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Basics of Qualitative Theory of Linear Fractional Difference Equations

Základy kvalitativní teorie lineárních zlomkových rovnic

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Introduction

The main topic of this doctoral thesis, the *discrete fractional calculus*, forms a discrete analogue of the *fractional calculus*, i.e. a mathematical discipline studying derivatives and integrals of non-integer orders (see, e.g. [31, 39, 41, 43]).

The theory of discrete fractional calculus originates from the works by Agarwal [1] and Díaz & Osler [21], where the first definitions of non-integer order differences and sums were introduced on the sets of points forming geometric and arithmetic sequences, respectively. Recently, both the cases were unified and generalized, because fractional operators were established on any set of points such that the distance of two neighbours (the so-called graininess function) is given by a linear function (see [18]). So far, the research in this field is mainly concentrated on methods for solving of difference equations involving fractional differences (the so-called fractional difference equations, in short FdEs; see, e.g. [7, 8, 38, 40]), while qualitative analysis of FdEs is just at the beginning.

This doctoral thesis summarizes the author’s papers [14–17] (written jointly with other authors) with regard to [32, 33]. The former group of papers deals with solutions of FdEs (often considered as discretizations of appropriate fractional differential equations, in short FDEs), their qualitative properties and potential consequences for some numerical methods, while the latter one concerns with numerical solving of particular problems.

The work is organized as follows. Chapter 1 recalls a few necessary notions and also states some original preliminary results. Chapter 2 is based on papers [14, 15]. It deals, among others, with properties of a higher-order scalar linear FdEs on the set with a general linear graininess function. In Chapter 3, originating from [16], the stability and asymptotic properties of a scalar two-term linear FdE are studied via its conversion to a Volterra difference equation. Further, Chapter 4 is based on the paper [17] and utilizes the discrete Laplace transform for an investigation of qualitative properties of linear fractional difference systems. Both Chapter 3 and 4 consider the set of equidistant points. At last, an introduction of fractional operators on a general time scale is proposed in Chapter 5.

This is a short version of the doctoral thesis. It includes the most important notions and results. All the omitted parts (definitions, proofs) can be found in the original work.

1 Preliminaries

In past few decades, several attempts to establish definitions of discrete fractional operators were performed (see, e.g. [1, 18, 21, 27]). Our approach to discrete fractional calculus is closest to [18] originating from the time scales theory.

By a time scale $\mathbb{T}$ we understand any non-empty closed subset of real numbers with ordering inherited from reals. For the sake of simplicity, we consider only nabla calculus
on isolated time scales which are of our interest. The presented definitions and properties were adopted, with a few minor adjustments, from [11, 12].

Until now the fractional calculus has not been satisfactory established in the time scales theory, at least not in a form exceeding a formal generalization of symbols. Hence, we focus solely on the isolated time scales, where this issue has been overcome, i.e. the time scales with a linear graininess function, namely

- $h\mathbb{Z} = \{nh; n \in \mathbb{Z}\}$, where $h > 0$,
- $\mathbb{T}_{(q,h)} = \{t_0 q^n + h \frac{q^n - 1}{q - 1}; n \in \mathbb{Z}\} \cup \{\frac{h}{1-q}\}$, where $t_0 > 0$, $q \geq 1$, $h \geq 0$, $q + h > 1$.

Note that if $q = 1$, $h > 0$ and $t_0 = h$, then $\mathbb{T}_{(q,h)} = h\mathbb{Z}$ and the cluster point $h/(1-q) = -\infty$ is not involved. If $h = 0$ and $q > 1$, then $\mathbb{T}_{(q,h)} = q\mathbb{Z} = \{t_0 q^n; n \in \mathbb{Z}\}$.

The fundamental notion of the nabla calculus is nabla derivative and integral. On an isolated time scale $\mathbb{T}$ we also speak of nabla difference and sum and introduce them by

\[
\nabla f(t) = \frac{f(t) - f(\rho(t))}{\nu(t)}, \quad t \in \mathbb{T}_\kappa, \tag{1.1}
\]

\[
\int_a^b f(t) \nabla t = \sum_{t \in (a,b]\cap \mathbb{T}} \nu(t) f(t), \tag{1.2}
\]

respectively, where $\rho(t)$ is the so-called backward jump operator, $\nu(t)$ the so-called backward graininess function ($\nu(t) = t - \rho(t)$) and $\mathbb{T}_\kappa$ is the original time scale with the minimum removed.

Considering fractional calculus it is essential to introduce an appropriate generalization of the time scales polynomials (also called monomials, see [11]), i.e. the time scale version of power functions $\hat{h}_\beta(t, s)$ ($\beta \in (-1, \infty)$). Regarding isolated time scales, the explicit formulas for these functions are known only for the cases $\mathbb{T} = h\mathbb{Z}$ and $\mathbb{T}_{(q,h)}$. To simplify the notation, we put $\tilde{q} = q^{-1}$ whenever considering the time scale $\mathbb{T}_{(q,h)}$.

**Definition 1.1.** Let $s, t \in \mathbb{T}$, $\beta \in (-1, \infty)$.

(i) If $\mathbb{T} = h\mathbb{Z}$ and $t = \sigma^n(s)$, then

\[
\hat{h}_\beta(t, s) = h^\beta \left(\frac{\beta + n - 1}{n - 1}\right) = (-1)^{n-1} h^\beta \left(\frac{-\beta - 1}{n - 1}\right). \tag{1.3}
\]

(ii) If $\mathbb{T} = \mathbb{T}_{(q,h)}$ and $t = \sigma^n(s)$, then

\[
\hat{h}_\beta(t, s) = \nu^\beta(t) \left[\frac{\beta + n - 1}{n - 1}\right]_{\tilde{q}} = (-1)^{n-1} \nu^\beta(s) \tilde{q}^{\beta(n)} \left[\frac{-\beta - 1}{n - 1}\right]_{\tilde{q}}. \tag{1.4}
\]

Note that the relations for power functions on $\mathbb{T}_{(q,h)}$ tend to those on $\mathbb{T} = h\mathbb{Z}$ for $q \to 1^+$. For more information about $q$-calculus we refer to, e.g. [30].
Now we are in a position to introduce fractional operators $a^\nabla_{(q,h)}^\alpha$ on $T_{(q,h)}$ and $a^\nabla_h^\alpha$ on $\mathbb{T} = h\mathbb{Z}$. We speak of the so-called fractional $(q,h)$-calculus and $h$-calculus, respectively. We recall the relevant definitions and basic properties known from [18] and present some original results published in [14,15]. Note that since $\mathbb{T} = h\mathbb{Z}$ is a special case of $T_{(q,h)}$, all the results derived on $T_{(q,h)}$ are valid on $\mathbb{T} = h\mathbb{Z}$ as well.

**Definition 1.2.** Let $\gamma \in \mathbb{R}_0^+$ and $\bar{a}, a, b \in T_{(q,h)}$ be such that $\bar{a} \leq a < b$. Then for a function $f : (\bar{a}, b]_{T_{(q,h)}} \to \mathbb{R}$ we define the fractional sum of order $\gamma \in \mathbb{R}_0^+$ with the lower limit $a$ as

$$a^\nabla_{(q,h)}^{-\gamma} f(t) = \int_a^t \hat{h}_{\gamma-1}(t, \rho(\tau)) f(\tau) \nabla \tau, \quad t \in [a, b]_{T_{(q,h)}} \cap (\bar{a}, b]_{T_{(q,h)}} \tag{1.5}$$

and for $\gamma = 0$ we put $a^\nabla_{(q,h)}^0 f(t) = f(t)$.

**Definition 1.3.** Let $\alpha \in \mathbb{R}_0^+$ and $\bar{a}, a, b \in T_{(q,h)}$, be such that $\bar{a} \leq a < b$. Then for a function $f : (\bar{a}, b]_{T_{(q,h)}} \to \mathbb{R}$ we define the Riemann-Liouville fractional difference of order $\alpha$ with the lower limit $a$ as

$$a^\nabla_{(q,h)}^\alpha f(t) = a^\nabla_{(q,h)}^{[\alpha]} a^\nabla_{(q,h)}^{-(\lfloor \alpha \rfloor - \alpha)} f(t), \quad t \in [\sigma(a), b]_{T_{(q,h)}} \cap (\bar{a}, b]_{T_{(q,h)}} \tag{1.6}$$

To obtain a representation of fractional $(q,h)$-sum more convenient for calculations, we expand the definition (1.5) with respect to (1.2) and (1.4). It yields

$$a^\nabla_{(q,h)}^{-\gamma} f(t) = \sum_{k=1}^n (-1)^{n-k} \nu^\gamma(\sigma^k(a)) \left[ \frac{-\gamma}{n-k} \right] \hat{q}^{(n-k+1)} f(\sigma^k(a)), \quad t = \sigma^n(a)$$

where $\gamma \in \mathbb{R}_0^+, t = \sigma^n(a)$ and $n = 1, 2, \ldots$. These relations along with the definition formula (1.6) provide a solid tool for evaluation of fractional differences. Nevertheless, sometimes it is suitable to utilize directly an expansion of the fractional difference, in particular

$$a^\nabla_{(q,h)}^\alpha f(t) = \sum_{k=1}^n (-1)^{n-k} \nu^{-\alpha}(\sigma^k(a)) \left[ \frac{\alpha}{n-k} \right] \hat{q}^{(n-k+1)} f(\sigma^k(a)), \quad t = \sigma^n(a)$$

where $\alpha \in \mathbb{R}_0^+ \setminus \mathbb{Z}^+$, $t = \sigma^n(a)$ and $n = \lfloor \alpha \rfloor + 1, \lfloor \alpha \rfloor + 2, \ldots$ (for more details we refer to [18, Propositions 1 and 3] with respect to (1.4)).

Next, we recall some assertions presented in the author’s joint paper [15]. In particular, we perform an extension of the power rule for fractional operators of $(q,h)$-calculus.

**Lemma 1.4.** Let $\gamma \in \mathbb{R}_0^+, \beta \in \mathbb{R} \setminus \mathbb{Z}^-$ and $a, t \in T_{(q,h)}$ be such that $t > a$. Then it holds

$$a^\nabla_{(q,h)}^{-\gamma} \hat{h}_{\beta}(t, a) = \hat{h}_{\gamma+\beta}(t, a).$$

Further, we formulate the assertion dealing with the Riemann-Liouville fractional difference of the power function.
Corollary 1.5. Let $\alpha \in \mathbb{R}^+$, $\beta \in \mathbb{R} \setminus \mathbb{Z}^-$ and $a, t \in T_{(q,h)}$ be such that $t > \sigma^{[\alpha]}(a)$. Then

$$a \nabla_{(q,h)}^\alpha \hat{h}_\beta(t,a) = \begin{cases} \hat{h}_{\beta-\alpha}(t,a), & \beta - \alpha \notin \{-1, \ldots, -[\alpha]\}, \\ 0, & \beta - \alpha \in \{-1, \ldots, -[\alpha]\}. \end{cases}$$

Now, we present a few properties regarding the nabla $h$-Laplace transform of a function $f : h\mathbb{Z} \to \mathbb{R}$. Employing [3] it is introduced as

$$\mathcal{L}\{f\}(z) = h \sum_{k=1}^\infty f(t_k)(1 - hz)^{k-1}, \quad \text{where } t_n = nh. \quad (1.8)$$

The following assertion is of the utmost importance for Chapter 4. Some of these, or similar relations have been derived in [7,8] or in the author’s joint paper [17].

Lemma 1.6. Let $\alpha, \gamma \in \mathbb{R}^+$, $\beta \in \mathbb{R} \setminus \mathbb{Z}^-$ and $f(t_n), g(t_n)$ be functions such that $\mathcal{L}\{f\}(z)$, $\mathcal{L}\{g\}(z)$ exist. Then it holds

(i) $\mathcal{L}\{\hat{h}_\beta(\cdot, 0)\}(z) = z^{-\beta-1},$

(ii) $\mathcal{L}\{f * g\}(z) = \mathcal{L}\{f\}(z) \cdot \mathcal{L}\{g\}(z),$ 

(iii) $\mathcal{L}\{0 \nabla_{h}^{-\gamma} f\}(z) = z^{-\gamma} \mathcal{L}\{f\}(z),$

(iv) $\mathcal{L}\{0 \nabla_{h}^\alpha f\}(z) = z^{\alpha} \mathcal{L}\{f\}(z) - \sum_{j=0}^{[\alpha]-1} z^j 0 \nabla_{h}^{\alpha-j-1} f(t_n)|_{n=0},$

where the convolution is given by $(f * g)(t_n) = \sum_{k=1}^{n} h f(t_{n-k+1}) g(t_k)$.

2 Basic theory of higher-order linear FdEs on $T_{(q,h)}$

In this chapter we deal with foundations of the theory of linear FdEs on $T_{(q,h)}$. In particular, we discuss the existence and uniqueness of their solution, the form of a general solution and eigenfunctions of the operator $a \nabla_{(q,h)}^\alpha$. Derived conclusions can be applied to $T = h\mathbb{Z}$ (for $q = 1$) and $T = q\mathbb{Z}$ (for $h = 0$) as well. The presented results were published in [15], some of them for $T = h\mathbb{Z}$ also in [14].

For the sake of simplicity, we introduce a restriction of $T_{(q,h)}$ by

$$\tilde{T}_{(q,h)}^a = \{ t \in T_{(q,h)} ; t \geq a > h/(1 - q) \}, \quad \text{where } a \in T_{(q,h)}.$$

2.1 An initial value problem

In this section, we are going to discuss the linear initial value problem

$$\sum_{j=1}^{[\alpha]} p_{[\alpha]-j+1}(t) a \nabla_{(q,h)}^{\alpha-j} y(t) + p_0(t) y(t) = 0, \quad t \in (\tilde{T}_{(q,h)}^a)_{\kappa([\alpha]+1)}, \quad (2.1)$$

$$a \nabla_{(q,h)}^{\alpha-j} y(t)|_{t=\sigma([\alpha])} = y_{a-j}, \quad j = 1, 2, \ldots, [\alpha], \quad (2.2)$$

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where \( \alpha \in \mathbb{R}^+ \). Further, we assume that \( p_j(t) \) \((j = 1, \ldots, [\alpha] - 1)\) are real functions on \((\mathbb{T}_a^\alpha)_R^{[\alpha]+1}, p_{[\alpha]}(t) \equiv 1\) on \((\mathbb{T}_a^\alpha)_R^{[\alpha]+1}\) and \( y_{\alpha-j} \) \((j = 1, \ldots, [\alpha])\) are real scalars.

It can be shown, that arbitrary values of \( y(\sigma(a)), y(\sigma^2(a)), \ldots, y(\sigma^{[\alpha]}(a)) \) determine uniquely the solution \( y(t) \) on \((\mathbb{T}_a^\alpha)_R^{[\alpha]+1}\). The following assertion implies that the values \( y_{\alpha-1}, y_{\alpha-2}, \ldots, y_{\alpha-[\alpha]} \), introduced by (2.2), keep the same property and, consequently, that the initial value problem is well-defined.

**Proposition 2.1.** Let \( \alpha \in \mathbb{R}^+ \) and \( y : (\mathbb{T}_a^\alpha)_R \to \mathbb{R} \) be a function. Then (2.2) represents a one-to-one mapping between the vectors \((y(\sigma(a)), y(\sigma^2(a)), \ldots, y(\sigma^{[\alpha]}(a)))\) and \((y_{\alpha-1}, y_{\alpha-2}, \ldots, y_{\alpha-[\alpha]})\).

The key notion connected to the problem of existence and uniqueness of solutions of dynamic equations on time scales is \( \nu \)-regressivity (see [11, 12]). We are going to follow this pattern and generalize this notion for the linear FdE (2.1).

**Definition 2.2.** Let \( \alpha \in \mathbb{R}^+ \). Then the equation (2.1) is called \( \nu \)-regressive provided the matrix

\[
A(t) = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 \\
0 & 0 & \cdots & 0 & 1 \\
-p_{[\alpha]}(t) & -p_{[\alpha]-1}(t) & \cdots & -p_{[\alpha]-2}(t)
\end{pmatrix}
\]

is \( \nu \)-regressive.

If \( \alpha \in \mathbb{Z}^+ \), then this introduction agrees with the definition of \( \nu \)-regressivity of a higher order linear dynamic equation usually employed in the time scales theory. Similarly, the proof of the existence and uniqueness for (2.1), (2.2) is a generalization of the ordinary time scales approach, i.e. it utilizes a conversion of the initial value problem with higher-order equation to the matrix initial value problem of order less or equal to one. Thus, we get

**Theorem 2.3.** Let (2.1) be \( \nu \)-regressive. Then the problem (2.1), (2.2) has a unique solution defined for all \( t \in (\mathbb{T}_a^\alpha)_R \).

The final goal of this section is to investigate the structure of the solutions of (2.1). We start with the following notion generalizing the classical Wronskian.

**Definition 2.4.** Let \( m \in \mathbb{Z}^+ \) and \( \gamma \in [0,1) \). For \( m \) functions \( y_j : (\mathbb{T}_a^\alpha)_R \to \mathbb{R} \) \((j = 1, 2, \ldots, m)\) we define the \( \gamma \)-Wronskian \( W_{\gamma}(y_1, \ldots, y_m)(t) \) for all \( t \in (\mathbb{T}_a^\alpha)_R \) as determinant of the matrix

\[
V_{\gamma}(y_1, \ldots, y_m)(t) = \begin{pmatrix}
a^{-\gamma}(q,a)y_1(t) & a^{-\gamma}(q,a)y_2(t) & \cdots & a^{-\gamma}(q,a)y_m(t) \\
a^{-\gamma}(q,a)y_1(t) & a^{-\gamma}(q,a)y_2(t) & \cdots & a^{-\gamma}(q,a)y_m(t) \\
\vdots & \vdots & \ddots & \vdots \\
a^{-\gamma}(q,a)y_1(t) & a^{-\gamma}(q,a)y_2(t) & \cdots & a^{-\gamma}(q,a)y_m(t)
\end{pmatrix}
\]
Remark 2.5. Note that $W_\gamma(y_1, \ldots, y_m)(t)$ coincides for $\gamma = 0$ with the classical definition. Moreover, it holds $W_\gamma(y_1, \ldots, y_m)(t) = W_0(a\nabla_{(q,h)}^{-\gamma}y_1, \ldots, a\nabla_{(q,h)}^{-\gamma}y_m)(t)$.

**Theorem 2.6.** Let functions $y_1(t), \ldots, y_\lceil \alpha \rceil(t)$ be solutions of the $\nu$-regressive equation (2.1) and let $W_{\lceil \alpha \rceil - \alpha}(y_1, \ldots, y_\lceil \alpha \rceil)(\sigma^{\lceil \alpha \rceil}(a)) \neq 0$. Then any solution $y(t)$ of (2.1) can be written as

$$y(t) = \sum_{k=1}^{\lceil \alpha \rceil} c_k y_k(t), \quad t \in (\tilde{T}_{(q,h)}^a)_{K},$$

where $c_1, \ldots, c_{\lceil \alpha \rceil}$ are real constants.

**Remark 2.7.** The formula (2.3) is essentially an expression of the general solution of (2.1).

### 2.2 Eigenfunctions of the Riemann-Liouville difference operator

Our main interest in this section is to find eigenfunctions of the fractional operator $a\nabla_{(q,h)}^{\alpha}$, $\alpha \in \mathbb{R}^+$. In other words, we wish to solve the equation (2.1) in a special form

$$a\nabla_{(q,h)}^{\alpha}y(t) = \lambda y(t), \quad \lambda \in \mathbb{R}, \quad t \in (\tilde{T}_{(q,h)}^a)_{K_{\lceil \alpha \rceil}+1}.$$  

(2.4)

Throughout this section we assume that the $\nu$-regressivity condition is ensured ($\lambda \nu^{\alpha}(t) \neq 1$).

Discussions on methods for solving of FdEs are just at the beginning. In particular, the discrete analogue of the Laplace transform seems to be the most developed method (see, e.g. [7, 8, 38]).

In this section, we describe the technique not utilizing the transform method, but directly originating from the role which is played by the Mittag-Leffler function in the continuous fractional calculus (see, e.g. [43]). More precisely, we introduce a $(q,h)$-analogue of the Mittag-Leffler function, which turns out to be very useful in description of solutions of (2.4). These our results generalize and extend those derived in [40] and [14].

**Definition 2.8.** Let $\eta, \beta, \lambda \in \mathbb{R}$. We introduce the $(q,h)$-Mittag-Leffler function $E_{\eta,\beta}^{x,\lambda}(t)$ by the series expansion

$$E_{\eta,\beta}^{x,\lambda}(t) = \sum_{k=0}^{\infty} \lambda^k \hat{H}_{\eta k+\beta-1}(t, s), \quad s, t \in (\tilde{T}_{(q,h)}^a), \quad t \geq s.$$ 

It is easy to check that the series on the right-hand side converges (absolutely) if $|\lambda| \nu^{\alpha}(t) < 1$. As it might be expected, the particular $(q,h)$-Mittag-Leffler function $E_{\eta,\beta}^{x,\lambda}(t)$ is the solution of the equation $a\nabla_{(q,h)}^{\alpha}y(t) = \lambda y(t)$, i.e. it coincides with the time scales exponential function.

The main properties of the $(q,h)$-Mittag-Leffler function are described by
Theorem 2.9. (i) Let $\gamma \in \mathbb{R}^+$ and $t \in (\mathbb{T}_q^a, b)_\kappa$. Then

$$a\nabla_{(q,h)}^{-\gamma} E_{\alpha,\beta}^{a,\lambda}(t) = E_{\alpha,\beta}^{a,\lambda}(t).$$

(ii) Let $\alpha \in \mathbb{R}^+$ and $\eta k + \beta - \alpha \notin \{0, -1, \ldots, -[\alpha] + 1\}$ for all $k \in \mathbb{Z}^+$. If $t \in (\mathbb{T}_q^a, b)_\kappa^{[\alpha]+1}$ then

$$a\nabla_{(q,h)}^{\alpha} E_{\alpha,\beta}^{a,\lambda}(t) = \begin{cases} E_{\alpha,\beta-\alpha}^{a,\lambda}(t), & \beta - \alpha \notin \{0, -1, \ldots, -[\alpha] + 1\}, \\ \lambda E_{\alpha,\beta-\alpha+\eta}^{a,\lambda}(t), & \beta - \alpha \in \{0, -1, \ldots, -[\alpha] + 1\}. \end{cases}$$

Remark 2.10. The assumption $\eta k + \beta - \alpha \notin \{0, -1, \ldots, -[\alpha] + 1\}$ for all $k \in \mathbb{Z}^+$ in Theorem 2.9(ii) may seem to be quite restrictive. Note that it is satisfied trivially for $\beta \in \mathbb{R}^+$ and $\eta + \beta > \alpha$ and, as shown in the following assertion, this is the case we are interested in.

Corollary 2.11. Let $\alpha \in \mathbb{R}^+$. Then the functions

$$E_{\alpha,\beta}^{a,\lambda}(t), \quad \beta = \alpha - [\alpha] + 1, \ldots, \alpha - 1, \alpha$$

(2.5)

define eigenfunctions of the Riemann-Liouville fractional difference operator $a\nabla_{(q,h)}^{\alpha}$ on each set $[\sigma(a), b] \cap (\mathbb{T}_q^a, b)_\kappa$, where $b \in (\mathbb{T}_q^a, b)_\kappa^{[\alpha]+1}$ is satisfying $|\lambda| \nu^a(b) < 1$.

Our final aim is to show that any solution of the equation (2.4) can be written as a linear combination of $(q, h)$-Mittag-Leffler functions (2.5).

Lemma 2.12. Let $\alpha \in \mathbb{R}^+$ and $\lambda \in \mathbb{R}$ be such that $|\lambda| \nu^a(\sigma^{[\alpha]}(a)) < 1$. Then

$$W_{[\alpha]-\alpha}(E_{a,a-[\alpha]+1}^{a,\lambda}, E_{a,a-[\alpha]+2}^{a,\lambda}, \ldots, E_{a,a}^{a,\lambda})(\sigma^{[\alpha]}(a)) = \prod_{k=1}^{[\alpha]} \frac{1}{1 - \lambda \nu^a(\sigma^k(a))} \neq 0.$$ 

Now we summarize the results of Theorem 2.6, Corollary 2.11 and Lemma 2.12 to obtain

Theorem 2.13. Let $y(t)$ be any solution of the equation (2.4) defined on $[\sigma(a), b] \cap (\mathbb{T}_q^a, b)_\kappa$, where $b \in (\mathbb{T}_q^a, b)_\kappa^{[\alpha]+1}$ is satisfying $|\lambda| \nu^a(b) < 1$. Then

$$y(t) = \sum_{j=1}^{[\alpha]} c_j E_{a,a-[\alpha]+j}^{a,\lambda}(t),$$

where $c_1, \ldots, c_{[\alpha]}$ are real constants.

3 Qualitative analysis of a scalar linear FdE on $T = \mathbb{Z}$

In this chapter we utilize the time scale of integers, i.e. $T = \mathbb{Z}$. We note that a possible extension of the following results to $T = h\mathbb{Z}$ with arbitrary $h > 0$ is only a technical matter.
We prefer the standard difference case due to close relations of studied problems to some parts of the qualitative theory of difference equations.

We investigate here stability and asymptotic properties of the linear FdE

\[ \sum_{h=1}^{\alpha} y(t) = \lambda y(t), \quad t = 2, 3, \ldots, \]  

where \( 0 < \alpha < 1, \lambda \neq 1 \) are real scalars.

Although the solution of this equation via the discrete Mittag-Leffler functions (Definition 2.8) was discussed in several papers (see, e.g. [9, 40] and [14, 15]), the asymptotic behaviour of these functions has not been described yet. Thus, we cannot employ the results from the previous chapter for the qualitative analysis of (3.1).

Hence, we consider (3.1) in the form of a Volterra equation of convolution type. This enables us to analyze its stability and asymptotics by use of standard tools of the Volterra difference equations theory, namely the \( Z \)-transform. All the presented results come from [16].

First we introduce a Volterra form of (3.1) which will be studied in the sequel. To agree with the notation used in the theory of difference equations, we denote the independent variable by \( n \) instead of \( t \) throughout this chapter.

**Proposition 3.1.** Let \( 0 < \alpha < 1 \) and \( \lambda \neq 1 \). Then \( y(n) \) is the solution of (3.1) if and only if \( x(n) = y(n+1) \) is the solution of

\[ x(n+1) = \frac{1}{1-\lambda} \sum_{j=0}^{n} (-1)^{n-j} \binom{\alpha}{n-j+1} x(j), \quad n = 0, 1, \ldots. \]  

**Remark 3.2.** The existence and uniqueness of the solution is guaranteed since the \( \nu \)-regressivity of (3.1) is ensured due to the assumption \( \lambda \neq 1 \). If it is not satisfied, then (3.1) admits only the identically zero solution via the starting value \( y(1) = 0 \). If \( y(1) \neq 0 \), then (3.1) has no solution.

### 3.1 Stability and asymptotic analysis

Apart from the classical definitions of stability and asymptotical stability, we utilize for the equation (3.2) also stronger notions of uniform stability and uniform asymptotical stability (see, e.g. [25]). After a stability discussion, we consider the asymptotically stable case, when the solutions \( x(n) \) of (3.2) are tending to zero as \( n \to \infty \), and describe the exact rate of their decay. An asymptotic result concerning the non-stable case will be derived as well.

In the case of uniform asymptotic stability we can formulate the necessary and sufficient conditions via analysis of the roots of characteristic equation (see [25, Theorem 2]).
Theorem 3.3. Let $0 < \alpha < 1$ and $\lambda \neq 1$. Then (3.2) is uniformly asymptotically stable if and only if

$$\lambda < 0 \quad \text{or} \quad \lambda > 2^\alpha. \quad (3.3)$$

For other choices of $\lambda$ we get via the $Z$-transform method

Lemma 3.4. Let $0 < \alpha < 1$ and $\lambda \neq 1$. Then (3.2) is

(i) asymptotically stable, if $\lambda = 0$,
(ii) not stable, if $0 < \lambda < 2^\alpha$.

Now, we reformulate these results for the RDE (3.1). Considering this equation, we are interested especially in its asymptotic stability. Proposition 3.1, Theorem 3.3 and Lemma 3.4 imply the following assertion.

Theorem 3.5. Let $0 < \alpha < 1$ and $\lambda \neq 1$. Then (3.1) is asymptotically stable if

$$\lambda \leq 0 \quad \text{or} \quad \lambda > 2^\alpha. \quad (3.4)$$

Remark 3.6. The condition (3.4) for the asymptotic stability of (3.1) is close to be not only sufficient, but also necessary. It remains to discuss the asymptotic stability of (3.1) with $\lambda = 2^\alpha$, which is still an open problem.

Further, we concern with the asymptotic properties of (3.2). First we note that a preliminary information on the decay rate of the solutions of (3.2) follows immediately from [25, Theorem 2] and Theorem 3.3.

Corollary 3.7. Let $0 < \alpha < 1$ and let either $\lambda < 0$ or $\lambda > 2^\alpha$. Then

$$x(n) \in \ell^1$$

for any solution $x(n)$ of (3.2).

Employing the general results of Appleby et al. [6], we obtain a precise description of asymptotics of (3.2) for a certain class of asymptotically stable cases.

Corollary 3.8. Let $0 < \alpha < 1$ and $|1 - \lambda| > 1$. Then

$$\lim_{n \to \infty} \frac{x(n)}{n^{-(1+\alpha)}} = \frac{\alpha(1-\lambda)}{\lambda^2 \Gamma(1-\alpha)} x(0)$$

for any solution $x(n)$ of (3.2).
The assumptions of Corollary 3.8 do not cover the cases \( \lambda = 0 \) and \( 2^\alpha < \lambda \leq 2 \), when the equation (3.2) is asymptotically stable as well. While the decay rate of the solution for \( 2^\alpha < \lambda \leq 2 \) remains an open problem, the case \( \lambda = 0 \) can be easily investigated. It is not difficult to derive that the exact form of the solutions \( x(n) \) of (3.2) with \( \lambda = 0 \) is

\[
x(n) = x(0)(-1)^n \left( \frac{-\alpha}{n} \right).
\]

Then, we can easily get the following asymptotic result.

**Corollary 3.9.** Let \( 0 < \alpha < 1 \) and \( \lambda = 0 \). Then

\[
\lim_{n \to \infty} \frac{x(n)}{n^{-(1-\alpha)}} = \frac{1}{\Gamma(\alpha)} x(0)
\]

for any solution \( x(n) \) of (3.2).

The derived asymptotic results can be reformulated for the rDE (3.1) as

**Corollary 3.10.** Let \( 0 < \alpha < 1 \) and let either \( \lambda \leq 0 \) or \( \lambda > 2 \). Then

\[
y(n) \sim \begin{cases} 
  \frac{K_1}{n^{1-\alpha}} & \text{as } n \to \infty, \\
  \frac{K_2}{n^{1+\alpha}} & \text{as } n \to \infty,
\end{cases} \quad K_1 = \frac{1}{\Gamma(\alpha)} y(1) \quad \text{if } \lambda = 0, \\
K_2 = \frac{\alpha(1-\lambda)}{\lambda^2 \Gamma(1-\alpha)} y(1) \quad \text{otherwise}
\]

for any solution \( y(n) \) of (3.1).

Now we turn our attention to the unstable case. Recently, Atici and Eloe [9] analyzed the closed form of the solutions of (3.1) based on discrete Mittag-Leffler functions and proved that if \( 1/2 \leq \alpha < 1 \) and \( 0 < \lambda < 1 \), then \( y(n) \to \infty \) as \( n \to \infty \) for any solution \( y(n) \) of (3.1) with \( y(1) > 0 \) (in our notation). We employ our approach based on analysis of the corresponding Volterra difference equation (3.2) to obtain a slightly stronger result.

**Theorem 3.11.** Let \( 0 < \alpha < 1 \), \( 0 < \lambda < 1 \) and let \( x(n) \) be a solution of (3.2) with \( x(0) > 0 \). Then

\[
\frac{\lambda^{1/\alpha} x(0)}{(1-\lambda^{1/\alpha})^n} < x(n) < \frac{x(0)}{(1-\lambda^{1/\alpha})^n}, \quad n = 1,2,\ldots.
\]

### 3.2 A connection to some recent results

Our stability investigation of (3.2) was based on analysis of the roots of the corresponding characteristic equation and their location with respect to the unit disk. In general, this direct approach is not practical just because of difficulties connected with the localization of the roots of a complex function resulting from the utilized \( Z \)-transform method. Therefore, the following explicit criterion for the asymptotic stability is usually applied.
Theorem 3.12 ([23, Theorem 6.18]). Consider the equation \( x(n+1) = \sum_{j=0}^{n} a(n-j)x(j) \). Suppose that \( a(n) \) does not change sign for \( n \in \mathbb{Z}_0^+ \) and
\[
\left| \sum_{n=0}^{\infty} a(n) \right| < 1.
\] (3.5)
Then the equation is uniformly asymptotically stable.

Till lately, it was an open question whether or not (3.5) is also necessary for the (uniform) asymptotic stability (see, e.g. [23, 24]). Considering (3.2), Theorem 3.12 implies the condition \(|1 - \lambda| > 1\), which is a weaker result than (3.3) yields. More precisely, if \(2^\alpha < \lambda \leq 2\), then Theorem 3.3 implies the uniform asymptotic stability of (3.2), although (3.5) does not hold.

Besides this contribution to the stability theory of Volterra difference equations, we can observe some other specific qualitative properties of the FdE (3.1):

(i) The stability result for \( \lambda = 0 \) and \( 0 < \alpha < 1 \) (see Proposition 3.4) does not agree with the limit (trivial) case \( \alpha = 1 \), when (3.1) is stable, but not asymptotically stable.

(ii) The qualitative behaviour for \( \lambda = 0 \) is qualitatively different from the behaviour for other values of \( \lambda \) corresponding to the asymptotic stable case \((\lambda < 0 \text{ or } \lambda > 2^\alpha)\).

(iii) An algebraic decay rate of the solutions \( y(n) \) of (3.1) with \(|1 - \lambda| > 1\) is equal to \(1 + \alpha\), which is the same as derived in [45] for the corresponding FDE.

4 Qualitative analysis of a vector linear FdE on \( \mathbb{T} = h\mathbb{Z} \)

The development of numerical methods for solving of FDEs is one of propulsion powers of the discrete fractional calculus. One of the simplest numerical methods, a generalization of the well-known Euler method, consists of reduction of involved functions to
\[
\mathbb{T} = h\mathbb{Z}_0^+ = \{ t_n = nh ; n \in \mathbb{Z}_0^+ \}, \quad h > 0
\]
and replacing of the continuous fractional operators by the corresponding discrete ones. For an illustration we refer to methods described in [22, 44], whose basic properties were studied, e.g. in [37].

The subject of this chapter is closely related to the qualitative analysis of a numerical method utilizing the fractional difference given by (1.7) for \( q = 1 \). This method, a generalization of the backward Euler method, was proposed in [33], where its convenience for solving of initial value problems with fractional-order initial conditions was discussed. Its application to the boundary value problems representing anomalous diffusion was introduced in [32].
The results presented in this chapter originate from the paper [17] and some of them can be viewed as a vector extension of the main results of the previous chapter. However, while proof techniques employed in Chapter 3 utilize tools from the theory of Volterra equations, assertions presented in this chapter are derived by use of original direct methods.

We show, among others, that the discrete system of linear FdEs can retain the key qualitative properties of the underlying continuous one regardless of the discretization stepsize (this property of backward discretizations is well-known for $\alpha = 1$ and we wish to confirm it also for $0 < \alpha < 1$). In particular, we formulate a direct discrete counterpart to the main result of [45] using the $h$-Laplace transform as a proof tool.

4.1 Problem formulation and its solution

We are going to discuss some basic qualitative properties of the fractional difference system

\[ 0^{\nabla_{\alpha}^n} y(t_n) = Ay(t_n), \quad 0 < \alpha \leq 1, \quad n \in \mathbb{Z}^+, \]  

(4.1)

where $A$ is a $d \times d$ constant matrix with real entries, $y(t_n)$ is $d$-vector. Since we utilize the $h$-Laplace transform as the main proof tool throughout this chapter, the initial condition is expected to be prescribed (see Lemma 1.6 (iv)) in the form

\[ 0^{\nabla_{\alpha}^n} y(t_n) \bigg|_{n=0} = y_0, \quad y_0 \in \mathbb{R}^d \]  

(4.2)

which requires some additional comments. Suppose that the solution of (4.1), (4.2) is on $T = h\mathbb{Z}_0^+$. Then Definition 1.2 automatically implies $0^{\nabla_{\alpha}^n} y(t_n) \bigg|_{n=0} = 0$. Now assume that the solution has the domain $h\mathbb{Z}^+$, i.e. its value at point $t_0 = 0$ is undefined. Then the formula (1.5) does not assign any value to the symbol $0^{\nabla_{\alpha}^n} y(t_n) \bigg|_{n=0}$ and the fractional difference $0^{\nabla_{\alpha}^n} y(t_n)$ is not defined for $n = 1$ (see (1.6)). Thus, the system (4.1), (4.2) seems not to be covered for $n = 1$. However, Definition 1.3 discusses a stand-alone function $f(t)$ given by its values, while the solution of the initial value problem is specified via its properties. In particular, a prescription of the initial condition by (4.2) provides an additional information allowing to interpret (4.1) even for $n = 1$. Indeed, the sequential expanding of (1.6) yields for (4.1), (4.2) the equation $0^{\nabla_{\alpha}^n} y(t_n) \bigg|_{n=1} - 0^{\nabla_{\alpha}^n} y(t_n) \bigg|_{n=0} = hAy(t_n)$, which utilizing (4.2) and expanding the first term by (1.5) leads to

\[ y(t_1) = h^{\alpha-1}(I - h^\alpha A)^{-1}y_0. \]  

(4.3)

This relation defines (under the regularity assumption of the matrix $I - h^\alpha A$) a one-to-one mapping between $y_0$ and $y(t_1)$. Summarizing this, in the frame of Riemann-Liouville approach we shall study the initial value problem (4.1), (4.2), where the meaning of (4.1) for $n = 1$ and the meaning of (4.2) are specified via (4.3).

Now, we discuss a condition guaranteeing the existence and uniqueness for (4.1), (4.2). If $\alpha = 1$ then this condition can be expressed via $\nu$-regressivity of $A$ (see [12]). This property can be extended to $0 < \alpha < 1$ as follows.
**Definition 4.1.** A matrix function $A : \mathbb{T} \to \mathbb{R}^{d \times d}$ is called $\nu^\alpha$-regressive if

$$
\det(I - \nu^\alpha(t)A(t)) \neq 0 \quad \text{for all } t \in \mathbb{T}_\kappa.
$$

**Remark 4.2.** We are interested in the case of a constant matrix $A$ on time scale $\mathbb{T} = h\mathbb{Z}_0^+$, i.e. (4.4) reduces to a single inequality $\det(I - h^\alpha A) \neq 0$.

**Proposition 4.3.** Let $A$ be $\nu^\alpha$-regressive. Then the initial value problem (4.1), (4.2) has a unique solution.

In this chapter, we utilize the $h$-Laplace transform for qualitative analysis of (4.1). Nevertheless, it can serve also as a useful tool for finding the solution of the initial value problem (4.1), (4.2). Doing this, we recall the $h$-Mittag-Leffler function in the matrix form

$$
E^A_{\eta,\beta}(t_n) = \sum_{k=0}^{\infty} A^k \tilde{h}_{\eta k+\beta-1}(t_n, 0), \quad \eta, \beta \in \mathbb{R}, \ A \in \mathbb{R}^{d \times d},
$$

where all eigenvalues $\lambda(A)$ are assumed to lie inside $B(0, h^{-\eta})$ (see, e.g. [40]). Using Lemma 1.6 (i) it can be shown that $\mathcal{L}\{E^A_{\eta,\beta}\}(z) = z^{\eta-\beta}(z^n I - A)^{-1}$ which enables us to prove

**Theorem 4.4.** Assume that all eigenvalues $\lambda(A)$ lie inside $B(0, h^{-\alpha})$. Then the initial value problem (4.1), (4.2) has the unique solution given by

$$
y(t_n) = E^A_{\alpha,\alpha}(t_n)y_0.
$$

# 4.2 Stability and asymptotic analysis

Our stability analysis of (4.1) is based on the investigation of the $h$-Laplace transform of the solution $\mathcal{L}\{y\}(z)$, in particular regarding its series expansion (1.8).

Before we formulate the main theorem of this section, we introduce the following preliminary assertion.

**Proposition 4.5.** Let $y(t_n)$ be a solution of (4.1), (4.2) and let $R$ be the set of all roots of the equation

$$
\det(z^n I - A) = 0.
$$

(i) If $\min_{z \in R} |z - h^{-1}| > h^{-1}$, then $y(t_n) \in \ell^1$.

(ii) If $\min_{z \in R} |z - h^{-1}| < h^{-1}$, then (4.1) is not stable.

Let $C = (c_{ij})$ be a matrix. By the symbol $|C|$ we shall understand the matrix given by $|C| = (|c_{ij}|)$. Further, we introduce the regions

$$
\mathcal{S}_{\alpha,h} = \left\{ z \in \mathbb{C} ; |\text{Arg}(z)| > \frac{\alpha \pi}{2} \text{ or } |z| > \frac{2^\alpha}{h^\alpha \cos^\alpha \left( \frac{\text{Arg}(z)}{\alpha} \right)} \right\}
$$
and the interior of its complement in \( \mathbb{C} \)

\[
U_{\alpha,h} = \left\{ z \in \mathbb{C} ; |\arg(z)| < \frac{\alpha \pi}{2} \text{ and } |z| < \frac{2^\alpha}{h^\alpha} \cos^\alpha \left( \frac{\arg(z)}{\alpha} \right) \right\}.
\]

Using this notation we have

**Theorem 4.6.** Let the matrix \( A \) be \( \nu^\alpha \)-regressive, \( 0 < \alpha \leq 1 \) and let \( y(t_n) \) be the solution of (4.1), (4.2).

(i) If all eigenvalues \( \lambda(A) \) satisfy \( \lambda(A) \in \mathcal{S}_{\alpha,h} \), then \( y(t_n) \in \ell^1 \), hence (4.1) is asymptotically stable. Moreover, if all eigenvalues of the matrix \( |(I - h^\alpha A)^{-1}| \) lie inside the open unit disk, then each component of \( y(t_n) \) tends to zero like \( O(n^{-(1+\alpha)}) \) as \( n \to \infty \).

(ii) If there exists an eigenvalue \( \lambda(A) \) such that \( \lambda(A) \in \mathcal{U}_{\alpha,h} \), then (4.1) is not stable.

**Remark 4.7.** The asymptotic stability region \( \mathcal{S}_\alpha \) of the differential system is given by \( \mathcal{S}_\alpha = \{ z \in \mathbb{C} ; |\arg(z)| > \frac{\alpha \pi}{2} \} \) (see [45]). We can see that \( \mathcal{S}_\alpha \subset \mathcal{S}_{\alpha,h} \) for any \( 0 < \alpha \leq 1 \) and any \( h > 0 \). Moreover, by Theorem 4.6, the discretization (4.1) preserves the decay rate of the exact solutions (at least in a part of asymptotic stability region \( \mathcal{S}_{\alpha,h} \)).

Theorem 4.6 does not solve the stability problem when some of eigenvalues \( \lambda(A) \) lie on the stability boundary. The following assertion demonstrates that all stability variants are possible in such the case.

**Theorem 4.8.** Let the matrix \( A \) be \( \nu^\alpha \)-regressive, \( 0 < \alpha \leq 1 \), let \( A \) has the zero eigenvalue \( \lambda_1(A) = 0 \) and let all its nonzero eigenvalues belong to \( \mathcal{S}_{\alpha,h} \). Denote \( \hat{r} \in \mathbb{Z}^+ \) the maximal size of the Jordan block corresponding to \( \lambda_1(A) \).
(i) If \( \hat{r} < \alpha^{-1} \), then (4.1) is asymptotically stable. Moreover, each component of all solutions \( y(t_n) \) of (4.1) tends to zero like \( O(n^{\hat{r} - 1}) \) as \( n \to \infty \).

(ii) If \( \hat{r} = \alpha^{-1} \), then (4.1) is stable, but not asymptotically stable.

(iii) If \( \hat{r} > \alpha^{-1} \), then (4.1) is not stable.

Remark 4.9. The case (i) never occurs when \( \alpha = 1 \). Similarly, the case (ii) may occur only when \( \alpha \) is reciprocal of a positive integer. Finally, if \( \hat{r} = 1 \), i.e. when algebraic and geometric multiplicities of the zero eigenvalue are equal, then (4.1) is asymptotically stable for all \( \alpha \in (0, 1) \) and \( y(t_n) = O(n^{\alpha - 1}) \) as \( n \to \infty \) for any solution \( y(t_n) \) of (4.1).

5 A possible extension of fractional calculus to a general time scale

In this chapter we abandon the study of FdEs on time scales with linear graininess and turn our attention to a more wide problem, namely establishing the fractional calculus in time scales theory. Our research performed in previous chapters motivates us to contribute to the discussion on this issue (see [4, 10]) and provide some comments on this matter.

The continuous and \((q,h)\)-calculus paradigms suggest to introduce the time scales definition of fractional integral of order \( \gamma > 0 \) as

\[
\hat{h}_\gamma(t,\tau) = \int_a^t \hat{h}_{\gamma-1}(t,\rho(\tau)) f(\tau) \nabla \tau
\]

and the Riemann-Liouville fractional derivative of order \( \alpha > 0 \) as

\[
a \nabla_\alpha f(t) = a \nabla^{[\alpha]} a \nabla^{-([\alpha]-\alpha)} f(t)
\]

However, considering a general time scale \( T \), (5.1) (and consequently (5.2)) is nothing but a symbolical expression. Its practical use requires a reasonable definition of power functions \( \hat{h}_\beta, \beta \in (-1, \infty) \). Such extensions are available only on some special time scales, namely \( \mathbb{R} \) and \( T_{(q,h)} \) (and its special cases \( h\mathbb{Z}, q\mathbb{Z} \)).

On this account, we propose to establish the power functions \( \hat{h}_\beta : T \times T \to \mathbb{R} (\beta > -1) \) in the frame of time scales theory as a family of functions satisfying

\[
\hat{h}_\beta * \hat{h}_\gamma(t,s) = \hat{h}_{\beta+\gamma+1}(t,s), \quad t \geq s, \beta, \gamma > -1,
\]

\[
\hat{h}_0(t,s) = 1, \quad t \geq s,
\]

\[
\hat{h}_\beta(t,t) = 0, \quad 0 < \beta < 1,
\]

where \( (\hat{h}_\beta * \hat{h}_\gamma)(t,s) = \frac{1}{\Gamma(\beta+\gamma+1)} \int_s^t (t-\tau)^{\beta-1} (\tau-s)^{\gamma-1} f(\tau) \nabla \tau \).

It can be shown that the system of conditions (5.3)-(5.5) implies many properties and assertions regarding the fractional calculus. In particular, the relation (5.3) often serves as
a unifying element of the corresponding proofs in continuous and \((q, h)\)-fractional calculus (the composition rules, the power rule, the role of a time scales Mittag-Leffler function).

Regarding the properties of the power functions themselves, the conditions (5.3)-(5.5) imply the derivative formula \(\nabla \hat{h}_\beta(t, s) = \hat{h}_{\beta-1}(t, s) \) \((\beta > 0)\) and seem to fulfill the Laplace transform property, i.e. \(\mathcal{L}\{\hat{h}_\beta\}(z) = z^{-\beta-1}\). The polynomials also satisfy (5.3)-(5.5), therefore they are included as a special case. Moreover, our approach generalizes and extends the introductions in [4,10].

Furthermore, this proposal provides many directions for the future research. Besides a construction of precise proofs of various properties and finding of power functions on particular time scales, it is especially important to perform an analysis of existence and uniqueness for the system of conditions (5.3)-(5.5). The introduction of power functions of negative orders opens a question of singular functions on time scales. Finally, establishing of the set of conditions satisfied by power functions on every time scale brings a possibility to incorporate entirely the fractional calculus into the time scales theory.

**Conclusions**

This doctoral thesis concerns with the fractional calculus on time scales, in particular with the FdEs on the time scale \(T(q,h)\) and its special cases.

The necessary theoretical background, such as basics of continuous fractional calculus, the time scales theory and discrete fractional calculus, is summarized in Chapter 1. It also contains some original preliminary results regarding the power functions in \((q, h)\)-calculus, the \(h\)-Laplace transform and properties of fractional operators on the time scale \(T(q,h)\) (we especially refer to \((q, h)\)-version of the power rule established in Lemma 1.4).

Author’s main results are presented in Chapters 2-4. The contributions to the field can be summarized into the following points:

- **Basic theory of linear FdEs on \(T(q,h)\)** - Basic properties were introduced for a quite general linear initial value problem. In particular, the existence and uniqueness was discussed (Theorem 2.3) and the form of a general solution was given (Theorem 2.6).
- **Eigenfunctions of the fractional difference operator on \(T(q,h)\)** - The \((q, h)\)-version of the Mittag-Leffler function was established (Definition 2.8) which enabled to introduce eigenfunctions of the Riemann-Liouville fractional difference operator (Corollary 2.11). Their relation to the solution of a linear two-term FdE was discussed (Theorem 2.13).
- **Qualitative theory** - The stability and asymptotic properties of a scalar linear two-term FdE on \(\mathbb{T} = \mathbb{Z}\) were investigated employing a connection to the Volterra difference equations theory (Theorem 3.5 and Corollary 3.10, respectively). A vector
analogue of these assertions considering the underlying set $\mathbb{T} = h\mathbb{Z}$ was proven via the properties of $h$-Laplace transform of the solution (Theorem 4.6).

The thesis is concluded by Chapter 5 which outlines a possible way of an extension of the fractional calculus to the time scales theory. This proposal implies some interesting consequences regarding the time scales theory and generates many other open questions providing many challenges for the future research.

We believe that the main results of this doctoral thesis upgraded the theory of discrete fractional calculus in several directions and thus contributed to its further development. In particular, the foundations of the theory of FdEs in $(q, h)$-calculus were established and the qualitative theory of FdEs in $h$-calculus was extended. Moreover, there were brought up some ideas contributing to discussions on some open problems in the theory of Volterra difference equations and the time scales theory.

Apart from the possible usage of our results in further theoretical development, our work can be employed in numerical analysis of FDEs and therefore, by an appropriate extension, used in many applications. It was pointed out that our approach to discrete fractional $h$-calculus can be taken as a discretization resulting in the backward fractional Euler method. In particular, the qualitative investigation of the vector initial value problem on $\mathbb{T} = h\mathbb{Z}$ is, among others, closely related to the numerical analysis of the corresponding continuous initial value problem.
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List of author’s publications


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Abstract

This doctoral thesis concerns with the fractional calculus on discrete settings, namely in the frame of the so-called \((q, h)\)-calculus and its special case \(h\)-calculus. First, foundations of the theory of linear fractional difference equations in \((q, h)\)-calculus are established. In particular, basic properties, such as existence, uniqueness and structure of solutions, are discussed and a discrete analogue of the Mittag-Leffler function is introduced via eigenfunctions of a fractional difference operator. Further, qualitative analysis of a scalar and vector test fractional difference equation is performed in the frame of \(h\)-calculus. The results of stability and asymptotic analysis enable us to specify the connection to other mathematical disciplines, such as continuous fractional calculus, Volterra difference equations and numerical analysis. Finally, a possible generalization of the fractional calculus to more general settings is outlined.