BRNO UNIVERSITY OF TECHNOLOGY

Faculty of Electrical Engineering and Communication

Department of Mathematics

Mgr. Vladislav Biba

GENERATED FUZZY IMPLICATIONS IN FUZZY DECISION MAKING

GENEROVANÉ FUZZY IMPLIKÁTORY VO FUZZY ROZHODOVANÍ

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Supervisor: RNDr. Dana Hliněná, Ph.D.

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Contents

1 Introduction

Fuzzy logic and fuzzy sets are basic framework when working with vague notions. In classical logic all assertions are either true or false (i.e have truth values 1 or 0 respectively). In case of fuzzy logic the truth value may be any value in the interval $[0, 1]$. Connected with fuzzy logic is the notion of fuzzy sets. Classical set is given by it's characteristic function with values 0 and 1. Likewise, a fuzzy set is given by it's membership function with values from interval $[0, 1]$. The advantage of this approach is illustrated in the simple example:

Let's turn out attention to the assertion "He is a tall man". Suppose that we want to construct a set $\mathcal{T}$ of all "tall men". Obviously, this decision depends on one's personal experience. (For example a professional basketball player and regular people probably have a different notion of "being tall".) Moreover, if we want to evaluate this assertions only by "true" or "false", we get the following paradox: a $180cm$ tall man may be considered "tall" (i.e is in the set $\mathcal{T}$) but a $179cm$ one is considered "not tall at all" (and belong to the set $\mathcal{T}'$).

In this example we are working with vague notions. It is therefore better to consider the characteristic function with all values from interval $[0, 1]$ not only two values 0 and 1. For example a $190cm$ tall man can be considered "tall", while $170cm$ one is "not tall at all", a $185cm$ tall man can be considered "tall" in the degree 0.75, etc.

This approach was introduced by Lotfi A. Zadeh in 1965 in the article *Fuzzy sets*. Fuzzy sets were at first used in control theory and fuzzy regulation and later it expanded to other sectors where the informations are incomplete or imprecise, such as economy, bioinformatics, medicine, genealogical research etc.

The truth value of some assertion can not be decided in the classical two-valued (Aristotle) logic. Such assertions are known as logical paradoxes. Recall the well-known liar's paradox, which is sometimes credited to Epimenides. One of the versions of this paradox is a statement "This statement is false." Hypothesis that previous sentence is true leads to the conclusion that it is false, which is a contradiction. On the other hand, hypothesis that the statement is false also lead to contradiction.

The need of working with the vague notions is evident in so-called "paradox of the heap": One grain of sand is not a heap. If you don't have a heap and add just one grain of sand, then you won't get a heap. Both these assertions are obvious, but using them one can conclude that no number of grains will make a heap, which is in a contradiction with our experience.

These limitations of classical logic was known long ago, however, multivalued logics were not proposed until the beginning of 20th century. The three-valued logic was proposed by polish mathematician and philosopher Jan Łukasiewicz around 1920. Later, Łukasiewicz together with Alfred Tarski extended this logic for $n \geq 2$. In 1932, Hans Reichenbach formulated a logic with infinitely many values.

2 Preliminaries

2.1 Fuzzy logic connectives

In this paragraph we briefly introduce basic definitions and properties of fuzzy logic connectives. First we turn our attention to the fuzzy negations, which are monotonic extensions of classical negations.

Definition 2.1 (see, e.g., Fodor and Roubens [7]) *A decreasing function* $N : [0,1] \to [0,1]$ *is called a* fuzzy negation *if* $N(0) = 1, N(1) = 0$. *A negation* N *is called*

1. strict *if it is strictly decreasing and continuous for arbitrary* $x \in [0,1]$,
2. strong *if it is an involution, i.e., if* $N(N(x)) = x$ *for all* $x \in [0,1]$.

A dual negation based on a negation N is given by

$$N^d(x) = 1 - N(1-x).$$

Lemma 2.2 *Let* $N : [0,1] \to [0,1]$ *be a strict negation. Then its dual negation,* N^d, *is also strict.*

Monotonic extension of the classical conjunction is called a *fuzzy conjunction.*

Definition 2.3 *An increasing mapping* $C : [0,1]^2 \to [0,1]$ *is called a* fuzzy conjunction *if, for any* $x, y \in [0,1]$, *it holds*

- $C(x,y) = 0 \quad$ *whenever* $\quad x = 0 \quad$ *or* $\quad y = 0$,
- $C(1,1) = 1$.

The special fuzzy conjunctions called *triangular norms* are widely used in applications to model a conjunction in multivalued logic or an intersection of fuzzy sets. Triangular norms were introduced by Schweizer and Sklar in [17] as a generalization of triangular inequality to probabilistic metric spaces.

Definition 2.4 (Klement, Mesiar and Pap [13]) *A* triangular norm *(t-*norm *for short) is a binary operation on the unit interval* $[0,1]$, *i.e., a function* $T : [0,1]^2 \to [0,1]$ *such that for all* $x, y, z \in [0,1]$, *the following four axioms are satisfied:*
(T1) Commutativity $T(x,y) = T(y,x)$,
(T2) Associativity $T(x, T(y,z)) = T(T(x,y), z)$,
(T3) Monotonicity $T(x,y) \le T(x,z)$ *whenever* $y \le z$,
(T4) Boundary Condition $T(x,1) = x$.

Example 2.5 *Four most common t-norms are:*

- *Minimum t-norm*
  $T_M(x,y) = \min(x,y)$,
- *Product t-norm*
  $T_P(x,y) = x \cdot y$,

- *Łukasiewicz t-norm*
  $T_L(x,y) = \max(0, x+y-1)$,
- *Drastic t-norm*
  $T_D(x,y) = \begin{cases} \min(x,y) & \textit{if } \max(x,y)=1, \\ 0 & \textit{othervise.} \end{cases}$

Remark 2.6 *Another interesting t-norms are given by*

$$T^s(x,y) = \log_s\left(1 + \frac{(s^x-1)(s^y-1)}{s-1}\right),$$

where $s \in]0,\infty[-\{1\}$ *and the limit cases are* $T^0 = T_M$, $T^1 = T_P$ *and* $T^\infty = T_L$. *The functions* $T^s : [0,1]^2 \to [0,1]$ *are called* Frank t-norms.

In the construction of fuzzy operators we use a generalization of inverse function which is called a *pseudo-inverse*:

Definition 2.7 *Let* $f : [0,1] \to [0,\infty]$ *be a decreasing and non-constant function. The function* $f^{(-1)} : [0,\infty] \to [0,1]$ *which is defined by*

$$f^{(-1)}(x) = \sup\{z \in \ [0,1]; f(z) > \ x\},$$

is called the pseudo-inverse *of the function* f, *with the convention* $\sup\emptyset = 0$.

A pseudo-inverse can be defined also for increasing functions:

Definition 2.8 *Let* $\varphi : [0,1] \to [0,\infty]$ *be an increasing and non-constant function. The function* $\varphi^{(-1)} : [0,\infty] \to [0,1]$ *which is defined by*

$$\varphi^{(-1)}(x) = \sup\{z \in \ [0,1]; \varphi(z) < \ x\},$$

is called the pseudo-inverse of the function φ, *with the convention* $\sup\emptyset = 0$.

Dual operator to a fuzzy conjunction is called a *fuzzy disjunction*. A fuzzy disjunction is the monotonic extension of classical disjunction:

Definition 2.9 *An increasing mapping* $D : [0,1]^2 \to [0,1]$ *is called a* fuzzy disjunction *if, for any* $x, y \in [0,1]$, *it holds*

- $D(x,y) = 1$ *whenever* $x = 1$ *or* $y = 1$,
- $D(0,0) = 0$.

The dual mapping to a t-norm is a *triangular conorm* (*t-conorm* for short). The t-conorms are used to model a union of fuzzy sets or as disjunctions in fuzzy logic. One possible definition is axiomatic:

Definition 2.10 *A* triangular conorm *(*t-conorm *for short) is a binary operation on the unit interval* $[0,1]$, *i.e., a function* $S : [0,1]^2 \to [0,1]$ *such that for all* $x, y, z \in [0,1]$, *the following four axioms are satisfied:*
(S1) Commutativity $S(x,y) = S(y,x)$,
(S2) Associativity $S(x, S(y,z)) = S(S(x,y), z)$,
(S3) Monotonicity $S(x,y) \le S(x,z)$ *whenever* $y \le z$,
(S4) Boundary Condition $S(x,0) = x$.

The original definition was given by Schweizer and Sklar:

Definition 2.11 (Schweizer and Sklar [17]) *Dual operator to t-norm is called a* triangular conorm (t-conorm)*, defined as* $S(x, y) = 1 - T(1 - x, 1 - y)$.

Of course, the mentioned definitions are equivalent, both are used in the literature.

Special classes of fuzzy conjunctions and disjunctions are called t-seminorms C and t-semiconorms D. We use these mappings as the truth functions for conjunctions and disjunctions in some parts of the thesis.

Definition 2.12 (Schweizer and Sklar [17])

(i) A t-seminorm *C is a fuzzy conjunction that satisfied the boundary condition*

$$C(1, x) = C(x, 1) = x \text{ for all } x \in [0, 1].$$

(ii) A t-semiconorm *D is a fuzzy disjunction that satisfied the boundary condition*

$$D(0, x) = D(x, 0) = x \text{ for all } x \in [0, 1].$$

In the literature, we can find several different definitions of fuzzy implications. We will use the following one, which is equivalent to the definition introduced by Fodor and Roubens in [7]. More information on this topic can be found in [3] and [16].

Definition 2.13 *A function $I : [0, 1]^2 \to [0, 1]$ is called a* fuzzy implication *if it satisfies the following conditions:*

(I1) I is decreasing in its first variable,

(I2) I is increasing in its second variable,

(I3) $I(1, 0) = 0$, $I(0, 0) = I(1, 1) = 1$.

Several classes of fuzzy implications are well-known. First one is based on a tautology $B \Rightarrow H \equiv \neg B \vee H$:

Definition 2.14 (Baczyński and Jayaram [2]) *A function $I : [0, 1]^2 \to [0, 1]$ is called an* (S, N)-implication *if there exists a t-conorm S and a fuzzy negation N such that*

$$I(x, y) = S(N(x), y), \qquad \forall x, y \in [0, 1].$$

If N is a strong negation, then I is called strong implication.

Other well-known approach to obtain a fuzzy implication uses residuation with respect to t-norm:

Definition 2.15 (Fodor and Roubens [7]) *A function $I : [0, 1]^2 \to [0, 1]$ is called an* R-implication *if there exists a t-norm T such that*

$$R_T(x, y) = \sup\{t \in [0, 1]; T(x, t) \le y\}, \qquad \forall x, y \in [0, 1].$$

In a classical logic there is no difference between (S, N)-implications and R-implications, but this property doesn't hold in fuzzy case in general.

The third well-known class of implications is the class of Q-implications (Q is short for quantum logic). This class is based on the tautology $(A \Rightarrow B) \equiv (\neg A \vee (A \wedge B))$. The Q-implication is therefore defined as

$$I^Q_{S,T}(x, y) = S(N(x), T(x, y)) \qquad \forall x, y \in [0, 1].$$

Fuzzy implications may possess several important properties. Note that some of these properties (namely (EP), (CP) and (LI)) are well-known tautologies in classical two-valued logic.

Definition 2.16 *A fuzzy implication $I : [0, 1]^2 \to [0, 1]$ satisfies:*

(NP) *the* left neutrality property*, or is called* left neutral*, if*

$$I(1, y) = y; \quad y \in [0, 1],$$

(EP) *the* exchange principle *if*

$$I(x, I(y, z)) = I(y, I(x, z)) \text{ for all } x, y, z \in [0, 1],$$

(IP) *the* identity principle *if*

$$I(x, x) = 1; \quad x \in [0, 1],$$

(OP) *the* ordering property *if*

$$x \leq y \iff I(x, y) = 1; \quad x, y \in [0, 1],$$

(CP) *the* contrapositive symmetry with respect to a given negation N *if*

$$I(x, y) = I(N(y), N(x)); \quad x, y \in [0, 1].$$

(LI) *the* law of importation with respect to a t-norm T *if*

$$I(T(x, y), z) = I(x, I(y, z)); \quad x, y, z \in [0, 1].$$

(WLI) *the* weak law of importation with respect to a commutative and increasing function $F : [0, 1]^2 \to [0, 1]$ *if*

$$I(F(x, y), z) = I(x, I(y, z)); \quad x, y, z \in [0, 1].$$

Definition 2.17 *Let $I : [0, 1]^2 \to [0, 1]$ be a fuzzy implication. The function $N_I : [0, 1] \to [0, 1]$ defined by $N_I(x) = I(x, 0)$ for all $x \in [0, 1]$, is called the* natural negation *of I.*

2.2 Fuzzy relations

A preference structure is a basic concept of preference modelling. In a classical preference structure (PS), a decision-maker makes one of three decisions for each pair (a, b) from the set $\boldsymbol{A}$ of all alternatives. His decision defines a triplet P, I, J of crisp binary relations on $\boldsymbol{A}$:

1) a is preferred to $b \Leftrightarrow (a, b) \in P$ (strict preference).

2) a and b are indifferent $\Leftrightarrow (a, b) \in I$ (indifference).

3) a and b are incomparable $\Leftrightarrow (a, b) \in J$ (incomparability).

A *preference structure (PS)* on a set $\boldsymbol{A}$ is a triplet (P, I, J) of binary relations on $\boldsymbol{A}$ such that

(ps1) I is reflexive, while P and J are irreflexive,

(ps2) P is asymmetric, while I and J are symmetric,

(ps3) $P \cap I = P \cap J = I \cap J = \emptyset$,

(ps4) $P \cup I \cup J \cup P^t = \mathbf{A} \times \mathbf{A}$ where $P^t(x, y) = P(y, x)$.

Using characteristic mappings [25] a minimal definition of (PS) can be formulated as a triplet (P, I, J) of binary relations on $\boldsymbol{A}$ such that

- I is reflexive and symmetric.
- $P(a, b) + P^t(a, b) + I(a, b) + J(a, b) = 1$ for all $(a, b) \in \boldsymbol{A}^{\mathbf{2}}$.

A preference structure can be characterized by the reflexive relation $R = P \cup I$ called the *large preference relation*. The relation R can be interpreted as

$$(a, b) \in R \Leftrightarrow a \text{ is prefered to } b \text{ or } a \text{ and } b \text{ are indifferent.}$$

It can be easily proved that

$$co(R) = P^t \cup J,$$

where $co(R)$ is the complement of R and

$$P = R \cap co(R^t),\ I = R \cap R^t,\ J = co(R) \cap co(R^t).$$

This allows us to construct a preference structure (P, I, J) from a reflexive binary operation R only.

We shall consider a continuous De Morgan triplet (T, S, N) consisting of a continuous t-norm T, continuous t-conorm S and a strong fuzzy negation N such that $T(x, y) = N(S(N(x), N(y)))$. The main problem lies in the fact that the completeness condition (ps4) can be written in many forms, e.g.:

$$co(P \cup P^t) = I \cup J, P = co(P^t \cup I \cup J), P \cup I = co(P^t \cup J).$$

Note that it was proved in [7, 25] that reasonable constructions of fuzzy preference structure (FPS) should use a nilpotent t-norm only. Since any nilpotent t-norm (t-conorm) is isomorphic to the Łukasiewicz t-norm (t-conorm), it is enough to restrict our attention to De Morgan triplet $(T_L, S_L, 1-x)$. Then we can define (FPS) as the triplet of binary fuzzy relations (P, I, J) on the set of alternatives $\boldsymbol{A}$ satisfying:

- I is reflexive and symmetric.
- $\forall (a,b) \in \boldsymbol{A^2}, \;\; P(a,b) + P^t(a,b) + I(a,b) + J(a,b) = 1.$

It has been mentioned, that it is possible to construct preference structure from a large preference relation R in the classical case, however, in fuzzy case this is not possible. This fact was proved by Alsina in [1] and later by Fodor and Roubens in [7]:

Proposition 2.18 (Fodor and Roubens [7], Proposition 3.1) *There is no continuous de Morgan triplet (T, S, N) such that $R = P \cup_S I$ holds with $P(a,b) = T(R(a,b), N(R(b,a)))$ and $I(a,b) = T(R(a,b), R(b,a))$.*

Because of this negative result, Fodor and Roubens (among others) proposed axiomatic construction. Assume that we deal with the Łukasiewicz triplet $(T_L, S_L, 1-x)$.

(R1) Independence of Irrelevant Alternatives:
For any two alternatives a, b the values of $P(a,b), I(a,b), J(a,b)$ depend only on the values $R(a,b), R(b,a)$. I.e., there exist functions $p, i, j : [0,1]^2 \to [0,1]$ such that, for any $a, b \in \mathbf{A}$,

$$P(a,b) = p(R(a,b), R(b,a)),$$
$$I(a,b) = i(R(a,b), R(b,a)),$$
$$J(a,b) = j(R(a,b), R(b,a)).$$

(R2) Positive Association Principle:
Functions $p(x, 1-y),\; i(x,y),\; j(1-x, 1-y)$ are increasing in x and y.

(R3) Symmetry:
$i(x,y)$ and $j(x,y)$ are symmetric functions.

(R4) $(P,\; I,\; J)$ is (FPS) for any reflexive relation R on a set $\boldsymbol{A}$ such that

$$S_L(P, I) = R, \quad S_L(P, J) = 1 - R^t.$$

It was proved ([7], Theorem 3.1) that for all $x, y \in [0,1]$ it holds:

$$T_L(x,y) \;\leq\; p(x, 1-y), i(x,y), j(1-x, 1-y) \;\leq\; T_M(x,y).$$

The mentioned triplet (p, i, j) is called *the monotone generator triplet.* Summarizing, the monotone generator triplet is a triplet (p, i, j) of mappings $[0,1]^2 \to [0,1]$ such that

(gt1) $p(x, 1-y),\; i(x,y),\; j(1-x, 1-y)$ are increasing in both coordinates,

(gt2) $T_L(x,y) \;\leq\; p(x, 1-y), i(x,y), j(1-x, 1-y) \;\leq\; T_M(x,y)$, (gt3) $i(x,y) = i(y,x)$,

(gt4) $p(x,y) + p(y,x) + i(x,y) + j(x,y) = 1$,

(gt5) $p(x,y) + i(x,y) = x$.

Using these properties, one may show that also $j(x,y) = j(y,x)$ and $p(x,y) + j(x,y) = 1 - y$. Therefore the axiom (R4) can be expressed as a system of functional equations:

(R4')

$$p(x,y) + i(x,y) = x,$$

$$p(x,y) + j(x,y) = 1 - y.$$

Definition 2.19 *Let $\varphi : [0,1] \to [0,1]$ be an order-automorphism. Then*

$$T_\varphi(x,y) = \varphi^{-1}(T(\varphi(x), \varphi(y))),$$

$$S_\varphi(x,y) = \varphi^{-1}(S(\varphi(x), \varphi(y))),$$

$$(N_s)_\varphi(x) = \varphi^{-1}(1 - \varphi(x)),$$

are called φ-transformations *of T, S, and N_s, respectively.*

Remark 2.20 *It is possible to formulate similar axioms in the framework of more general De-Morgan triplet $(T_L)_\varphi, (S_L)_\varphi, (N_s)_\varphi)$, which is a $\varphi-$transformation of $(T_L, S_L, 1-x)$. The solution is then expressed as $(p,i,j)_\varphi$.*

2.3 Aggregation deficit and its properties

In [20] there was introduced a new operator, called an *aggregation deficit R_D, which is based on a disjunction D.* We recall its definition and important properties; their proofs can be found in [20]. The motivation is following. Assume the truth value $TV(\boldsymbol{A}) = a$. We would like to know conditions on truth values $TV(\boldsymbol{B}) = b$ and $TV(\boldsymbol{C}) = c$ such that they aggregate together with a or $1-a$ to have $D(c,a) \geq x$ and $D(b, 1-a) \geq y$. In order to obtain this aggregation deficit, R_D is defined by the next inequalities:

$$x \leq D(c,a) \quad \text{and} \quad y \leq D(b, 1-a).$$

$$c \geq R_D(a,x) \quad \text{and} \quad b \geq R_D(1-a, y).$$

This leads naturally to the following definition.

Definition 2.21 (Smutná-Hliněná and Vojtáš [20]) *Let D be a fuzzy disjunction. The aggregation deficit is defined by*

$$R_D(x,y) = \inf\{z \in [0,1]; D(z,x) \geq y\}.$$

Remark 2.22 *Note that one easily verifies the hybrid monotonicity of the aggregation deficit R_D. Let D_1 and D_2 be the disjunctions such that $\forall x, y \in [0,1]; D_1(x,y) \leq D_2(x,y)$. Then $R_{D_1}(x,y) \geq R_{D_2}(x,y)$ for every x, y. This follows from the fact that the aggregation deficit R_D is decreasing in its first argument.*

Let $D : [0,1]^2 \to [0,1]$ be a t-semiconorm. Then $R_D(x,y) \leq y$ for $(x,y) \in [0,1]^2$. If $x \geq y$, then $R_D(x,y) = 0$. It means, that for any aggregation deficit R_D it holds that $R_D \leq R_{S_M}$. More, if the partial mappings of disjunction D are infimum-morphism ($\inf\limits_{a \in M} D(x,a) = D(x, \inf\limits_{a \in M} a)$, where M is subset of interval $[0,1]$) then $x \geq y$ if and only if $R_D(x,y) = 0$. It follows from boundary condition and monotonicity of t-semiconorm D. Consider an aggregation deficit R_D, then the partial mapping $R_D(.,1)$ is negation on $[0,1]$. The aggregation deficit R_S of t-conorm S coincides with residual coimplication J_S, which was introduced by Bernard De Baets in [5] for different purpose.

3 Actual progress

We turn our attention to the investigation of properties under which the fuzzy implications are $(S, N)-$implications or $R-$implications. The following characterization of $(S, N)-$implications is from [2].

Theorem 3.1 (Baczyński and Jayaram [2], Theorem 5.1) *For a function $I : [0, 1]^2 \to [0, 1]$, the following statements are equivalent:*

- *I is an (S, N)-implication generated from some t-conorm and some continuous (strict, strong) fuzzy negation N.*
- *I satisfies (I2), (EP) and N_I is a continuous (strict, strong) fuzzy negation.*

For $R-$implications we have the following characterization, which is from [7].

Theorem 3.2 (Fodor and Roubens [7], Theorem 1.14) *For a function $I : [0, 1]^2 \to [0, 1]$, the following statements are equivalent:*

- *I is an R-implication based on some left-continuous t-norm T.*
- *I satisfies (I2), (OP), (EP), and $I(x, .)$ is a right-continuous for any $x \in [0, 1]$.*

At the moment we know a lot of families of generated fuzzy implications. We recall some classes of generated fuzzy implications which were proposed in various papers. Recently, several possibilities have occurred how to generate implications using appropriate one-variable functions.

We list the well-known of them and their properties. Yager [26] introduced two new families of fuzzy implications, called f-generated and g-generated fuzzy implications, respectively, and discussed their properties as listed in [7] or [6]. Also Jayaram in [12] discussed f-generated fuzzy implications with respect to three classical logic tautologies, such as distributivity, the law of importation and the contrapositive symmetry.

Proposition 3.3 (Yager [26]) *If $f : [0, 1] \to [0, \infty]$ is a strictly decreasing and continuous function with $f(1) = 0$, then the function $I : [0, 1]^2 \to [0, 1]$ defined by*

$$I(x, y) = f^{-1}(x \cdot f(y)), \quad x, y \in [0, 1], \tag{1}$$

with the understanding $0 \cdot \infty = 0$, is a fuzzy implication.

The function f is called an f–generator and the fuzzy implication represented by (1) is called an $f-$ fuzzy implication.

Yager [26] has also proposed another class of implications called the g-generated implications. In a similar way as in the part about $f-$fuzzy implications we present their properties.

Proposition 3.4 (Yager [26], p. 197) *If $g : [0, 1] \to [0, \infty]$ is a strictly increasing and continuous function with $g(0) = 0$, then the function $I : [0, 1]^2 \to [0, 1]$ defined by*

$$I(x, y) = g^{(-1)}\left(\frac{1}{x} \cdot g(y)\right), \quad x, y \in [0, 1], \tag{2}$$

with the understanding $\frac{1}{0} = \infty$ and $0 \cdot \infty = \infty$, is a fuzzy implication.

The function g is called a g–generator and the fuzzy implication represented by (2) is called a g–implication.

The f- and g-generators can be seen as the continuous additive generators of t-norms and t-conorms, respectively. A new family of fuzzy implications called the h-generated implications has been proposed by Jayaram in [11], where h can be seen as a multiplicative generator of a continuous Archimedean t-conorm. We present its definitions and a few of its properties. More details can be found in [4].

Proposition 3.5 (Jayaram [11]) *If $h : [0,1] \to [0,1]$ is a strictly decreasing and continuous function with $h(0) = 1$, then the function $I : [0,1]^2 \to [0,1]$ defined by*

$$I(x,y) = h^{(-1)}\left(x \cdot h(y)\right), \quad x,y \in [0,1], \tag{3}$$

is a fuzzy implication.

The function h is called an h–generator and the fuzzy implication represented by (3) is called an h–generated implication.

Smutná in [19] introduced generated fuzzy implications I_f, I^g and I^g_N. The implications I_f are generated with using strictly decreasing functions, the implications I^g are generated with using strictly increasing functions.

Proposition 3.6 (Smutná [19]) *Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing function such that $f(1) = 0$. Then the function $I_f : [0,1]^2 \to [0,1]$ which is given by*

$$I_f(x,y) = f^{(-1)}(f(y^+) - f(x)),$$

where $f(y^+) = \lim\limits_{t \to y^+} f(t)$ and $f(1^+) = f(1)$, is a fuzzy implication.

Construction of the fuzzy implications I^g is described in the following proposition:

Proposition 3.7 (Smutná [19]) *Let $g : [0,1] \to [0,\infty]$ be a strictly increasing function such that $g(0) = 0$. Then the function $I^g(x,y) : [0,1]^2 \to [0,1]$ which is given by*

$$I^g(x,y) = g^{(-1)}(g(1-x) + g(y)), \tag{4}$$

is a fuzzy implication.

Implications I^g may be further generalized. This generalization is based on the replacement of the standard negation by an arbitrary fuzzy negation.

Proposition 3.8 (Smutná [19]) *Let $g : [0,1] \to [0,\infty]$ be a strictly increasing function such that $g(0) = 0$ and N be a fuzzy negation. Then I^g_N, defined by*

$$I^g_N(x,y) = g^{(-1)}(g(N(x)) + g(y)),$$

is a fuzzy implication.

Fuzzy implications are closely related to the generators of a strict preference. The following proposition can be found in [7]. Fodor and Roubens supposed general triplet $(T_\varphi, S_\varphi, N_\varphi)$:

Proposition 3.9 (Fodor and Roubens [7], Proposition 3.5) *Let $S : [0,1]^2 \to [0,1]$ be any continuous t-conorm and $N : [0,1] \to [0,1]$ be a strong fuzzy negation. If $(p,i,j)_\varphi$ is a solution of the system*

$$S(p(x,y), i(x,y)) = x,$$
$$S(p(x,y), j(x,y)) = N(y),$$

then $I^{\to}(x,y) = N_\varphi(p(x,y))$ is a fuzzy implication such that

$$I^{\to}(1,x) = x \qquad \forall x \in [0,1],$$
$$I^{\to}(x,0) = N_\varphi(x) \quad \forall x \in [0,1].$$

Since we are dealing with Łukasiewicz triplet $(T_L, S_L, 1-x)$, this proposition can be simplified:

Proposition 3.10 *Let (p,i,j) be a solution of the system in (R4'), then $I^{\to}(x,y) = 1 - p(x,y)$ is a fuzzy implication and*

$$I^{\to}(1,x) = x \qquad \forall x \in [0,1],$$
$$I^{\to}(x,0) = 1 - x \quad \forall x \in [0,1].$$

The generator of indifference i and t-norms has common properties (both are symmetric and increasing mappings from $[0,1]^2$ to $[0,1]$). The following theorem shows that we can use some continuous t-norms in defining mapping i:

Theorem 3.11 (Fodor and Roubens [7]) *Assume that $p(x,y) = T_1(x, N_\varphi(y))$ and $i(x,y) = T_2(x,y)$, where T_1 and T_2 are continuous t-norms. Then $(p,i,j)_\varphi$ satisfies (R4) if and only if there exists a number $s \in [0,\infty]$ such that*

$$T_1(x,y) = \varphi^{-1}\left(T^s(\varphi(x),\varphi(y))\right),$$
$$T_2(x,y) = \varphi^{-1}\left(T^{1/s}(\varphi(x),\varphi(y))\right),$$

where T^s and $T^{1/s}$ belong to the Frank family.

In Pavelka's language of evaluated expressions, we would like to achieve the following: from $(\boldsymbol{C} \vee_D \boldsymbol{A}, x)$ and $(\boldsymbol{B} \vee_D \neg\boldsymbol{A}, y)$ to infer $(\boldsymbol{C} \vee_D \boldsymbol{B}, f_{\vee_D}(x,y))$ where $f_{\vee_D}(x,y)$ should be the best promise, we can give the truth function of disjunction $\vee_D$ and x and y. In the previous section we have mentioned the construction of new fuzzy operator, which is called the aggregation deficit. The formulation of a result on sound and complete full resolution is based on the aggregation operators. Smutná - Hliněná and Vojtáš in [20] investigated the *resolution truth function* $f_{R_D} : [0,1]^2 \to [0,1]$, which is defined by

$$f_{R_D}(x,y) = \inf_{a\in[0,1]}\{D(R_D(a,x), R_D(1-a,y))\}.$$

Theorem 3.12 (Smutná-Hliněná and Vojtáš [20]) *Assume the truth evaluation of proposition variables is a model of $(\boldsymbol{C} \vee_D \boldsymbol{A}, x)$ and $(\boldsymbol{B} \vee_D \neg\boldsymbol{A}, y)$. Then*

$$TV(\boldsymbol{C} \vee_D \boldsymbol{B}) \geq f_{R_D}(x,y).$$

4 The doctoral thesis objectives

The topic of the thesis is the study of generated fuzzy implications and their applications. There are two well-known families of implications - (S, N)–implications and R–implications. The main part of my research is studying connections between several classes generated fuzzy implications and families of (S, N)–implications and R–implications. Another hot topic of my research is the investigation of generators of fuzzy preference structures. This leads to search for the special conditions of the mentioned generated fuzzy implications. And the last direction of my thesis is devoted to fuzzy resolution, particularly to modelling of fuzzy modus ponens.

At this point my research can continue in several directions. Unless unforeseen circumstances occur, it is quite probable that the thesis will explore some of the following research directions:

- Investigation of some classes of generated implications
- Generated fuzzy implications as the generators of fuzzy preference structures
- Generated fuzzy implications in fuzzy resolution

5 Generated fuzzy implications

In [19] Smutná introduced generated implications I_f (for original description see the Theorem 3.6). This class of implications were later studied by Hliněná and Biba in [37].

In the article [29] the original description of I_f was slightly modified (and notation was changed to I_f^*). However, for continuous functions f both the definitions are equivalent. This later article presents new results concerning I_f^* implications as well as stronger versions of some previous results.

In my thesis we focus on I_f^* implications, their properties and intersections with classes of $(S,N)-$ and $R-$ implications. Implications I_f^* are described in the following way:

Proposition 5.1 (Biba, Hliněná, Kalina, and Kráľ [29]) *Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing function such that $f(1) = 0$. Then the function $I_f^*(x,y) : [0,1]^2 \to [0,1]$ which is given by*

$$I_f^*(x,y) = f^{(-1)} \left(\max\{0, f(y) - f(x)\}\right) \tag{5}$$

is a fuzzy implication.

In our investigation we use the following technical result of pseudo-inverse functions.

Proposition 5.2 (Hliněná and Biba [38]) *Let c be a positive real number. Then the pseudo-inverse of a positive multiple of any monotone function $f : [0,1] \to [0,\infty]$ satisfies*

$$(c \cdot f)^{(-1)}(x) = f^{(-1)}\left(\frac{x}{c}\right).$$

It is well-known that generators of continuous Archimedean t-norms are unique up to a positive multiplicative constant, and this is also valid for the f generators of I_f^* implications.

Proposition 5.3 (Biba, Hliněná, Kalina, and Kráľ [29], Hliněná and Biba [37]) *Let c be a positive constant and $f : [0,1] \to [0,\infty]$ be a strictly decreasing function. Then the implications I_f^* and $I_{c\cdot f}^*$ which are based on functions f and $c\cdot f$, respectively, are identical.*

Corollary 5.4 (Biba, Hliněná, Kalina, and Kráľ [29]) *Let $f : [0,1] \to [0,\infty]$ be bounded and strictly decreasing function such that $f(1) = 0$. Let $f^*(x) = \frac{f(x)}{f(0)}$. Then $I_f^* = I_{f^*}^*$ and also $f^*(0) = 1$. Hence, if f is a bounded function we can always assume that $f(0) = 1$.*

We study the properties of implications I_f^* under which they are (S,N)– or R–implications. For more details one can find the full version of the thesis. Using theorems 3.2, 3.1 we are able to partially characterize class of I_f^* implications:

Theorem 5.5 (Biba, Hliněná, Kalina, and Kráľ [29]) *Let $f : [0,1] \to [0,\infty]$ be a continuous strictly decreasing function such that $f(1) = 0$. Then I_f^* is an R–implication given by a continuous t-norm. Moreover, if $f(0) < \infty$ then I_f^* is an (S,N)–implication.*

The class of implications I^g was introduced by Smutná in [19] (original description is in the Theorem 3.7). This result was presented without proof, full proof can be found in [30].

We study properties of implications I^g and we also study the intersections between implications I^g and classes of $(S,N)-$ and $R-$ implications. Again, for more details one can find in the full version of thesis.

In the case of implications I_f^*, functions f and $(c \cdot f)$ give the same implication. This is also true for the g generators of implications I^g, since Lemma 5.2 holds for all monotone functions.

Proposition 5.6 *Let c be a positive constant and $g : [0,1] \to [0,\infty]$ be a strictly increasing function. Then the implications I^g and $I^{c \cdot g}$ which are based on functions g and $c \cdot g$, respectively, are identical.*

Corollary 5.7 *Let $g : [0,1] \to [0,\infty]$ be bounded and strictly increasing function such that $g(0) = 0$. Let $g^*(x) = \frac{g(x)}{g(1)}$. Then $I^g = I^{g^*}$ and also $g^*(1) = 1$. Hence, if g is a bounded function we can always assume that $g(1) = 1$.*

From the Theorem 3.1 we get the following relation between I^g implications and $(S,N)-$implications:

Theorem 5.8 *Let $g : [0,1] \to [0,\infty]$ be a strictly increasing function continuous on $]0,1]$ such that $g(0) = 0$. Then I^g is an (S,N)-implication which is strong.*

Analogically, using the Theorem 3.2 we get the following relation between implications I^g and $R-$implications:

Theorem 5.9 *Let $g : [0,1] \to [0,c]$ be a continuous and strictly increasing bounded function such that $g(0) = 0$. If $\forall x \in [0,1]$; $g(1-x) = g(1^-) - g(x)$, then I^g is an R-implication based on some left-continuous t-norm T.*

The implications I^g can be generalized by substituting the standard negation N_s by the arbitrary one (see the Theorem 3.8). This class was also introduced by Smutná in [19] and it was studied by Biba and Hliněná in [31] and [37]. Several results obtained for I^g implications are valid also for I_N^g implications.

The following two theorems follow from Theorems 3.2 and 3.1:

Theorem 5.10 (Biba and Hliněná [31]) *Let $g : [0,1] \to [0,c]$ be a continuous and strictly increasing function such that $g(0) = 0$, and negation N be right-continuous. If for all $x \in [0,1]$; $g(N(x)) = g(1^-) - g(x)$, then I_N^g is an R-implication given by some left-continuous t-norm T.*

Theorem 5.11 (Biba and Hliněná [31]) *Let $g : [0,1] \to [0,\infty]$ be a continuous and strictly increasing function such that $g(0) = 0$ and $N : [0,1] \to [0,1]$ be a continuous fuzzy negation. Then I_N^g is an (S,N)-implication where S is the t-conorm generated by g.*

A new class of generated fuzzy implications can be obtained by combining the previous approaches. We use a strictly decreasing function and a formula similar to Formula (4).

If we compose a strictly decreasing function f with a fuzzy negation N then $g(x) = f(N(x))$ is again an increasing function (though not necessarily strictly increasing). This allows us to generalize the fuzzy implications I^g. This class was introduced in [28] and it was studied in [32].

Theorem 5.12 (Biba, Hliněná, Kalina and Král' [28] without proof, Biba and Hliněná [32]) *Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing function with $f(1) = 0$ and $N : [0,1] \to [0,1]$ be a fuzzy negation. Then the function $I_f^N : [0,1]^2 \to [0,1]$ defined by*

$$I_f^N(x,y) = N\left(f^{(-1)}\left(f(x) + f(N(y))\right)\right), \tag{6}$$

is a fuzzy implication.

The following theorem describes the relationship between the generated fuzzy implications I_N^f and (S,N)– or R–implications.

Theorem 5.13 (Biba and Hliněná [32]) *Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing and continuous function such that $f(1) = 0$. Let $N : [0,1] \to [0,1]$ be a strong negation. Then I_f^N is (S,N)–implication. Moreover, if f is bounded function and $N(x) = f^{-1}(f(0) - f(x))$, then I_f^N is an R–implication as well.*

Note, that mentioned fuzzy implications are not the only generalizations of fuzzy implications I_N^g. Considering Formula (6) and Lemma 5.14, we can see that N might be replaced by $N^{(-1)}$ if it is a fuzzy negation. Still, there are at least two fuzzy negations (in general different from N) which are related to N. Namely, $N^{(-1)}$ and N^d. Hence we have the following two additional possibilities how to generate fuzzy implications.

If we apply the pseudo-inverse to a negation N we get the following assertion.

Lemma 5.14 (Biba, Hliněná, Kalina, and Král' [28]) *Let $N : [0,1] \to [0,1]$ be a fuzzy negation. Then $N^{(-1)}$ is a fuzzy negation if and only if*

$$N(x) = 0 \quad \Leftrightarrow \quad x = 1. \tag{7}$$

Theorem 5.15 (Biba, Hliněná, Kalina and Král' [28]) *Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing function with $f(1) = 0$, and $N : [0,1] \to [0,1]$ be a fuzzy negation such that (7) is fulfilled for N. Then the function $I_f^{(N,N^{(-1)})} : [0,1]^2 \to [0,1]$ defined by*

$$I_f^{(N,N^{(-1)})}(x,y) = N^{(-1)}\left(f^{(-1)}\left(f(x) + f(N(y))\right)\right), \tag{8}$$

is a fuzzy implication.

Theorem 5.16 (Biba, Hliněná, Kalina and Král' [28]) *Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing function with $f(1) = 0$ and $N : [0,1] \to [0,1]$ be a fuzzy negation. Then function $I_f^{(N,N^d)} : [0,1]^2 \to [0,1]$ defined by*

$$I_f^{(N,N^d)}(x,y) = N^d\left(f^{(-1)}\left(f(x) + f(N(y))\right)\right), \tag{9}$$

is a fuzzy implication.

6 Preference structures given by generated fuzzy implications

In this section we study the construction of fuzzy preference structures from fuzzy implications. We use the fuzzy implications I_f and I^g mentioned in the previous chapter. The following results can be found in the article [30] by Biba and Hliněná. The inspiration for this investigation was the article [23] by Šabo and Strežo.

First we turn our attention to the fuzzy implications I_f. In the next example, we deal with the Łukasiewicz triplet $(T_L, S_L, 1-x)$.

Example 6.1 (Biba and Hliněná [30]) *Let $f(x) = N_s(x)$. Note that fuzzy negation N_s satisfies assumptions of Proposition 5.1. We obtain fuzzy implication $I_{N_s}(x,y) = \min(1-x+y, 1)$. For function p we have*

$$p(x,y) = 1 - I_{N_s}(x,y) = \max(x-y, 0).$$

In order to satisfy (R4'), mappings i, j must be $i(x,y) = \min(x,y)$ and $j(x,y) = \min(1-x, 1-y)$. Obviously i and j are symmetric functions. Therefore (R3) is satisfied.

Now, we turn our attention to the properties (gt1)–(gt5). Axioms (R3) and (R4') imply properties (gt3) and (gt5). More, from (R3) and (R4') we have

$$p(x,y) + p(y,x) + i(x,y) + j(x,y) = p(x,y) + i(x,y) + p(y,x) + j(y,x) = x + 1 - x = 1.$$

Therefore property (gt4) again follows from (R3) and (R4').

It is obvious that in this example the properties (gt1) and (gt2) are satisfied, too. Therefore triplet (p,i,j) is the monotone generator triplet.

The following proposition shows that the fuzzy implications I_{T_L} is the only one we can use:

Proposition 6.2 (Biba and Hliněná [30]) *Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing function such that $f(1) = 0$ and $p(x,y) = 1 - I_f(x,y)$. Then triplet (p,i,j), where $i(x,y) = x - p(x,y)$ and $j(x,y) = 1 - y - p(x,y)$, satisfies (R3) and (R4') if and only if $I_f(x,y) = I_{T_L}$.*

Proposition 6.3 (Biba and Hliněná [30]) *Let $I : [0,1]^2 \to [0,1]$ be a fuzzy implication satisfying (OP), and $p(x,y) = 1 - I(x,y)$. Then the triplet (p,i,j) satisfies (R3) and (R4') if and only if $I(x,y) = I_{T_L}$.*

For the triplet $((T_L)_\varphi, (S_L)_\varphi, (N_s)_\varphi)$ and fuzzy implications I_f we get a result similar to Proposition 6.2:

Proposition 6.4 (Biba and Hliněná [30]) *Let φ be an order-automorphism. Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing function such that $f(1) = 0$, and*

$$I_f(x,y) = \begin{cases} 1 & x \le y, \\ f^{(-1)}(f(y^+) - f(x)) & x > y. \end{cases}$$

Then the system $(p,i,j)_\varphi$ where $p(x,y) = (N_s)_\varphi(I_f(x,y))$ satisfies (R3), and (R4') if and only if $I_f(x,y) = \min(\varphi^{-1}(1 - \varphi(x) + \varphi(y)), 1)$.

Now we turn our attention to the fuzzy implications I^g and I^g_N. The partial mapping of $I^g(x,0)$ is $I^g(x,0) = 1-x$, and for an arbitrary fuzzy negation N we have $I^g_N(x,0) = N(x)$. On the other hand, Proposition 3.10 gives that $I^{\rightarrow}(x,0) = 1-x$, therefore we will investigate function $p(x,y) = 1 - I^g(x,y)$. Using (R4'), we get $i(x,y) = I^g(x,y) + x - 1$ and $j(x,y) = I^g(x,y) - y$. From (R3), the function i is symmetric, which leads to the equality

$$I^g(x,y) - I^g(y,x) = y - x \qquad \forall x, y \in [0,1]. \tag{10}$$

If this equality is fulfilled for some fuzzy implication I, then the described triplet (p,i,j) is a generator triplet.

We are looking for functions g, such that fuzzy implications I^g satisfy the equality (10).

We present special class of fuzzy implications with the equality (10). This class of fuzzy implications is not isomorphic with I_{T_L} for arbitrary $s \in]0,\infty[-\{1\}$.

Proposition 6.5 (Biba and Hliněná [30]) *Let* $s \in]0,\infty[-\{1\}$ *and* $g_s(x) = \ln \frac{s-1}{s^{1-x}-1}$. *Then the fuzzy implication* I^{g_s} *satisfies equality* $I(x,y) - I(y,x) = y - x$.

Corollary 6.6 (Biba and Hliněná [30]) *Let* $s \in]0,\infty[-\{1\}$. *If*

$$I^{g_s}(x,y) = 1 - \log_s \left(\frac{(s^x - 1)\cdot(s^{1-y}-1)}{s-1} + 1 \right),$$

then there exists a triplet of generators (p,i,j), *such that* $p(x,y) = 1 - I^{g_s}(x,y)$.

Remark 6.7 *Note that described triplet of generators is same as triplet in Theorem 3.11 in case when* $\varphi(x) = x$.

We have investigated the case, when $p(x,y) = 1 - I^g(x,y)$. A more general formula is $p(x,y) = N^{(-1)}(I^g_N(x,y))$. In this case, the condition for the generator of triplet is

$$N^{(-1)}(I^g_N(y,x)) - N^{(-1)}(I^g_N(x,y)) = y - x.$$

In this chapter we described construction methods of monotone generators for fuzzy preference structures with use of generated fuzzy implications. It is possible that there exists other solutions of the equality (10). We plan to describe this solutions in the future.

7 Modus ponens

In the last chapter of the thesis we study a many-valued case of modus ponens with clause-based rules and we compare the results with estimations of modus ponens via implicative rules. This part is based on results of work [20]. In the second part we propose a discrete case of many-valued modus ponens. Results presented in this section are found in [36].

For implicative rules, the following estimation of modus ponens is in [8] and [9]

$$\frac{(\boldsymbol{B}, b), (\boldsymbol{B} \rightarrow \boldsymbol{H}, r)}{\boldsymbol{H}, f_{\rightarrow}(b, r)}.$$

We know that the implication ($\boldsymbol{B} \rightarrow \boldsymbol{H}$) is true to degree r (at least). Therefore $\boldsymbol{H}$ must be true to some degree h such that $I(b, h) \geq r$. We need to find the least value h with this property in order to guarantee that $TV(\boldsymbol{H}) \geq h$. Let I be the truth function of implication $\rightarrow$, then truth function $f_{\rightarrow}$ is residual conjunction of implication I (note mnemonic body-head-rule notation of variables)

$$f_{\rightarrow}(b, r) = C_I(b, r) = \inf\{h \in [0, 1]; I(b, h) \geq r\}.$$

7.1 Modus ponens for clause based rules

To be consistent with body-head-rule notation of [14], we will use it also here for clausal rules.

Another possibility is to calculate the lower bound on the truth value of $\boldsymbol{H}$ using aggregation deficit.

Example 7.1 (Hliněná and Biba [36]) *To have a sound clause based modus ponens, we make following observation. Let $D : [0, 1]^2 \rightarrow [0, 1]$ be a commutative disjunction. If for all $b, r \in [0, 1]$*

$$(\boldsymbol{B}, b) \text{ and } (\neg \boldsymbol{B} \vee_D \boldsymbol{H}, r) \text{ should imply } (\boldsymbol{H}, g_D(b, r)),$$

then using Theorem 2.2

$$r \leq D(1 - b, h) \implies r \leq D(h, 1 - b) \implies h \geq R_D(1 - b, r).$$

Hence the best possible estimate for h is

$$g_D(b, r) = \inf_{b' \geq b} R_D(1 - b', r).$$

Since the aggregation deficit R_D is decreasing in the first argument, hence $\inf_{b' \geq b} R_D(1 - b', r) = R_D(1 - b, r)$, it means that

$$g_D(b, r) = R_D(1 - b, r).$$

Remark 7.2 *Note that the truth value of $\boldsymbol{H}$ depends on the truth functions of disjunction and negation. Therefore, on a very formal level, one would write $g_{\vee_D \neg_N}$. To make the notation shorter we omit the symbols of disjunction and negation, since it they do not bear any additional information. Because we deal only with the standard negation N_s in this section, symbol N is omitted as well. We thus use g_D.*

For commutative disjunctions we get:

Theorem 7.3 (Hliněná and Biba [36])

1. *Let* $D_1 \leq D_2$, *then* $g_{D_1} \geq g_{D_2}$.

2. *Let* D *be a t-semiconorm, then* $g_D \leq g_{S_M}$.

3. *Function* g_D *is increasing in both arguments.*

4. *Let* $D : [0,1]^2 \to [0,1]$ *be a commutative* $t-$*semiconorm. For function* g_D *we get* $g_D(1,1) = 1, g_D(0,x) = g_D(x,0) = 0$. *It means, the function* g_D *is the fuzzy conjunction.*

Remark 7.4 (Hliněná and Biba [36]) *If a commutative* $t-$*semiconorm* D *possesses the properties*

$$D(x,y) = 1 \text{ for all } x,y \in [0,1], \text{ such that } x+y=1$$

$$D(x,y) < 1 \text{ for all } x,y \in [0,1], \text{ such that } x+y<1$$

then g_D *is a* $t-$*seminorm. These properties guarantee that the boundary condition* $g_D(x,1) = x$ *is satisfied for all* $x \in [0,1]$. *The second boundary condition,* $g_D(1,x) = x$, *is satisfied for arbitrary commutative* $t-$*semiconorm* D. *Note that, for example,* $t-$*conorm* S_L *possesses these properties.*

Estimation for clause rules and implicative rules are in some cases identical:

Theorem 7.5 (Hliněná and Biba [36]) *Let* $g_D : [0,1]^2 \to [0,1]$ *be truth function based on* R_D, *where* D *is a commutative disjunction and* $C_I : [0,1]^2 \to [0,1]$ *be a truth function based on* I, *where* $I(b,h) = D(h, 1-b)$. *Then*

$$C_I(b,r) = g_D(b,r)$$

for all $b, r \in [0,1]$.

7.2 Discrete many valued modus ponens

Assume users will evaluate preference on attributes X and Y with fuzzy or linguistic values x and y. In this part we will estimate modus ponens via discrete connectives. It is known ([24] and `http://en.wikipedia.org/wiki/Likert_scale`), that people are not able to sort according to quality to more than 7 ± 2 categories. In accordance with this fact we use coefficients k, l as follows:

$$k \in \{5,6,7,8,9\} \quad \text{and} \quad l \in \{5,6,7,8,9\}.$$

And for m (the number of roundings) we take $m = k * l$, which provides us with good ordering of results. The meaning of these coefficients will be come obvious in the next definition of a discrete fuzzy conjunction:

Definition 7.6 *Let $C : [0,1]^2 \to [0,1]$ be a fuzzy conjunction, $k \in \{5,6,7,8,9\}, l \in \{5,6,7,8,9\}$ and $m = k * l$. Mapping $C^m_{k,l} : [0,1]^2 \to [0,1]$ which is defined as follows*

$$C^m_{k,l}(x,y) = \frac{\left\lceil m \cdot C\left(\frac{\lceil k\cdot x\rceil}{k}, \frac{\lceil l\cdot y\rceil}{l}\right)\right\rceil}{m}$$

is called a discrete fuzzy conjunction.

Obviously this mapping is a fuzzy conjunction. However it is not a t-seminorm. Commutative or associative conjunction C may lead to $C^m_{k,l}$ without these properties. Note, that if a conjunction C is commutative, then the discrete conjunction $C^m_{k,k}$ is commutative, too. Dual mapping to the discrete conjunction is given by a similar equality.

Theorem 7.7 (Hliněná and Biba [36]) *Let $C : [0,1]^2 \to [0,1]$ and $D : [0,1]^2 \to [0,1]$ be the dual conjunction and disjunction which are continuous, $k \in \{5,6,7,8,9\}, l \in \{5,6,7,8,9\}$ and $m = k*l$. Then the dual discrete fuzzy disjunction to $C^m_{k,l}$ is the mapping $D^m_{k,l} : [0,1]^2 \to [0,1]$ such that*

$$D^m_{k,l}(x,y) = \frac{\left\lfloor m \cdot D\left(\frac{\lfloor k\cdot x\rfloor}{k}, \frac{\lfloor l\cdot y\rfloor}{l}\right)\right\rfloor}{m}. \tag{11}$$

Since the functions $\lceil x\rceil$ and $\lfloor x\rfloor$ are left- and right-continuous, respectively, discrete cojunctors and disjunctors also possess same properties:

Theorem 7.8 (Hliněná and Biba [36]) *Let $C : [0,1]^2 \to [0,1]$ be a continuous fuzzy conjunction. Then the discrete fuzzy conjunction $C^m_{k,l}$ is left-continuous and the discrete fuzzy disjunction $D^m_{k,l}$ is right-continuous.*

Remark 7.9 (Hliněná and Biba [36]) *Let $C : [0,1]^2 \to [0,1]$ be a fuzzy conjunction and $D : [0,1]^2 \to [0,1]$ be a fuzzy disjunction. Then the following inequalities hold:*

- $C \le C^m_{k,l}$,
- $D \ge D^m_{k,l}$.

The discrete aggregation deficit is denoted by R^*_D. By definition, the aggregation deficit R^*_D is given by the formula

$$R^*_D(x,y) = \inf\{z \in [0,1]; \left\lfloor m \cdot D\left(\frac{\lfloor k\cdot z\rfloor}{k}, \frac{\lfloor l\cdot x\rfloor}{l}\right)\right\rfloor \ge m\cdot y\}.$$

Theorem 7.10 (Hliněná and Biba [36]) *Let $D : [0,1]^2 \to [0,1]$ be a continuous fuzzy disjunction and $D^m_{k,l}$ be a discrete fuzzy disjunction. Let $R_D : [0,1]^2 \to [0,1]$ and $R^*_D : [0,1]^2 \to [0,1]$ be the aggregation deficits given by D and $D^m_{k,l}$ respectively. Then the following equality holds:*

$$R^*_D(x,y) = \frac{\left\lceil k \cdot R_D\left(\frac{\lfloor l\cdot x\rfloor}{l}, \frac{\lceil m\cdot y\rceil}{m}\right)\right\rceil}{k}.$$

Corollary 7.11 (Hliněná and Biba [36]) *Let $g_D : [0,1]^2 \to [0,1]$ and $g_D^* : [0,1]^2 \to [0,1]$ be the estimaties of modus ponens with commutative disjunctions D and $D^m_{k,k}$ respectively. Then the following equality holds:*

$$g_D^*(b,r) = \frac{\left\lceil k \cdot g_D\left(\frac{\lceil k \cdot b\rceil}{k}, \frac{\lceil m \cdot r\rceil}{m}\right)\right\rceil}{k}.$$

Since $f_{\to}(b,r) = C_{I_D}(b,r)$, it may seem that one can obtain discrete operator $f^*_{\to}$ simply from conjunction C_{I_D} using Definition 7.6 However, this is not a correct procedure - residual conjunction to I^*_D is different.

Theorem 7.12 (Hliněná and Biba [36]) *Let $D : [0,1]^2 \to [0,1]$ be a continuous disjunction. Let $I^*_D : [0,1]^2 \to [0,1]$ be a material implication given by discrete disjunction $D^m_{k,l}$. Then the discrete residual conjunction to I^*_D is given by*

$$C_{I^*_D}(b,r) = \frac{\left\lceil k \cdot C_{I_D}\left(\frac{\lceil l \cdot b\rceil}{l}, \frac{\lceil m \cdot r\rceil}{m}\right)\right\rceil}{k}.$$

If conjunction $C_{I^*_D}$ is obtained using disjunction D without non-trivial zero divisors, then either $C_{I^*_D}(b,1) = 0$ or $C_{I^*_D}(b,1) = 1$. It is generalized in the following theorem:

Theorem 7.13 (Hliněná and Biba [36]) *Let $D^m_{k,l}$ be a discrete disjunction without non-trivial zero divisors, then $C_{I^*_D}(0,1) = 0$ and $C_{I^*_D}(b,1) = 1$ for all $b > 0$.*

8 Conclusions

The thesis shows new results concerning various classes of fuzzy implications given by one-variable functions. We described the properties of these classes of generated implications and also the intersections with already known classes of $(S, N)-$ and $R-$ implications. We also described the possibility of defining fuzzy preference structures using fuzzy implications. Some possibilities of generating a fuzzy implication are given without closer study, properties of these classes is not fully known to-day.

The thesis contains also some new results about many-valued modus ponens rule. The most interesting of results are those about discrete case of many-valued modus ponens. Many-valued modus ponens is used in fuzzy inference, fuzzy regulation etc. The discrete modus ponens can be used especially in cases when the input is not physical parameter (which can be measured with good precision), but instead, input is a qualitative characteristic (see "Likert scale"). The possible application of this discrete many-valued modus ponens is in the decision-making process. In my future work I would like to continue research in discrete modus ponens and, particularly, its possible applications in multicriteria decision making.

References

[1] ALSINA, C.: On a family of connectivities for fuzzy sets *Fuzzy Sets and Systems* 16 (1985), p. 231-235, ISSN 0165-0114.

[2] BACZYŃSKI, M., BALASUBRAMANIAM, J.: On the characterizations of (S,N)-implications *Fuzzy Sets and Systems* 158 (2007), p. 1713–1727, ISSN 0165-0114.

[3] BACZYŃSKI, M., BALASUBRAMANIAM, J.: *Fuzzy implications* Studies in Fuzziness and Soft Computing, Vol. 231, Springer, Berlin 2008, ISBN 978-3-540-69080-1

[4] BACZYŃSKI, M., JAYARAM, B.: Yager's classes of fuzzy implications: some properties and intersections *Kybernetika*, vol. 43 (2007), p. 157–182, ISSN 0023-5954

[5] DE BAETS, B.: Coimplicators, the forgotten connectives, *Tatra Mt. Math. Publ. 12* (1997), 229-240, ISSN 1210-3195

[6] DUBOIS, D., PRADE, H.: Fuzzy sets in approximate reasoning. Part I. Inference with possibility distributions, *Fuzzy Sets and Systems*, 40 (1991), 143–202, ISSN 0165-0114

[7] FODOR, J. C., ROUBENS, M. *Fuzzy Preference Modelling and Multicriteria Decision Support*, Kluwer Academic Publishers, Dordrecht (1994). ISBN 0-7923-3116-8

[8] GOTTWALD, S.: *Fuzzy Sets and Fuzzy Logic*, Vieweg, Braunschweig 1993, ISBN 3528053119

[9] HÁJEK, P.: Metamathematics of fuzzy logic. Kluwer Acad. Publ., Dordrecht 1999, ISBN 0792352386

[10] JAYARAM, B. MESIAR, R.: $I-$fuzzy equivalence relations and I-fuzzy partitions, *Inform. Sci.* 179(9) (2009), 1278–1297, ISSN 0020-0255

[11] JAYARAM, B.: Contrapositive symmetrization of fuzzy implications - Revisited, *Fuzzy Sets and Systems 157* (2006), 2291–2310, ISSN 0165-0114

[12] JAYARAM, B.: Yager's new class of implications J_f and some classical tautologies, *Information Sciences*, 177(3) (2007), 930 – 946, ISSN 0020-0255

[13] KLEMENT, E. P., MESIAR, R., PAP, E. *Triangular Norms*, Kluwer, Dordrecht 2000, ISBN 0-7923-6416-3

[14] KRAJČI, S., LENCSES, R., VOJTÁŠ, P.: A comparison of fuzzy and annotated logic programming, *Fuzzy Sets and Systems 144* (2004), 173-192, ISSN 0165-0114

[15] LING, C. M.: Representation of associative functions In: *Publ. Math. Debrecen 12* (1965) 189-212, ISSN 0033-3883

[16] MAS, M., MONSERRAT, M., TORRENS, J., TRILLAS, E.: A survey on fuzzy implication functions *IEEE Transactions on Fuzzy Systems*, 15 (2007), p. 1107–1121, ISSN 1063-6706

[17] SCHWEIZER, B., SKLAR, A. *Probabilistic Metric Spaces*, North Holland, New York 1983, ISBN 0-444-00666-4

[18] SMETS, P., MAGREZ, P.: Implication in fuzzy logic In: *Int. J. Approx. Reasoning 1* (1987) 327–347, ISSN: 0888-613X

[19] SMUTNÁ, D.: On many valued conjunctions and implications *Journal of Electrical Engineering*, 10/s, vol. 50 (1999), p. 8–10, ISSN 1335-3632.

[20] SMUTNÁ - HLINĚNÁ, D., VOJTÁŠ, P.: Graded many valued resolution with aggregation, *Fuzzy Sets and Systems 143* (2004), 157-168, ISSN 0165-0114

[21] SMUTNÁ, D., VOJTÁŠ, P.: New connectives for (full)fuzzy resolution, *Advances in Soft Computing*, Physica-Verlag, Heidelberg (2000), 146-151, ISBN 3-7908-1322-2
[22] STREŽO, P. Rozhodovacie procesy vo viachodnotovej logike, rigorózna práca, UMB Banská Bystrica, 2009
[23] ŠABO, M., STREŽO, P.: On reverses of some binary operators *Kybernetika*, vol. 41 (2005), p. 435–450, ISSN 0023-5954
[24] VOJTÁŠ, P.: Fuzzy reasoning with tunable t-operators, *J. Advanced Comp. Intelligence 2* (1998), 121-127, ISSN 1883-8014
[25] VAN DE WALLE, B., DE BAETS, B., KERRE, E.: A comparative study of completeness conditions in fuzzy preference structures. *In: Proc. IFSA'97, vol. III*, Prague, 74–79, ISBN 80-200-0633-8
[26] YAGER, R., R.: On some new classes of implication operators and their role in approximate reasoning *Inform. Sci. 167* (2004), 193–216, ISSN 0020-0255
[27] ZADEH, L., A.: Similarity relations and fuzzy orderings. *Inform. Sci. 3* (1971), 177-200, ISSN 0020-0255

Selected publications of the author

[28] BIBA, V., HLINĚNÁ, D., KALINA, M., KRÁL', P.: I - fuzzy equivalences and I - partitions, In *Zittau 17th Fuzzy Colloquium Conference Proceedings* (2010) 89-96, ISBN 978-3-9812655-4-5
[29] BIBA, V., HLINĚNÁ, D., KALINA, M., KRÁL', P.: Implicators generated by strictly decreasing functions and known classes of implicators *submited to Inf.Sci.*, ISSN 0020-0255
[30] BIBA, V., HLINĚNÁ, D.: Generated fuzzy implications and fuzzy preference structures, *accepted in Kybernetika*, ISSN 0023-5954
[31] BIBA, V., HLINĚNÁ, D. Intersection of generated S, N-implications and R-implications *XXVIII Internation colloqium on the management of the educational process*, Brno (2010), ISBN 978-80-7231-779-0
[32] BIBA, V., HLINĚNÁ, D. A note on some classes of generated fuzzy implications In *Aplimat 2012, 11th international conference.* Bratislava (2012), p. 321-330, ISBN 978-80-89313-58-7
[33] BIBA, V. A note on generated implicators In *XXVII Internation colloqium on the management of the educational process*, Brno (2009), p. 1-9, ISBN 978-80-7231-650-2
[34] BIBA, V. Generated implicators In *ICSC - Seventh International Conference on Soft Computing Applied in Computer and Economic Environments*, Hodonín (2009), p. 175-180, ISBN 978-80-7314-163-9
[35] BIBA, V. Generated fuzzy implicators and fuzzy preference structures In *Aplimat 2010, 9th international conference.* Bratislava (2010), p. 397-405, ISBN 978-80-89313-47-1
[36] HLINĚNÁ, D., BIBA, V.: Evaluating many valued modus ponens, *accepted in Kybernetika*, ISSN 0023-5954
[37] HLINĚNÁ, D., BIBA, V. Generated fuzzy implications and known classes of implications, *Acta Universitatis Matthiae Belii* 16 (2009), p. 25-34, ISBN 978-80-8083-955-0
[38] HLINĚNÁ, D., BIBA, V. Generated fuzzy implicators and their properties In *Aplimat 2009, 8th international conference.* Bratislava (2009), p. 365-374, ISBN 978-80-89313-47-1

Curriculum Vitae

Name:	Vladislav Biba
Date of birth:	september 3th, 1981
Country:	Slovakia
Nationality:	Slovak
Contact:	v_biba@post.sk

Education

2000–2005	Matej Bel University in Banská Bystrica, Faculty of Natural Sciences, Teacher training for general subjects (mathematics and physics), Master study
2005–2008	Matej Bel University in Banská Bystrica, Faculty of Natural Sciences, Mathematical statistics and financial mathematics, Bachelor study
2008–2012	Brno University of Technology, Faculty of Electrical Engineering and Communication, Mathematics in electrical engineering, Ph.D. study

Languages

English, Slovak, Czech

Abstract

The Ph.D. thesis is focused on implications in many-valued logic. The main topic of interest are fuzzy implications generated by one-variable functions. This approach is known to-day mainly in case of t-norms, which are used to model a conjunction in many-valued logics. Several possibilities of construction of fuzzy implications via one-variable functions are given. Properties of these classes of generated implications and their intersections with known classes of $(S, N)-$ and $R-$ implications are studied.

The second topic of interest is a construction of fuzzy preference structures (FPS for short) using these generated fuzzy implications. Fuzzy preference structures present one of well-known apparatuses to model preference when working with vague notions. Our approach to construction of FPS utilizes the connection between fuzzy preference relations and fuzzy implications.

The last part is focused on a many-valued case of the modus ponens rule. Modus ponens is the most frequent rule of inference and it is used for example in artificial intelligence. There are two possible definitions of modus ponens, one with implicative rules and other with clause-based rules. In the case of many-valued logic, it is necessary to distinct between these two definitions, therefore we study them separately. One possible approach to define many-valued discrete case of modus ponens rule is also studied.

Abstrakt

Dizertačná práca sa zaoberá implikátormi vo viachodnotovej logike. Hlavným objektom záujmu sú implikátory vytvorené pomocou funkcie jednej premennej, čo je prístup známy hlavne v prípade t-noriem, ktoré modelujú konjunkciu vo fuzzy logike. Opísaných je niekoľko možností konštrukcie fuzzy implikátora pomocou funkcie jednej premennej. Skúmané sú vlastnosti takto vygenerovaných implikátorov a takisto prienik tried generovaných implikátorov so známymi triedami $(S, N)-$ a $R-$ implikátorov.

Ďalej sa práca zaoberá možnostou konštrukcie fuzzy preferenčných štruktúr s pomocou uvedených implikátorov. Fuzzy preferenčné štruktúry sú jedným z využívaných nástrojov pri modelovaní preferencie v práci s vágnymi pojmami. Prezentovaný prístup konštrukcie využíva vzťah medzi reláciou preferencie a fuzzy implikátormi.

V poslednej časti sa zaoberáme viachodnotovou podobou pravidla modus ponens. Modus ponens je najčastejšie využívaným pravidlom odvodzovania a nachádza využitie napr. v umelej inteligencii. Modus ponens je možné definovať s využitím implikatívnych alebo klauzálnych pravidiel. V prípade viachodnotovej logiky musíme tieto dve možnosti rozlišovať, skúmané sú preto obidve. Takisto je ukázaný jeden z možných prístupov pri definovaní viachodnotovej diskrétnej podoby tohoto pravidla.