OPTIMALITY CONDITIONS FOR SCALAR LINEAR DIFFERENTIAL SYSTEM

Hanna Demchenko
Doctoral Degree Programme (3. ročník), FEEC BUT
E-mail: xdemch02@stud.feet.vutbr.cz

Supervised by: Josef Diblík
E-mail: diblik@feec.vutbr.cz

Abstract: In the contribution, for scalar linear differential system

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = Ax(t) + Bu(t),$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$, $x(t) \in R^n$ and $u(t) \in R^m$ is a control function, a problem of minimizing a function

$$I[x(t),u(t)] = \int_{t_0}^{\infty} \left(x^T(t)Cx(t) + u^T(t)Du(t)\right)\mathrm{d}t$$

where $C \in R^{n\times n}$ is a symmetric, positive definite matrix and D is a diagonal control matrix, $D = \operatorname{diag}\{d_j\}$, $d_j > 0$, $j = 1,\dots,m.$, is considered. To solve the problem, Malkin's approach and Lyapunov's second method are utilized.

Keywords: optimization problem, control function, Lyapunov function.

1 INTRODUCTION

Consider a process, controlled by means of a vector-function $u\colon [t_0,\infty) \to \mathbb{R}^m$, and assume that it can be represented by a system of differential equations of delayed type

$$x'(t) = f(t,x(t),u(t)), \quad t \geq t_0, \tag{1}$$

where $t_0 \in \mathbb{R}$, $f\colon D \to \mathbb{R}^n$,

$$D := \{(t,x,u) \in [t_0,\infty) \times \mathbb{R}^n \times \mathbb{R}^m, \|x\| < M, \|u\| < M\},$$

M is a positive constant, n, $m \in \mathbb{N}$,

$$\|x(s)\| := \max_{i=1,\dots,n} \{|x_i(s)|\}, \; s \in [t_0,\infty).$$

We will assume that

1. $f(t,\theta_n,\theta_m) = \theta_n$, $t \geq t_0$, where θ_n,θ_m are n or m dimensional zero vectors.
2. f is locally Lipschitzian in every bounded neighborhood of each point $(t,x,u) \in D$.

The problem under consideration is following. We need to determine a function (control function) $u\colon [t_0,\infty) \to \mathbb{R}^m$ such that zero solution $x(t) = \theta_n$, $t \geq t_0$ of system (1) will be asymptotically stable and for an arbitrary solution $x = x(t)$, $t \geqslant t_0$ of system (1) satisfying $\|x\| < M$, the integral

$$\int_{t_0}^{\infty} \omega(t,x(t),u(t))\,\mathrm{dt} \tag{2}$$

exists and attains minimum value. Here $\omega\colon D \to \mathbb{R}$ is a positive-definite function.

Define an auxiliary function $B\colon D_1 \to \mathbb{R}$,

$$D_1 := \{(v,t,x,u) \in \mathbb{R} \times [t_0,\infty) \times \mathbb{R}^n \times \mathbb{R}^m,\ \|x\| < M,\ \|u\| < M\},$$

by formula

$$B(V,t,x,u) := \frac{\mathrm{d}V(t,x)}{\mathrm{d}t} + \omega(t,x,u), \tag{3}$$

where V is a Lyapunov function.

To solve the problem, we use the following theorem presented in [3, Theorem IV] (we omit its proof).

Theorem 1.

Assume that, for the system of differential equations (1), there exists a positive definite function V having an infinitesimal upper bound and a vector-function $u_0\colon [t_0,\infty) \to \mathbb{R}^m$, $\|u_0(t)\| \le M$, $t \ge t_0$ such that

i) Function $\omega(t,x,u_0)$ is positive-definite for every $t \ge t_0$, $\|x| < M$.

ii) Identity $B(V,t,x,u_0) \equiv 0$ holds on $[t_0,\infty)$ for every solution $x\colon [t_0,\infty) \to \mathbb{R}^n$ of system (1) where $u = u_0$.

iii) Inequality $B(V,t,x,u) \ge 0$ holds on $[t_0,\infty)$ for every solution $x\colon [t_0,\infty) \to \mathbb{R}^n$ of system (1) and every vector-function $u\colon [t_0,\infty) \to \mathbb{R}^m$ with $\|u(t)\| < M$, $t \in [t_0,\infty)$.

Then, the function u_0 is a solution of the problem (1), (2) and

$$\int_{t_0}^{\infty} \omega(t,x(t),u_0(t))\mathrm{d}t = \min_u \left[\int_{t_0}^{\infty} \omega(t,x(t),u(t))\mathrm{d}t\right] = V(t_0,x(t_0)).$$

2 SCALAR LINEAR DIFFERENTIAL SYSTEM

Consider system:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = Ax(t) + Bu(t), \tag{4}$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$, $x(t) \in R^n$, $u(t) \in R^m$. We need to find a control function $u = u_0(t)$ for which the system is asymptotically stable and an integral quality criterion

$$\int_{t_0}^{\infty} \left(x^T(t)Cx(t) + u^T(t)Du(t)\right)\mathrm{d}t \tag{5}$$

takes a minimum value provided that $C \in R^{n\times n}$ is a symmetric, positive definite matrix and D is a diagonal control matrix, $D = \mathrm{diag}\{d_j\}$, $d_j > 0$, $j = 1,\dots,m$.

Theorem 2. Assume that there exists a positive definite symmetric matrix H satisfying the matrix equation

$$A^T H + HA + C - HBD^{-1}B^T H = \Theta, \tag{6}$$

then for problem (4)-(5) the optimal stabilization control function exists and equals

$$u_0(t) = -D^{-1}B^T Hx(t) \tag{7}$$

Moreover, equation (4) with $u(t) = u_0(t)$, i.e.,

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = Ax(t) + Bu_0(t),$$

is asymptotically stable and

$$V(t_0,x(t_0)) = \int_{t_0}^{\infty} \left(x^T(t)Cx(t)+u_0^T(t)Du_0(t)\right) \mathrm{d}t = \min_u \int_{t_0}^{\infty} \left(x^T(t)Cx(t)+u^T(t)Du(t)\right) \mathrm{d}t.$$

Proof. We utilize Theorem 1. Define a Lyapunov function

$$V(t,x(t)) = x^T(t)Hx(t),$$

where H is $n \times n$ positive-definite symmetric matrix. Then, in accordance with the condition *ii*) of Theorem 1 we analyse the expression B given by (3), i.e.,

$$B(V,t,x(t),u_0) = [Ax(t)+Bu_0(t)]^T Hx(t)+x^T(t)H[Ax(t)+Bu_0(t)]+x^T(t)Cx(t)+u_0^T(t)Du_0(t) = 0.$$

Simplifying the last expression, we get

$$B(V,t,x(t),u_0) = x^T(t)[A^TH+HA+C]x(t)+u_0^T(t)B^THx(t)+x^T(t)HBu_0(t)+u_0^T(t)Du_0(t) = 0. \quad (8)$$

Looking for an extremum of (8) we get

$$B'_{u_0}(V,t,x(t),u_0(t)) = 2B^THx(t)+2Du_0(t) = 0,$$

i.e.,

$$u_0(t) = -D^{-1}B^THx(t),$$

which is the minimum of the function B because $B''_{u_0u_0} = 2D$ is positive-definite matrix.

For (8) to be valid, i.e.,

$$x^T(t)\left[A^TH+HA+C-[D^{-1}B^TH]^TB^TH-HBD^{-1}B^TH+[D^{-1}B^TH]^TDD^{-1}B^TH\right]x(t) =$$

$$= x^T(t)\left[A^TH+HA+C-HBD^{-1}B^TH-HBD^{-1}B^TH+HBD^{-1}B^TH\right]x(t) =$$

$$= x^T(t)\left[A^TH+HA+C-HBD^{-1}B^TH\right]x(t) = 0.$$

we obtain

$$A^TH+HA+C-HBD^{-1}B^TH = \Theta.$$

Thus, for the control function (7) and used Lyapunov function, the system (4) is asymptotically stable and the quality criterion (5) has a minimum value.

Example 1. Consider the system (4) with quality criterion (5).

Let matrices have the form $A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

By (7), the optimal control function will be in the form

$$u_1^0(t) = -(h_1+h_2)x_1-(h_2+h_3)x_2,$$

$$u_2^0(t) = -(h_1+h_2)x_1-(h_2+h_3)x_2. \quad (9)$$

We need to find matrix H. In our case we can compute expression (6), i.e.,

$$A^TH+HA+C-HBD^{-1}B^TH =$$

$$= \begin{pmatrix} -4h_1+2h_2+3-2(h_1+h_2)^2 & h_1-4h_2+h_3-2(h_1+h_2)(h_2+h_3) \\ h_1-4h_2+h_3-2(h_1+h_2)(h_2+h_3) & 2h_2-4h_3+3-2(h_2+h_3)^2 \end{pmatrix} = \Theta.$$

which means that

$$\begin{cases} -4h_1+2h_2+3-2(h_1+h_2)^2=0, \\ h_1-4h_2+h_3-2(h_1+h_2)(h_2+h_3)=0, \\ 2h_2-4h_3+3-2(h_2+h_3)^2=0. \end{cases}$$

To solve it we can, for example, add the first and the third equations to each other and the second multiplied by 2. We obtain

$$-2h_1-4h_2-2h_3+6-2[(h_1+h_2)+(h_2+h_3)]^2 = -2[h_1+2h_2+h_3]+6-2[h_1+2h_2+h_3]^2=0.$$

If put

$$h_1+2h_2+h_3=K, \tag{10}$$

then we have

$$K^2+K-3=0$$

and $K=-1\pm\sqrt{13}$.

After substracting the first equation from the third, we obtain

$$4h_1-4h_3+2(h_1+h_2)^2-2(h_2+h_3)^2=4(h_1-h_3)+2(h_1+2h_2+h_3)(h_1-h_3)=2(h_1-h_3)(2+K)=0$$

and

$$h_1=h_3.$$

Using the last equation to (10) we find

$$h_1+h_2=\frac{1}{2}K. \tag{11}$$

For the second equation from the system, we obtain

$$2(h_1-2h_2)-(h_1+h_2)^2=0 \Rightarrow h_1-2h_2=\frac{1}{4}K^2. \tag{12}$$

From (11) and (12) we find that

$$h_1=h_3=\frac{1}{3}K+\frac{1}{12}K^2,$$

$$h_2=\frac{1}{6}K-\frac{1}{12}K^2.$$

For $K=-1-\sqrt{13}$ matrix H is not positive definite, so by (9) the optimal stabilization control function will be

$$u_1^0(t)=\frac{1-\sqrt{13}}{2}(x_1(t)+x_2(t)).$$

$$u_2^0(t)=\frac{1-\sqrt{13}}{2}(x_1(t)+x_2(t)).$$

3 CONCLUTION

In the contribution we apply Lyapunov's direct method to solve optimal stabilization problems for scalar linear differential system. This method permits to find a control function in the form of a feedback such that the zero solution of given equation or system will be asymptotically stable and, simultaneously, an integral quality criterion attains a minimum value.

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