POWERS OF CERTAIN AUTOMATON AND CREATED APPROXIMATION SPACE

David Staněk
Doctoral Degree Programme (2), FEEC BUT
E-mail: xstane41@stud.feec.vutbr.cz

Supervised by: Jan Chvalina
E-mail: chvalina@feec.vutbr.cz

Abstract: This paper focuses on the $n-$th power of a special case of an automaton $\mathbb{A}$ formed by homogeneous product of mentioned automaton $\mathbb{A}$. Moreover, the number of isomorphic embeddings of $\mathbb{A}$ into $\mathbb{A}^n$ based on the number of components of $\mathbb{A}^n$ is discussed.

Keywords: Approximation space, automata, homogeneous product,information system, set approximation.

1 INTRODUCTION

There are several known ways of defining products of automata on the Cartesian product of the state sets of two given automata in [4]. In this paper a homogeneous product of automata is discussed and demonstrated on a special case of an automaton $\mathbb{A}$ with finite state set followed by examples of automata $\mathbb{A}^2$ and $\mathbb{A}^3$.

Further, we answer questions concerning all possible isomorphic embeddings of a special automaton $\mathbb{A}$ into its powers $\mathbb{A}^n$.

Finally, examples of rough sets on the state spaces of automaton $\mathbb{A}^3$, with respect to the ρ related sets, are included.

2 USED CONCEPTS

For understanding of the following text it is crucial to know or recall some basic concepts of automata theory and rough set theory.

Definition 1. An ordered pair $(M,\circ)$ is called *groupoid* with a non-empty set M and a binary operation $\circ : M \times M \to M$.

Definition 2. By a quasi-automaton $\mathbb{A}$ (or automaton without output) we mean an ordered triad $\mathbb{A} = (S,M,\delta)$, where sets S and M are state set and input alphabet in the given order. Moreover, $\delta : S \times M \to S$ is the transition function (or next state function) satisfying conditions:

- $\delta(s,e) = s$ - neutral element axiom;
- $\delta(\delta(s,m),n) = \delta(s,m \cdot n)$ - mixed associativity axiom

for all $s \in S$ and $m,n \in M$.

Remark: The concept of a quasi-automaton is in agreement with [1, 5].

Definition 3. An ordered triad $\mathbb{B} = (T,M,\delta')$, where $T \subset S$ and δ' is restriction of δ to $T \times M$, is a *subautomaton* of $\mathbb{A}$, written as $\mathbb{B} \ll \mathbb{A}$, if and only if $\delta'(t,m) \in T, \forall t \in T, \forall m \in M$.

Definition 4. Let $\mathbb{A} = (S,M,\delta)$ and $\mathbb{B} = (T,M,\sigma)$ be two automata with same input set M and different state sets S,T (in general). Then the *homogeneous product* $\mathbb{A} \times \mathbb{B}$ is the automaton $(S \times T, M, \delta \times \sigma)$ with

$$(\delta \times \sigma)([s,t],m) = (\delta[s,m],\sigma[t,m]);\ s \in S,\ t \in T,\ m \in M.$$

Definition 5. Let $\mathbb{A} = (A,M,\delta)$ and $\mathbb{B} = (B,M,\sigma)$ be two automata with the same input alphabet M. Then we say that $\varphi : A \to B$ is a *homomorphism* of $\mathbb{A}$ into $\mathbb{B}$ if for arbitrary $a \in A$ and $m \in M$

$$\varphi(\delta(a,m)) = \sigma(\varphi(a),m)$$

holds. If φ is *one-to-one* mapping, then φ is an *isomorphism*, and the automata $\mathbb{A}$ and $\mathbb{B}$ are said to be *isomorphic*.

The following definition is taken from [6].
Definition 6. By *information system* we mean a pair $\mathcal{S} = (O,AT)$, where O is non-empty finite set of objects and AT is non-empty finite set of attributes, such that $a : O \to V_a$ for any $a \in AT$, where V_a is called the *value set* of a.

Definition 7. Let U be a non-empty set - called an *universal set* - and Θ be an equivalence relation on U. The ordered pair (U,Θ) is called the *approximation space.* For $x \in X$ a symbol $[x]_\Theta$ denotes a block (class) of the equivalence Θ containing the element x, i.e. $[x]_\Theta = \{y \in X; [x,y] \in \Theta\}$. Denote:

- $\underline{\mathrm{Ap}}(X) = \{x \in U; [x]_\Theta \subset X\}$ - the lower approximation of X;
- $\overline{\mathrm{Ap}}(X) = \{x \in U; [x]_\Theta \cap X \neq \emptyset\}$ the upper approximation of X.

Definition 8. Now we define a mapping $\mathrm{Apr} : \mathcal{P}(U) \to \mathcal{P}(U) \times \mathcal{P}(U)$ by the rule: For $X \in \mathcal{P}(U)$ we put $\mathrm{Apr}(X) = (\underline{\mathrm{Ap}}(X),\overline{\mathrm{Ap}}(X))$, and the pair $(\underline{\mathrm{Ap}}(X),\overline{\mathrm{Ap}}(X))$ is called a *rough set* in (U,Θ) if there exists $X \in \mathcal{P}(U)$ with the property $(A,B) = \mathrm{Apr}(X)$. The notation $\mathcal{P}(U)$ denotes the power set of U. (In some papers the name *rough set* is given to the upper approximation $\overline{\mathrm{Ap}}(X)$ only.)
The **Definition 7** and **Definition 8** are based on [2, 3, 7, 8, 10].

It is possible to define set approximations using *information systems* as in [6].

Definition 9. Let $X \subseteq O$ and $A \subseteq AT$. $\underline{\mathrm{Ap}}(X)$ is a *lower approximation* of X if and only if

$$\underline{\mathrm{Ap}}(X) = \{x \in O | S_A(x) \subseteq X\} = \{x \in X | S_A(x) \subseteq X\}.$$

$\overline{\mathrm{Ap}}(X)$ is *upper approximation* of X if and only if

$$\overline{\mathrm{Ap}}(X) = \{x \in O | S_A(x) \cap X \neq \emptyset\} = \cup\{S_A(x) | x \in X\}.$$

Here $S_A(x)$ stands for maximal set of objects which are possibly indiscernible by A with x.

Remark: According to the paper [9] we define upper closure in this way.

Definition 10. The *upper closure* u_δ^+ of the set $Q \in \mathcal{P}(S)$ we define in the following way

$$u_\delta^+(Q) = \{\delta(q,m) | q \in Q; m \in M\}.$$

Definition 11. Let us define a binary relation on $\mathcal{P}(S)$, where S is non-empty set, such that two sets $R,T \in \mathcal{P}(S)$ are in relation ρ if and only if $u_\delta^+(R) = u_\delta^+(T)$.

It is easy to see that the relation ρ, taken from [9],is equivalence relation on $\mathcal{P}(S)$ since it is reflexive, symmetric and transitive.

Definition 12. By *orbital structure* we mean the oriented graph where states of an automaton are nodes and arrows are defined using the transition function, i.e. $s \to t$ means that for a suitable input symbol a we have $\delta(s,a) = t$.

3 HOMOGENEOUS SQUARE POWER OF A SPECIAL AUTOMATON

Let us consider the following example.

Example 1 *Let $(M,\circ)$ be groupoid with state set $M = \{0,1,2\}$ and the binary operation $\circ$ defined as follows:*

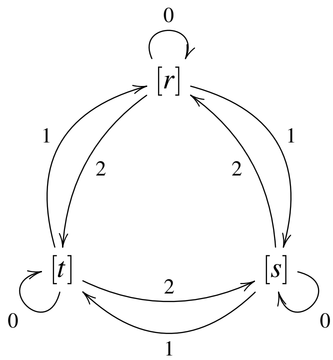
$$x \circ y = x + y \ (mod\ 3)$$

for all $x,y \in M$. Since commutativity and associativity of $(M,\circ)$ are obvious, the algebra is a commutative semigroup, moreover, it has a neutral element 0. Hence algebra $(M,\circ)$ is a commutative monoid.

Now consider the set $S = \{r,s,t\}$ and a function $\sigma : S \times M \to S$ defined this way:

$$\sigma(r,0) = \sigma(t,1) = \sigma(s,2) = r;$$
$$\sigma(r,1) = \sigma(t,2) = \sigma(s,0) = s;$$
$$\sigma(r,2) = \sigma(t,0) = \sigma(s,1) = t.$$

One can see, that the condition of neutral element axiom holds and the mixed associativity axiom condition is fulfilled by associativity of addition of residual classes. Then the triad $\mathbb{A} = (S,M,\sigma)$ is a quasi-automaton with the state set S, the input alphabet M and the transition function σ. An orbital structure of the automaton $\mathbb{A}$ is

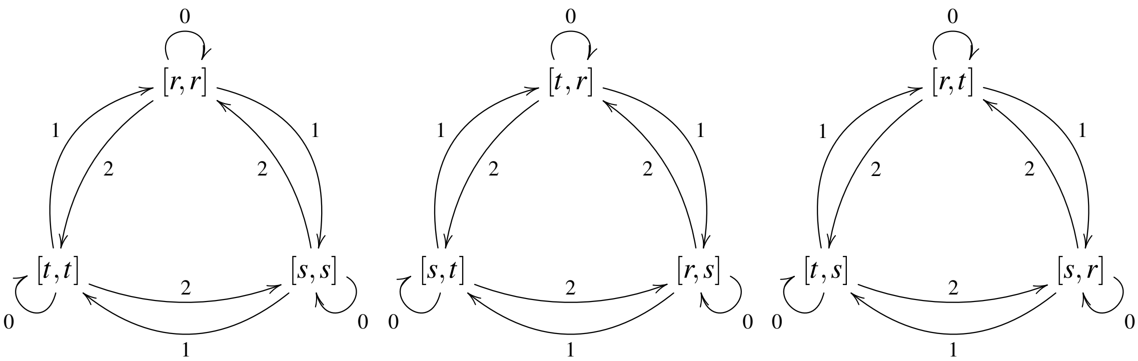


Here notations $[r],[t],[s]$ stand for state r,t,s respectively.

Construction. Now we construct homogeneous product $\mathbb{A} \times \mathbb{A}$ with respect to the **Definition 4.** For $\mathbb{A} \times \mathbb{A}$ we obtain $(S \times S, M, \sigma \times \sigma)$ with

$$\delta([s_1,s_2],m) = (\sigma[s_1,m],\sigma[s_2,m])$$

Then the corresponding orbital structure of $\mathbb{A}^2 = \mathbb{A} \times \mathbb{A}$ is



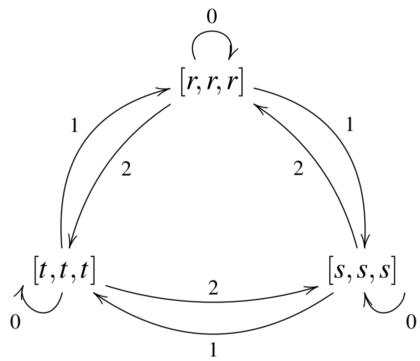
With respect to orbital structures of previous automata a natural question arises. As it is obviously possible to find isomorphic embedding φ of the automaton $\mathbb{A}$ into its square power $\mathbb{A}^2$, how many such mappings exist? The answer is simple, as the orbital structure of $\mathbb{A}$ is identical to any of $\mathbb{A}^2$ components, then quantity of such mappings is equal to permutation of three-element set. Moreover, automaton $\mathbb{A}^2$ has three different components, hence, the final result is multiplied by three, i.e. there exist 18 different mappings of automaton $\mathbb{A}$ into automaton $\mathbb{A}^2$.

Another question may be how many isomorphic embeddings φ exist for $\varphi : \mathbb{A} \to \mathbb{A}^n$, where $n \in \mathbb{N}$. It does not take too much time to realize that the answer depends on the total amount of components of $\mathbb{A}^n$. We may assume that each component of $\mathbb{A}^n$ consists of three elements $[s_1, s_2, \ldots, s_n] \in \mathbb{A}^n$ because any part s_i of $[s_1, s_2, \ldots, s_n]$ is a cycle around three states of original automaton $\mathbb{A}$. It implies that an arbitrary component of $\mathbb{A}$ is isomorphic to the component of $\mathbb{A}$.

Then the amount of components of $\mathbb{A}^n$ is the number of all states of $\mathbb{A}^n$ divided by three due to the three existing element components. After these considerations it is not hard to find out that the number of states of $\mathbb{A}^n$ equals to 3^n, and number of all components equals to 3^{n-1}.

The above mentioned considerations of the number of isomorphic embeddings $\varphi : \mathbb{A} \to \mathbb{A}^2$ holds, i.e. we are able to map $\mathbb{A}$ into any component of $\mathbb{A}^n$ in six ways (permutations of a three-element set). Then finally the number of all possible isomorphic embeddings $\varphi\mathbb{A} \to \mathbb{A}^n$ equal $3! \cdot 3^{n-1} = 2 \cdot 3^n$.

The orbital structure of $\mathbb{A}^3 = \mathbb{A} \times \mathbb{A} \times \mathbb{A}$ consists of 27 elements (states) divided into nine components, each made of three elements. In order to save space only one of these components is shown in this contribution.



Other components creating triads of elements $[s_1, s_2, s_3] \in S \times S \times S$ are:

$$\{[r,r,s],[s,s,t],[t,t,r]\},\{[r,r,t],[s,s,r],[t,t,s]\},\{[r,s,r],[s,t,s],[t,r,t]\},\{[r,t,r],[s,r,s],[t,s,t]\},$$
$$\{[s,r,r],[t,s,s],[r,t,t]\},\{[t,r,r],[r,s,s],[s,t,t]\},\{[r,s,t],[s,t,r],[t,r,s]\},\{[r,t,s],[s,r,t],[t,s,r]\}.$$

According to a **Theorem** in [9] we have:
Theorem. *Let $\mathbb{A} = (S, M, \delta)$ be a quasi-automaton. For any pair $a, b \subset S$ we have:*
$A\rho B$ if and only if either $A = B$ or for any state $s \in A \triangle B$ there exists a state $t \in A \cap B$ such that $\delta(t, m) = s$ for suitable element $m \in M$.

With respect to the definition of relation ρ one can see that the above mentioned equivalence relation builds natural blocks of equivalence on state sets of automata $\mathbb{A}^n$, here $n \in \mathbb{N}$, identical to their components.

4 EXAMPLE OF ROUGH SETS ON THE STATE SPACE OF THE AUTOMATON $\mathbb{A}^3$

Example 2 *Let the state set $(S \times S \times S)$ of the automaton $\mathbb{A}^3$ be a universal set and the relation ρ be the equivalence relation on the set $(S \times S \times S)$. Then the ordered pair $((S \times S \times S), \rho)$ is an approximation space and for the lower approximation of the set $\{[r,r,r]\} \in \mathcal{P}(S \times S \times S)$ we have $\underline{Ap}(\{[r,r,r]\}) = \emptyset$. This is due to the fact that there does not exist any block of equivalence $[[r,r,r]]_\rho$ of*

approximation space $(S \times S \times S)$ *fully included in* $\{[r,r,r]\}$.
For $W = \{[r,r,s],[s,s,t],[t,t,r],[s,s,s]\} \in \mathcal{P}(S \times S)$ *the lower approximation is non-empty. Since the block of equivalence* $[[r,r,s]]_\rho$ *is fully included in w, the lower approximation is* $\underline{Ap}(W) = \{[r,r,s],[s,s,t],[t,t,r]\}$.

The upper approximation of the mentioned sets are $\overline{Ap}(\{[r,r,r]\}) = \{[r,r,r],[s,s,s],[t,t,t]\}$ *and* $\overline{Ap}(W) = \{[r,r,r],[s,s,s],[t,t,t],[r,r,s],[s,s,t],[t,t,r]\}$.

And finally, corresponding rough sets are $Apr(\{[r,r,r]\}) = (\emptyset,\{[r,r,r],[s,s,s],[t,t,t]\})$ *and* $Apr(W) = (\{[r,r,s],[s,s,t],[t,t,r]\},\{[r,r,r],[s,s,s],[t,t,t],[r,r,s],[s,s,t],[t,t,r]\})$.

5 CONCLUSIONS

In this contribution we recall basic concepts of the automata theory and definition of homogeneous product of automata. Moreover, basic terms of rough sets theory are included.

The paper focused on special case of automaton and its powers with respect to the definition of product mentioned above. Further, we answer some questions concerning the number of individual components in $\mathbb{A}^n$ and number of isomorphic embeddings of automaton $\mathbb{A}$ into $\mathbb{A}^n$ in dependence of n.

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