SEMI-GLOBAL SOLUTIONS TO MIXED-TYPE FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract: The purpose of this paper is to give sufficient conditions for existence of right semi-global solutions to mixed-type functional differential equations. We also give an example to illustrate applicability of the result.

Keywords: semi-global solutions, mixed-type functional differential equations, delayed argument, advanced argument, monotone iterative method

1 INTRODUCTION

For $r > 0$ let $\mathrm{C}_r := \mathrm{C}([0,r],\mathbb{R}^n)$ be the Banach space of continuous functions from the interval $[0,r]$ to $\mathbb{R}^n$ equipped with the supremum norm

$$\|\psi\|_r = \sup_{\alpha\in[0,r]} |\psi(\alpha)|, \qquad \psi \in \mathrm{C}([0,r],\mathbb{R}^n),$$

where $|\cdot|$ is the maximum norm in $\mathbb{R}^n$.

For a function $y = y(t)$, continuous on an interval $[t-d,t]$, $t \in \mathbb{R}$, $d > 0$ we define a delayed-type function $y_t \in \mathrm{C}_d$ by formula $y_t(\tau) = y(t-\tau)$ where $\tau \in [0,d]$. Similarly, for a function $y = y(t)$, continuous on an interval $[t,t+a]$, $t \in \mathbb{R}$, $a > 0$, we define an advanced-type function $y^t \in \mathrm{C}_a$ by formula $y^t(\sigma) = y(t+\sigma)$ where $\sigma \in [0,a]$. Throughout the rest of the paper we assume that $d > 0$ and $a > 0$ are fixed.

In this paper we will consider a system of mixed-type functional differential equations

$$\dot{y}(t) = f\left(t, y_t, y^t\right), \tag{1}$$

where $f\colon \mathcal{J} \times \mathrm{C}_d \times \mathrm{C}_a \to \mathbb{R}^n$ is a continuous quasi-bounded functional which satisfies a local Lipschitz condition with respect to the second and the third argument. For definitions of quasi-boundedness, etc., we refer to [3].

Let t_0 be fixed, $\mathcal{J} := [t_0,\infty)$ and $\mathcal{J}_d := [t_0-d,\infty)$. A continuous function $y\colon \mathcal{J}_d \to \mathbb{R}^n$ is a right semi-global solution of (1) if it is continuously differentiable on $\mathcal{J}$ and satisfies (1) on $\mathcal{J}$.

By $\mathbb{R}^n_{\geqslant 0}$ ($\mathbb{R}^n_{>0}$) we denote the set of all componentwise nonnegative (positive) vectors v in $\mathbb{R}^n$, i.e., $v = \left(v^1,\dots,v^n\right)$ with $v^i \geqslant 0$ ($v^i > 0$) for $i = 1,\dots,n$. For $u,v \in \mathbb{R}^n$, we denote $u \leqslant v$ if $v-u \in \mathbb{R}^n_{\geqslant 0}$, $u \ll v$ if $v-u \in \mathbb{R}^n_{>0}$, and $u < v$ if $u \leqslant v$ and $u \neq v$. In order to avoid unnecessary additional definitions, we use, whenever the meaning is not ambiguous, the same symbols $\mathbb{R}^n_{\geqslant 0}$ $\left(\mathbb{R}^n_{>0}\right)$ to denote relevant subsets of the set $\mathbb{R}^n$.

2 MAIN RESULT

Below we will look for a solution of system (1) in the form

$$y(t) = I(k,\lambda)(t), \tag{2}$$

where I is a mapping, $I\colon \mathbb{R}^n_{>0} \times \mathrm{C}(\mathcal{J}_d, \mathbb{R}^n) \to \mathrm{C}(\mathcal{J}_d, \mathbb{R}^n)$,

$$I(k,\lambda) = (I_1(k,\lambda), I_2(k,\lambda), \ldots, I_n(k,\lambda))$$

defined as

$$I_i(k,\lambda)(t) := k_i \exp\left(\int_{t_0-d}^{t} \lambda_i(s)\,\mathrm{d}s\right),$$

for $i = 1,\ldots,n$ and $t \in \mathcal{J}_d$.

Substituting (2) into (1) we have

$$(\operatorname{diag}(I(k,\lambda)(t)))\,\lambda(t) = f\left(t, I(k,\lambda)_t, I(k,\lambda)^t\right),$$

for $t \in \mathcal{J}$, by diag we denote a diagonal matrix. Consequently,

$$\lambda(t) = (\operatorname{diag}(I(k,\lambda)(t)))^{-1} \cdot f\left(t, I(k,\lambda)_t, I(k,\lambda)^t\right). \tag{3}$$

Note that the matrix $(\operatorname{diag}(I(k,\lambda)(t)))^{-1}$ exists because the matrix $(\operatorname{diag}(I(k,\lambda)(t)))$ is regular. Equation (3) is an operator equation with respect to λ. A function $\lambda \in \mathrm{C}(\mathcal{J}_d, \mathbb{R}^n)$ is called a solution of equation (3) on $\mathcal{J}_d$ if (3) is valid for all $t \in \mathcal{J}$.

Let us define an operator

$$T : \mathrm{C}(\mathcal{J}_d, \mathbb{R}^n) \to \mathrm{C}(\mathcal{J}_d, \mathbb{R}^n),$$

where

$$(T\lambda)(t) = (\operatorname{diag}(I(k,\lambda)(t)))^{-1} \cdot f\left(t, I(k,\lambda)_t, I(k,\lambda)^t\right) \tag{4}$$

for $t \in \mathcal{J}$.

Theorem 1. *Let us assume that the following holds:*

(i) For any $M \geqslant 0$, $\theta > t_0 + a$ there exists a constant K, such that for all $t, t' \in [t_0, \theta - a]$ and for any continuous function $\lambda\colon [t_0 - d, \theta] \to \mathbb{R}^n$ with $|\lambda| \leqslant M$,

$$\left|(T\lambda)(t) - (T\lambda)(t')\right| \leqslant K\left|t - t'\right|. \tag{5}$$

(ii) There exist $k \in \mathbb{R}^n_{>0}$ and continuous functions $\mathcal{L}, \mathcal{U}\colon \mathcal{J}_d \to \mathbb{R}^n$ satisfying here $\mathcal{L}(t) \leqslant \mathcal{U}(t)$, and

$$\mathcal{L}(t) \leqslant (T\mathcal{L})(t),$$
$$\mathcal{U}(t) \geqslant (T\mathcal{U})(t)$$

on $\mathcal{J}$ and

$$\mathcal{L}(t) \leqslant (T\mathcal{L})(t_0),$$
$$\mathcal{U}(t) \geqslant (T\mathcal{U})(t_0)$$

on $[t_0 - d, t_0]$.

(iii) For any continuous functions $\lambda, \mu\colon \mathcal{J}_d \to \mathbb{R}^n$ the inequality $\lambda(t) \leqslant \mu(t)$, $t \in \mathcal{J}_d$ implies

$$(T\lambda)(t) \leqslant (T\mu)(t)$$

for $t \in \mathcal{J}$.

Then there exists a right semi-global solution $y\colon \mathcal{J}_d \to \mathbb{R}^n$ of (1) satisfying

$$I(k,\mathcal{L})(t) \leqslant y(t) \leqslant I(k,\mathcal{U})(t) \tag{6}$$

for $t \in \mathcal{J}_d$ and such that

$$y(t_0 - d) = k.$$

Proof. We need to show, that the equation (3), i.e.,

$$\lambda(t) = (T\lambda)(t) := \left(\operatorname{diag}\left(I(k,\lambda)(t)\right)^{-1}\right) \cdot f\left(t, I(k,\lambda)_t, I(k,\lambda)^t\right), \quad t \in \mathcal{J}$$

has a solution $\lambda \in C(\mathcal{J}_d, \mathbb{R})$ which satisfies $\mathcal{L}(t) \leqslant \lambda(t) \leqslant \mathcal{U}(t)$ for $t \in \mathcal{J}_d$.

For $\theta > t_0 + a$, we denote by

$$L_\theta := \mathrm{C}\left([t_0 - d, \theta], \mathbb{R}^n\right)$$

the Banach space of the continuous functions from $[t_0 - d, \theta]$ into $\mathbb{R}^n$ equipped with the maximum norm. Further, we introduce the closed, normal cone

$$\mathcal{K}_\theta := C\left([t_0 - d, \theta], \mathbb{R}^n_{\geqslant 0}\right)$$

of the continuous functions from $[t_0 - d, \theta]$ into $\mathbb{R}^n_{\geqslant 0}$. The cone defines a partial ordering in L_θ: for $\lambda, \mu \in L_\theta$, we say that $\lambda \leqslant \mu$ if and only if $\mu - \lambda \in \mathcal{K}_\theta$.

Let us define an operator $T_\theta\colon L_\theta \to L_\theta$ by

$$(T_\theta\lambda)(t) = \begin{cases} (T\lambda)(t_0) & t \in [t_0 - d, t_0), \\ (T\lambda)(t) & t \in [t_0, \theta - a), \\ (T\lambda)(\theta - a) & t \in [\theta - a, \theta]. \end{cases}$$

The operator T_θ is well-defined and, according to condition (iii), monotone increasing. Further, we define

$$\nu_\theta(t) := \begin{cases} \mathcal{L}(t) & t \in [t_0 - d, \theta - a), \\ \mathcal{L}(\theta - a) & t \in [\theta - a, \theta], \end{cases}$$

$$\mu_\theta(t) := \begin{cases} \mathcal{U}(t) & t \in [t_0 - d, \theta - a), \\ \mathcal{U}(\theta - a) & t \in [\theta - a, \theta]. \end{cases}$$

Then, we construct a monotone and bounded sequences

$$\nu_\theta \leq T_\theta\nu_\theta \leq T_\theta^2\nu_\theta \leq \cdots \leq T_\theta^2\mu_\theta \leq T_\theta\mu_\theta \leq \mu_\theta.$$

Now, we are going to show that T_θ is compact and therefore there exist limit functions $\underline{\lambda}_\theta$ and $\overline{\lambda}_\theta$ such that

$$\nu_\theta \leqslant \underline{\lambda}_\theta = T_\theta\underline{\lambda}_\theta \leqslant T_\theta\overline{\lambda}_\theta = \overline{\lambda}_\theta \leqslant \mu_\theta. \tag{7}$$

Let M be a bounded subset of L_θ. We need to prove that $T_\theta M$ is relatively compact subset of L. According to Arzela-Ascoli Theorem, it is enough to show that $T_\theta M$ is bounded and equicontinuous. Because of the definition of the operator T_θ the equicontinuity has to be checked in the following six cases:

1. $t,t' \in [t_0 - r_1, t_0)$
2. $t,t' \in [t_0, \theta - r_2)$
3. $t,t' \in [\theta - r_2, \theta]$
4. $t \in [t_0 - r_1, t_0), t' \in [t_0, \theta - r_2)$
5. $t \in [t_0 - r_1, t_0), t' \in [\theta - r_2, \theta]$
6. $t \in [t_0, \theta - r_2), t' \in [\theta - r_2, \theta]$

For example, if $t \in [t_0 - d, t_0), t' \in [t_0, \theta - a)$, using the inequality (5) we obtain

$$\left|(T_\theta\lambda)(t) - (T_\theta\lambda)(t')\right| = \left|(T\lambda)(t_0) - (T\lambda)(t')\right| \leqslant K|t_0 - t'| \leqslant K|t - t'|.$$

The remaining estimations can be obtained in a similar way. Now, we may conclude that T_θ is equicontinuous.

Next, the boundedness of T_θ is guaranteed due to the quasi-boundedness of f and the fact that

$$(I_i(k,\lambda)(t))^{-1} = k_i^{-1}\exp\left(-\int_{t_0-d}^{t}\lambda_i(s)\,\mathrm{d}s\right) \leqslant k_i^{-1}\exp\left(M\cdot(\theta - t_0 + d)\right), \tag{8}$$

for $i = 1,\ldots,n$ and $|\lambda| \leqslant M$.

So we have shown that T_θ is compact. Therefore the sequences $\left(T_\theta^m \nu_\theta\right)_{m=0}^{\infty}$ and $\left(T_\theta^m \mu_\theta\right)_{m=0}^{\infty}$ have limit functions $\underline{\lambda}_\theta$ and $\overline{\lambda}_\theta$ satisfying inequality (7).

It is easy to see that

$$(T_\theta\lambda)|_{[t_0-d,\theta-a]} = (T_\Theta\lambda)|_{[t_0-d,\theta-a]}$$

for $\Theta \geqslant \theta$ and $\lambda \in L_\Theta$. Therefore,

$$\underline{\lambda}_\theta|_{[t_0-d,\theta-a]} = \underline{\lambda}_\Theta|_{[t_0-d,\theta-a]},$$
$$\overline{\lambda}_\theta|_{[t_0-d,\theta-a]} = \overline{\lambda}_\Theta|_{[t_0-d,\theta-a]}$$

for $\Theta \geqslant \theta$.

Let us define the functions $\underline{\lambda}, \overline{\lambda} \in C(\mathcal{J}_d, \mathbb{R}^n)$

$$\underline{\lambda}(t) := \begin{cases} \underline{\lambda}_\theta(t) & t \in [t_0 - d, \theta - a), \\ \underline{\lambda}_{\Theta(t)}(t) & t \in [\theta - a, \infty) \end{cases}$$

and

$$\overline{\lambda}(t) := \begin{cases} \overline{\lambda}_\theta(t) & t \in [t_0 - d, \theta - a), \\ \overline{\lambda}_{\Theta(t)}(t) & t \in [\theta - a, \infty) \end{cases}$$

where $\Theta(t) = t + a$. The defined functions $\overline{\lambda}$ and $\underline{\lambda}$ satisfy

$$\mathcal{L}(t) \leqslant \underline{\lambda}(t) \leqslant \overline{\lambda}(t) \leqslant \mathcal{U}(t),\ t \in \mathcal{J}_d, \tag{9}$$

$$\underline{\lambda}(t) = (T\underline{\lambda})(t)$$

and

$$\overline{\lambda}(t) = \left(T\overline{\lambda}\right)(t)$$

for $t \in \mathcal{J}$.

The proof will be completed by choosing, for example, $\lambda = \underline{\lambda}$ and the searched solution will be $y = I(k, \underline{\lambda})$. The inequality (6) holds because of (9).

□

3 EXAMPLE

Consider a linear equation

$$\dot{y}(t) = -\left(2 - \frac{2}{\pi}\arctan t\right)\cdot y(t-2) + \left(3 - \mathrm{e}^{-t^2}\right)\cdot y\left(t + \frac{1}{10}\right). \tag{10}$$

In this case

$$f(t, y_t, y^t) = -\left(2 - \frac{2}{\pi}\arctan t\right)\cdot y(t-2) + \left(3 - \mathrm{e}^{-t^2}\right)\cdot y\left(t + \frac{1}{10}\right).$$

Then, according to (4), the corresponding operator T is defined as

$$(T\lambda)(t) = -\left(2 - \frac{2}{\pi}\arctan t\right)\exp\left(\int_t^{t-2}\lambda(s)\,\mathrm{d}s\right) + \left(3 - \mathrm{e}^{-t^2}\right)\exp\left(\int_t^{t+1/10}\lambda(s)\,\mathrm{d}s\right).$$

The operator T is equicontinuous and monotone increasing. (Details, how to prove it, may be found in [1], [2]). That means, assumptions (i) and (iii) of Theorem 1 are valid. Set $\mathcal{L} = 1$ and $\mathcal{U} = 10$. Then, for every $t \in \mathbb{R}$ holds

$$(T\mathcal{L})(t) = -\left(2 - \frac{2}{\pi}\arctan t\right)\cdot \mathrm{e}^{-2} + \left(3 - \mathrm{e}^{-t^2}\right)\cdot \mathrm{e}^{1/10} \geqslant -3\mathrm{e}^{-2} + 2\mathrm{e}^{1/10} \geqslant 1 = \mathcal{L},$$

$$(T\mathcal{U})(t) = -\left(2 - \frac{2}{\pi}\arctan t\right)\cdot \mathrm{e}^{-20} + \left(3 - \mathrm{e}^{-t^2}\right)\cdot \mathrm{e} \leqslant \left(3 - \mathrm{e}^{-t^2}\right)\cdot \mathrm{e} \leqslant 10 = \mathcal{U}.$$

So, assumption (ii) of Theorem 1 holds. These expressions were calculated online by Wolfram Alpha software (see [4]).

Therefore, by Theorem 1, for every fixed $t_0 \in \mathbb{R}$ there exists a solution of equation (10) on $[t_0 - 2, \infty)$ such that

$$k \cdot \exp(t - t_0 + 2) \leqslant y(t) \leqslant k \cdot \exp(10(t - t_0 + 2)).$$

Moreover, this solution satisfies $y(t_0 - 2) = k$.

4 CONCLUSION

In this paper we have discussed existence of right semi-global solutions to mixed-type functional differential equations and formulated conditions under which such solutions exist. Moreover, upper and lower bound for solutions are derived.

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