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OPTIMIZATION IN FINANCE

FINANČNÍ OPTIMALIZACE

MASTER'S THESIS
DIPLOMOVÁ PRÁCE

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Pursuant to Act no. 111/1998 concerning universities and the BUT study and examination rules, you have been assigned the following topic by the institute director Master's Thesis:

Optimization in Finance

Concise characteristic of the task:

Student will use his knowledge of probability theory, mathematical statistics and mathematical programming and he will deepen his knowledge in the area of mathematical–economical models and models of financial mathematics. He will select models of stochastic programming, which he will apply in the area of optimal allocation of financial resources in terms of maximum return and low risk. He will implement the built models in the modeling language and he will realize the test calculations for real data.

Goals Master's Thesis:

Acquisition and deepening of knowledge of financial mathematics and mathematical programming.
Creation of mathematical models of optimal investment allocation.
Study of theoretical properties of models and their transformation.
Choice of the effective algorithms.
Software implementation of the models, interconnection with data and realization of computational experiments.

Recommended bibliography:

BIRGE, John R. and LOUVEAUX, François. Introduction to Stochastic Programming. Springer Verlag, 1997. ISBN: 978-1-4614-0236-7.

KALL, Peter and WALLACE, Stein W. Stochastic Programming. New York: John Wiley & Sons, 1993. ISBN 978-0471951582.

RUSZCZYNSKI, Andrzej et al. Handbooks in Operations Research and Management Science, vol. 10: Stochastic Programming. Amsterdam: Elsevier, 2003. ISBN 978-0-444-50854-6.

WALLACE, Stein W. and KING, Alan. Modeling with Stochastic Programming. Springer Verlag, 2012. ISBN 978-0-387-87816-4.

GAMS Modelling Language Manuals, GAMS, 2015.

Deadline for submission Master's Thesis is given by the Schedule of the Academic year 2019/20

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L. S.

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prof. RNDr. Josef Šlapal, CSc.   doc. Ing. Jaroslav Katolický, Ph.D.
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Abstract

This thesis presents two Models of portfolio optimization, namely the Markowitz Mean Variance Optimization Model and the Rockefeller and Uryasev CVaR Optimization Model. It then presents an application of these models to a portfolio of clean energy assets for optimal allocation of financial resources in terms of maximum returns and low risk. This is done by writing GAMS programs for these optimization problems. An in-depth analysis of the results is conducted, and we see that the difference between both models is not very significant even though these results are data-specific.

SOWUNMI, Ololade. *Financial optimization* [online]. Brno, 2020 [cit. 2020-06-15]. Available from: https://www.vutbr.cz/en/studenti/zav-prace/detail/125383 . Thesis. Brno University of Technology, Faculty of Mechanical Engineering, Department of Mathematics. Thesis supervisor Pavel Popela.

I declare that I have worked on this thesis independently under the supervision of RNDr. Pavel Popela Ph.D. and using the sources listed in the bibliography.

SOWUNMI OLOLADE RIDWAN

Acknowledgement

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1 Introduction

This chapter begins with basic preliminary definitions, and concepts of capital, investments and interests, it then gives basic concepts of mathematical programming. The final section gives the outline of the following chapters of the thesis.

1.1 Basics of Capital, Investments and Interests

Optimization in finance deals chiefly with allocating capital into various investments vehicles, the goal is to accumulate the maximum returns and interests on these investments. We take a brief look into capital, investments and interests, see [8] for details.

First, capital can be defined as accumulated wealth spent for the sole aim of generating more wealth via an economic activity. To generate more wealth, individuals and companies have to partly spend and partly save their wealth. In the latter case, they postpone their immediate consumption in favour of investments, see [12] for details.

Therefore, investments may be defined as a decision to postpone the consumption of wealth in order to maximize the expected utility (level of satisfaction) of the investor. Note that the outcome of an investment is almost always uncertain, and the reward for this uncertainty is usually paid in the form of interest. For more details, see [8].

Interest can then be said to be the price of investments, or the return received by a capital on its investments.This price/return is calculated through interest rates. see [8] for more details.

Interest rate is a quantitative measure of interest expressed as a proportion of a sum of money in question (the capital) that is paid over a specified time period. If the initial amount of money is PV (present value) and the interest rate is r for the given time period, then the interest paid at the end of the period is $PV \cdot r$, and the accumulated money at the end of the period, known as FV (future value) is given as

$$FV = PV(1 + r). \tag{1.1}$$

If we consider T periods, where $T > 1$, and the same interest rate r for one period, then there are two approaches towards handling the interests, they are the simple interest method and the compound interests method. Under the simple interest method, only the interests for the present value is received at the end of any period, so after T periods, the future value is

$$FV_T = PV(1 + rT). \tag{1.2}$$

Under the compound interest method, the interest after each period is added to the previous principal and the interest for the next period is calculated from this increased value of the principal. The corresponding future value is

$$FV_T = PV(1 + r)^T. \tag{1.3}$$

For a thorough discussion on interest rates, see [8].

1.2 Basics of Mathematical Programming

In dealing with decision problems, such as the one we intend to deal with in this thesis, formulating and analysing optimization models would be the focal point of our work.

Remark 1. Optimization Model: An optimization model (also known as "mathematical program") represents possible choices as decision variables, and seek values that maximize or minimize objective functions of the decision variables subject to constraints on variable values expressing the limits on possible decision choices. [19]

From this definition, we see that the goal of mathematical programming is to find optimal solutions of the objective function with respect to given constraints.

Now,we give a general description of a mathematical program. Given a function $f(\mathbf{x})$: $\mathbb{R}^n \to \mathbb{R}$ and a set $S \subset \mathbb{R}^n$, the problem of finding an $\mathbf{x}^* \in \mathbb{R}^n$ that solves (1.4) below is called a mathematical program.

$$\begin{aligned} \min_{\mathbf{x}} \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{x} \in S \end{aligned} \tag{1.4}$$

The vector $\mathbf{x} = (x_1, x_2, ..., x_n)^T$ describes the decision variables, the function f is called the objective function, while the set S is known as the feasible set. In most cases, the feasible set S is described explicitly using functional constraints (equalities and inequalities). For example, S may be given as

$$S = \{\mathbf{x} : h_j(\mathbf{x}) = 0; j \in J, g_k(\mathbf{x}) \leq 0; k \in K\} \tag{1.5}$$

where J and K are the index sets for equality and inequality constraints, and the functions f, $h_j \; \forall \; j$, $g_k \; \forall k$ are real functions.

Then the general mathematical program (1.4) takes the form

$$\begin{aligned} \min_{\mathbf{x}} \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & h_j(\mathbf{x}) = 0, j \in J \\ & g_k(\mathbf{x}) \leq 0, k \in K \end{aligned} \tag{1.6}$$

A vector $\mathbf{x}^*$ is called optimal, or a solution of the problem (1.6), if it has the smallest objective value among all vectors that satisfy the constraints.

The mathematical program (1.6) is called
- Linear if the set $S \subset \mathbb{R}^n$ and all functions $f, h_j \forall j, g_k \forall k$ are linear.
- Nonlinear if at least one if the functions $f, h_j \forall j, g_k \forall k$ is nonlinear.
- Quadratic if the objective function is a quadratic function and constraints are linear.
- Integer when some or all of the variables are required to have integer values.

An important class of mathematical programs are convex programs, in which S is a convex set and the objective and constraint functions are convex, that is they satisfy the inequality $f(\alpha\mathbf{x} + \beta\mathbf{y}) \leq \alpha f(\mathbf{x}) + \beta f(\mathbf{y}) \; \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\forall \alpha, \beta \in \mathbb{R}$ with $\alpha + \beta = 1, \alpha \geq 0, \beta \geq 0$.

For a detailed discussion on mathematical programming and the classes of programs, see [19], [7] and [4].

The functions f, h_j and g_k may depend on parameters, and based on the fact that these parameters are known or not, the approach to mathematical programs may be classified into two; namely Deterministic Programming and Stochastic Programming.

1.2.1 Deterministic Programming

Deterministic programming is the approach used for solving mathematical programs for which all the parameters and coefficients (in objective functions and constraints) are assumed to be known, there is neither uncertainty nor randomness. As an example, we consider a linear program below, where we find a solution to (1.7) below

$$
\begin{aligned}
\min_{\mathbf{x}} \quad & \mathbf{c^T x} \\
\text{s.t.} \quad & A(\mathbf{x}) = \mathbf{b} \\
& (\mathbf{x}) \geq 0
\end{aligned}
\tag{1.7}
$$

Where $\mathbf{x}$ is a vector of decisions, and $\mathbf{c}$, A and $\mathbf{b}$ are known parameters. Model (1.7) is an example of a linear deterministic program.

1.2.2 Stochastic Programming

Frequently, in the real world, the parameters of the optimization problem we intend to solve correspond to quantities that will only be realized in the future, or cannot be known exactly at the time the problem must be formulated and solved. A model involving financial quantities such as returns and risk is a very good example. Such models are generally known as problems with data uncertainty and to solve these problems, there are two different approaches; stochastic programming and robust programming. See [15], [3] and [8] for more details.

Stochastic programming is an approach used when the data uncertainty is random and can be explained by some probability distribution, while robust optimization is used when one wants a solution that behaves well in all possible realizations of the uncertain data.[7] For the purpose of this thesis, we focus our attention on the stochastic programming approach. In stochastic programming, the uncertain parameters are modelled as random variables with known probability distributions.

This is described by the triplet $(\Omega, \mathcal{A}, P)$ called a probability space. It comprises of the following [7]
- A sample space Ω which represent the set of all outcome of a random experiment. An arbitrary outcome is denoted by ω. In stochastic programming, Ω is often a finite set $\omega_1, \omega_2, ..., \omega_s$. As an example, in the throwing of a die; $\Omega = 1, 2, 3, 4, 5, 6$.
- A class $\mathcal{A}$ of subsets of Ω. An event is a combination of different outcomes. In the example of the throw of a die, $\mathcal{A}$ contains events such as an even number, a number greater than 3, and every possible combination of the outcomes. We take note that $\mathcal{A}$ is closed under standard set operations, the set Ω belongs to $\mathcal{A}$ and if $A_i \in \mathcal{A}, i \in N$, then $\cup A_i \in \mathcal{A}$.
- Lastly, to complete the definition of a probability space, we need a probability measure P. This is a real-valued function defined on $\mathcal{A}$ and it satisfies the following condition.
  1. $0 \leq P(A) \leq 1, \forall A \in \mathcal{A}$

2. $P(\emptyset) = 0$,and $P(\Omega) = 1$

3. If $A_1, A_2, ...$ is a sequence of disjoint sets in $\mathcal{A}$ and if $\cup A_i \in \mathcal{A}$, then $P(\cup A_i) = \sum P(A_i)$.

In our example of the throwing of a die, the probability of having any of the numbers is $1/6$.

Random variables associated with a probability space can be defined as all variables that are influenced by the random events in $\mathcal{A}$. Mathematically, The mapping $\xi : \Omega \to \mathbb{R}$ is called a random variable if $\forall\ x \in \mathbb{R}$, $\{\omega \in \Omega : \xi(\omega) < x\} \in \mathcal{A}$ holds. An accurate probabilistic description of the random variables is assumed available.

A general stochastic program has the following form [8]

$$
\begin{aligned}
\min_{\mathbf{x}} \quad & f(\mathbf{x}, \xi) \\
\text{s.t.} \quad & h_j(\mathbf{x}, \xi) = 0, j \in J \\
& g_k(\mathbf{x}, \xi) \leq 0, k \in K
\end{aligned}
\tag{1.8}
$$

where $\mathbf{x} \in S \subset \mathbb{R}^n$, $\xi = (\xi_1, \xi_2, ..., \xi_s)$, $\xi : \Omega \to \mathbb{R}^s$ is a finite dimensional random vector, formed by random variables of the probability space $(\Omega, \mathcal{A}, P)$. For more details on probability, see [6].

1.3   Outline of the thesis

This thesis is focused on optimization in finance, and it contains three main chapters (excluding the introduction and the conclusion).

Chapter 2 introduces the mathematical models of portfolio optimization that we work with in this thesis, it begins with a discussion of some basic notions in portfolio optimization, and then moves to risk and return; the two major properties of a portfolio. The rest of the chapter is then focused on the Markowitz Mean-Variance optimization model and the Rockefeller and Uryasev CVaR Optimization model. [18], [22] and [23].

In chapter 3, we discuss the applications of these models to obtain an optimal portfolio of clean energy assets, we include additional constraints on the sizes of the assets and see how the portfolio readjusts according to these constraints.

Chapter 4 discusses and compares the results of both models, comparing the efficient frontier of the optimal portfolios obtained. We then make important deductions from the available results and graphs.

2 Mathematical Models of Portfolio Optimization

This chapter focuses on discussing the formulation of two models of portfolio optimization. It begins with section 2.1 which discusses basic notions that will be used throughout the chapter. Section 2.2 focuses on risk and returns with a focus on coherent risk measures. Section 2.3 discusses the Markowitz mean variance optimization models while section 2.4 discusses the Rockefeller Uryasev type models.

2.1 Basic Notions in Portfolio Optimization

To begin the discussion of the mathematical models of portfolio optimization, we introduce some definition and key concepts. [16]

Definition 2.1. Portfolio Optimization: A portfolio is a group of assets held for investment purposes. If we consider n alternative investments and assume that the total wealth available for investments is 1, then a portfolio is described by the decision vector $\mathbf{x} = (x_1, x_2, ..., x_n)$ where x_i represents the fraction of the unit wealth invested in the ith asset. The vector $\mathbf{x} \in S \subset \mathbb{R}^n$ represents the decision vector where S represents the set of admissible portfolios.

The process of selecting the best portfolio $\mathbf{x}^* \in S$ that provides maximum return with minimum risk is called Portfolio Optimization.

The return of the investments on each asset is uncertain, therefore the investor's focus is on how to avoid a loss of wealth. Therefore in this thesis, our focus is on minimizing loss.

Losses are random variables, therefore we will discuss some statistical measures needed to cover the basics of portfolio optimization.

Definition 2.2. Expectation [16]:The expectation, sometimes called expected value or mean, of a random variable ξ is defined as

$$E[\xi] = \int_{-\infty}^{\infty} x f(x) dx \quad \text{in the continuous case} \tag{2.1}$$

or

$$E[\xi] = \sum_{k=-\infty}^{\infty} k P(\xi = k) \quad \text{in the discrete case} \tag{2.2}$$

where $f(x)$ is the probability density function of ξ and $P(\xi = k)$ is the probability mass function.[16]

The expectation is denoted by us as μ, so we have that $\mu = E[\xi]$. $E[\xi]$ can be described as the center value around which possible values of ξ disperse. [16]

Definition 2.3. Variance: The variance of a random variable ξ is defined as

$$\text{var}(\xi) = E[(\xi - E[\xi])^2] \tag{2.3}$$

The variance is often denoted by σ^2, but since it is given in square units, we use the standard deviation (denoted $\sigma = \sqrt{\text{var}(\xi)}$). The standard deviation does not contain additional information, but is easier to interpret as σ is given in the same units as μ. [16]

The standard deviation σ (or variance σ^2) is a measure of dispersion of ξ around μ. Small values of σ indicate that ξ is concentrated around μ, while large values of σ mean that values of ξ is strongly dispersed away from μ.

Definition 2.4. Covariance: The covariance of two random variables ξ_1 and ξ_2 is defined as

$$\text{cov}(\xi_1, \xi_2) = E[(\xi_1 - E[\xi_1])(\xi_2 - E[\xi_2])] \tag{2.4}$$

Covariance measures how strongly the random variables ξ_1 and ξ_2 move together. Important to note is that $\text{cov}(\xi_1, \xi_1) = \text{var}(\xi_1)$. Also, if ξ_1 and ξ_2 are independent, their covariance is 0.[16]

Like the variance, the covariance is also given in square units, so to give a measure of dependency between two random variables, ξ_1 and ξ_2, we define the correlation coefficient.

Definition 2.5. Correlation Coefficient:The correlation coefficient of two random variables ξ_1 and ξ_2 is defined as

$$\rho_{12} = \frac{\text{cov}(\xi_1, \xi_2)}{\sqrt{(\text{var}(\xi_1))}\sqrt{(\text{var}(\xi_2))}} \tag{2.5}$$

ρ always takes values between -1 and 1 and is therefore easier to interprete than covariance. If $\rho_{12} = 0$ then ξ_1 and ξ_2 are independent, if $|\rho_{12}|$ is close to 1, then there is a strong linear dependence between ξ_1 and ξ_2.[16]

Another important concept throughout this thesis is the Value at Risk and the Conditional Value at Risk. We give the definitions at this point, and go into details later in the chapter.

Definition 2.6. Value at Risk (VaR): Let ξ be a random variable representing loss. Given a parameter $0 < \alpha < 1$, the α-VaR of ξ is

$$\text{VaR}_\alpha(\xi) = \min\{c : P(\xi \le c) \ge \alpha\}. \tag{2.6}$$

Given the definition above, we can have the following equivalent interpretation:

First,the $\text{VaR}_\alpha(\xi)$ is the minimum loss that will not be exceeded with probability α.

Secondly, the $\text{VaR}_\alpha(\xi)$ is the α-quantile of the distribution of ξ, see [14].

Now, we discuss the definition of CVaR, but we give only the CVaR definition of a continuous random variable to create a more intuitive introduction into the concept.

Definition 2.7. Conditional Value at Risk (CVaR) in the continuous case [14]: Let ξ be a continuous random variable representing loss. Given a parameter $0 < \alpha < 1$, the α-VaR of ξ is

$$\text{CVaR}_\alpha(\xi) = E[\xi : \xi \ge \text{VaR}_\alpha(\xi)] \tag{2.7}$$

Other names for CVAR in [14] include Expected Shortfall, Average Value at Risk, Tail Conditional Expectation. Figure 1 shows the VaR and CVaR for a continuous random variable ξ.[14]
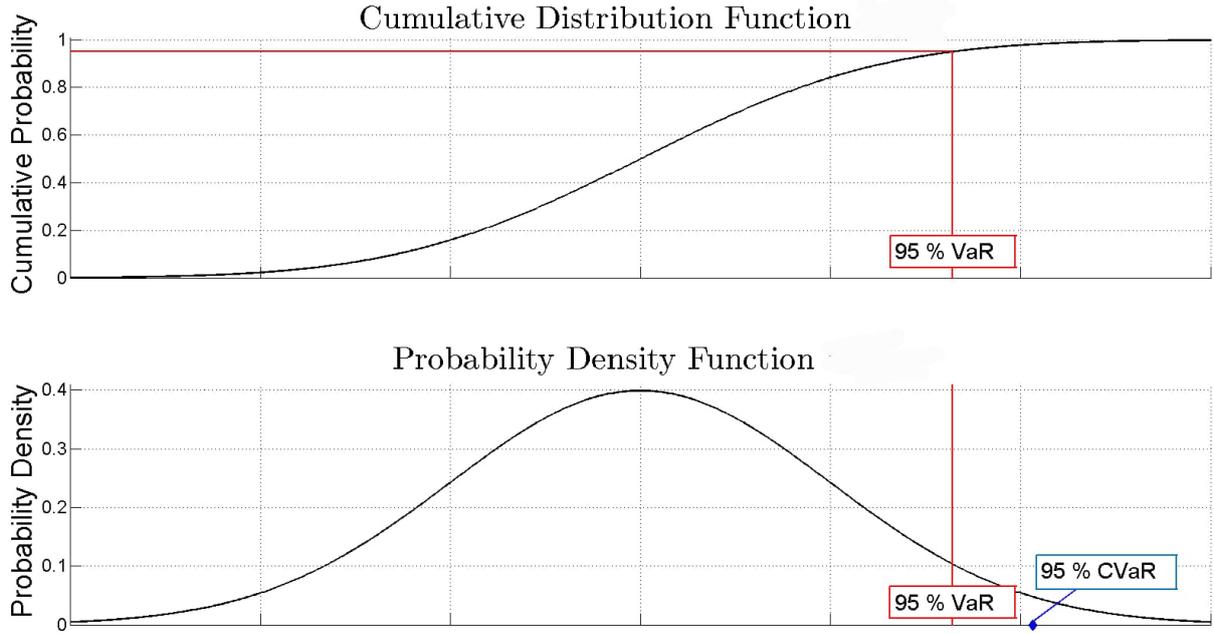
Figure 1: VaR_α and CVaR_α of a random variable representing loss [14]

2.2 Risk and Return

Let S be the set of admissible portfolios, such that the decision vector $\mathbf{x} \in S \subset \mathbb{R}^n$. The returns on each asset i are random, therefore the losses an investor makes on a portfolio can be represented by the random loss vector $\mathbf{r} \in \mathbb{R}^n$, (where we represent losses as the negative values of returns) where r_i is a random variable representing the loss distribution of the ith asset. See [23] for more details.

2.2.1 Expected Return of an asset

Let r_{it} be the loss on the ith asset at a particular scenario/time t, (for example r_{23} is the loss on the 2nd asset at the 3rd scenario), and let r_i be the random variable representing the loss distribution of the ith asset, then the expected return of the asset i is given as

$$\hat{r_i} = E[r_i] = \sum_{t=1}^{m} r_{it} P(r_i = r_{it}) \tag{2.8}$$

where m is the number of observations, and $P(r_i = r_{it})$ is the probability function. [9]

2.2.2 Expected Return of a portfolio

Let $\mathbf{r} \in \mathbb{R}^n$ be the random loss vector representing the loss distribution of the portfolio $\mathbf{x} \in S \subset \mathbb{R}^n$, then the expected return on the portfolio is given as

$$\mathbf{x^T}\hat{r} \quad \text{where} \quad \hat{r} = E[\mathbf{r}] \tag{2.9}$$

[9]

2.2.3 Coherent Risk Measures

Risk, as it pertains to this thesis can be defined as a measure of how volatile an asset's return is, or how volatile a portfolio returns is. Exposure to this volatility can lead to a loss in investments, which is why we focus on loss as the random variable. Each model that we investigate in this thesis measures this volatility in different ways, i.e they utilize a different measure of risk. In order to utilize these measures, they need to obey a set of axioms as stated by Artzner et al. [1]. Any risk measure which satisfies these axioms is said to be coherent. The four axioms are Monotonicity, Translation equivalence, Sub-additivity and Positive Homogeneity.

We define these axioms below but note that in these definitions, ξ_1 and ξ_2 are random variables representing loss, $c \in \mathbb{R}$ is a scalar representing loss and ρ is a risk measure i.e. it maps a random variable ξ_1 or(ξ_2) to a set of real numbers according to the risk associated with ξ_1 or ξ_2.

Definition 2.8. Monotonicity [1]: A risk measure ρ is monotone, if for all ξ_1, ξ_2:

$$\xi_1 \leq \xi_2 \Rightarrow \rho(\xi_1) \leq \rho(\xi_2). \tag{2.10}$$

This means that if a random loss variable ξ_1 is less than ξ_2, then the risk measure $\rho(\xi_1)$ will always be less than $\rho(\xi_2)$.

Definition 2.9. Translation Equivariance [1]: A risk measure ρ is translation equivariant, if for all ξ_1, c:

$$\rho(\xi_1 + c) = \rho(\xi_1) + c. \tag{2.11}$$

Definition 2.10. Sub-additivity [1]: A risk measure ρ is sub-additive, if for all ξ_1, ξ_2:

$$\rho(\xi_1 + \xi_2) \leq \rho(\xi_1) + \rho(\xi_2). \tag{2.12}$$

Definition 2.11. Positive Homogeneity [1]: A risk measure ρ is positively homogenuous, if for all ξ_1, $\lambda \geq 0$:

$$\rho(\lambda\xi_1) = \lambda\rho(\xi_1). \tag{2.13}$$

Remark 2. :We can interpret the above axioms in a more intuitive form as follows [14].

Monotonicity: Higher losses mean higher risk.
Translation Equivariance: Increasing (or decreasing) the loss increases(decreases) the risk by the same amount.
Sub-additivity: Diversification decreases risk.
Positive Homogeneity: Doubling the portfolio size doubles the risk.

2.3 Markowitz Mean Variance Optimization model

Markowitz is the father of modern portfolio theory. Before Markowitz pioneered research into this field, investment decisions were made merely by an investor's belief, but the award winning work of Markowitz on portfolio theory [18] emphasized that assets could not be selected only on characteristics that were unique to the asset, rather an investor had to consider how each asset co-moved with all other assets. Furthermore, taking these co-movements into account resulted in an ability to construct a portfolio that had the

same expected return and less risk than a portfolio constructed by ignoring the interactions between assets. [10]

This model is called the mean-variance model because Markowitz formulated the portfolio problem as a choice of the mean and variance of a portfolio of assets.

To discuss this model, we define the expected returns and the risk measure as was used by Markowitz [18].

Definition 2.12. Expected Return of a portfolio [9]: Let a portfolio $\mathbf{x} \in S \subset \mathbb{R}^n$ be such that each asset i has an expected return $\hat{r}_i$ as defined in subsections 2.2.1 and 2.2.2, and let x_i be the fraction of the wealth assigned to the ith asset, then the expected return of a portfolio is given as

$$R = E[\mathbf{x}^{\mathbf{T}}\mathbf{r}] = \sum_{i=1}^{n} x_i \cdot \hat{r}_i \tag{2.14}$$

This is the weighted average of expected asset returns.

The risk measure utilized by Markowitz is the variance and covariance, which is represented by the covariance matrix $\Sigma \in \mathbb{R}^{n \times n}$ of the random loss vector $\mathbf{r}$. We define Σ below

Definition 2.13. Covariance Matrix [14]: The covariance matrix of the random loss vector is defined as

$$\Sigma = \begin{bmatrix} \text{var}(r_1) & \text{cov}(r_1, r_2) & . & . & . & \text{cov}(r_1, r_n) \\ \text{cov}(r_2, r_1) & \text{var}(r_2) & . & . & . & \text{cov}(r_2, r_n) \\ . & . & . & & . \\ . & . & . & & . \\ . & . & . & & . \\ \text{cov}(r_n, r_1) & \text{cov}(r_n, r_2) & . & . & . & \text{var}(r_n) \end{bmatrix} \tag{2.15}$$

where $\text{var}(r_i) = \sigma_i^2$ is the variance as defined in (2.3) and $\text{Covariance}(r_i, r_j)$ is the covariance as defined in (2.4).

Definition 2.14. Risk of a Portfolio: In the Markowitz model, the risk of a portfolio $\mathbf{x}$ is given as the variance or standard deviation of the random loss vector $\mathbf{r}$. The variance and standard deviation is given in the equation below

$$\text{var}\left(\mathbf{x}^T\mathbf{r}\right) = \mathbf{x}^T\mathbf{\Sigma}\mathbf{x} \quad \text{and} \quad \sigma\left(\mathbf{x}^T\mathbf{r}\right) = \sqrt{\text{var}\left(\mathbf{x}^T\mathbf{r}\right)} = \sqrt{\mathbf{x}^T\mathbf{\Sigma}\mathbf{x}}. \tag{2.16}$$

Note that the risk of the portfolio is less than or equal to the weighted average of asset

risks. This was derived in [14] and it is shown below.

$$\sigma\left(\mathbf{x}^T\mathbf{r}\right) = \sqrt{\operatorname{var}\left(\mathbf{x}^T\mathbf{r}\right)} = \sqrt{\mathbf{x}^T\mathbf{\Sigma}\mathbf{x}}$$

$$= \sqrt{\sum_{i=1}^{N} x_i^2\sigma_i^2 + \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} 2\rho_{ij}x_ix_j\sigma_i\sigma_j}$$

$$= \sqrt{\sum_{i=1}^{N} x_i^2\sigma_i^2 + \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} 2x_ix_j\sigma_i\sigma_j - \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} 2\left(1-\rho_{ij}\right)x_ix_j\sigma_i\sigma_j}$$

$$= \sqrt{\left(\sum_{i=1}^{N} x_i\sigma_i\right)^2 - \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} 2\left(1-\rho_{ij}\right)x_ix_j\sigma_i\sigma_j}$$

$$\leq \sqrt{\left(\sum_{i=1}^{N} x_i\sigma_i\right)^2} = \sum_{i=0}^{N} x_i\sigma_i$$

For a robust discussion of the expected return and variance of a portfolio, see [9].

2.3.1   Markowitz Type Model

Using the principal idea of mean variance portfolio theory, namely holding constant variance, maximize expected return and holding constant expected return, minimize variance, we can design three Markowitz type models. The inputs for the models are $\hat{\mathbf{r}}$, the expected return of asset 1,...,n and the covariance matrix Σ.

Definition 2.15. Maximum Return Portfolio Model: This model is designed to find the portfolio with the maximum expected return, with a prescribed threshold (V_0) for the variance of the portfolio. It is modelled as follows:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{x}^T\hat{\mathbf{r}} \\ \text{s.t.} \quad & \mathbf{x}^T\mathbf{\Sigma}\mathbf{x} \leq V_0 \\ & \mathbf{x} \in S \end{aligned} \tag{2.17}$$

where the set S of admissible portfolios is given as

$$S = \{\mathbf{x} \in \mathbb{R}^n : x_i \geq 0 \qquad \forall i \in \{1,2,...,n\}, \sum_{i=1}^{n} x_i = 1\} \tag{2.18}$$

Definition 2.16. Minimum Variance Portfolio Model: This model finds the portfolio with the minimum variance with an expected return that will not go lesser than a prescribed threshold $-R_0$, where R_0 is positive. (Recall that the expected returns is defined in terms of loss, therefore a negative expected returns implies negative loss which implies profit for the portfolio). It is modelled as

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{x}^T\mathbf{\Sigma}\mathbf{x} \\ \text{s.t.} \quad & \mathbf{x}^T\hat{\mathbf{r}} \leq -R_0 \\ & \mathbf{x} \in S \end{aligned} \tag{2.19}$$

where the parameters are as defined earlier and the set S of admissible portfolios remain the same.

Definition 2.17. Risk Adjusted Return Portfolio Model: This is a multi-objective optimization problem in which we maximize the expected returns while simultaneously minimizing the risk involved. This is modelled as

$$\begin{aligned}
\min_{\mathbf{x}} \quad & \mathbf{x}^T \hat{\mathbf{r}} + \frac{\delta}{2}\mathbf{x}^T \mathbf{\Sigma} \mathbf{x} \\
\text{s.t.} \quad & \mathbf{x} \in S
\end{aligned} \tag{2.20}$$

Here, the constant δ serves as a risk aversion constant.

For a robust discussion on these models, see [7], [14] and [18].

These Markowitz type models lead to the formulation of an efficient frontier from which investors could choose the preferred portfolio, depending on individual risk return preferences. For fixed probability beliefs $(\hat{\mathbf{r}}, \mathbf{\Sigma})$, the investor has a choice of various combinations of the portfolio returns R and the portfolio variance σ^2 depending on the choice of portfolio, where R is as defined in definition 2.12. Suppose that the set of all obtainable (R, σ^2) combination were as in figure 2. The R-σ^2 rule states that the investor would (or should) want to select one of these portfolios which give rise to the (R, σ^2) combination indicated as efficient in the figure, i.e. those with minimum σ^2 for given R or more, or maximum R for given σ^2 or less.[18]

Definition 2.18. Efficient Portfolio: A feasible portfolio $\mathbf{x} \in S$ is called efficient if it has the maximal expected return among all portfolios with the same variance, or alternatively, if it has the maximum variance among all portfolios that have at least a certain expected return.[9]

To explain this we take an example, in figure 2 we see that between portfolio A and B, portfolio B is preferable because it offers a higher return for the same risk as A, If we also compare portfolio C and portfolio A, we see that portfolio C is preferable because it presents lower risk compared to A. From here, we see that an efficient set of portfolios cannot include interior points. Now we eliminate portfolio D and F from further consideration since portfolio E has the lower risk as compared to F and the higher return as compared to D. Portfolio C cannot be eliminated as there is no other portfolio that has less risk for the same return or more return for the same risk. We call portfolio C the global minimum variance portfolio. Also note that Portfolio B cannot be eliminated as there is no other portfolio that has the same return and less risk or the same risk and more return, we call this the maximum return portfolio. Thus the efficient set consists of the envelope curve of all portfolios that lie between the global minimum variance portfolio and the maximum return portfolio. This set of portfolios is called the efficient frontier and it is shown in figure 3. [9]

2.3.2 Theoretical Properties of Markowitz Type Model

The models (2.19) and (2.20) are quadratic programming problems since the objective function is a quadratic equation while model (2.18) has a convex quadratic constraint and is therefore not a quadratic program, therefore it can be solved using general non-linear programming solution techniques. We discuss mainly the theoretical properties of models (2.19) and (2.20) as explained in [7].
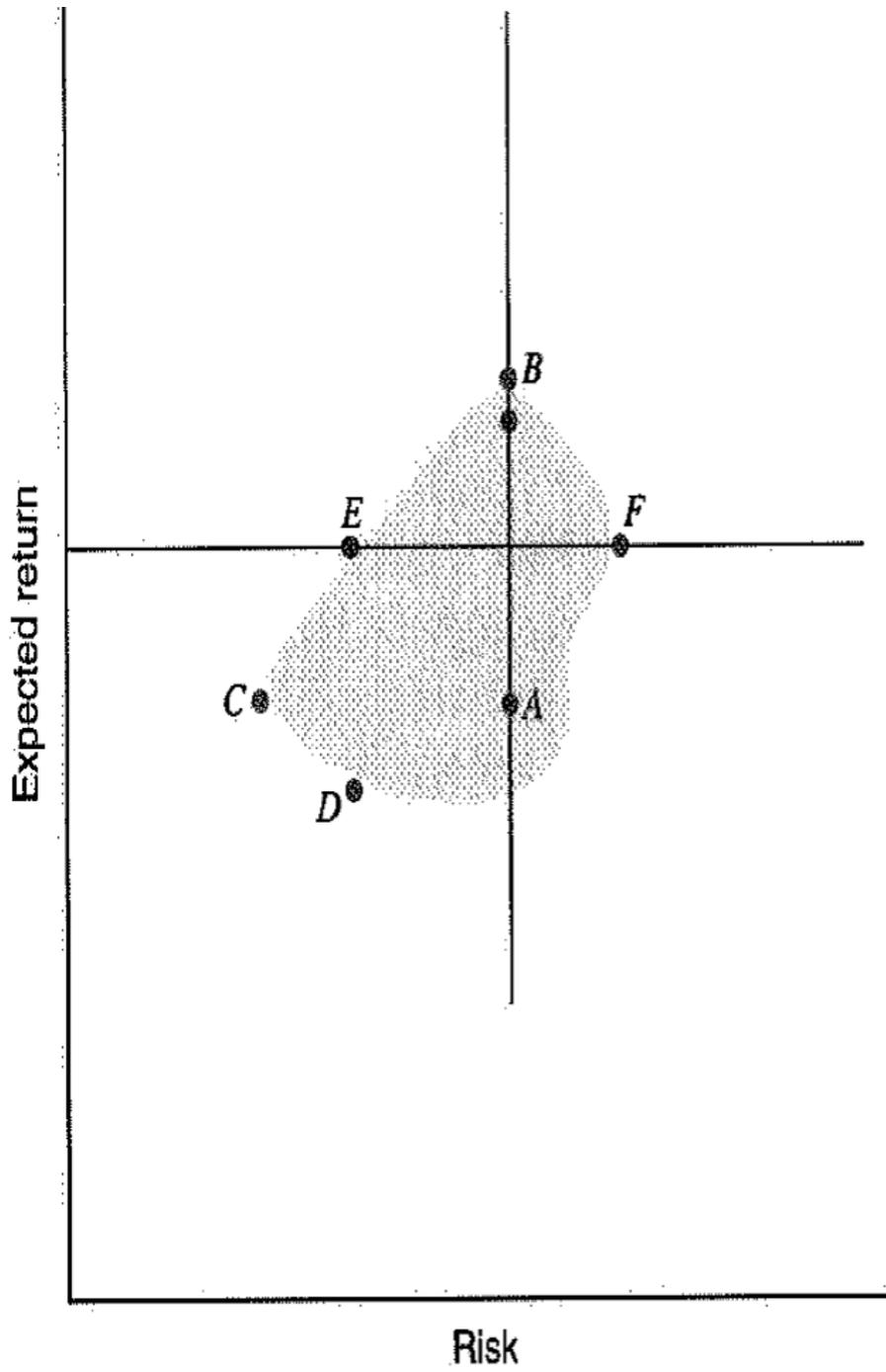
Figure 2: All possible portfolios [9]
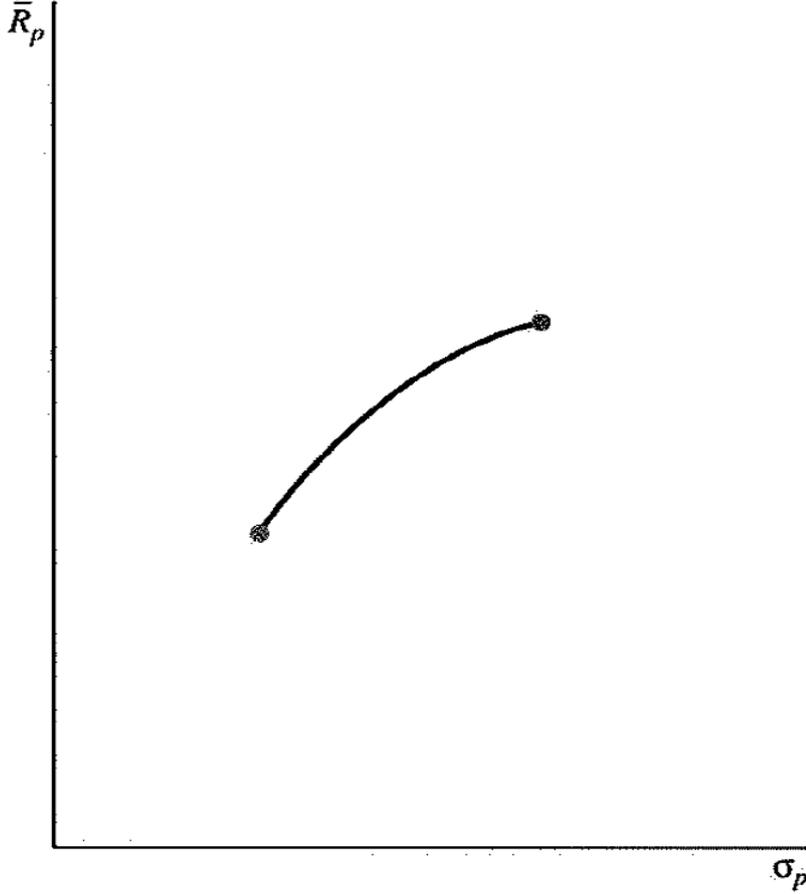
Figure 3: Efficient frontier [9]

1. First since the covariance matrix Σ is always positive definite, i.e. $\mathbf{x}^T\Sigma\mathbf{x} \geq 0$ $\forall \mathbf{x}$, then the model (2.19) and (2.20) are convex quadratic optimization problem, therefore there exists a unique portfolio in S that has the minimum variance. This is because the local optimal solutions are also global optimal solutions. Figure 4 below shows a contrast between the contours of a quadratic function with a positive semi-definite Σ and those of an indefinite Σ to illustrate this property.

2. Applying the Karush Kahn Tucker conditions (see [7] for details) to a quadratic programming problem, we obtain the following set of necessary conditions for optimality.

Theorem 2.19. *Suppose that $\mathbf{x}^*$ is a local optimal solution of the quadratic program given as*

$$
\begin{aligned}
\min_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} + \tfrac{1}{2}\mathbf{x}^T\Sigma\mathbf{x} \\
s.t. \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\
& \mathbf{x} \geq 0
\end{aligned} \tag{2.21}
$$

Also assume that Σ is a positive semi-definite matrix. Then there exist vectors $\mathbf{y}$ and $\mathbf{s}$ such that the following conditions hold

$$
\begin{aligned}
\mathbf{A}^{\mathbf{T}}\mathbf{y} - \Sigma\mathbf{x} \; + \; & \mathbf{s} = \mathbf{c} \\
& \mathbf{s} \geq 0 \\
& x_i s_i = 0 \quad \forall i
\end{aligned} \tag{2.22}
$$

Furthermore, $\mathbf{x}^*$ is a global optimal solution.
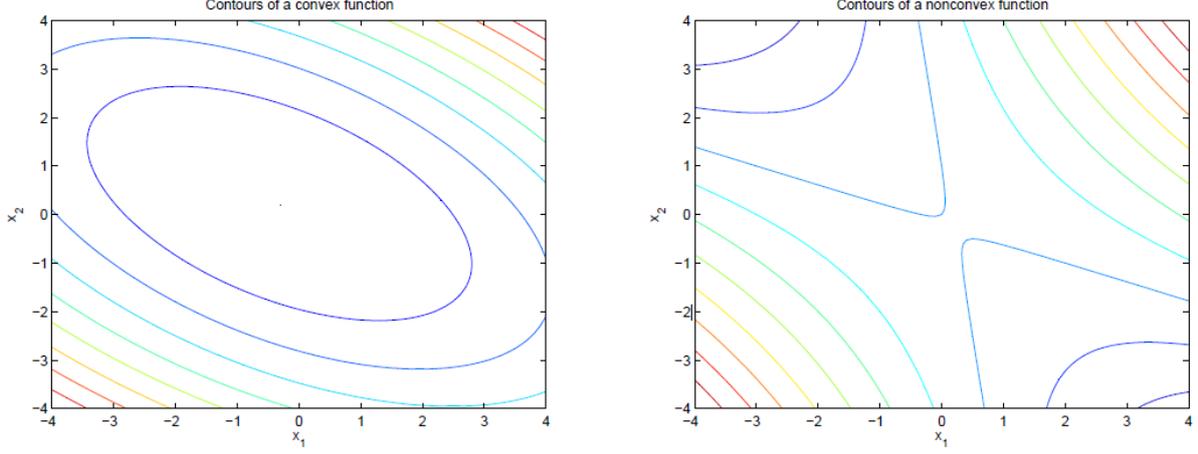
Figure 4: Contours of positive semidefinite and indefinite quadratic functions [7]

This is applicable to model (2.20).

3. It has been established that model (2.19) is a convex quadratic programming problem, therefore the first order conditions are both necessary and sufficient for optimality. This condition is presented below.

$\mathbf{x}^*$ is an optimal solution of model (2.19) if and only if there exists $\lambda^* \in \mathbb{R}, \Upsilon_E \in \mathbb{R}^n$ and $\Upsilon_I \in \mathbb{R}^n$ satisfying the following conditions

$$
\begin{aligned}
&\sum_{i=1}^{N} x_i^* - \lambda^* \widehat{r} - \sum_{i=1}^{N} \Upsilon_E - \sum_{i=1}^{N} \Upsilon_I = 0 \\
&\mathbf{x}^{*T}\widehat{\mathbf{r}} \leq -R_0, \quad \sum_{i=1}^{N} x_i^* = 1, \quad \mathbf{x}^* \geq 0 \\
&\lambda^* \geq 0, \quad \lambda^*(\mathbf{x}^{*T}\widehat{\mathbf{r}} + R_0) = 0 \\
&\Upsilon_I \geq 0, \quad \Upsilon_I^T(\sum_{i=1}^{N} x_i^* - 1) = 0
\end{aligned}
\tag{2.23}
$$

2.3.3   A hypothetical example

To see how the portfolio risk changes for different expected returns, we take an hypothetical example from [14] and solve the model (2.19) for different values of R_0 (expected minimum return) and calculate the resulting portfolio risk (standard deviation). We make use of the risk/return pairs to draw the efficient frontier. For a sample portfolio of three assets with expected return and covariance matrix

$$
\widehat{\mathbf{r}} = \begin{bmatrix} -0.1073 \\ -0.0737 \\ -0.0627 \end{bmatrix} \quad \text{and} \quad \mathbf{\Sigma} = \begin{bmatrix} 0.02778 & 0.00387 & 0.00021 \\ 0.00387 & 0.01112 & -0.00020 \\ 0.00021 & -0.00020 & 0.00115 \end{bmatrix}
$$

The efficient frontier is shown in figure 5.

2.4   Rockefeller and Uryasev CVaR Optimization model

The Markowitz model revolutionalized the modern portfolio theory, but due to the choice of variance as the risk measure, the model has some limitations which leaves a vacuum for better models. One major limitation of the mean variance model is the fact that the
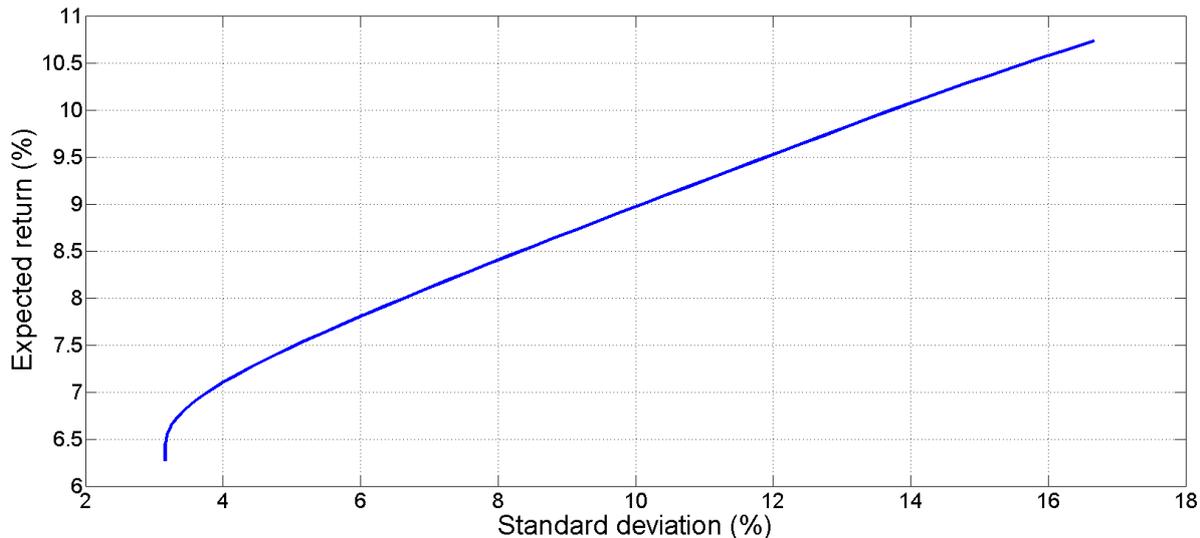
Figure 5: Efficient frontier for a sample portfolio [14]

variance is not a coherent risk measure as it is not monotone, to show this, we take an example from [14]: "Consider two random variables, ξ_1 and ξ_2 (both representing loss) which are normally distributed, but with different μ and σ^2: $\xi_1 \sim N(\mu_{\xi_1} = 0, \sigma^2_{\xi_1} = 2)$ and $\xi_2 \sim N(\mu_{\xi_2} = 10, \sigma^2_{\xi_2} = 1)$. The probability that ξ_1 is bigger than ξ_2 is insignificantly small. To be precise, $P(\xi_2 \leq \xi_1) = 3.9 \times 10^{-9}$. Hence, it is nearly impossible that the loss of ξ_1 will exceed the loss of ξ_2. However ξ_1 has a higher variance than ξ_2, i.e $\text{var}(\xi_1) = 2 \geq \text{var}(\xi_2) = 1$, and would therefore be considered riskier if the risk were measured by the variance." This and other limitations make it preferable for investors to optimize their portfolios with regards to other risk measures. We investigate two risk measures in this section.

2.4.1 Value at Risk (VaR) and Conditional Value at Risk (CVaR)

The Value-at-Risk (VaR) as defined in Definition 2.6 answers the all-important question: What is the maximum loss with a specified confidence level? It is based on the common sense fact that investors worry more about managing losses.

As an example, if we say the VaR of a portfolio is CZK100,000 with a 95 percent level of confidence over the next one year, this means that, under normal market conditions, there is a 5 percent chance that the portfolio will lose over CZK100,000 in the next one year. The VaR is preferred to the variance and is a widely used risk measure but it has undesirable properties.
Two major adverse properties of the VaR are [17]:

- First, the lack of Sub-additivity: This implies that the VaR (risk) of a portfolio of n assets may be greater than the sum of the individual VaR (risks) of the assets that make up the portfolio. This is counter intuitive as risk management is a major objective of portfolio optimization.
- The second undesirable property is the Non-convexity of VaR: This implies that it has multiple local extrema as illustrated in a plot of the VaR shown in figure 6. This means that the VaR is difficult to optimize when calculated using scenarios.[17]
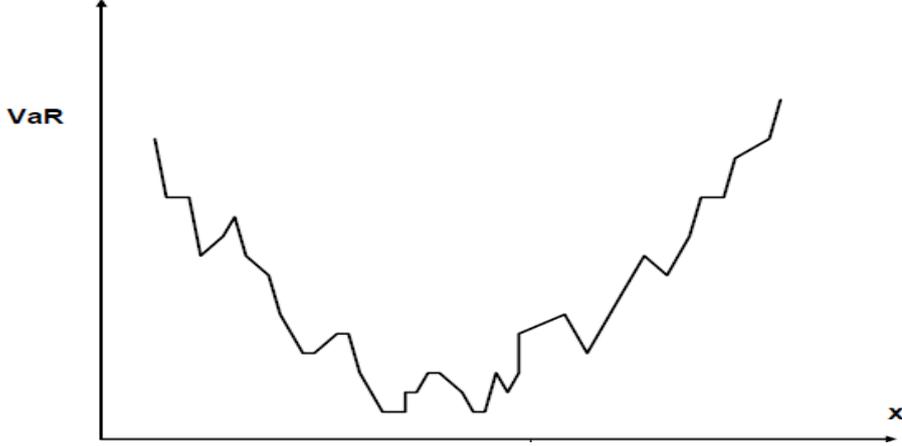
Figure 6: Multiple extrema of the VaR [17]

These two properties shows that the VaR is not a coherent risk measure. An alternative measure of risk is the Conditional Value-at-Risk (CVaR). The CVaR as defined in definition 2.7 is a coherent measure of risk as it is sub-additive and convex. It has also been shown that it can be optimized using linear programming (LP).

2.4.2 Minimization of the Conditional Value at Risk

This subsection discusses the approach for the simultaneous minimization of the CVaR and the calculation of the VaR as outlined in [14].

Let $f(\mathbf{x}, \mathbf{r})$ be a loss function depending upon the decision vector $\mathbf{x}$ and a random loss vector $\mathbf{r}$ where the vector $\mathbf{x} \in S \subset \mathbb{R}^n$. We denote by $\Psi(\mathbf{x}, c)$, the cumulative distribution function of losses, so that for any given decision $\mathbf{x} \in S$ and random asset losses $\mathbf{r} \in \mathbb{R}^n$, and loss distribution $f(\mathbf{x}, \mathbf{r})$

$$\Psi(\mathbf{x}, c) = P(f(\mathbf{x}, \mathbf{r}) \geq c) \quad \text{in the general case and} \tag{2.24}$$

$$\Psi(\mathbf{x}, c) = \int_{r:f(\mathbf{x},\mathbf{r}) \geq c)} p(\mathbf{r}) d\mathbf{r} \quad \text{in the continuous case} \tag{2.25}$$

where $p(\mathbf{r})$ in (2.25) is the probability density function for a continuous $\mathbf{r}$. The function $\Psi(\mathbf{x}, c)$ can be interpreted as the probability that the losses do not exceed threshold c.

With the definition of $\Psi(\mathbf{x}, c)$, as the probability that losses do not exceed a threshold, VaR_α and CVaR_α of a portfolio $\mathbf{x}$ can then be denoted as

$$\text{VaR}_\alpha(\mathbf{x}) = \text{VaR}_\alpha(f(\mathbf{x}, \mathbf{r})) = \min\{c : \Psi(\mathbf{x}, c) \geq \alpha\} \quad \text{and} \tag{2.26}$$

$$\text{CVaR}_\alpha(\mathbf{x}) = \text{CVaR}_\alpha(f(\mathbf{x}, \mathbf{r})) = E_r[f(\mathbf{x}, \mathbf{r}) | f(\mathbf{x}, \mathbf{r}) \geq \text{VaR}_\alpha(\mathbf{x})] \tag{2.27}$$

For clarity, this definition means that the VaR_α is the minimum threshold c, such that the probability that losses will not exceed this threshold is at least α and the CVaR_α is the expected loss function that is greater than the VaR_α. In order to easily handle CVaR, Uryasev and Rockefeller [20] defined a simpler function

$$F_\alpha(\mathbf{x}, c) = c + \frac{1}{1 - \alpha} E[(f(\mathbf{x}, \mathbf{r}) - c)^+] \tag{2.28}$$

where E[.] is the expectation and $(t)^+ = max\{0.t\}$. Based on (2.28), they formulated theorem 2.20 which states that

Theorem 2.20. *As a function of c, $F_\alpha(\mathbf{x}, c)$ is convex and continuously differentiable. The $CVaR_\alpha$ of the loss associated with any $\mathbf{x} \in S$ can be determined from the formula*

$$CVaR_\alpha(\mathbf{x}) = min_{c \in R} F_\alpha(\mathbf{x}, c) \tag{2.29}$$

Furthermore, let $\Phi_\alpha^*(\mathbf{x}) = arg\ min_c F_\alpha(\mathbf{x}, c)$ i.e. $\Phi_\alpha^*(\mathbf{x})$ is a set of minimizers of $F_\alpha(\mathbf{x}, c)$. Then

$$VaR_\alpha(\mathbf{x}) = min\{c : c \in \Phi_\alpha^*(\mathbf{x})\} \tag{2.30}$$

And following from equation 2.29 and 2.30, the following equation always holds

$$CVaR_\alpha(\mathbf{x}) = F_\alpha(\mathbf{x},\ VaR_\alpha(\mathbf{x})) \tag{2.31}$$

The proof of Theorem 2.20 is given in the appendix of [20]. Based on Theorem 2.20, Rockefeller and Uryasev stated another theorem useful for the computational calculation to find a CVaR optimal portfolio $\mathbf{x}^* \in S$.

Theorem 2.21. *Let S be a convex set of feasible portfolios $\mathbf{x}$ and assume that $f(\mathbf{x}, \mathbf{r})$ is convex in $\mathbf{x}$. Then minimizing the $CVaR_\alpha$ of the loss associated with the portfolio $\mathbf{x} \in S$ is equivalent to minimizing $F_\alpha(\mathbf{x}, c)$ over all $(\mathbf{x}, c) \in S \times R$ in the sense that*

$$min_{\mathbf{x} \in S} CVaR_\alpha(\mathbf{x}) = min_{(\mathbf{x},c) \in S \times R} F_\alpha(\mathbf{x}, c) \tag{2.32}$$

where moreover a pair $(\mathbf{x}^*, c^*)$ achieves the right hand side minimum if and only if $(\mathbf{x}^*$ achieves the left hand side minimum and $c^* \in \Phi_\alpha^*(\mathbf{x})$. Therefore in circumstances where the interval $\Phi_\alpha^*(\mathbf{x})$ reduces to a single point c (as is typical), the minimization of $F_\alpha(\mathbf{x}, c)$ produces a pair $(\mathbf{x}^*, c^*)$ such that $(\mathbf{x}^*$ minimizes the $CVaR_\alpha$ and c^* gives the corresponding VaR_α.

Theorem 2.21 not only gives a way to express the CVaR minimization problem in a tractable form, but also allows to calculate $CVaR_\alpha$ without having to calculate VaR_α first. More remarkably, finding the CVaR by using theorem 2.21 gives the corresponding VaR as a by-product. [20]

Applying theorem 2.21 with equation (2.28), the portfolio $\mathbf{x}$ that minimizes the Conditional Value at Risk of a portfolio at the confidence level α can be expressed as

$$min_{\mathbf{x} \in S} CVaR_\alpha(\mathbf{x}) = min_{(\mathbf{x},c) \in S \times R} \left(c + \frac{1}{1-\alpha} E[(f(\mathbf{x}, \mathbf{r}) - c)^+] \right) \tag{2.33}$$

2.4.3 CVaR Optimization Models

In this section, we outline the optimization problem discussed in [22] into two major types

1. Maximum Return with constraints on CVaR: Here, the mean loss $\mu(\mathbf{x}) = E[f(\mathbf{x}, \mathbf{r})]$ is minimized subject to the constraints $\mathbf{x} \in S$ and CVaR constraints level α. In this case, the optimization problem can be stated as follows

$$
\begin{aligned}
min_{\mathbf{x}} \quad & \mu(\mathbf{x}) \\
\text{s.t.} \quad & \mathbf{x} \in S \\
& CVaR_\alpha(\mathbf{x}) \leq C_\alpha
\end{aligned}
\tag{2.34}
$$

where C_α represents some constants that constrain the CVaR at confidence level α.

As a consequence of (2.32), the last constraints can be replaced by constraints

$$F_\alpha(\mathbf{x}, c_1) \leq C_\alpha. \tag{2.35}$$

Indeed if this constraint is satisfied for some c_1 then it is satisfied for the minimal values $\min_{c_1} F_\alpha(\mathbf{x}, c_1) = \text{CVaR}_\alpha(\mathbf{x})$.

Optimization with these constraints assumes that the CVaR values are properly restricted.

2. Minimizing CVaR with finite number of scenarios(with constraint on return): Now we consider the case in which an analytical representation of the density function $p(\mathbf{r})$ is not available, but we have J scenarios, $r_j, j = 1, ..., J$, sampled from the density $p(\mathbf{r})$. For instance, we may have historical observations of prices for assets of the portfolio. In this case, the function $F_\alpha(\mathbf{x}, c)$ can be calculated approximately as follows

$$\widetilde{F}(\mathbf{x}, c)_\alpha = c + v \sum_{i=1}^{J} (f(\mathbf{x}, r_j) - c)^+ \quad \text{where} \quad v = \frac{1}{J(1 - \alpha)} \quad \text{and} \quad t^+ = \max(0, t) \tag{2.36}$$

For the problem we seek to solve, $f(\mathbf{x}, r)$ is linear with respect to $\mathbf{x}$, so these problems can be solved using linear programming techniques.

To do this, we replace in $\widetilde{F}(\mathbf{x}, c)_\alpha$ the terms $(f(\mathbf{x}, r_j) - c)^+$ by auxiliary variables z_j and we impose constraints

$$z_j \geq f(\mathbf{x}, r_j) - c \quad \text{and} \quad z_j \geq 0 \tag{2.37}$$

It is also of immense importance to state the condition that the mean portfolio losses are less than or equal to $-R_0$ (i.e the mean positive return is bigger than or equal to R_0).

The mean portfolio losses is given as $\mu(\mathbf{x}) = \frac{1}{J} \sum_{i=1}^{J} f(\mathbf{x}, r_j)$ and the constraints on the mean loss is formulated as follows

$$\frac{1}{J} \sum_{i=1}^{J} f(\mathbf{x}, r_j) \leq -R \tag{2.38}$$

With the explanation above, the CVaR optimization problem can be given as

$$\begin{aligned}
\min_{c,z} \quad & c + v \sum_{i=1}^{J} z_j \\
\text{s.t.} \quad & z_j \geq f(\mathbf{x}, r_j) - c \quad \text{for} \quad j \in \{1, ..., J\} \\
& z_j \geq 0 \quad \text{for} \quad j \in \{1, ..., J\} \\
& \frac{1}{J} \sum_{i=1}^{J} f(\mathbf{x}, r_j) \leq -R \\
& \mathbf{x} \in S
\end{aligned} \tag{2.39}$$

3 Application of Portfolio Optimization to clean energy assets

In Chapter 2, we discussed the Markowitz mean variance model and the Rockefeller-Uryasev CVaR model. In this chapter, we apply both models to a portfolio of clean energy assets, with the objective of optimal allocation of financial resources in terms of maximum returns and low risk.

This chapter begins with the basic introductions into the clean energy industry, followed by the methods with which we apply each model to the data available.

3.1 Background on Clean Energy Industry

For a long time now, fossil fuels have been a major source of the energy that the world thrives on, but in recent times, there has been an increase in the search for alternative sources. The reasons for this are numerous, but we mention a few. First, the global resources of oil and coal are limited, and it is possible that we are already experiencing the peak of oil extraction. This implies that fossil fuels will become increasingly expensive to continue to recover and use as major fuel sources. A second important reason is the fact that the handling and burning of fossil fuels are known to emit certain gases, such as carbon dioxide, sulphur dioxide and nitrogen oxide, we know that these gases affect the climate and air quality and constitutes one of the major causes of climate change. These two major concerns have paved the way for a search for renewable sources of energy.[5]

Renewable sources of energy are considered to be those that are replaceable on a human time scale and have manageable environmental impacts.[5]

Renewable energy is classified into four by sources
1. Solar energy coming directly from the sun can be captured passively for its thermal properties, or it can be captured actively for electricity, one of such ways is via photovoltaic cells.
2. Biomass: Energy from biomass comes from vegetation, solid wastes, and the gases they produce during decomposition. Biomass and produced gases are burned for heat, which in turn may be used to make electricity in conjunction with a turbine and generator.
3. Hydro energy comes from the physical energy in the flow of water, and this powers turbines, thereby creating electricity.
4. Wind Energy: Like hydro energy, the physical energy in the flow of wind is used to power turbines, creating energy in the process.

The diverse renewable energy sources are harnessed through a variety of different technologies, and this leaves a huge opportunity for investment. [5]

Investment opportunities truly abound in the clean energy sector, as we can see from the graph in figure 7 (a graph taking from the World Energy Investment report 2019)[13], investment in renewable energy has gone up by 55 percent since 2010.

This shows that more investors are considering joining fight against climate change by investing in the clean energy sector. Now there are different ways of investing in this
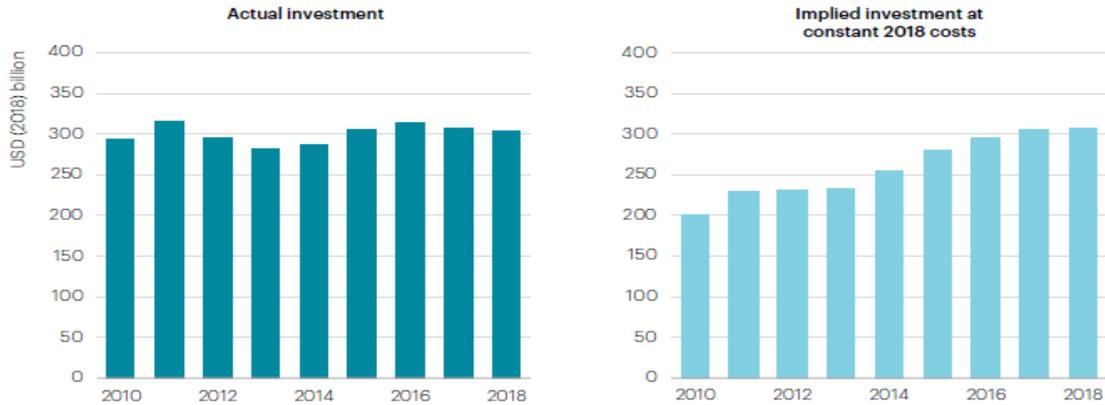
Figure 7: Investment in renewable power-actual spend versus implied investments at constant at constant 2018 cost levels [13]

industry, we list them below:

1. Direct Investments in Renewable Energy Projects:This involves investing directly in green energy projects via different channels including but not limited to renewable energy bonds, debentures, and renewable energy funds which are focused only on clean energy assets. Note that this presents a very low exposure to risk as well as low returns.[2]

2. Exchange Traded Funds: This is a slightly riskier approach, albeit with scope for higher returns. An ETF functions like an individual stock, but they are actually a combination of dozens of different stocks. This provides huge diversification across multiple green energy companies. But a drawback of this method is the exposure to larger renewable energy companies which presents more risk.[2]

3. Buy Shares of Renewable Energy Companies: This is for investors who are interested in high risk and potentially high returns. It involves buying shares of individual renewable energy companies.[2]

For the purpose of our thesis, we will run a simulation of the third type of investing as we seek to apply risk management to the potential high risk involved. To do this, we have chosen five clean energy companies through which we will invest in this industry, the companies are;

1. Vestas Wind Technology
2. SMA Solar Technology
3. Tesla Motors
4. Siemens
5. Biomas Secure Power Inc.

3.2   Methodology

We will present here an application of both models to a portfolio of stocks of renewable energy companies. It is important to note that this is designed as a demonstration of the models rather than a practical recommendation for investments.

As discussed earlier, we have selected five companies (listed in 3.1) on which we have based this experiment.

We obtained the historical data of the share prices of these companies from yahoo finance (finance.yahoo.com). In our case study, we used historical daily prices from June 2010 through to March 2020 for the scenario generations, fixing the optimization period to three (3) months. Let us denote q_{ij} as the share price of asset i at scenario j (which is the closing prices of asset i for the jth period), the $q_{i(j+1)}$ is the share price of asset i after three (3) months. Therefore we can present the scenario set for the random variable r_i (the loss associated with asset i at the end of the period) which is given as

$$r_{ij} = \frac{q_{i(j-1)} - q_{ij}}{q_{i(j-1)}} \quad \text{where} \quad j \in \{1, ..., J\} \tag{3.1}$$

where q_{i0} is the stock price on June 30, 2010, and $q_{i(j-1)}$ and q_{ij} are the closing prices for J consecutive business periods. Table 1 below gives us the data of these closing prices, while table 2 presents the loss r_{ij} associated with asset i for each scenario.

Vestas Wind	SMA Solar	Tesla	Biomass	Siemens
34.220001	84.56	23.83	0.0185	71.7276
27.5	81.03	20.41	0.025	75.031998
24.200001	69.50	26.5	0.015	89.829102
30.9	88.44	27.75	0.0185	93.714897
16.035	76.82	29.129999	0.015	91.767097
12.36	39.37	24.389999	0.018	66.0103
8.428	43.17	28.559999	0.011	71.650101
7.61	33.98	37.240002	0.01	73.249001
4.43	26.99	31.290001	0.008	64.091599
5.501	27.22	29.280001	0.008	75.206398
4.38	19.01	33.869999	0.008	79.654297
6.118	18.67	37.889999	0.008	81.427597
10.81	22.93	107.360001	0.008	75.245201
18.84	25.33	193.369995	0.01	89.059998
21.455	22.96	150.429993	0.011	99.290001
29	38.50	208.449997	0.0259	97.699997
37.150002	27.45	240.059998	0.031	96.449997
30.85	20.90	242.679993	0.0203	94.370003
30.370001	15.30	222.410004	0.02	93.75
38.810001	11.96	188.770004	0.0025	100.699997
44.360001	20.05	268.26001	0.0031	90.349998
46.209999	38.62	248.399994	0.005	79.940002
65.169998	51.72	240.009995	0.015	89.879997
62.040001	45.93	229.770004	0.0136	93.150002
60.709999	44.20	212.279999	0.008	91.809998
73.489998	28.39	204.029999	0.003	104.199997
61	25.12	213.690002	0.004	116.800003
76.019997	23.68	278.299988	0.01687	128.399994
80.720001	26.30	361.609985	0.0137	120.349998
75.589996	33.17	341.100006	0.0095	119.199997
56.900002	36.00	311.350006	0.0085	116.150002
57.959999	45.98	266.130005	0.008	103.519997
53.540001	35.48	342.950012	0.015	113.220001
57.639999	20.18	264.769989	0.007	110.339996
66.099998	16.59	332.799988	0.006	97.379997
75.080002	18.25	279.859985	0.01362	95.940002
76.900002	25.58	223.460007	0.0121	104.599998
71.699997	26.02	240.869995	0.0045	98.25
90.779999	34.56	418.329987	0.0097	116.540001
75.68	26.12	524	0.0043	77.470001

Table 1: Table of closing prices from June 2010 to March 2020

With these tables, we are set to give an estimate of the expected return and the risk of the portfolio. For each model, this is computed distinctly. We then make use of the GAMS software for the implementation of each model.

Vestas Wind	SMA Solar	Tesla	Biomass	Siemens
0.196376412	0.041745506	0.143516576	-0.351351351	-0.04606871
0.119999964	0.142292978	-0.298383146	0.4	-0.197210582
-0.276859451	-0.272517986	-0.047169811	-0.233333333	-0.04325764
0.481067961	0.131388512	-0.049729694	0.189189189	0.020784316
0.229186155	0.487503254	0.162718852	-0.2	0.280675731
0.318122977	-0.096393193	-0.170971717	0.388888889	-0.085438197
0.097057428	0.212788139	-0.303921684	0.090909091	-0.022315391
0.417871222	0.205709241	0.159774454	0.2	0.125017432
-0.241760722	-0.008336421	0.064237774	0	-0.173420529
0.203781131	0.30148815	-0.156762221	0	-0.059142561
-0.396803653	0.017885324	-0.118689109	0	-0.022262452
-0.766917293	-0.22817354	-1.833465395	0	0.07592507
-0.742830712	-0.104448321	-0.8011363	-0.25	-0.183597051
-0.138800425	0.093385982	0.222061349	-0.1	-0.114866418
-0.351666278	-0.676829268	-0.385694387	-1.354545455	0.016013737
-0.281034552	0.287012987	-0.151643087	-0.196911197	0.012794269
0.169582817	0.238615665	-0.010913917	0.34516129	0.021565516
0.015559125	0.267942584	0.083525588	0.014778325	0.006569916
-0.277905819	0.218300654	0.151252189	0.875	-0.074133301
-0.143004377	-0.676421405	-0.421094476	-0.24	0.102780529
-0.041704192	-0.926184539	0.074032712	-0.612903226	0.115218553
-0.410300788	-0.339202486	0.033776164	-2	-0.124343192
0.048028189	0.111948956	0.042664852	0.093333333	-0.036381899
0.021437814	0.037666013	0.076119618	0.411764706	0.014385443
-0.210508964	0.357692308	0.038863765	0.625	-0.134952612
0.169955073	0.11535752	-0.047345993	-0.333333333	-0.120921366
-0.246229459	0.057336253	-0.302353809	-3.2175	-0.099314989
-0.06182589	-0.110876452	-0.299353218	0.187907528	0.062694676
0.063553084	-0.261026616	0.056718508	0.306569343	0.009555472
0.247254862	-0.085330921	0.087217823	0.105263158	0.025587207
-0.018629121	-0.277399639	0.145238478	0.058823529	0.108738741
0.076259456	0.228360157	-0.288655941	-0.875	-0.093701742
-0.07657822	0.431228861	0.227963319	0.533333333	0.025437246
-0.146773059	0.17789891	-0.256939993	0.142857143	0.117455134
-0.13585483	-0.100060277	0.159074534	-1.27	0.01478738
-0.024240809	-0.401643836	0.201529268	0.111600587	-0.090264705
0.067620349	-0.017200938	-0.077910979	0.628099174	0.060707439
-0.266108826	-0.32820907	-0.736745945	-1.155555556	-0.186157771
0.166336188	0.244212963	-0.252599661	0.556701031	0.335249697

Table 2: Table of the scenario set for loss variable r_i

3.2.1 Application of the Markowitz model to the data

Here, we make use of the historical losses (table 2) as a scenario input to the model, but without any assumptions about the distribution. First we recall the Markowitz model

from Section 2.3.1

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{x}^T \widehat{\mathbf{r}} \\ \text{s.t.} \quad & \mathbf{x}^T \Sigma \mathbf{x} \le V_0 \\ & \mathbf{x} \in S \end{aligned} \tag{3.2}$$

where the set S of admissible portfolios is given as

$$S = \{\mathbf{x} \in \mathbb{R}^n : x_i \ge 0 \quad \forall i \in \{1, 2, ..., n\}, \sum_{i=1}^{n} x_i = 1\} \tag{3.3}$$

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{x}^T \Sigma \mathbf{x} \\ \text{s.t.} \quad & \mathbf{x}^T \widehat{\mathbf{r}} \le -R_0 \\ & \mathbf{x} \in S \end{aligned} \tag{3.4}$$

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{x}^T \widehat{\mathbf{r}} + \tfrac{\delta}{2} \mathbf{x}^T \Sigma \mathbf{x} \\ \text{s.t.} \quad & \mathbf{x} \in S \end{aligned} \tag{3.5}$$

- $\mathbf{x}$ represents the portfolio and x_i represents the weight of asset i in the portfolio.
- $\widehat{\mathbf{r}} = E[r_i]$ is the expected rate of loss of investment i.
- Σ is the covariance matrix as defined in Definition 2.13, where $\mathrm{var}(r_i)$ is the variance of the loss associated with investment i, and $\mathrm{cov}(r_i, r_k)$ is the covariance pertaining to investment i and k.

To compute the expectations and covariance matrix in the equation above, we have the following parameters:

$\widehat{\mathbf{r}} = E[r_i] = \frac{1}{J} \sum_{j=1}^{J} r_{ij}$ where J is the number of scenarios,

and $\sigma_{ik} = \frac{1}{J} \sum_{j=1}^{J} (r_{ij} - E[r_i])(r_{kj} - E[r_k])$, where r_{ij} is the returns for asset i in scenario j and r_{kj} is the returns for asset k in the same scenario j.

When we supply GAMS with all these information, we have the expected losses and covariance matrix of loss to be the following, (for clarity, we include the investment related information)

$$\widehat{\mathbf{r}} = \begin{bmatrix} Vestas & -0.06 \\ SMA & -0.01 \\ Tesla & -0.13 \\ Biomass & -0.16 \\ Siemens & -0.01 \end{bmatrix}$$

and

$$\Sigma = \begin{bmatrix} & Vestas & SMA & Tesla & Biomass & Siemens \\ Vestas & 0.0733 & 0.0270 & 0.0540 & 0.0648 & 0.0090 \\ SMA & 0.0270 & 0.0945 & 0.0246 & 0.0790 & 0.0011 \\ Tesla & 0.0540 & 0.0246 & 0.1349 & 0.0427 & 0.0022 \\ Biomass & 0.0648 & 0.0790 & 0.0427 & 0.5853 & 0.0201 \\ Siemens & 0.0090 & 0.0011 & 0.0022 & 0.0201 & 0.0133 \end{bmatrix}$$

From the results above we see that the Biomass shares have the highest expected returns (synonymous to the lowest expected losses -0.16) but with a variance of 0.5853, this presents a very high risk probability. SMA Solar and Siemens shares with an expected return of 0.01 both have the lowest expected return, but the risk associated with these

assets is also very low, with Siemens being the lowest with a variance of 0.0133. Tesla has an expected return of 0.13 and a variance of 0.1349, while Vestas Wind shares gives an expected return of 0.06 against a variance of 0.0733. With this, an investor who is fully focused on high returns without much care to the risk involved will invest in the Biomass stocks, while a risk-averse investor will go for the Siemens stocks.

Now if we consider how the return will change as the risk threshold changes, we make use of model (2.17) and we have the result in table 3. From this table, we notice that

Vestas	SMA	Tesla	Biomass	Siemens	Variance V_0	Portfolio Expected Returns
0.00	0.01	0.27	0.04	0.68	0.02	0.05
0.00	0.00	0.50	0.11	0.39	0.05	0.08
0.00	0.00	0.65	0.15	0.20	0.08	0.11
0.00	0.00	0.76	0.18	0.06	0.11	0.12
0.00	0.00	0.68	0.32	0.00	0.14	0.14
0.00	0.00	0.40	0.60	0.00	0.25	0.14
0.00	0.00	0.25	0.75	0.00	0.35	0.15
0.00	0.00	0.14	0.86	0.00	0.45	0.15

Table 3: Results for the Markowitz model (2.17) for varying variance V_0

as we increase the variance of the portfolio, the expected return also increases. There is another important observation, and that is the fact that in order to maximize the return for a small variance (low level of risk) the model presents us with the option of investing a large fraction of our wealth into Siemens; the asset with the lowest risk,and a very small amount in the most risky asset-Biomass. We notice that as we increase the risk, the fraction of wealth in Biomass steadily increases. This shows that for an investor willing to take on risk, the model will push to maximize the assets with the highest returns. Now if we consider risk averse investors, who are more interested in minimizing risk while

Vestas	SMA	Tesla	Biomass	Siemens	Expected Returns R_0	Portfolio Variance
0.00	0.06	0.17	0.01	0.76	0.03	0.01
0.00	0.00	0.36	0.06	0.58	0.06	0.03
0.00	0.00	0.55	0.12	0.34	0.09	0.06
0.00	0.00	0.74	0.17	0.09	0.12	0.10
0.00	0.00	0.22	0.78	0.00	0.15	0.38
0.00	0.00	0.00	1.00	0.00	0.16	0.59

Table 4: Results for the Markowitz model (2.19) for varying Expected Return R_0

setting a minimum expected return R_0, we make use of model (2.19), which gives us the result shown in Table 4. In table 4 we notice, as in table 3 that the variance of the

portfolio increases as we increase the expected return, but we also notice that because of the quadratic term in the objective function, an investor can increase the expected return on his portfolio with little additional risk if the portfolio has a low variance to begin with. For example, increasing the expected return from 0.03 to 0.06 increases the variance from 0.01 to 0.03. However, the more expected return an investor demands, the higher the increase in risk, in table 4 increasing the expected return from 0.12 to 0.15 increases the risk from 0.10 to 0.38.

Now for the risk adjusted return model (2.20), the results are presented in table 5. We see that as the risk averse constant δ increases, the portfolio returns decreases. When $\delta = 0$ (i.e. the investor doesn't care about risk), the total wealth is invested in Biomass and as δ increases, the model optimizes the portfolio to manage the risk associated with the investments.

Vestas	SMA	Tesla	Biomass	Siemens	Risk Averse constant δ	Portfolio Expected Return
0.00	0.00	0.00	1.00	0.00	0.00	0.16
0.00	0.00	0.60	0.40	0.00	0.20	0.12
0.00	0.00	0.73	0.27	0.00	0.40	0.11
0.00	0.00	0.77	0.23	0.00	0.60	0.09
0.00	0.00	0.79	0.21	0.00	0.80	0.08

Table 5: Results for the Markowitz model (2.20) for varying risk averse constant δ

3.2.2 Markowitz model: Additional constraints

A pivotal observation in subsection 3.2.1 is that the models focus on investing in the shares of Tesla, Biomass and sometimes Siemens. This is because the model seeks to maximize the expected returns. This is a very risky thing to do, as this exposes the investor to the risk associated to these stocks, thereby causing a possible loss of wealth. To correct this, we introduce another constraint to models (2.17), (2.19) and (2.20). The constraint is given as

$$x_i \leq v, i \in \{1, .., n\} \tag{3.6}$$

This equation implies that investment in each asset must not exceed a set constant v, where $v \in (0, 1]$. With the addition of this constraint, the new models become

$$
\begin{aligned}
\min_{\mathbf{x}} \quad & \mathbf{x}^T \widehat{\mathbf{r}} \\
\text{s.t.} \quad & \mathbf{x}^T \mathbf{\Sigma} \mathbf{x} \leq V_0 \\
& \mathbf{x} \in S \\
& x_i \leq v, i \in \{1, .., n\}
\end{aligned}
\tag{3.7}
$$

where the set S of admissible portfolios is given as

$$S = \{x \in \mathbb{R}^n : x_i \geq 0 \quad \forall i \in \{1, 2, ..., n\}, \sum_{i=1}^{n} x_i = 1\} \tag{3.8}$$

$$\min_{\mathbf{x}} \quad \mathbf{x}^T \mathbf{\Sigma} \mathbf{x}$$
$$\text{s.t.} \quad \mathbf{x}^T \hat{\mathbf{r}} \leq -R_0$$
$$\mathbf{x} \in S$$
$$x_i \leq v, i \in \{1, .., n\} \tag{3.9}$$

$$\min_{\mathbf{x}} \quad \mathbf{x}^T \hat{\mathbf{r}} + \frac{\delta}{2} \mathbf{x}^T \mathbf{\Sigma} \mathbf{x}$$
$$\text{s.t.} \quad \mathbf{x} \in S$$
$$x_i \leq v, i \in \{1, .., n\} \tag{3.10}$$

Table 6 shows the results for model (3.7) when $v = 0.3$, and we see that a better diversification of the portfolio. We see that with this diversification, the highest variance (risk) we are allowed to get exposed to is 0.11 compared to 0.45 in table 3. Table 7 shows the results for model (3.9) where $v = 0.3$ while table 8 shows the results for model (3.10) $v = 0.4$. Comparing sections 3.2.1 and 3.2.2 , we see that diversification is an important part of risk management and even though the expected returns of the portfolio is lesser in 3.2.2, the risk exposure of the portfolio is well managed.

Vestas	SMA	Tesla	Biomass	Siemens	Variance V_0	Portfolio Expected Returns
0.24	0.24	0.09	0.00	0.30	0.02	0.03
0.26	0.00	0.30	0.14	0.30	0.05	0.08
0.30	0.00	0.30	0.24	0.16	0.08	0.09
0.30	0.10	0.30	0.30	0.00	0.11	0.10

Table 6: Results for the Markowitz model (3.7) for varying variance V_0

Vestas	SMA	Tesla	Biomass	Siemens	Expected Returns R_0	Portfolio Variance
0.30	0.29	0.11	0.00	0.30	0.04	0.03
0.21	0.15	0.30	0.04	0.30	0.06	0.04
0.30	0.00	0.30	0.22	0.18	0.09	0.07
0.30	0.10	0.30	0.30	0.00	0.10	0.11

Table 7: Results for the Markowitz model (3.9) for varying expected returns R_0

Vestas	SMA	Tesla	Biomass	Siemens	Risk averse constant δ	Portfolio Expected Returns
0.20	0.00	0.40	0.40	0.00	0.00	0.12
0.20	0.00	0.40	0.40	0.00	0.20	0.11
0.20	0.00	0.40	0.40	0.00	0.40	0.09
0.26	0.00	0.40	0.34	0.00	0.60	0.08
0.34	0.00	0.40	0.26	0.00	0.80	0.07

Table 8: Results for the Markowitz model (3.10) for varying risk averse constant δ

3.2.3 Application of the CVaR model to the data

To begin the application of the CVaR models to the data, we recall the two models; first the maximum returns with a constraint on the CVaR model given as

$$
\begin{aligned}
\min_{\mathbf{x}} \quad & \mu(\mathbf{x}) \\
\text{s.t.} \quad & \mathbf{x} \in S \\
& \widetilde{F}(\mathbf{x}, c)_\alpha \le C_\alpha
\end{aligned}
\tag{3.11}
$$

and the minimal CVaR with constraint on the expected returns model given as

$$
\begin{aligned}
\min_{c,z} \quad & c + \frac{1}{J(1-\alpha)} \sum_{i=1}^{J} z_j \\
\text{s.t.} \quad & z_j \ge f(\mathbf{x}, r_j) - c \quad \text{for} \quad j \in \{1, ..., J\} \\
& z_j \ge 0 \quad \text{for} \quad j \in \{1, ..., J\} \\
& \mu(\mathbf{x}) \le -R_0 \\
& \mathbf{x} \in S
\end{aligned}
\tag{3.12}
$$

- where $\mathbf{x}$ represents the portfolio and x_i represents the weight of asset i in the portfolio
- $f(\mathbf{x}, r_j) = \mathbf{x}^T r_j$ is the loss function for scenario $j \in \{1, ..., J\}$
- $\mu(\mathbf{x}) = \frac{1}{J} \sum_{i=1}^{J} f(\mathbf{x}, r_j)$

and the set S of admissible portfolios remains the same as in equation (3.3).

From [23], Uryasev et al gave a theorem that shows that the models (3.11) and (3.12) produce the same efficient frontier once the following condition is satisfied.

Theorem 3.1. *Let us consider the function $\phi(\mathbf{x})$ and $\mu(\mathbf{x})$ dependent on the decision vector $\mathbf{x}$, and the following three problems*
P1

$$
\begin{aligned}
\min_{\mathbf{x}} \quad & \phi(\mathbf{x}) - \delta\mu(\mathbf{x}) \\
\textit{s.t.} \quad & \mathbf{x} \in S \\
& \delta \ge 0
\end{aligned}
\tag{3.13}
$$

P2

$$
\begin{aligned}
\min_{\mathbf{x}} \quad & \phi(\mathbf{x}) \\
\textit{s.t.} \quad & \mu(\mathbf{x}) \le -\rho \\
& \mathbf{x} \in S
\end{aligned}
\tag{3.14}
$$

P3

$$\min_{\mathbf{x}} \quad -\mu(\mathbf{x})$$
$$s.t. \quad \phi(\mathbf{x}) \leq \omega \tag{3.15}$$
$$\mathbf{x} \in S$$

Suppose that constraints $\mu(\mathbf{x}) \leq -\rho$ and $\phi(\mathbf{x}) \leq \omega$ have internal points, varying the parameters δ, ρ and ω traces the efficient frontier for the problems 1-3, accordingly. If $\phi(\mathbf{x})$ is convex, $\mu(\mathbf{x})$ is concave and the set S is convex, then the three problems 1-3 generate the same efficient frontier.

The proof of theorem 3.1 is available in the appendix of [23].

Now, considering that the loss function $f(\mathbf{x}, \mathbf{r})$ is linear with respect to $\mathbf{x}$, and theorem 2.20 implies that the CVaR risk function is convex with respect to $\mathbf{x}$. Also, from models (3.11) and (3.12), we see that the expected returns $\mu(\mathbf{x})$ is linear, and the constraints are also linear. We see that the conditions of theorem 3.1 are satisfied for the $\text{CVaR}_\alpha(\mathbf{x})$ and the mean $\mu(\mathbf{x})$. Therefore models (3.11) and (3.12) generate the same efficient frontier. This means that for our data, it suffices to make use of one of the two models in the software implementation. We choose to work with model (3.12) for the implementation.

Just as we did for the Markowitz model, we make use of the historical losses (table 2) as a scenario input to the model, without any assumptions about the distribution. Here, we consider how the CVaR will vary with an increase in the return threshold R_0. Note that for every R_0, we find the optimal portfolio $\mathbf{x}^*$ and the VaR c^* that produces the minimum objective function which is the CVaR, this means that for every R_0, our optimization model is over $S \times \mathbb{R}$. The result can be found in table 9.

From the table, for each R_0 we point out the optimum portfolio as the one in which the CVaR is the minimum, and the corresponding c is the VaR, e.g for $R_0 = 0.05$, the VaR is 0.15. Another observation that we make from the result is that for each R_0, as we increase the constant c, the portfolio is reshuffled so as to maximize the more profitable assets until it becomes infeasible. We also see that as R_0 increases, the model seeks to invest in the most profitable shares, which are the shares of Tesla and Biomass, but an investment in these stocks alone come with a minimum CVaR of 0.61 and a VaR of 0.50. This means that in a period of three months, there is a 5percent chance of losing 50 percent of the initial wealth. For an expected return of 0.20, the model chooses a portfolio of Biomass stocks alone, with a CVaR of 2.36. We see that the risk is enormous and dangerous even.

To reduce this risk, we introduce equation (3.6) as an additional constraint to the model. The new model we have is

$$\min_{c,z} \quad c + \frac{1}{J(1-\alpha)} \sum_{i=1}^{J} z_j$$
$$s.t. \quad z_j \geq f(\mathbf{x}, r_j) - c \quad \text{for} \quad j \in \{1, ..., J\}$$
$$z_j \geq 0 \quad \text{for} \quad j \in \{1, ..., J\}$$
$$\mu(\mathbf{x}) \leq -R_0 \tag{3.16}$$
$$x_i \leq v, i \in \{1, .., n\}$$
$$\mathbf{x} \in S$$

Table 10 shows the results we have when $v = 0.3$. We see that a spread of the risk over the various assets (diversification) presents a lesser risk, for example; for an expected return of 0.2, the CVaR is 0.5 compared to 2.36 when this constraint did not exist in the model.

Vestas	SMA	Tesla	Biomass	Siemens	Expected Returns R_0	c	CVaR
0.16	0.00	0.37	0.03	0.44	0.05	0.10	0.20
0.02	0.00	0.35	0.11	0.52	0.05	0.15	0.17
0.04	0.00	0.35	0.14	0.46	0.05	0.20	0.20
0.04	0.00	0.35	0.14	0.46	0.05	0.25	0.25
0.20	0.00	0.59	0.09	0.12	0.10	0.10	0.24
0.00	0.00	0.69	0.07	0.24	0.10	0.15	0.19
0.00	0.00	0.71	0.05	0.23	0.10	0.20	0.20
0.00	0.00	0.71	0.05	0.23	0.10	0.25	0.25
0.00	0.00	0.22	0.78	0.00	0.15	0.30	0.85
0.00	0.00	0.22	0.78	0.00	0.15	0.40	0.68
0.00	0.00	0.22	0.78	0.00	0.15	0.50	0.61
0.00	0.00	0.22	0.78	0.00	0.15	0.55	0.63
0.00	0.00	0.00	1.00	0.00	0.20	0.10	2.36
0.00	0.00	0.00	1.00	0.00	0.20	0.20	2.46
0.00	0.00	0.00	1.00	0.00	0.20	0.30	2.56
0.00	0.00	0.00	1.00	0.00	0.20	0.40	2.66

Table 9: Results for the CVaR model (3.12) for varying expected return R_0

Vestas	SMA	Tesla	Biomass	Siemens	Expected Returns R_0	c	CVaR
0.22	0.04	0.30	0.14	0.30	0.05	0.10	0.26
0.28	0.00	0.30	0.17	0.30	0.05	0.15	0.19
0.16	0.00	0.30	0.24	0.30	0.05	0.20	0.20
0.16	0.00	0.30	0.24	0.30	0.05	0.25	0.25
0.30	0.00	0.00	0.29	0.11	0.10	0.10	0.51
0.30	0.00	0.00	0.29	0.11	0.10	0.15	0.30
0.30	0.00	0.00	0.29	0.11	0.10	0.20	0.23
0.30	0.00	0.00	0.29	0.11	0.10	0.25	0.25
0.30	0.10	0.30	0.30	0.00	0.15	0.10	0.60
0.30	0.10	0.30	0.30	0.00	0.15	0.15	0.53
0.30	0.10	0.30	0.30	0.00	0.15	0.20	0.50
0.30	0.10	0.30	0.30	0.00	0.15	0.25	0.52
0.30	0.10	0.30	0.30	0.00	0.20	0.10	0.60
0.30	0.10	0.30	0.30	0.00	0.20	0.15	0.53
0.30	0.10	0.30	0.30	0.00	0.20	0.20	0.50
0.30	0.10	0.30	0.30	0.00	0.20	0.25	0.52

Table 10: Results for the CVaR model (3.16) for varying expected return R_0

4 Comparison between the mean-variance and CVaR models

In the previous chapter, we applied a dataset of share prices of five different clean energy assets to the models we discussed in the second chapter and we derived some results. In this chapter, we compare these results and make distinctions and similarities between the two models.

Section 4.1 discusses the efficient frontier as a tool for this comparison. We then discuss the efficient frontier of the models without the constraint on the size of each asset while section 4.2 discusses the efficient frontier of the models with the constraint.

This chapter concludes with the section 4.3, which involves finding the CVaR of the mean-variance optimal portfolio and the variance of the CVaR optimal portfolio. This will give us an insight into the better of the two models.

4.1 Efficient frontier of the models without the constraint $x_i \geq v$

Definition 2.18 succinctly defined an efficient portfolio, and we saw that to explain this, the use of an efficient frontier was inevitable. This is because an efficient frontier helps us to compare between various portfolios and choose the best according to our preferences. Now we seek to draw the efficient frontier for the results we obtained in chapter 3 after the application of the dataset to the model.

Figure 8 gives the efficient frontier of the models (3.2) to (3.5), the Markowitz type models without the constraint on the individual stocks, $x_i \geq v$. From the efficient frontier, we see that an investor seeking maximum return for a minimal risk, the portfolio with a return of 14 percent will be the optimal portfolio.
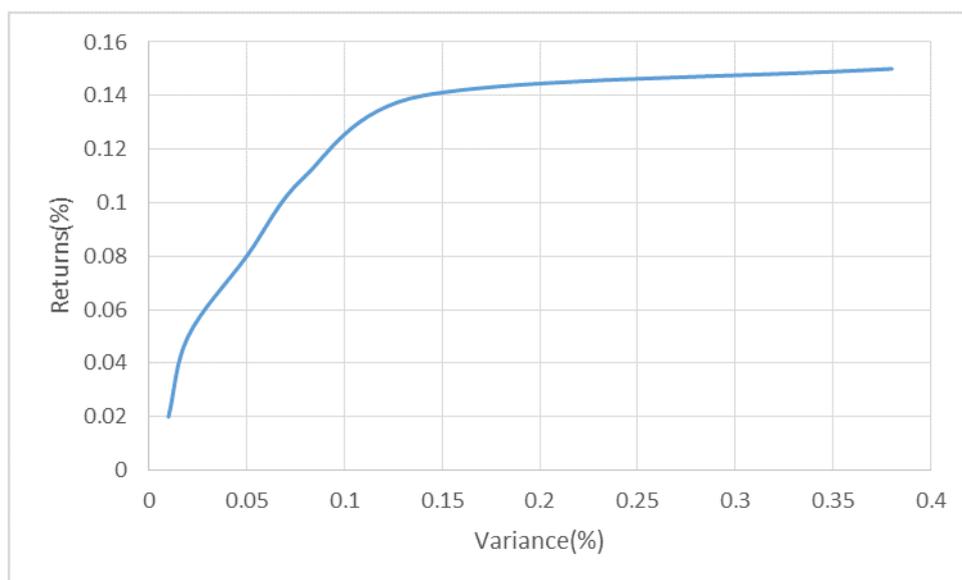
Figure 8: Efficient frontier of the unconstrained Markowitz model

Figure 9 gives the efficient frontier of model (3.12), it shows the return-CVaR plot for a confidence level of 0.95.
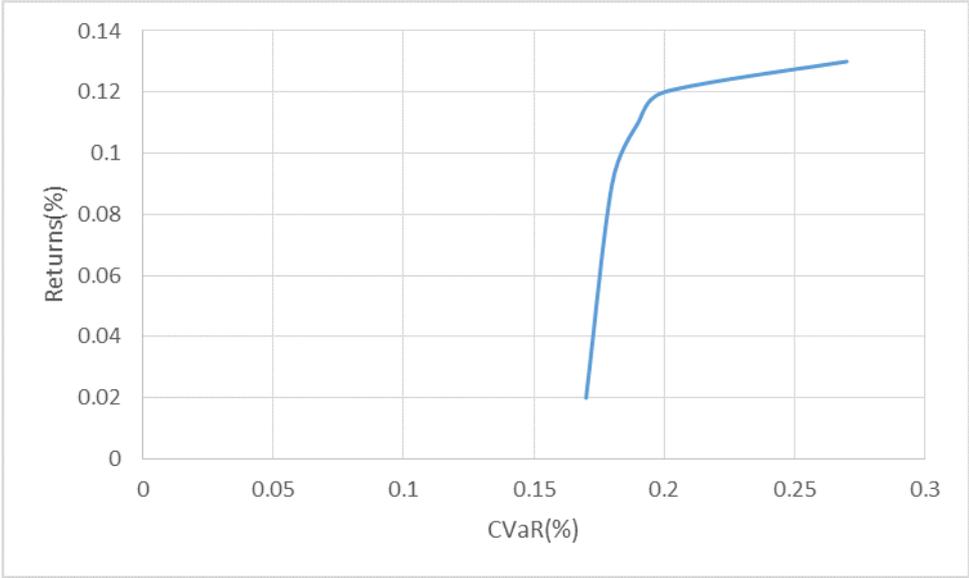
Figure 9: Efficient frontier of the unconstrained CVaR model with a 0.95 confidence level

4.2 Efficient frontier of the models with the constraint $x_i \geq v$

After we have observed the efficient frontier for the models in which the size of each asset is not constrained, we also want to observe the efficient frontier in which the sizes are constrained.

Figure 10 shows the efficient frontier for models (3.7) to (3.10), the Markowitz type models with constraints on the sizes of individual stocks.
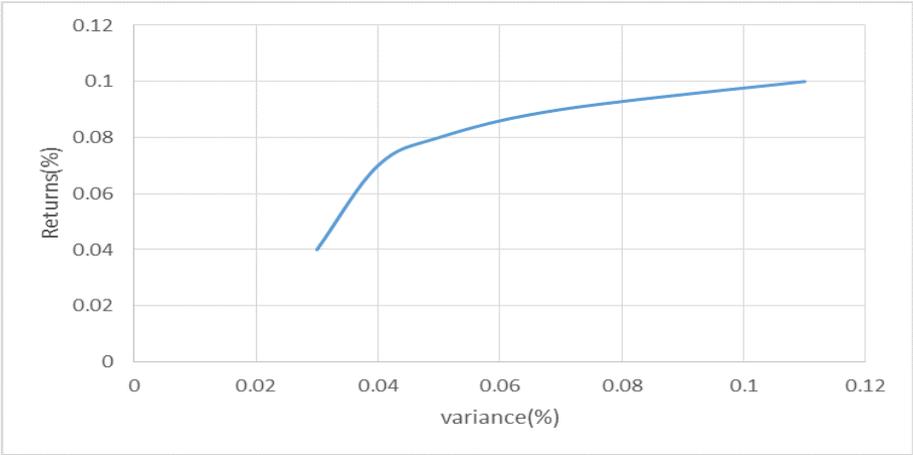
Figure 10: Efficient frontier of the constrained Markowitz model

Figure 11 shows the efficient frontier of the constrained CVaR model (3.16) with a confidence level of 0.95
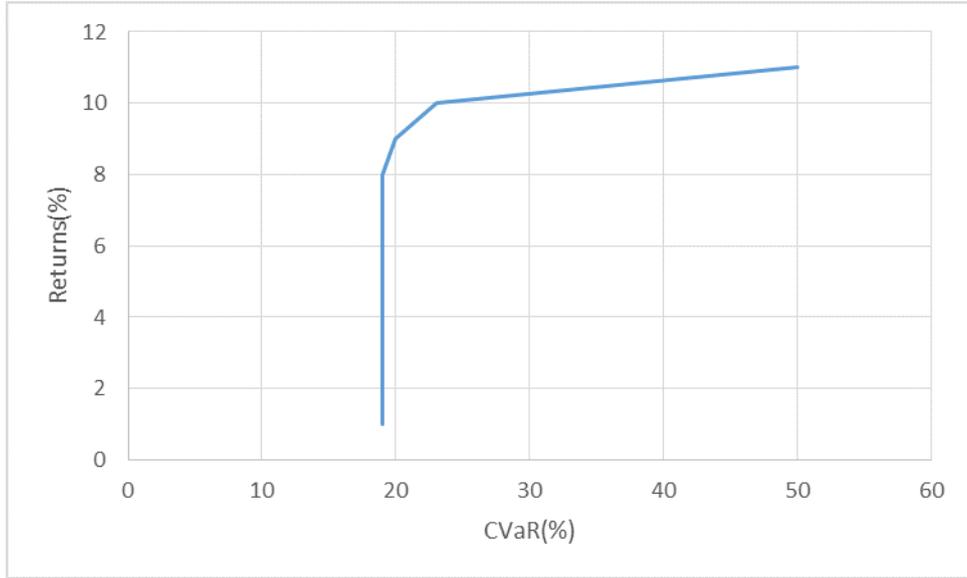
Figure 11: Efficient frontier of the constrained CVaR model

4.3 Concluding Remarks

Figure 12 displays the CVaR efficient portfolios, with a confidence level of 0.95 for the unconstrained CVaR model (orange line denoted as cvarop i.e. CVaR optimized portfolios). We compute the CVaR of the mean variance optimal portfolio(blue lines denoted as mvop i.e. mean-variance optimized portfolios). This is done by making use of the equations (2.28) and (2.31), and the results from the mean-variance model (Tables 3 and 4). The Matlab Code can be found in the appendix but for clarity, we make an illustration below.

From Table 4, for an expected return of 0.03, the optimal portfolio is $\mathbf{x} = [0.00, 0.06, 0.17, 0.01, 0.76]$. We recall Equation (2.28) which is given below

$$F_\alpha(\mathbf{x}, c) = c + \frac{1}{1 - \alpha} E[(f(\mathbf{x}, \mathbf{r}) - c)^+] \tag{4.1}$$

where E[.] is the expectation and $(t)^+ = \max\{0.t\}$. To make use of this equation, we compute

$$E[(f(\mathbf{x}, \mathbf{r}) - c)^+] = \frac{1}{J} \sum_{j=1}^{J} (f_j(\mathbf{x}, \mathbf{r}) - c)^+ \tag{4.2}$$

where $f(\mathbf{x}, \mathbf{r}) = \mathbf{x}^T \mathbf{r}$, where $\mathbf{r}$ is the Table 2. With this we calculate $F_\alpha(\mathbf{x}, c)$ for different values of c, and the c for which $F_\alpha(\mathbf{x}, c)$ is minimum is the VaR_α and the corresponding CVaR_α is calculated from equation (2.31) recalled below.

$$\text{CVaR}_\alpha(\mathbf{x}) = F_\alpha(\mathbf{x}, \text{VaR}_\alpha(\mathbf{x})) \tag{4.3}$$

With this computation, we can get the corresponding CVaR values for the Markowitz mean-variance optimal portfolios. (For this illustration, the VaR is 0.23, while the CVaR

is 0.2507). We then compare the CVaR of the efficient portfolios of both models.
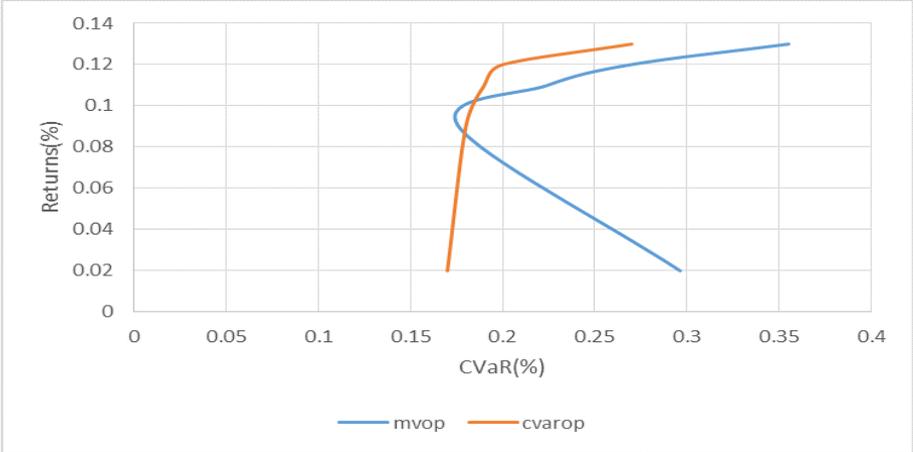
Figure 12: A comparison of the risk (CVaR) of the unconstrained models

From the figure above, we see that for a given return, the mean-variance optimal portfolio has a higher CVaR level than the efficient CVaR portfolio. This makes the CVaR model the better one if CVaR is the risk measure we choose to make use of.

Similarly,for the constrained models, we have figure 13 below.
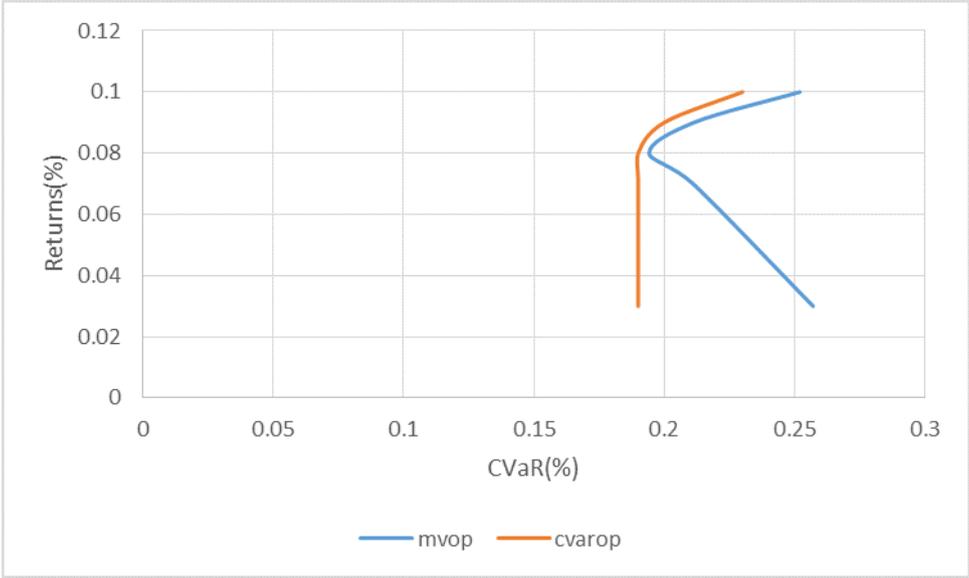
Figure 13: A comparison of the risk (CVaR) of the constrained models

Figure 14 and 15 displays the efficient frontier for the mean variance efficient portfolios (for unconstrained and constrained models respectively). For each given return, we compute the variance of the CVaR optimal portfolios with $\alpha = 0.95$ as the confidence level. Similar to what we did above we compare the risk (variance) of the mean variance efficient portfolio (blue lines denoted as mvop i.e. mean-variance optimized portfolios) to that of

the CVaR efficient portfolio ((orange line denoted as cvarop i.e. CVaR optimized portfolios)) for a given return. From these figures, we see that for a given return, the CVaR optimal portfolio has a slightly higher variance than the efficient mean-variance portfolio, but the difference is really small.
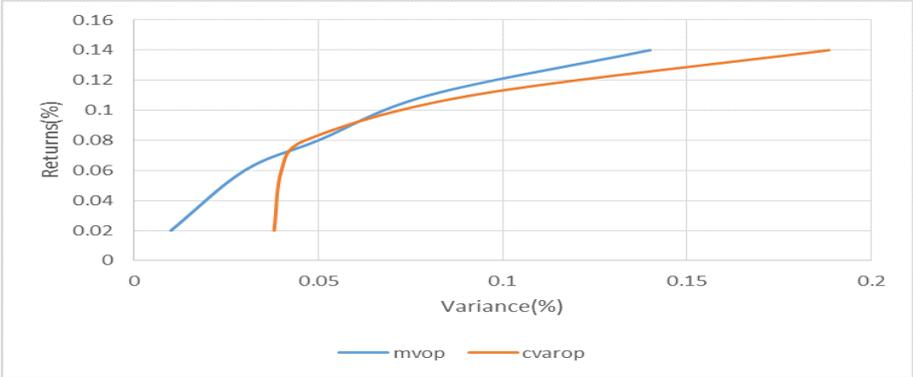
Figure 14: A comparison of the risk (variance) of the unconstrained models
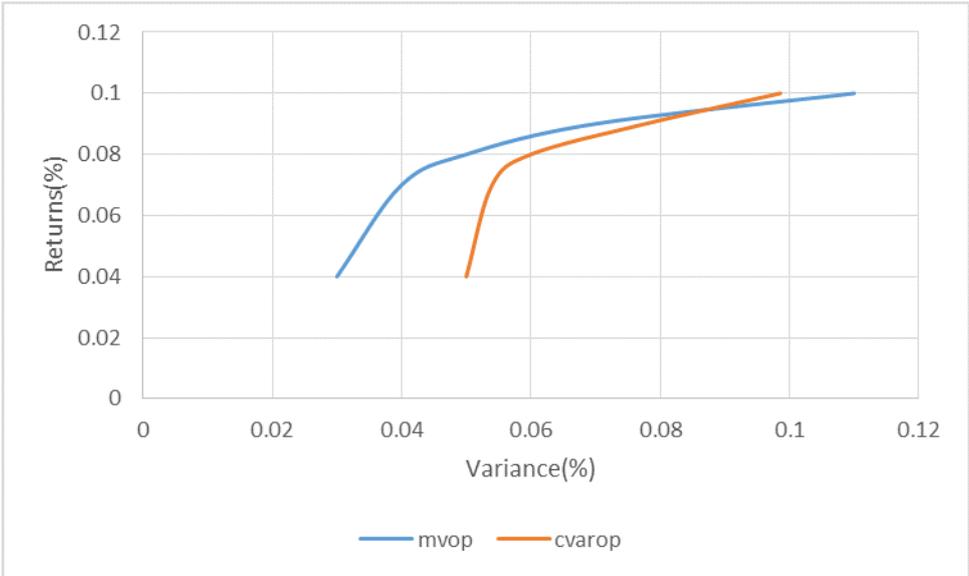
Figure 15: A comparison of the risk (variance) of the constrained models

From figures 14 and 15, we see that the difference between the mean-variance and CVaR models is not very significant if the risk measure we choose to use is the variance, but we reiterate that these results are data-specific, and is due to the distribution of the historical returns used in our case study.

5   Conclusion

This thesis covered two different mathematical models of optimal investment allocation, with a focus on the clean energy industry (renewable energy). The Markowitz Mean Variance model and the Rockefeller and Uryasev CVAR model were the models this thesis focused on, first we introduced these models, then we went on to apply them on a portfolio of clean energy assets to obtain various observations and returns.

Chapter 2 started with the basic notion of portfolio optimization, defining expectation and the measures of risk that is utilized in our model. It then discusses the properties of a coherent risk measure. Section 2.3 is focused on the Markowitz Mean Variance Optimization, the three equivalent types of the markowitz model and the theoretical properties of the model. Section 2.4 then focuses on the Rockefeller and Uryasev CVAR Optimization, first it is shown that the Value-at-Risk is not coherent, and the CVAR is shown to be a worthy replacement. We then proceed to discuss the approach for the simultaneous minimization of the CVAR and the calculation of the VaR. A discussion of the CVAR Optimization models concludes this section, in which the two CVAR Optimization type models were discussed.

In Chapter 3, we apply these models to a portfolio of clean energy assets, it begins with a background on the clean energy industry, the four major classes of renewable energy and the different methods of investing in these classes. Section 3.2 then focuses on the methodology. The historical daily prices of five renewable energy companies from June 2010 to March 2020 is provided, then expected return and the variance is calculated and both models are applied to the data. During the experimentation, an extra constraint is applied on the size of each asset, and we observe that this reduces the exposure to the market risk of individual assets.

Chapter 4 then gives a comparison between the markowitz model and the CVaR model. First the efficient frontiers of the optimal portfolio from each model is presented. Then we calculate the CVaR of the optimal mean variance portfolio and vice versa to compare and contrast between both models.

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A   Appendix
Gams Code

A.1   Gams Code for the Markowitz model (2.17) to (2.20)

```
Sets
        i alternative investments / Vestas,Sma,Tesla,Biomass,Siemens /
        j periods / 1*39 /
        m         / 1 * 5 /;
alias (i,k);
Parameters

Table R(j,i) returns of investments i in j years in percentage
```

	Vestas	Sma	Tesla	Biomass	Siemens
1	0.196376412	0.041745506	0.143516576	-0.351351351	-0.04606871
2	0.119999964	0.142292978	-0.298383146	0.4	-0.197210582
3	-0.276859451	-0.272517986	-0.047169811	-0.233333333	-0.04325764
4	0.481067961	0.131388512	-0.049729694	0.189189189	0.020784316
5	0.229186155	0.487503254	0.162718852	-0.2	0.280675731
6	0.318122977	-0.096393193	-0.170971717	0.388888889	-0.085438197
7	0.097057428	0.212788139	-0.303921684	0.090909091	-0.022315391
8	0.417871222	0.205709241	0.159774454	0.2	0.125017432
9	-0.241760722	-0.008336421	0.064237774	0	-0.173420529
10	0.203781131	0.30148815	-0.156762221	0	-0.059142561
11	-0.396803653	0.017885324	-0.118689109	0	-0.022262452
12	-0.766917293	-0.22817354	-1.833465395	0	0.07592507
13	-0.742830712	-0.104448321	-0.8011363	-0.25	-0.183597051
14	-0.138800425	0.093385982	0.222061349	-0.1	-0.114866418
15	-0.351666278	-0.676829268	-0.385694387	-1.354545455	0.016013737
16	-0.281034552	0.287012987	-0.151643087	-0.196911197	0.012794269
17	0.169582817	0.238615665	-0.010913917	0.34516129	0.021565516
18	0.015559125	0.267942584	0.083525588	0.014778325	0.006569916
19	-0.277905819	0.218300654	0.151252189	0.875	-0.074133301
20	-0.143004377	-0.676421405	-0.421094476	-0.24	0.102780529
21	-0.041704192	-0.926184539	0.074032712	-0.612903226	0.115218553
22	-0.410300788	-0.339202486	0.033776164	-2	-0.124343192
23	0.048028189	0.111948956	0.042664852	0.093333333	-0.036381899
24	0.021437814	0.037666013	0.076119618	0.411764706	0.014385443
25	-0.210508964	0.357692308	0.038863765	0.625	-0.134952612
26	0.169955073	0.11535752	-0.047345993	-0.333333333	-0.120921366
27	-0.246229459	0.057336253	-0.302353809	-3.2175	-0.099314989
28	-0.06182589	-0.110876452	-0.299353218	0.187907528	0.062694676
29	0.063553084	-0.261026616	0.056718508	0.306569343	0.009555472
30	0.247254862	-0.085330921	0.087217823	0.105263158	0.025587207
31	-0.018629121	-0.277399639	0.145238478	0.058823529	0.108738741
32	0.076259456	0.228360157	-0.288655941	-0.875	-0.093701742
33	-0.07657822	0.431228861	0.227963319	0.533333333	0.025437246

```
34 -0.146773059      0.17789891      -0.256939993      0.142857143      0.117455134
35 -0.13585483      -0.100060277      0.159074534      -1.27      0.01478738
36 -0.024240809      -0.401643836      0.201529268      0.111600587      -0.090264705
37  0.067620349      -0.017200938      -0.077910979      0.628099174      0.060707439
38 -0.266108826      -0.32820907      -0.736745945      -1.155555556      -0.186157771
39  0.166336188      0.244212963      -0.252599661      0.556701031      0.335249697
 ;
Scalar n number of periods of historical data /39/;
Scalar V0 risk threshold /0.02/;
scalar xi / 0.03 /;
Parameters E(i)     expected return of investment i in percent
          V(i,k)  covariance matrix of alternative investments;
          E(i)=sum(j, R(j,i))/n                  ;
          V(i,k)= sum(j, (R(j,i)-E(i))*(R(j,k)-E(k)))/n  ;
Variables
        x(i)     fraction of unit capital invested in ith investment
        z        objective

Positive Variable x;

Equations
        returns  the objective is to maximise this
        risk     minimize the variance
        capital  observe capital limit;

returns..     z =e= sum(i,E(i)*x(i));
risk..        sum(i,sum(k,x(i)*V(i,k)*x(k))) =l= V0;
capital..     sum(i,x(i)) =e= 1;
file printout /printout.txt/ ;
put printout ;
model greenhouse /returns, risk, capital /;
put @1 "m", @4 "i",@10" x(i)", @20 "E(i)", @35" V(i,i)",@50 "risk", @75 "returns"/;
loop(m,
solve greenhouse using NLP minimizing z ;


loop(i,put @1 m.tl,@5 i.tl, @6 x.L(i), @18   E(i), @30 V(i,i),@50 risk.L, @75 z.L /);
V0= V0 + xi;);
put "covariance" /;
loop((i,k), put i.tl, @12, k.tl, @24, V(i,k):8:4 /);



display z.L, x.L, V0, risk.L;



Scalar R0 least return /0.01/;
scalar phi / 0.02 /;
```

```
Variables
        x(i)    fraction of unit capital invested in ith investment
        z       objective
Positive Variable x ;

Equations
        returns2  we set in infimum for the returns
        risk2     the objective is to minimize the variance
        capital  observe capital limit;
risk2..         z =e= sum(i,sum(k,x(i)*V(i,k)*x(k)));
returns2..      sum(i,E(i)*x(i)) =l= -R0;
model greenhouse2 /risk2, returns2, capital /;
put @1 "m ", @4 "i",@10" x(i)",@20" E(i)", @35 " V(i,i)", @50 "risk", @75 "returns"/;
loop(m,
solve greenhouse2 using NLP minimizing z ;
loop(i,put @1 m.tl, @5 i.tl, @6 x.L(i),@18 E(i),@30 V(i,i),@50 z.L, @75 returns2.L/);
R0= R0 + phi;);
display z.L, x.L, R0, returns2.L;




Scalar lambda measure of risk averse nature of investor /0.0/;
scalar delta / 0.2 /;
Variables
        x(i)    fraction of unit capital invested in ith investment
        z       objective
Positive Variable x ;

Equations
        risk_return  our objective is to minimize risk and maximise return simult.
        capital  observe capital limit;
risk_return.. z =e=  (sum(i,E(i)*x(i)))+lambda*0.5*sum(i,sum(k,x(i)*V(i,k)*x(k)));
model greenhouse3 /risk_return, capital /;
put @1 "m", @4" i",@10 "x(i)", @20 " E(i)",@35 "V(i,i)",@50 "lambda", @75 "return"/;
loop(m,
solve greenhouse3 using NLP minimizing z ;
loop(i,put @1 m.tl, @5 i.tl, @6 x.L(i), @18   E(i), @30 V(i,i),@50 lambda,@75 z.L/);
lambda = lambda + delta;);
display z.L, x.L;
```

A.2 Gams Code for the Markowitz model (3.7) to (3.10)

```
Sets
        i alternative investments / Vestas,Sma,Tesla,Biomass,Siemens /
        j periods / 1*39 /
        m        / 1 * 5 /;
alias (i,k);
Parameters
```

Table R(j,i) returns of investments i in j years in percentage

	Vestas	Sma	Tesla	Biomass	Siemens
1	0.196376412	0.041745506	0.143516576	-0.351351351	-0.04606871
2	0.119999964	0.142292978	-0.298383146	0.4	-0.197210582
3	-0.276859451	-0.272517986	-0.047169811	-0.233333333	-0.04325764
4	0.481067961	0.131388512	-0.049729694	0.189189189	0.020784316
5	0.229186155	0.487503254	0.162718852	-0.2	0.280675731
6	0.318122977	-0.096393193	-0.170971717	0.388888889	-0.085438197
7	0.097057428	0.212788139	-0.303921684	0.090909091	-0.022315391
8	0.417871222	0.205709241	0.159774454	0.2	0.125017432
9	-0.241760722	-0.008336421	0.064237774	0	-0.173420529
10	0.203781131	0.30148815	-0.156762221	0	-0.059142561
11	-0.396803653	0.017885324	-0.118689109	0	-0.022262452
12	-0.766917293	-0.22817354	-1.833465395	0	0.07592507
13	-0.742830712	-0.104448321	-0.8011363	-0.25	-0.183597051
14	-0.138800425	0.093385982	0.222061349	-0.1	-0.114866418
15	-0.351666278	-0.676829268	-0.385694387	-1.354545455	0.016013737
16	-0.281034552	0.287012987	-0.151643087	-0.196911197	0.012794269
17	0.169582817	0.238615665	-0.010913917	0.34516129	0.021565516
18	0.015559125	0.267942584	0.083525588	0.014778325	0.006569916
19	-0.277905819	0.218300654	0.151252189	0.875	-0.074133301
20	-0.143004377	-0.676421405	-0.421094476	-0.24	0.102780529
21	-0.041704192	-0.926184539	0.074032712	-0.612903226	0.115218553
22	-0.410300788	-0.339202486	0.033776164	-2	-0.124343192
23	0.048028189	0.111948956	0.042664852	0.093333333	-0.036381899
24	0.021437814	0.037666013	0.076119618	0.411764706	0.014385443
25	-0.210508964	0.357692308	0.038863765	0.625	-0.134952612
26	0.169955073	0.11535752	-0.047345993	-0.333333333	-0.120921366
27	-0.246229459	0.057336253	-0.302353809	-3.2175	-0.099314989
28	-0.06182589	-0.110876452	-0.299353218	0.187907528	0.062694676
29	0.063553084	-0.261026616	0.056718508	0.306569343	0.009555472
30	0.247254862	-0.085330921	0.087217823	0.105263158	0.025587207
31	-0.018629121	-0.277399639	0.145238478	0.058823529	0.108738741
32	0.076259456	0.228360157	-0.288655941	-0.875	-0.093701742
33	-0.07657822	0.431228861	0.227963319	0.533333333	0.025437246
34	-0.146773059	0.17789891	-0.256939993	0.142857143	0.117455134
35	-0.13585483	-0.100060277	0.159074534	-1.27	0.01478738
36	-0.024240809	-0.401643836	0.201529268	0.111600587	-0.090264705
37	0.067620349	-0.017200938	-0.077910979	0.628099174	0.060707439

```
38 -0.266108826    -0.32820907    -0.736745945    -1.155555556    -0.186157771
39  0.166336188     0.244212963    -0.252599661     0.556701031     0.335249697
 ;
Scalar n number of periods of historical data /39/;
Scalar V0 risk threshold /0.02/;
scalar xi / 0.03 /;
Parameters E(i)     expected return of investement i in percent
           V(i,k)   covariance matrix of alternative investments;
           E(i)=sum(j, R(j,i))/n                   ;
           V(i,k)= sum(j, (R(j,i)-E(i))*(R(j,k)-E(k)))/n   ;
Variables
        x(i)     fraction of unit capital invested in ith investment
        z        objective

Positive Variable x;
                x.up(i) =0.30;
Equations
        returns   the objective is to maximise this
        risk      minimize the variance
        capital   observe capital limit;

returns..     z =e= sum(i,E(i)*x(i));
risk..        sum(i,sum(k,x(i)*V(i,k)*x(k))) =l= V0;
capital..     sum(i,x(i)) =e= 1;
file printout /printout.txt/ ;
put printout ;
model greenhouse /returns, risk, capital /;
put @1 "m",@4 "i",@10 "x(i)",@20  "E(i)",@35  "V(i,i)",@50 "risk",@75  "returns" /;
loop(m,
solve greenhouse using NLP minimizing z ;


loop(i,put @1 m.tl,@5 i.tl ,@6 x.L(i) ,@18   E(i),@30  V(i,i),@50 risk.L,@75 z.L /);
V0= V0 + xi;);
put "covariance" /;
loop((i,k), put i.tl, @12, k.tl, @24, V(i,k):8:4 /);


display z.L, x.L, V0, risk.L;



Scalar R0 least return /0.01/;
scalar phi / 0.02 /;
Variables
        x(i)     fraction of unit capital invested in ith investment
        z        objective
Positive Variable x ;
```

```
                            x.up(i) =0.30;
Equations
          returns2  we set in infimum for the returns
          risk2      the objective is to minimize the variance
          capital   observe capital limit;
risk2..          z =e= sum(i,sum(k,x(i)*V(i,k)*x(k)));
returns2..        sum(i,E(i)*x(i)) =l= -R0;
model greenhouse2 /risk2, returns2, capital /;
put @1 "m",@4 "i", @10 " x(i)",@20" E(i)",@35 " V(i,i) ",@50 "risk", @75 "returns"/;
loop(m,
solve greenhouse2 using NLP minimizing z ;
loop(i,put @1 m.tl,@5 i.tl, @6 x.L(i),@18 E(i), @30 V(i,i),@50 z.L,@75 returns2.L /);
R0= R0 + phi;);
display z.L, x.L, R0, returns2.L;




Scalar lambda measure of risk averse nature of investor /0.0/;
scalar delta / 0.2 /;
Variables
          x(i)     fraction of unit capital invested in ith investment
          z        objective
Positive Variable x ;
                   x.up(i) =0.40;
Equations
          risk_return  our objective is to minimize risk and maximise return simult.
          capital   observe capital limit;
risk_return.. z =e= (sum(i,E(i)*x(i)))+lambda*0.5*sum(i,sum(k,x(i)*V(i,k)*x(k)));
model greenhouse3 /risk_return, capital /;
put @1 "m",@4 "i",@10" x(i)", @20 " E(i)",@35 "V(i,i)",@50 " lambda", @75 "return"/;
loop(m,
solve greenhouse3 using NLP minimizing z ;
loop(i,put @1 m.tl, @5 i.tl, @6 x.L(i), @18   E(i), @30 V(i,i),@50 lambda,@75 z.L /);
lambda = lambda + delta;);
display z.L, x.L;
```

A.3 Gams Code for the CVAR model (3.12)

```
Sets
        i alternative investments / Vestas,Sma,Tesla,Biomass,Siemens /
        j scenarios / 1*39 /
        m         / 1 * 10 /
        n         / 1 * 6 /;
Parameters
```

Table R(j,i) returns of investments i in j years in percentage

	Vestas	Sma	Tesla	Biomass	Siemens
1	0.196376412	0.041745506	0.143516576	-0.351351351	-0.04606871
2	0.119999964	0.142292978	-0.298383146	0.4	-0.197210582
3	-0.276859451	-0.272517986	-0.047169811	-0.233333333	-0.04325764
4	0.481067961	0.131388512	-0.049729694	0.189189189	0.020784316
5	0.229186155	0.487503254	0.162718852	-0.2	0.280675731
6	0.318122977	-0.096393193	-0.170971717	0.388888889	-0.085438197
7	0.097057428	0.212788139	-0.303921684	0.090909091	-0.022315391
8	0.417871222	0.205709241	0.159774454	0.2	0.125017432
9	-0.241760722	-0.008336421	0.064237774	0	-0.173420529
10	0.203781131	0.30148815	-0.156762221	0	-0.059142561
11	-0.396803653	0.017885324	-0.118689109	0	-0.022262452
12	-0.766917293	-0.22817354	-1.833465395	0	0.07592507
13	-0.742830712	-0.104448321	-0.8011363	-0.25	-0.183597051
14	-0.138800425	0.093385982	0.222061349	-0.1	-0.114866418
15	-0.351666278	-0.676829268	-0.385694387	-1.354545455	0.016013737
16	-0.281034552	0.287012987	-0.151643087	-0.196911197	0.012794269
17	0.169582817	0.238615665	-0.010913917	0.34516129	0.021565516
18	0.015559125	0.267942584	0.083525588	0.014778325	0.006569916
19	-0.277905819	0.218300654	0.151252189	0.875	-0.074133301
20	-0.143004377	-0.676421405	-0.421094476	-0.24	0.102780529
21	-0.041704192	-0.926184539	0.074032712	-0.612903226	0.115218553
22	-0.410300788	-0.339202486	0.033776164	-2	-0.124343192
23	0.048028189	0.111948956	0.042664852	0.093333333	-0.036381899
24	0.021437814	0.037666013	0.076119618	0.411764706	0.014385443
25	-0.210508964	0.357692308	0.038863765	0.625	-0.134952612
26	0.169955073	0.11535752	-0.047345993	-0.333333333	-0.120921366
27	-0.246229459	0.057336253	-0.302353809	-3.2175	-0.099314989
28	-0.06182589	-0.110876452	-0.299353218	0.187907528	0.062694676
29	0.063553084	-0.261026616	0.056718508	0.306569343	0.009555472
30	0.247254862	-0.085330921	0.087217823	0.105263158	0.025587207
31	-0.018629121	-0.277399639	0.145238478	0.058823529	0.108738741
32	0.076259456	0.228360157	-0.288655941	-0.875	-0.093701742
33	-0.07657822	0.431228861	0.227963319	0.533333333	0.025437246
34	-0.146773059	0.17789891	-0.256939993	0.142857143	0.117455134
35	-0.13585483	-0.100060277	0.159074534	-1.27	0.01478738
36	-0.024240809	-0.401643836	0.201529268	0.111600587	-0.090264705
37	0.067620349	-0.017200938	-0.077910979	0.628099174	0.060707439

```
38 -0.266108826     -0.32820907     -0.736745945      -1.155555556       -0.186157771
39  0.166336188      0.244212963     -0.252599661       0.556701031        0.335249697
 ;
Scalar K number of scenarios of historical data /39/;
scalar b / 0.95 /;
Scalar R0 another risk threshold /0.15/;
scalar c / 0.1 /;
Scalar xi /0.05/;
Parameters E(i)     expected return of investement i in percent
          Q         the divisor                               ;
          E(i)=sum(j, R(j,i))/K               ;
          Q=1/(K*(1-b));
Variables
        x(i)    fraction of unit capital invested in ith investment
        f(j)    loss function for each scenario
        p(j)    the positive difference
        z       objective

Positive Variable x;

Positive Variable p;
Equations
        returns  the objective is to maximise this
        CVaR     minimize the variance
        capital  observe capital limit
        loss_function(j)  calculate the loss function for each scenario
        positiveconstraint(j)  to calculate the cvar;
loss_function(j)..     f(j) =e=sum(i,x(i)*R(j,i));
positiveconstraint(j)..       p(j) =g= f(j)-c;
returns..       sum(i,E(i)*x(i)) =l= -R0;
CVaR..       z =e= c+(Q*sum(j,p(j)));
capital..     sum(i,x(i)) =e= 1;
file outsider /outsider.txt/ ;
put outsider ;
model cvarmodel /all  /;
loop(m,
solve cvarmodel using NLP minimizing z ;
put @8 "c", @15 "   cvar ", @40 "  R0", @50 "    assets  ",@75 "x", @80 "m" /;
loop(i,put @1c, @15  z.L , @36   R0, @55 i.tl, @67 x.L(i), @80 m.tl /);
c= c + xi;);
display z.L, x.L, f.L, p.L, R0, CVaR.L;
```

A.4 Gams Code for the CVAR model (3.16)

```
Sets
        i alternative investments / Vestas,Sma,Tesla,Biomass,Siemens /
        j scenarios / 1*39 /
        m        / 1 * 10 /
        n        / 1 * 6 /;
Parameters
```

Table R(j,i) returns of investments i in j years in percentage

	Vestas	Sma	Tesla	Biomass	Siemens
1	0.196376412	0.041745506	0.143516576	-0.351351351	-0.04606871
2	0.119999964	0.142292978	-0.298383146	0.4	-0.197210582
3	-0.276859451	-0.272517986	-0.047169811	-0.233333333	-0.04325764
4	0.481067961	0.131388512	-0.049729694	0.189189189	0.020784316
5	0.229186155	0.487503254	0.162718852	-0.2	0.280675731
6	0.318122977	-0.096393193	-0.170971717	0.388888889	-0.085438197
7	0.097057428	0.212788139	-0.303921684	0.090909091	-0.022315391
8	0.417871222	0.205709241	0.159774454	0.2	0.125017432
9	-0.241760722	-0.008336421	0.064237774	0	-0.173420529
10	0.203781131	0.30148815	-0.156762221	0	-0.059142561
11	-0.396803653	0.017885324	-0.118689109	0	-0.022262452
12	-0.766917293	-0.22817354	-1.833465395	0	0.07592507
13	-0.742830712	-0.104448321	-0.8011363	-0.25	-0.183597051
14	-0.138800425	0.093385982	0.222061349	-0.1	-0.114866418
15	-0.351666278	-0.676829268	-0.385694387	-1.354545455	0.016013737
16	-0.281034552	0.287012987	-0.151643087	-0.196911197	0.012794269
17	0.169582817	0.238615665	-0.010913917	0.34516129	0.021565516
18	0.015559125	0.267942584	0.083525588	0.014778325	0.006569916
19	-0.277905819	0.218300654	0.151252189	0.875	-0.074133301
20	-0.143004377	-0.676421405	-0.421094476	-0.24	0.102780529
21	-0.041704192	-0.926184539	0.074032712	-0.612903226	0.115218553
22	-0.410300788	-0.339202486	0.033776164	-2	-0.124343192
23	0.048028189	0.111948956	0.042664852	0.093333333	-0.036381899
24	0.021437814	0.037666013	0.076119618	0.411764706	0.014385443
25	-0.210508964	0.357692308	0.038863765	0.625	-0.134952612
26	0.169955073	0.11535752	-0.047345993	-0.333333333	-0.120921366
27	-0.246229459	0.057336253	-0.302353809	-3.2175	-0.099314989
28	-0.06182589	-0.110876452	-0.299353218	0.187907528	0.062694676
29	0.063553084	-0.261026616	0.056718508	0.306569343	0.009555472
30	0.247254862	-0.085330921	0.087217823	0.105263158	0.025587207
31	-0.018629121	-0.277399639	0.145238478	0.058823529	0.108738741
32	0.076259456	0.228360157	-0.288655941	-0.875	-0.093701742
33	-0.07657822	0.431228861	0.227963319	0.533333333	0.025437246
34	-0.146773059	0.17789891	-0.256939993	0.142857143	0.117455134
35	-0.13585483	-0.100060277	0.159074534	-1.27	0.01478738
36	-0.024240809	-0.401643836	0.201529268	0.111600587	-0.090264705
37	0.067620349	-0.017200938	-0.077910979	0.628099174	0.060707439

```
38 -0.266108826    -0.32820907    -0.736745945    -1.155555556    -0.186157771
39  0.166336188     0.244212963   -0.252599661     0.556701031     0.335249697
 ;
Scalar K number of scenarios of historical data /39/;
scalar b / 0.95 /;
Scalar R0 another risk threshold /0.15/;
scalar c / 0.1 /;
Scalar xi /0.05/;
Parameters E(i)    expected return of investement i in percent
           Q        the divisor                              ;
           E(i)=sum(j, R(j,i))/K              ;
           Q=1/(K*(1-b));
Variables
        x(i)    fraction of unit capital invested in ith investment
        f(j)    loss function for each scenario
        p(j)    the positive difference
        z       objective

Positive Variable x;
               x.up(i) =0.30;
Positive Variable p;
Equations
        returns   the objective is to maximise this
        CVaR      minimize the variance
        capital   observe capital limit
        loss_function(j)  calculate the loss function for each scenario
        positiveconstraint(j)  to calculate the cvar;
loss_function(j)..      f(j) =e=sum(i,x(i)*R(j,i));
positiveconstraint(j)..       p(j) =g= f(j)-c;
returns..       sum(i,E(i)*x(i)) =l= -R0;
CVaR..        z =e= c+(Q*sum(j,p(j)));
capital..     sum(i,x(i)) =e= 1;
file outsider /outsider.txt/ ;
put outsider ;
model cvarmodel /all  /;
loop(m,
solve cvarmodel using NLP minimizing z ;
put @8 "c", @15 "   cvar ", @40 "  R0", @50 "    assets  ",@75 "x", @80 "m" /;
loop(i,put @1c, @15  z.L , @36   R0, @55 i.tl, @67 x.L(i), @80 m.tl /);
c= c + xi;);
display z.L, x.L, f.L, p.L, R0, CVaR.L;
```

B Matlab Code for the CVaR computation of the Mean-Variance optimal portfolios

```
x=[0.00  0.06   0.17   0.01   0.76]; %this is x, the row matrix for the portfolio

r=[0.196376412     0.041745506     0.143516576    -0.351351351    -0.04606871;
   0.119999964     0.142292978    -0.298383146     0.4            -0.197210582;
  -0.276859451    -0.272517986    -0.047169811    -0.233333333    -0.04325764;
   0.481067961     0.131388512    -0.049729694     0.189189189     0.020784316;
   0.229186155     0.487503254     0.162718852    -0.2             0.280675731;
   0.318122977    -0.096393193    -0.170971717     0.388888889    -0.085438197;
   0.097057428     0.212788139    -0.303921684     0.090909091    -0.022315391;
   0.417871222     0.205709241     0.159774454     0.2             0.125017432;
  -0.241760722    -0.008336421     0.064237774     0              -0.173420529;
   0.203781131     0.30148815     -0.156762221     0              -0.059142561;
  -0.396803653     0.017885324    -0.118689109     0              -0.022262452;
  -0.766917293    -0.22817354     -1.833465395     0               0.07592507;
  -0.742830712    -0.104448321    -0.8011363      -0.25           -0.183597051;
  -0.138800425     0.093385982     0.222061349    -0.1            -0.114866418;
  -0.351666278    -0.676829268    -0.385694387    -1.354545455     0.016013737;
  -0.281034552     0.287012987    -0.151643087    -0.196911197     0.012794269;
   0.169582817     0.238615665    -0.010913917     0.34516129      0.021565516;
   0.015559125     0.267942584     0.083525588     0.014778325     0.006569916;
  -0.277905819     0.218300654     0.151252189     0.875          -0.074133301;
  -0.143004377    -0.676421405    -0.421094476    -0.24            0.102780529;
  -0.041704192    -0.926184539     0.074032712    -0.612903226     0.115218553;
  -0.410300788    -0.339202486     0.033776164    -2              -0.124343192;
   0.048028189     0.111948956     0.042664852     0.093333333    -0.036381899;
   0.021437814     0.037666013     0.076119618     0.411764706     0.014385443;
  -0.210508964     0.357692308     0.038863765     0.625          -0.134952612;
   0.169955073     0.11535752     -0.047345993    -0.333333333    -0.120921366;
  -0.246229459     0.057336253    -0.302353809    -3.2175         -0.099314989;
  -0.06182589     -0.110876452    -0.299353218     0.187907528     0.062694676;
   0.063553084    -0.261026616     0.056718508     0.306569343     0.009555472;
   0.247254862    -0.085330921     0.087217823     0.105263158     0.025587207;
  -0.018629121    -0.277399639     0.145238478     0.058823529     0.108738741;
   0.076259456     0.228360157    -0.288655941    -0.875          -0.093701742;
  -0.07657822      0.431228861     0.227963319     0.533333333     0.025437246;
  -0.146773059     0.17789891     -0.256939993     0.142857143     0.117455134;
  -0.13585483     -0.100060277     0.159074534    -1.27            0.01478738;
  -0.024240809    -0.401643836     0.201529268     0.111600587    -0.090264705;
   0.067620349    -0.017200938    -0.077910979     0.628099174     0.060707439;
  -0.266108826    -0.32820907     -0.736745945    -1.155555556    -0.186157771;
   0.166336188     0.244212963    -0.252599661     0.556701031     0.335249697;];
                   %r is the matrix of returns of the asset for each scenario
C= transpose(r);
D=x*C; %f(x,r)
```

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```matlab
D;
Q=1/(39*0.05);
s=32;

for k=1:s      %iteration for c
    result = 0;
    for i=1:39
        newValues=D(1,i)-(k/100);
        if newValues >=0
            result = result + newValues;
        else
            newValues=0;
            result = result + newValues;
        end
    end
    h=(k/100)+Q*result;
    disp(k)
    disp(h)
end
```