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ON AN EQUATION RELATED TO NONADDITIVE ENTROPIES IN INFORMATION THEORY

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Abstract. The paper provides the general solutions of a sum form functional equation containing three unknown mappings with some of the solutions related to the nonadditive entropies in information theory.

1. Introduction

For $n = 1, 2, \ldots,$ let $\Gamma_n = \left\{(p_1, \ldots, p_n) : p_i \geq 0, i = 1, \ldots, n; \sum\limits_{i=1}^{n} p_i = 1\right\}$ denote the set of all n-component finite discrete complete probability distributions with nonnegative elements.

Behara and Nath [1] considered the functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} p_i^{\alpha} \sum_{j=1}^{m} f(q_j) + \sum_{j=1}^{m} q_j^{\beta} \sum_{i=1}^{n} f(p_i) \tag{1.1}$$

with $f : I \to \mathbb{R}$ an unknown mapping, $I = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$, $\mathbb{R}$ denoting the set of all real numbers; $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; $n = 1, 2, \ldots;$ $m = 1, 2, \ldots;$ α and β being fixed positive real powers; $0^{\alpha} := 0$, $0^{\beta} := 0$ and $1^{\alpha} := 1$, $1^{\beta} := 1$. They found the continuous solutions of (1.1).

Losonczi and Maksa ([3], p. 263) considered the functional equation (1.1) with $f : I \to \mathbb{R}$ an unknown mapping, $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; $n \geq 3$ and $m \geq 2$ being fixed integers; $1 \neq \alpha \in \mathbb{R}$, $1 \neq \beta \in \mathbb{R}$; $0^{\alpha} := 0$, $0^{\beta} := 0$; $1^{\alpha} := 1$, $1^{\beta} := 1$; and they found the general solutions of (1.1) without imposing any regularity condition on the mapping $f : I \to \mathbb{R}$.

The object of this paper is to determine the general solutions of the functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} g(p_i) \sum_{j=1}^{m} h(q_j) + \sum_{j=1}^{m} q_j^{\beta} \sum_{i=1}^{n} h(p_i) \tag{A}$$

with $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; $n \geq 3$ and $m \geq 3$ being fixed integers; β a fixed positive real power satisfying the conventions $0^{\beta} := 0$, $1^{\beta} := 1$; f, g, h being unknown real-valued mappings each with domain I.

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If $g(x) = x$ and $\beta = 1$, then (A) reduces to the equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} h(p_i) + \sum_{j=1}^{m} h(q_j)$$

which has been discussed in [5].

If $g(x) = x^\alpha$ for all $x \in I$, α being a fixed positive real power; $0^\alpha := 0$, $1^\alpha := 1$, then (A) reduces to the functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} p_i^\alpha \sum_{j=1}^{m} h(q_j) + \sum_{j=1}^{m} q_j^\beta \sum_{i=1}^{n} h(p_i) \tag{1.2}$$

which is a Pexider-type generalization of (1.1). Notice that (1.2) reduces to (1.1) when $h(x) = f(x)$ for all $x \in I$. Some results concerning the functional equation (1.2) will be presented elsewhere.

Now, we mention below some definitions and results needed for the development of subsequent sections of this paper.

A mapping $a : \mathbb{R} \to \mathbb{R}$ is said to be additive if $a(x + y) = a(x) + a(y)$ for all $x \in \mathbb{R}$, $y \in \mathbb{R}$.

A mapping $M : I \to \mathbb{R}$ is said to be multiplicative if $M(xy) = M(x)M(y)$ for all $x \in I$, $y \in I$.

A mapping $\ell : I \to \mathbb{R}$ is said to be logarithmic if $\ell(0) = 0$ and $\ell(xy) = \ell(x)+\ell(y)$ for all $x \in]0, 1]$, $y \in]0, 1]$, $]0, 1] = \{x \in \mathbb{R} : 0 < x \leq 1\}$.

Result 1.1 ([4]). *Let a mapping $\phi : I \to \mathbb{R}$ satisfy the functional equation $\sum_{i=1}^{n} \phi(p_i) = c$ for all $(p_1, \ldots, p_n) \in \Gamma_n$, $n \geq 3$ a fixed integer and c a given real constant. Then, there exists an additive mapping $A : \mathbb{R} \to \mathbb{R}$ such that $\phi(p) = A(p) - \frac{1}{n}A(1) + \frac{c}{n}$ for all $p \in I$.*

Result 1.2 ([6]). *Let $n \geq 3$, $m \geq 3$ be fixed integers. If mappings $H : I \to \mathbb{R}$, $G : I \to \mathbb{R}$ satisfy the equation*

$$\sum_{i=1}^{n}\sum_{j=1}^{m} H(p_i q_j) = \sum_{i=1}^{n} G(p_i) \sum_{j=1}^{m} H(q_j) + \sum_{i=1}^{n} H(p_i) \sum_{j=1}^{m} q_j^\beta + (m - n)H(0) \sum_{j=1}^{m} q_j^\beta + m(n - 1)H(0) \tag{B}$$

for all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; $\beta \neq 1$ a fixed positive real power such that $0^\beta := 0$ and $1^\beta := 1$. Then, any general solution of (B) is of the form

$$\left.\begin{array}{ll} \text{(i)} & H(p) = [H(1) + (m - 1)H(0)]p^\beta + b_1(p) + H(0); b_1(1) = -mH(0) \\ \text{(ii)} & G(p) = a_1(p) + G(0);\ a_1(1) = -nG(0) \end{array}\right\} \tag{β_1}$$

or

$$\left.\begin{array}{ll} \text{(i)} & H(p) = p^\beta \ell(p) - \bar{b}(p) + H(0);\ \bar{b}(1) = mH(0) \\ \text{(ii)} & G(p) = p^\beta + \bar{a}(p) + G(0);\ \bar{a}(1) = -\,nG(0) \end{array}\right\} \tag{β_2}$$

or

$$\left.\begin{array}{ll}\text{(i)} & H(p)=b_2(p)+H(0);\ b_2(1)=-mH(0)\\ \text{(ii)} & G \textit{ an arbitrary real-valued mapping}\end{array}\right\}\qquad(\beta_3)$$

or

$$\left.\begin{array}{ll}\text{(i)} & H(p)=d[a(p)-p^\beta]+H(0);\ a(1)=1-\frac{m}{d}H(0)\\ \text{(ii)} & G(p)=a(p)+\bar{B}(p)+G(0);\ \bar{B}(1)=-nG(0)+\frac{m}{d}H(0)\end{array}\right\}\qquad(\beta_4)$$

or

$$\left.\begin{array}{ll}\text{(i)} & H(p)=d[M(p)-b(p)-p^\beta]+H(0);\ b(1)=\frac{m}{d}H(0)\\ \text{(ii)} & G(p)=M(p)-b(p)+\bar{B}(p)+G(0);\ \bar{B}(1)=-nG(0)+\frac{m}{d}H(0)\end{array}\right\}\qquad(\beta_5)$$

where $M : I \to \mathbb{R}$ *is a nonadditive multiplicative mapping with* $M(0) = 0$, $M(1) = 1$; $\ell : I \to \mathbb{R}$ *is a logarithmic mapping;* d *is an arbitrary nonzero real constant;* $a_1 : \mathbb{R} \to \mathbb{R}$, $b_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$, $\bar{a} : \mathbb{R} \to \mathbb{R}$, $\bar{b} : \mathbb{R} \to \mathbb{R}$, $a : \mathbb{R} \to \mathbb{R}$, $b : \mathbb{R} \to \mathbb{R}$ *and* $\bar{B} : \mathbb{R} \to \mathbb{R}$ *are additive mappings.*

Note. If $G(p) = p^\alpha$, $\alpha > 0$, $\alpha \neq 1$, $p \in I$, then (B) reduces to the functional equation

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m} H(p_i q_j) &= \sum_{i=1}^{n} p_i^\alpha \sum_{j=1}^{m} H(q_j) + \sum_{i=1}^{n} H(p_i) \sum_{j=1}^{m} q_j^\beta \\ &\quad + (m-n)H(0)\sum_{j=1}^{m} q_j^\beta + m(n-1)H(0), \qquad \text{(C)}\end{aligned}$$

which may be regarded as an enlargement of (1.1) (with H in place of f). Its solutions will be presented elsewhere.

2. The main result

The main result of this paper is the following:

Theorem 2.1. *Let* $n \geq 3$, $m \geq 3$ *be fixed integers and* $f : I \to \mathbb{R}$, $g : I \to \mathbb{R}$, $h : I \to \mathbb{R}$ *be mappings which satisfy the functional equation (A) for all* $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; $\beta \neq 1$ *being a fixed positive real power such that* $0^\beta := 0$, $1^\beta := 1$. *Then, any general solution* (f, g, h) *of (A) is of the form*

$$\left.\begin{array}{ll}\text{(i)} & f(p)=[h(1)+(n-1)h(0)]p^\beta+A_1(p)+f(0)\\ \text{(ii)} & g(p)=A_2(p)+g(0)\\ \text{(iii)} & h(p)=[h(1)+(n-1)h(0)]p^\beta+\bar{A}(p)+h(0)\end{array}\right\}\qquad(\alpha_1)$$

or

$$\left.\begin{array}{ll}\text{(i)} & f(p)=[g(1)+(n-1)g(0)]\{[h(1)+(m-1)h(0)]p^\beta+A_3(p)\}\\ & \qquad +[h(1)+(n-1)h(0)]p^\beta+A_1(p)+f(0)\\ \text{(ii)} & g \textit{ an arbitrary real-valued mapping}\\ \text{(iii)} & h(p)=[h(1)+(m-1)h(0)]p^\beta+A_3(p)+h(0)\end{array}\right\}\qquad(\alpha_2)$$

or

$$\left.\begin{array}{rl}\text{(i)} & f(p)=\{[g(1)+(n-1)g(0)]\ell(p)+[f(1)+(nm-1)f(0)]\}p^{\beta} \\ & \quad -[g(1)+(n-1)g(0)]\bar{b}(p)+A_1(p)+f(0) \\ \text{(ii)} & g(p)=[g(1)+(n-1)g(0)][p^{\beta}+\bar{a}(p)]+g(0) \\ \text{(iii)} & h(p)=\{\ell(p)+[h(1)+(m-1)h(0)]\}p^{\beta}-\bar{b}(p)+h(0)\end{array}\right\} \quad (\alpha_3)$$

with $[g(1)+(n-1)g(0)]\neq 0$ *or*

$$\left.\begin{array}{rl}\text{(i)} & f(p)=\{[f(1)+(nm-1)f(0)]-d[g(1)+(n-1)g(0)]\}p^{\beta} \\ & \quad +d[g(1)+(n-1)g(0)]a(p)+A_1(p)+f(0) \\ \text{(ii)} & g(p)=[g(1)+(n-1)g(0)][a(p)+\bar{B}(p)]+g(0) \\ \text{(iii)} & h(p)=\{[h(1)+(m-1)h(0)]-d\}p^{\beta}+da(p)+h(0)\end{array}\right\} \quad (\alpha_4)$$

with $[g(1)+(n-1)g(0)]\neq 0$ *or*

$$\left.\begin{array}{rl}\text{(i)} & f(p)=d[g(1)+(n-1)g(0)][M(p)-b(p)]+\{[f(1)+(nm-1)f(0)] \\ & \quad -d[g(1)+(n-1)g(0)]\}p^{\beta}+A_1(p)+f(0) \\ \text{(ii)} & g(p)=[g(1)+(n-1)g(0)][M(p)-b(p)+\bar{B}(p)]+g(0) \\ \text{(iii)} & h(p)=d[M(p)-b(p)]+\{[h(1)+(m-1)h(0)]-d\}p^{\beta}+h(0)\end{array}\right\} \quad (\alpha_5)$$

with $[g(1)+(n-1)g(0)]\neq 0$. *Moreover,*

$$\left.\begin{array}{rl}\text{(i)} & [g(1)+(n-1)g(0)-1][h(p)-h(0)] \\ & =[h(1)+(m-1)h(0)][g(p)-g(0)]-[h(1)+(n-1)h(0)]p^{\beta}+B(p) \\ \text{(ii)} & f(1)+(nm-1)f(0)=[g(1)+(n-1)g(0)][h(1)+(m-1)h(0)] \\ & \quad +[h(1)+(n-1)h(0)];\end{array}\right\} \quad (\gamma)$$

$M: I\to\mathbb{R}$ *is a nonadditive multiplicative mapping with* $M(0)=0$ *and* $M(1)=1$*;* $\ell: I\to\mathbb{R}$ *is a logarithmic mapping;* $d\neq 0$ *is an arbitrary real constant;* $A_i:\mathbb{R}\to\mathbb{R}$ $(i=1,2,3)$, $\bar{b}:\mathbb{R}\to\mathbb{R}$, $\bar{a}:\mathbb{R}\to\mathbb{R}$, $a:\mathbb{R}\to\mathbb{R}$, $b:\mathbb{R}\to\mathbb{R}$, $\bar{A}:\mathbb{R}\to\mathbb{R}$, $B:\mathbb{R}\to\mathbb{R}$,

$\bar{B}:\mathbb{R}\to\mathbb{R}$ *are additive mappings such that*

$$\left.\begin{array}{rl}
\text{(i)} & A_1(1) = -nmf(0) + m[g(1)+(n-1)g(0)]h(0) \\
\text{(ii)} & A_2(1) = -ng(0) \\
\text{(iii)} & A_3(1) = -mh(0) \\
\text{(iv)} & a(1) = 1 - \frac{m}{d}h(0) \\
\text{(v)} & b(1) = \frac{m}{d}h(0) \\
\text{(vi)} & \bar{a}(1) = -n[g(1)+(n-1)g(0)]^{-1}g(0) \\
\text{(vii)} & \bar{b}(1) = mh(0) \\
\text{(viii)} & \bar{B}(1) = -n[g(1)+(n-1)g(0)]^{-1}g(0) + \dfrac{m}{d}h(0) \\
\text{(ix)} & B(1) = nh(0) + n[h(1)+(m-1)h(0)]g(0) \\
 & \qquad\qquad -m[g(1)+(n-1)g(0)]h(0) \\
\text{(x)} & \bar{A}(1) = -nh(0).
\end{array}\right\} \qquad (\delta)$$

Proof. Putting $p_1 = 1$, $p_2 = \ldots = p_n = 0$ and $q_1 = 1$, $q_2 = \ldots = q_m = 0$ in (A), (γ)(ii) follows. Now, let us put $p_1 = 1$, $p_2 = \ldots = p_n = 0$ in (A). We obtain

$$\sum_{j=1}^{m}\{f(q_j)-[g(1)+(n-1)g(0)]h(q_j)-[h(1)+(n-1)h(0)]q_j^\beta\} = -m(n-1)f(0).$$

By Result 1.1, there exists an additive mapping $A_1:\mathbb{R}\to\mathbb{R}$ such that

$$f(p) = [g(1)+(n-1)g(0)][h(p)-h(0)] + [h(1)+(n-1)h(0)]p^\beta + A_1(p) + f(0) \tag{2.1}$$

with $A_1(1)$ given by (δ)(i). From equations (A) and (2.1), it follows that

$$\begin{aligned}
&[g(1)+(n-1)g(0)]\sum_{i=1}^{n}\sum_{j=1}^{m}h(p_iq_j) + [h(1)+(n-1)h(0)]\sum_{i=1}^{n}\sum_{j=1}^{m}(p_iq_j)^\beta \\
&\quad = \sum_{i=1}^{n}g(p_i)\sum_{j=1}^{m}h(q_j) + \sum_{i=1}^{n}h(p_i)\sum_{j=1}^{m}q_j^\beta + m(n-1)[g(1)+(n-1)g(0)]h(0).
\end{aligned} \tag{2.2}$$

The substitutions $q_1 = 1$, $q_2 = \ldots = q_m = 0$ in (2.2) give

$$\begin{aligned}
\sum_{i=1}^{n}h(p_i) &= [g(1)+(n-1)g(0)]\sum_{i=1}^{n}h(p_i) - [h(1)+(m-1)h(0)]\sum_{i=1}^{n}g(p_i) \\
&\quad + [h(1)+(n-1)h(0)]\sum_{i=1}^{n}p_i^\beta + (m-n)[g(1)+(n-1)g(0)]h(0).
\end{aligned} \tag{2.3}$$

Let us write (2.3) in the form

$$\begin{aligned}
&\sum_{i=1}^{n}\{[g(1)+(n-1)g(0)-1]h(p_i) - [h(1)+(m-1)h(0)]g(p_i) \\
&\qquad + [h(1)+(n-1)h(0)]p_i^\beta\} = -(m-n)[g(1)+(n-1)g(0)]h(0).
\end{aligned}$$

By Result 1.1, there exists an additive mapping $B : \mathbb{R} \to \mathbb{R}$ such that (γ)(i) holds with $B(1)$ given by (δ)(ix).

From equations (2.2) and (2.3), we obtain

$$[g(1)+(n-1)g(0)]\sum_{i=1}^{n}\sum_{j=1}^{m}h(p_iq_j)$$
$$=\sum_{i=1}^{n}g(p_i)\sum_{j=1}^{m}\{h(q_j)-[h(1)+(m-1)h(0)]q_j^{\beta}\}$$
$$+[g(1)+(n-1)g(0)]\left\{\sum_{i=1}^{n}h(p_i)\sum_{j=1}^{m}q_j^{\beta}+(m-n)h(0)\sum_{j=1}^{m}q_j^{\beta}+m(n-1)h(0)\right\}. \tag{2.4}$$

Case 1. $g(1)+(n-1)g(0)=0$.

In this case, equation (2.4) reduces to the equation

$$\sum_{i=1}^{n}g(p_i)\sum_{j=1}^{m}\{h(q_j)-[h(1)+(m-1)h(0)]q_j^{\beta}\}=0 \tag{2.5}$$

valid for all $(p_1,\ldots,p_n)\in\Gamma_n$, $(q_1,\ldots,q_m)\in\Gamma_m$; $n\geq 3$, $m\geq 3$ being fixed integers.

Consider the situation of $\sum_{i=1}^{n}g(p_i)=0$ for all $(p_1,\ldots,p_n)\in\Gamma_n$. By Result 1.1, there exists an additive mapping $A_2 : \mathbb{R}\to\mathbb{R}$ such that (α_1)(ii) follows with $A_2(1)$ given by (δ)(ii). Also, from (α_1)(ii) and (2.1); (α_1)(i) follows with $A_1(1) = -nmf(0)$ which can be obtained from $(\delta)(i)$ when $g(1)+(n-1)g(0)=0$. Now, let us put $q_1=1$, $q_2=\ldots=q_m=0$ in (A) and use (α_1)(i), (α_1)(ii). We obtain $\sum_{i=1}^{n}\{h(p_i)-[h(1)+(n-1)h(0)]p_i^{\beta}\}=0$. By Result 1.1, there exists an additive mapping $\bar{A}:\mathbb{R}\to\mathbb{R}$ such that (α_1)(iii) holds with $\bar{A}(1)$ given by (δ)(x). So, the solution (α_1) has been obtained with $A_1(1)=-nmf(0)$. Now, consider the situation of $\sum_{j=1}^{m}\{h(q_j)-[h(1)+(m-1)h(0)]q_j^{\beta}\}=0$ for all $(q_1,\ldots,q_m)\in\Gamma_m$. Making use of Result 1.1, it follows that there exists an additive mapping $A_3 : \mathbb{R}\to\mathbb{R}$ with $A_3(1)$ given by (δ)(iii), such that (α_2)(iii) holds. Now, from (α_2)(iii) and (2.5), one can easily conclude that g is an arbitrary real-valued mapping with $g(1)+(n-1)g(0)=0$. Also, from (2.1), it follows that $f(p)=[h(1)+(n-1)h(0)]p^{\beta}+A_1(p)+f(0)$ with $A_1(1)=-nmf(0)$. This solution is included in (α_2).

Case 2. $g(1)+(n-1)g(0)\neq 0$.

In this case, let us write (2.4) in the form

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\{h(p_iq_j)-[h(1)+(m-1)h(0)](p_iq_j)^{\beta}\}$$
$$=\sum_{i=1}^{n}\{[g(1)+(n-1)g(0)]^{-1}g(p_i)\}\sum_{j=1}^{m}\{h(q_j)-[h(1)+(m-1)h(0)]q_j^{\beta}\}$$

$$+\sum_{i=1}^{n}\{h(p_i)-[h(1)+(m-1)h(0)]p_i^{\beta}\}\sum_{j=1}^{m}q_j^{\beta}$$
$$+(m-n)h(0)\sum_{j=1}^{m}q_j^{\beta}+m(n-1)h(0). \tag{2.6}$$

Define the mappings $G : I \to \mathbb{R}$ and $H : I \to \mathbb{R}$ as

$$G(p) = [g(1)+(n-1)g(0)]^{-1}g(p) \tag{2.7}$$

and

$$H(p) = h(p) - [h(1)+(m-1)h(0)]p^{\beta} \tag{2.8}$$

for all $p \in I$. From (2.7) and (2.8), it is easy to conclude that

$$G(0) = \frac{g(0)}{g(1)+(n-1)g(0)}, \quad g(1)+(n-1)g(0) \neq 0 \tag{2.9}$$
$$G(1)+(n-1)G(0) = 1 \tag{2.10}$$
$$H(0) = h(0) \tag{2.11}$$

and

$$H(1)+(m-1)H(0) = 0. \tag{2.12}$$

Also, from (2.6), (2.7) and (2.8), equation (B) follows. Therefore, in Result 1.2, we have to consider only those solutions which satisfy both (2.10) and (2.12). We reject (β_1) because, in this case, (2.10) does not hold. Both (2.10) and (2.12) hold in (β_2), (β_4) and (β_5). In (β_3), (2.12) holds but (β_3) can be accepted provided we consider only those arbitrary mappings G which also satisfy (2.10). This requirement is met by any arbitrary mapping g which satisfies the condition $[g(1)+(n-1)g(0)] \neq 0$. This fact is obvious from (2.7). Keeping in view these observations, (α_2) with $[g(1)+(n-1)g(0)] \neq 0$ and (δ)((i), (iii)) follows from (β_3), (2.7), (2.8), (2.9), (2.11) and (2.1). Moreover,

(a_1) The solution (α_3) with $[g(1)+(n-1)g(0)] \neq 0$ and (γ)(ii), (δ)((i), (vi), (vii)) follows from (β_2), (2.7), (2.8), (2.9), (2.11) and (2.1);

(a_2) The solution (α_4) with $[g(1)+(n-1)g(0)] \neq 0$ and (γ)(ii), (δ)((i), (iv), (viii)) follows from (β_4), (2.7), (2.8), (2.9), (2.11) and (2.1);

(a_3) The solution (α_5) with $[g(1)+(n-1)g(0)] \neq 0$ and (γ)(ii), (δ)((i), (v), (viii)) follows from (β_5), (2.7), (2.8), (2.9), (2.11) and (2.1).

Making use of (γ)((i), (ii)), it can be verified that (α_1) to (α_5) are, indeed, the solutions of (A) . □

3. Comments

The object of this section is to comment upon various solutions, mentioned in Theorem 2.1, from the point of view of information theory.

Behara and Nath [1] have defined the entropy $H_n^{(\alpha,\beta)}(p_1,\ldots,p_n)$ of type (α,β) $(H_n^{(\alpha,\beta)} : \Gamma_n \to \mathbb{R},\ n = 1,2,\ldots)$ as

$$H_n^{(\alpha,\beta)}(p_1,\ldots,p_n) = \begin{cases} (2^{1-\alpha} - 2^{1-\beta})^{-1}\left(\sum\limits_{i=1}^{n} p_i^{\alpha} - \sum\limits_{i=1}^{n} p_i^{\beta}\right) & \text{if } \alpha \neq \beta \\ -\,2^{\beta-1}\sum\limits_{i=1}^{n} p_i^{\beta}\log_2 p_i & \text{if } \alpha = \beta \end{cases} \tag{3.1}$$

where $0^{\beta}\log_2 0 := 0$ and α, β are fixed positive real powers which satisfy $0^{\alpha} := 0$, $0^{\beta} := 0$, $1^{\alpha} := 1$, $1^{\beta} := 1$. Havrda and Charvát [2] have defined the entropies of degree β, $0 < \beta \in \mathbb{R}$, $\beta \neq 1$ as

$$H_n^{\beta}(p_1,\ldots,p_n) = (1 - 2^{1-\beta})^{-1}\left[1 - \sum_{i=1}^{n} p_i^{\beta}\right] \tag{3.2}$$

with $H_n^{\beta} : \Gamma_n \to \mathbb{R}$, $n = 1,2,\ldots$ and $0^{\beta} := 0$, $1^{\beta} := 1$. Both the entropies mentioned in (3.1) and (3.2) are nonadditive.

I. In the solution (α_1), $\sum\limits_{i=1}^{n} g(p_i) = 0$ follows. Then, equation (A) reduces to

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{j=1}^{m} q_j^{\beta}\sum_{i=1}^{n} h(p_i). \tag{3.3}$$

The functional equation (3.3) has been discussed by Nath and Singh [6].

II. In the solution (α_2), the mapping g is arbitrary. So, there are three possibilities: g is additive or g is multiplicative or g is logarithmic.

Let us consider the case of g being additive. If $g(p) \equiv 0$, then g is certainly additive. In this case, (A) reduces to (3.3). In general, when g is additive, (A) reduces to

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = g(1)\sum_{j=1}^{m} h(q_j) + \sum_{j=1}^{m} q_j^{\beta}\sum_{i=1}^{n} h(p_i). \tag{3.4}$$

If $g(1) = 0$, we get the functional equation (3.3) again. If $g(1) = 1$, then (3.4) reduces to the equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{j=1}^{m} h(q_j) + \sum_{j=1}^{m} q_j^{\beta}\sum_{i=1}^{n} h(p_i). \tag{3.5}$$

Consider the case of g being multiplicative. Here, we discuss some particular cases.

(i) $g(x) \equiv 0$. In this case, we again get (3.3).

(ii) $g(x) \equiv 1$. In this case, (A) reduces to

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = n\sum_{j=1}^{m} h(q_j) + \sum_{j=1}^{m} q_j^{\beta}\sum_{i=1}^{n} h(p_i)$$

where $n = 1,2$ are not admissible because we assume $n \geq 3$ throughout the paper.

(iii) $g(x) = x$ for all $x \in I$. In this case, equation (A) reduces to (3.5).

(iv) $g(p) = \begin{cases} p^\delta & \text{if } 0 < p \le 1 \\ 0 & \text{if } p = 0 \end{cases}$ where $\delta \in \mathbb{R}$. Some particular values of δ seem to be of interest from the point of view of applications. If $\delta = 1$, then $g(p) = p$, which has already been discussed above. If $\delta = \alpha$, $\alpha > 0$, $\alpha \neq 1$, then (A) reduces to (1.2). If $\delta = \beta$, $\beta > 0$, $\beta \neq 1$, then (A) reduces to

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} p_i^\beta \sum_{j=1}^{m} h(q_j) + \sum_{j=1}^{m} q_j^\beta \sum_{i=1}^{n} h(p_i).$$

If $\delta = 0$, then $g(p)$ can be written as $g(p) = \begin{cases} 1 & \text{if } 0 < p \le 1 \\ 0 & \text{if } p = 0 \end{cases}$. Therefore, equation (A) reduces to

$$\sum_{i=1}^{n}\sum_{j=1}^{m} g(p_i q_j) = N(P) \sum_{j=1}^{m} h(q_j) + \sum_{j=1}^{m} q_j^\beta \sum_{i=1}^{n} h(p_i)$$

where $N(P)$ denotes the number of non-zero elements in the probability distribution $P = (p_1, \ldots, p_n) \in \Gamma_n$.

(v) Consider the situation $g(p) = \begin{cases} 0 & \text{if } 0 \le p < 1 \\ 1 & \text{if } p = 1 \end{cases}$. In this case, $\sum_{i=1}^{n} g(p_i) = 1$ if $p_i = 1$ for exactly one i, $1 \le i \le n$ and 0 if $0 \le p_i < 1$, $i = 1, 2, \ldots, n$.

Finally, we consider the case of $g(p) = \ell(p)$ where ℓ is a logarithmic mapping. If $\ell(p) \equiv 0$, then (A) reduces to (3.3).

Notice that, if $g(p) \equiv 0$, then g is additive, multiplicative and logarithmic. In each case, the functional equation (3.3) arises. Such a discussion concerning $g(p) \equiv 0$ is needed for the sake of completeness in relation to solution (α_2).

III. From the solution (α_3), it can be easily derived that for all $(p_1, \ldots, p_n) \in \Gamma_n$,

$$\sum_{i=1}^{n} f(p_i) = [g(1) + (n-1)g(0)] \sum_{i=1}^{n} p_i^\beta \ell(p_i) + [f(1) + (nm-1)f(0)] \sum_{i=1}^{n} p_i^\beta - n(m-1)f(0);$$

$$\sum_{i=1}^{n} g(p_i) = [g(1) + (n-1)g(0)] \sum_{i=1}^{n} p_i^\beta;$$

$$\sum_{i=1}^{n} h(p_i) = \sum_{i=1}^{n} p_i^\beta \ell(p_i) + [h(1) + (m-1)h(0)] \sum_{i=1}^{n} p_i^\beta + (n-m)h(0).$$

Let us choose the logarithmic mapping $\ell : I \to \mathbb{R}$ as

$$\ell(p) = \begin{cases} \lambda \log_2 p & \text{if } 0 < p \le 1 \\ 0 & \text{if } p = 0 \end{cases} \tag{3.6}$$

where $\lambda \neq 0$ is an arbitrary real constant. Then, making use of (3.1), (3.2) and (3.6), we have

$$\sum_{i=1}^{n} f(p_i) = -[g(1) + (n-1)g(0)]\lambda 2^{1-\beta} H_n^{(\beta,\beta)}(p_1, \ldots, p_n)$$

$$+ [f(1) + (nm-1)f(0)][-(1-2^{1-\beta})H_n^{\beta}(p_1,\ldots,p_n)+1] \\ - n(m-1)f(0);$$

$$\sum_{i=1}^{n} g(p_i) = [g(1)+(n-1)g(0)][-(1-2^{1-\beta})H_n^{\beta}(p_1,\ldots,p_n)+1];$$

$$\sum_{i=1}^{n} h(p_i) = -\lambda 2^{1-\beta} H_n^{(\beta,\beta)}(p_1,\ldots,p_n) + [h(1)+(m-1)h(0)] \\ \times [-(1-2^{1-\beta})H_n^{\beta}(p_1,\ldots,p_n)+1] + (n-m)h(0).$$

Thus, we observe that the mappings f and h are related to both the entropies $H_n^{(\beta,\beta)}$ and H_n^{β} whereas the mapping g is related only to the entropies H_n^{β}.

IV. From the solution (α_4), for all $(p_1,\ldots,p_n) \in \Gamma_n$, it can be derived that

$$\sum_{i=1}^{n} f(p_i) = \{[f(1)+(nm-1)f(0)] - d[g(1)+(n-1)g(0)]\} \\ \times [-(1-2^{1-\beta})H_n^{\beta}(p_1,\ldots,p_n)+1] + d[g(1)+(n-1)g(0)] \\ - n(m-1)f(0);$$

$$\sum_{i=1}^{n} g(p_i) = [g(1)+(n-1)g(0)];$$

$$\sum_{i=1}^{n} h(p_i) = \{[h(1)+(m-1)h(0)] - d\}[-(1-2^{1-\beta})H_n^{\beta}(p_1,\ldots,p_n)+1] \\ + d + (n-m)h(0).$$

Thus, we observe that the mappings f and h are related to entropies H_n^{β} whereas the mapping g is not related to any of the entropies given by (3.1) and (3.2).

V. If we choose a mapping $M : I \to \mathbb{R}$ defined as $M(p) = p^{\alpha}$, $p \in I$, $\alpha \in \mathbb{R}$, $\alpha > 0$, $\alpha \neq 1$, $\alpha \neq \beta$, then, from solution (α_5), we obtain

$$\sum_{j=1}^{m} f(q_j) = d[g(1)+(n-1)g(0)](2^{1-\alpha}-2^{1-\beta})H_m^{(\alpha,\beta)}(q_1,\ldots,q_m) \\ - [f(1)+(nm-1)f(0)](1-2^{1-\beta})H_m^{\beta}(q_1,\ldots,q_m) \\ + [f(1)+(m-1)f(0)];$$

$$\sum_{j=1}^{m} g(q_j) = [g(1)+(n-1)g(0)][-(1-2^{1-\alpha})H_m^{\alpha}(q_1,\ldots,q_m)+1] + (m-n)g(0);$$

$$\sum_{j=1}^{m} h(q_j) = d(2^{1-\alpha}-2^{1-\beta})H_m^{(\alpha,\beta)}(q_1,\ldots,q_m) \\ - [h(1)+(m-1)h(0)][(1-2^{1-\beta})H_m^{\beta}(q_1,\ldots,q_m)-1].$$

Here, observe that the mappings in the solution (α_5) are related to the entropies of type (α,β) when $\alpha \neq \beta$; the entropies of degree α and the entropies of degree β.

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