),$$

$$\mathcal{S}_{\mathcal{SV}} = \mathcal{IS} = \mathcal{IS}_{\infty,\infty},$$
$$\mathcal{S}_{\mathcal{RV}}(-\delta-1) = \mathcal{DS} = \mathcal{DS}_{0,0}.$$

(iv) If $\delta > -1$ *and* $\sum_{j=k}^{\infty} G_j < \infty$, *then*

$$\mathcal{S} = \mathcal{S}_{\mathcal{SV}} \cup \mathcal{S}_{\mathcal{RV}}(-\delta-1),$$

$$\mathcal{S}_{\mathcal{SV}} = \mathcal{IS} = \mathcal{IS}_{B,\infty},$$
$$\mathcal{S}_{\mathcal{RV}}(-\delta-1) = \mathcal{DS} = \mathcal{DS}_{0,B}.$$

Theorem 4.9. *Let* $p \in \mathcal{RV}(\delta)$, $r \in \mathcal{RV}(\delta+2)$, $\delta \neq -1$, $\lim_{k\to\infty} \frac{k^2 p_k}{r_k} = A \in \left(0, \left(\frac{\delta+1}{2}\right)^2\right)$ *and* $\vartheta_1 < \vartheta_2$ *be the real roots of* (1.4). *Then*

$$\mathcal{S} = \mathcal{IS}_{\infty,0} = \mathcal{S}_{\mathcal{RV}}(\vartheta_1) \cup \mathcal{S}_{\mathcal{RV}}(\vartheta_2)$$

provided $\delta < -1$ *and*

$$\mathcal{S} = \mathcal{DS}_{0,\infty} = \mathcal{S}_{\mathcal{RV}}(\vartheta_1) \cup \mathcal{S}_{\mathcal{RV}}(\vartheta_2)$$

provided $\delta > -1$.

Proof. From Corollary 3.5 we know that

$$\mathcal{S} = \mathcal{S}_{\mathcal{RV}}(\vartheta_1) \cup \mathcal{S}_{\mathcal{RV}}(\vartheta_2).$$

Let $\delta < -1$ and by Lemma 1.5 we have $0 < \vartheta_1 < \vartheta_2$. Take $y \in \mathcal{S}_{\mathcal{RV}}(\vartheta_1)$. Then, clearly, $y \in \mathcal{IS}$ and

$$\lim_{k\to\infty} y_k = \infty.$$

Further, $r\Delta y \in \mathcal{RV}(\delta + 2 + \vartheta_1 - 1) = \mathcal{RV}(-\vartheta_2)$ and so

$$\lim_{k\to\infty} r_k \Delta y_k = 0$$

, since $-\vartheta_2 < 0$. Thus, $y \in \mathcal{IS}_{\infty,0}$. The same holds for $y \in \mathcal{S}_{\mathcal{RV}}(\vartheta_2)$.

Let $\delta > -1$ and by Lemma 1.5 we have $\vartheta_1 < \vartheta_2 <$. Take $y \in \mathcal{S}_{\mathcal{RV}}(\vartheta_1)$. Then, clearly, $y \in \mathcal{DS}$ and

$$\lim_{k\to\infty} y_k = 0.$$

Moreover, $r\Delta y \in \mathcal{RV}(\delta + 2 + \vartheta_1 - 1) = \mathcal{RV}(-\vartheta_2)$ and from $-\vartheta_2 < 0$ we get

$$\lim_{k\to\infty} r_k \Delta y_k = -\infty.$$

Hence, $y \in \mathcal{DS}_{0,\infty}$. The same holds for $y \in \mathcal{S}_{\mathcal{RV}}(\vartheta_2)$. □

Theorem 4.10. *Let $-p \in \mathcal{RV}(\delta)$, $r \in \mathcal{RV}(\delta + 2)$, $\delta \neq -1$, $\lim_{k\to\infty} \frac{k^2 p_k}{r_k} = A \in (-\infty, 0)$ and $\vartheta_1 < \vartheta_2$ be the real roots of* (1.4). *Then*

$$\mathcal{S} = \mathcal{S}_{\mathcal{RV}}(\vartheta_1) \cup \mathcal{S}_{\mathcal{RV}}(\vartheta_2),$$

$$\mathcal{S}_{\mathcal{RV}}(\vartheta_1) = \mathcal{DS} = \mathcal{DS}_{0,0},$$

$$\mathcal{S}_{\mathcal{RV}}(\vartheta_2) = \mathcal{IS} = \mathcal{IS}_{\infty,\infty}.$$

Proof. From Corollary 3.5 we know that

$$\mathcal{S} = \mathcal{S}_{\mathcal{RV}}(\vartheta_1) \cup \mathcal{S}_{\mathcal{RV}}(\vartheta_2)$$

and by Lemma 1.5 we have $\vartheta_1 < 0 < \vartheta_2$. Take $y \in \mathcal{S}_{\mathcal{RV}}(\vartheta_1)$. Then, clearly, $y \in \mathcal{DS}$ and

$$\lim_{k\to\infty} y_k = 0.$$

Further $r\Delta y \in \mathcal{RV}(\delta + 2 + \vartheta_1 - 1) = \mathcal{RV}(-\vartheta_2)$ and since $-\vartheta_2 < 0$, we get

$$\lim_{k\to\infty} r_k \Delta y_k = 0.$$

Hence, $y \in \mathcal{DS}_{0,0}$.

Take $y \in \mathcal{S}_{\mathcal{RV}}(\vartheta_2)$. Then, clearly, $y \in \mathcal{IS}$ and

$$\lim_{k\to\infty} y_k = \infty.$$

Moreover $r\Delta y \in \mathcal{RV}(\delta + 2 + \vartheta_2 - 1) = \mathcal{RV}(-\vartheta_1)$ and so

$$\lim_{k\to\infty} r_k \Delta y_k = \infty.$$

as $-\vartheta_2 < 0$. Hence, $y \in \mathcal{IS}_{\infty,\infty}$. □

4.4 Further remarks

In this section we present several remarks concerning our results and provide some directions for a future research.

Relation between the indices of regular variation of r and p

In Theorem 4.3, Theorem 4.7 and Theorem 4.8 we have assumed that $p \in \mathcal{RV}(\delta)$ and $r \in \mathcal{RV}(\delta + 2)$. Let us show that this relation between indices of regular variation is natural when one deals with slowly varying solutions which tend to zero or infinity.

Let $|p| \in \mathcal{RV}(\delta), r \in \mathcal{RV}(\gamma)$ and $y \in \mathcal{S}_{\mathcal{SV}}$. Assume, for instance, $\delta > -1$. Then $\sum_{j=k}^{\infty} p_j = \infty$ and summing (1.1) from n to $k-1$ yields

$$r_k \Delta y_k = r_n \Delta y_n - \sum_{j=n}^{k-1} p_j y_{j+1} \sim - \sum_{j=n}^{k-1} p_j y_{j+1} \text{ as } k \to \infty.$$

Since $p_k y_{k+1} \in \mathcal{RV}(\delta)$, we get $|\Delta y_k| \in \mathcal{RV}(\delta + 1 - \gamma)$. Combining that with $y_k \to \infty$ or $y_k \to\ 0$ as $k \to \infty$ yields $y \in \mathcal{RV}(\delta + 2 - \gamma)$. Now, if $\gamma \neq \delta + 2$, then $y \notin \mathcal{SV}$.

In general, one can study asymptotic properties of $\mathcal{RV}$ solutions of (1.1) under condition $\gamma \geq \delta + 2$. But allowing $\gamma > \delta + 2$ would require many other nontrivial computations. Analysis of regularly varying solutions of (1.1) under condition $\gamma < \delta + 2$ is meaningless, as under this condition, by Corollary 3.5, (1.1) has no regularly varying solution, since $|k^2 p_k / r_k| \to \infty$ as $k \to \infty$.

Recessive and dominant solutions

A concept of recessive and dominant solutions plays an important role in qualitative theory of difference equations. For nonoscillatory (1.1) there exists a positive solution y, called *recessive solution*, such that for any linearly independent solution x, called *dominant solution*, one has

$$\lim_{k \to \infty} \frac{y_k}{x_k} = 0.$$

Other characterizations of are for example the summation characterization

$$\sum_{j=k}^{\infty} \frac{1}{r_j y_j y_{j+1}} = \infty$$

and

$$\sum_{j=k}^{\infty} \frac{1}{r_j x_j x_{j+1}} < \infty$$

or

$$\frac{\Delta y_k}{y_k} < \frac{\Delta x_k}{x_k}$$

for large k.

In Chapter 3, we have established that (1.1) has a fundamental set of solutions $y \in \mathcal{NRV}(\vartheta_1)$, $x \in \mathcal{NRV}_{\mathbb{Z}}(\vartheta_2)$, $\vartheta_1 < \vartheta_2$. It can be proved that

y is a recessive solution and x is a dominant solution.

Indeed, take $y \in \mathcal{S}_{\mathcal{RV}}(\vartheta_1)$. Then

$$\frac{1}{r_k y_k y_{k+1}} \in \mathcal{NRV}(-\gamma - 2\vartheta_1).$$

From Lemma 1.5 we have $2\vartheta_1 > -1+\gamma$, and so $-\gamma - 2\vartheta_1 > -\gamma - 1 + \gamma = -1$. Hence,

$$\sum_{j=k}^{\infty} \frac{1}{r_j y_j y_{j+1}} < \infty$$

and y is recessive. The same idea holds for $x \in \mathcal{S}_{\mathcal{RV}}(\vartheta_2)$, only this time we Lemma 1.5 yields $2\vartheta_2 < -1+\gamma$ and consequently

$$\sum_{j=k}^{\infty} \frac{1}{r_j x_j x_{j+1}} = \infty$$

and x is dominant.

"Critical case" $\delta = -1$.

We have not considered a case $\delta = -1$ (or $\gamma = 1$) in our analysis. This case corresponds to the double root of (1.4) that results in the fact that two linearly independent solutions of (1.1) will be of the same index of regular variation. A slight change in the approach in the proof of the existence theorem should result in the regular variation of the solution space also in the double root case. Asymptotic formulae and the classification can be obtained via the transformation into a "noncritical" (in the sense of $\delta \neq -1$) equation on a suitable time scale.

Three term recurrence equation

In the literature the three term recurrence equation

$$a_k y_{k+2} + b_k y_{k+1} + c_k y_k = 0. \tag{4.24}$$

on $[m,\infty)_{\mathbb{Z}}$ is frequently discussed. If $c \neq 0$, (4.24) can be be written as (1.1) with

$$r_k = c_k \prod_{j=m}^{k-1} \frac{a_j}{c_{j+1}}$$

and

$$p_k = (a_k + b_k + c_k) \prod_{j=m}^{k-1} \frac{a_j}{c_{j+1}}.$$

Using this relation between (4.24) and (1.1), it is apparent how our results can be applied to (4.24). For example, some of our conditions that were important in the analysis of (1.1) for (4.24) read as follows

$$\lim_{k\to\infty} \frac{k^2 p_k}{r_k} = \lim_{k\to\infty} \frac{k^2(a_k + b_k + c_k)}{c_k}$$

and from

$$\Delta r_k = \prod_{j=m}^{k-1} \frac{a_j}{c_{j+1}} (a_k - c_k)$$

we have

$$r_k \in \mathcal{RV}(\gamma) \text{ if and only if } \frac{k(a_k - c_k)}{c_k} \to \gamma \text{ as } k \to \infty$$

and asymptotic formula (4.6) yields

$$y_k = A \exp\left\{ \sum_{j=m}^{k-1} \frac{1+o(1)}{\delta+1} \cdot \frac{j(a_j+b_j+c_j)}{c_j} \right\}$$

as $k \to \infty$.

CONCLUSSION

The main aim of this thesis was to demonstrate the usefulness of the Karamata sequences in asymptotic theory and to derive new results.

We have given a basic information about the difference equation (1.1), we have highlighted the importance of this equation and we have discussed a basic classification of its solutions. We have also introduced a concept of a time scale and time scale calculus.

We have recalled the concept of a discrete regular variation and a concept of a regular variation on time scales. We have gathered properties of regularly varying sequences that are useful in asymptotic theory of difference equations.

Next, the regular variation of the solution space of the difference equation (1.1) has been discussed and the new results were proved. By means of Theorem 3.1 we have obtained the condition under which (3.1) has a regularly varying solution space. This theorem generalizes [12] for arbitrary p, relaxing the sign condition $p < 0$, and [9] in the sense of generalized domain. By means of a transformation of independent variable, we have transformed a "difficult" difference equation (1.1) into a "simpler" dynamic equation (3.1). This allowed us to apply (new) Theorem 3.1 and we acquired the condition, under which the solution space of (1.1) consists of regularly varying sequences. This fact, presented in Theorem 3.2 and Theorem 3.4, generalize [9] in case $r \not\equiv 1$. The relation (3.46) is an improvement over [11] in case $p > 0$.

We have established asymptotic formulae for solutions of (1.1). These formulae are of great value, since, in general, (1.1) is not analytically solvable. Paper [10] is concerned with asymptotic formulae for case $p < 0$. Theorem 4.3 and Theorem 4.5 unify and complete this results. By means of Theorem 4.7 and Theorem 4.9 we have discussed the classification of the solutions of (1.1) and we have completed [10] in case $p > 0$.

The main contributions of this thesis are the demonstration of usefulness of discrete theory of regularly varying sequences, the examinations of difference equations under new settings, the completion and generalization of abovementionted results and the presentation of useful techniques, such as Riccati technique, reciprocity principle, linear transformation or transformation from one time scale to another.

It is possible to build upon our results using the directions for a future research mentioned at the end of the last chapter.

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