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THE T_0-REFLECTION IN THE CATEGORY V-PRETOP

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Abstract. A V-pretopological space is a pair (X, a) where X is a nonempty set and a is a $\mathcal{P}(X)$ self map satisfying $a(\emptyset) = \emptyset$, $A \subseteq a(A)$ and $a(A \cup B) = a(A) \cup a(B)$ for any $A, B \in \mathcal{P}(X)$. It is well known that the category **Top** of topological spaces is a reflective subcategory in the category **V-PreTop** whose objects are pretopological spaces of type V. In the present paper we give the construction of the T_0-reflection in the category **V-PreTop**. Hence, some new separation axioms are introduced and characterized. Finally, the orthogonal of some subcategories are studied.

Introduction

During the 1990s, computer science and the increase in computer computing power allowed the development of technologies based on pretopology for the classification of data or image analysis in a very relevant way. Recently, the simulation of complex systems and complex phenomena in the context of environmental and health problems or congestion in the context of aerial control has demonstrated the relevance of pretopology as a tool for computer modeling.

In this paper pretopological spaces are studied in terms of Čech closure operators and are of interest to topologists and computer scientists approaching topology in terms of operators.

Recall that a topological space (X, τ) can be defined as a pair (X, a) where a is a selfmapping on $\mathcal{P}(X)$ (the family of all subsets of X) satisfying the following properties:

(1) $a(\emptyset) = \emptyset$;
(2) $A \subseteq a(A)$, for all $A \in \mathcal{P}(X)$;
(3) $a(A \cup B) = a(A) \cup a(B)$, for all $A, B \in \mathcal{P}(X)$;
(4) $a(a(A)) = a(A)$, for all $A \in \mathcal{P}(X)$.

So, giving such a pair (X, a) there exists a unique topology τ on X, such that $a(A) = \overline{A}^{\tau}$. Conversely, it is clear that the closure operator on a topological space satisfies all previous conditions from (1) to (4). The operator a is called the Kuratowski operator.

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Generalizing this concept, such a pair (X, a) satisfying (1), (2) and (3) is called a pretopological space of type V and we call it simply a V-pretopological space.

Regarding the terminology of [4], in a V-pretopological space (X, a), a is called a *closure operation* or simply a *closure* for X (see [4, Definition 14A.1, page 237]). In particular if a is a Kuratowski operator, then a is called a *topological closure operation* and (X, a) is called a *topological space* or simply, a *T-space*.

It is well known that the category **Top** is a full reflective subcategory of the category **V-PreTop** whose objects are V-pretopological spaces and arrows all V-continuous maps. Recall that in this paper we call a map f from (X, a) to (Y, b) *V-continuous* if $a(f^{-1}(B)) = f^{-1}(B)$, for any subset B in Y such that $b(B) = B$ (see for example [4, Definition 16B.1, page 272]).

In the first section of this paper, we generalize some classical separation axioms in the category **V-PreTop**.

The second section is devoted to the construction of the T_0-reflection of a V-pretopological space. Hence, the construction of the T_0-reflection in **Top** was given by Herrlich and Strecker in [6]. It was given by Künzi and Richmond in the category **PreOrdTop** (see [7]). Mirhosseinkhani [9] has been interested in the T_0-reflection in the category **GenTop**. Recently, Lazaar, Mhemdi and Abbassi gave, in [1], the construction of the T_0-reflection in the category **PreTop**.

In the third section based on paper [3], we introduce and characterize some new separation axioms named $T_{(0,j)}$ for $j \in \{1, 2\}$. Some results in [3] are deduced.

Finally, in the fourth section, we are interested in some categorical properties on the categories **Top**, **V-PreTop**, **V-PreTop$_0$** and **Top$_0$**.

1. Some classical separation axioms in V-PreTop

This section is devoted to introducing some basic definitions and results needed throughout this paper.

Definitions 1.1. Let (X, a) be a V-pretopological space and A be a subset of X. A is called a *V-closed* (*V-open*) if $a(A) = A$ (A^c V-closed). We denote by $VC(X)$ ($VO(X)$) the family of all V-closed (V-open) subsets of X.

Remark 1.2.
1. Let $f : (X, a) \longrightarrow (Y, b)$ be a map between two V-pretopological spaces. Then, f is V-continuous if and only if $f^{-1}(B) \in VC(X)$ ($VO(X)$) for every $B \in VC(Y)$ ($VO(Y)$).
2. Let (X, a) be a V-pretopological space and $x \in X$. We denote by $vc(x)$ the intersection of all V-closed subsets of X containing x. It is clear that $vc(x)$ is the smallest V-closed subset of X containing x.

Regarding the classical separation axioms in **Top**, it is natural to introduce the following separation axioms in **V-PreTop**.

Definitions 1.3. Let (X, a) be a V-pretopological space.

1. (X, a) is said to be T_0 if, for any $x \neq y \in X$, there exists an $A \in VC(X)$ which contains one of the points x, y but not the other.
2. (X, a) is said to be T_1 if, for all $x \neq y$ in X, there exists an $A \in VC(X)$ which contains x but not y.

3. (X, a) is said to be T_2 if, for all $x \neq y$ in X, there exist two disjoint V-open subsets B_1, B_2 of X containing x and y, respectively.

Proposition 1.4. *Let (X, a) be a V-pretopological space. The following properties hold.*

1. *X is T_0 if and only if, for all $x, y \in X$, we have $vc\,(x) = vc\,(y) \Longrightarrow x = y$.*
2. *The following statements are equivalent:*
   (a) *(X, a) is a T_1 space;*
   (b) *For all $x \in X$, we have $vc\,(x) = \{x\}$;*
   (c) *For all $x \in X$, we have $a(\{x\}) = \{x\}$.*

Proof. 1. Straightforward.

2. (a) $\Longleftrightarrow$ (b) Let $y \neq x \in X$. Since (X, a) is T_1, there exists an $A \in VC\,(X)$ such that $x \in A$ and $y \notin A$. Now, $y \notin vc\,(x)$ and thus $vc\,(x) = \{x\}$. Conversely, let $y \neq x \in X$. Then, $y \notin \{x\} = vc\,(x)$ and thus there exists an $A \in VC(X)$ containing x but not y. Therefore, (X, a) is T_1.

(c) $\Longleftrightarrow$ (b) suppose that $vc\,(x) = \{x\}$. Then, by Remark 1.2, $\{x\}$ is V-closed and thus $a(\{x\}) = \{x\}$. Conversely, it is clear that, if $a(\{x\}) = \{x\}$, then $\{x\} \in VC(X)$, containing x, thus $vc(x) = \{x\}$. □

Remark 1.5. The following implications are immediate in **V-PreTop**.

$$T_2 \Longrightarrow T_1 \Longrightarrow T_0.$$

The inverse implications are not true in general. Indeed, in **Top**, which is a particular case of **V-PreTop**, T_0, T_1, and T_2 coincide with T_0, T_1 and T_2, respectively, and it is well known that:

$$T_0 \not\Rightarrow T_1 \not\Rightarrow T_2.$$

Examples 1.6. Regarding Remark 1.5, it is of interest to study some examples in **V-PreTop**, rather than in **Top**, in which there are no inverse implications.

1. A non-topological V-pretopological space which is T_0, not T_1. Consider $X = \{0, 1, 2\}$ and a from $\mathcal{P}(X)$ to itself defined by:

$$a(\{0\}) = \{0, 1\};$$
$$a(\{1\}) = \{1, 2\};$$
$$a(\{2\}) = \{2\}.$$

Clearly, $a(a(\{0\})) = X \neq a(\{0\})$, which implies that (X, a) is not a topological space. The family $VC(X) = \{\emptyset, \{2\}, \{1, 2\}, X\}$, so (X, a) is a T_0 pretopological space which is not T_1.

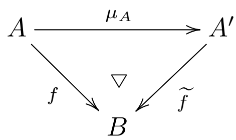
2. It is clear that, if (X, a) is finite, then: (X, a) is T_1 if and only if $a(A) = A$ for any subset A in X and thus (X, a) is a topological space which is T_2. So the following question is immediate: *Is there a pretopological space not topological which is T_1, not T_2?*

2. The T_0-reflection in V-PreTop

Definition 2.1. A subcategory $\mathcal{A}$ of $\mathcal{B}$ is called *reflective* in $\mathcal{B}$ when the inclusion functor $I : \mathcal{A} \longrightarrow \mathcal{B}$ has a left adjoint $F : \mathcal{B} \longrightarrow \mathcal{A}$. This functor F is called a reflector and the adjunction $(I, F, \varphi) : \mathcal{B} \longrightarrow \mathcal{A}$ a reflection of $\mathcal{B}$ in its subcategory $\mathcal{A}$.

Generally, to show that a full subcategory $\mathcal{A}$ of $\mathcal{B}$ is reflective, we use the characterization given by MacLane in [8, page 89]:

$\mathcal{A}$ is reflective in $\mathcal{B}$ if and only if, for each object A in $\mathcal{B}$, there exists an object A' in $\mathcal{A}$ and an arrow $\mu_A : A \longrightarrow A'$ such that, for each object B in $\mathcal{A}$ and $f : A \longrightarrow B$ in $\mathcal{B}$, there exists a unique arrow $\tilde{f} : A' \longrightarrow B$ in $\mathcal{A}$ such that the following diagram



is commutative.

In our case, denote by $\mathbf{V\text{-}PreTop_0}$ the full subcategory of **V-PreTop** with T_0-spaces as objects. Our first main result of this paper is the following.

Theorem 2.2. $\mathbf{V\text{-}PreTop_0}$ *is reflective in* **V-PreTop***.*

Proof. Let (X, a) be a V-pretopological space. We define on X the relation $\sim$ by: $x \sim y$ if and only if $vc\,(x) = vc\,(y)$. The previous relation is an equivalence relation on X. We denote by ${}^{X}/\!\sim$ the quotient set and by $\mu_X : X \longrightarrow {}^{X}/\!\sim$ the canonical surjection.

Let $\widehat{a}$ be the operator from ${}^{X}/\!\sim$ to itself defined by:

$$\widehat{a}\,(A) = \mu_X\left(a\left(\mu_X^{-1}\,(A)\right)\right).$$

It is clear that $\widehat{a}\,(\emptyset) = \emptyset$, $A \subseteq \widehat{a}\,(A)$ and $\widehat{a}\,(A \cup B) = \widehat{a}\,(A) \cup \widehat{a}\,(B)$, for all $A, B \in {}^{X}/\!\sim$.

Then, $({}^{X}/\!\sim, \widehat{a})$ is a V-pretopological space.

To show that $({}^{X}/\!\sim, \widehat{a})$ is a T_0- V-pretopological space, consider two points $x, y \in X$ such that $\mu_X\,(x) \neq \mu_X\,(y)$. There exists an $A \in VC(X)$ such that, for example, $x \in A$ and $y \notin A$. Thus, $\mu_X\,(x) \in \mu_X\,(A)$ and $\mu_X\,(y) \notin \mu_X\,(A)$. Now one can see easily that $\mu_X^{-1}(\mu_X(A)) = A$.

So, $\widehat{a}\,(\mu_X\,(A)) = \mu_X\left(a\left(\mu_X^{-1}\,(\mu_X\,(A))\right)\right) = \mu_X\,(a\,(A)) = \mu_X(A)$.

This is sufficient to prove the separation of $\mu_X\,(x)$ and $\mu_X\,(y)$ by the V-closed subset $\mu_X(A)$ of $VC\,({}^{X}/\!\sim)$.

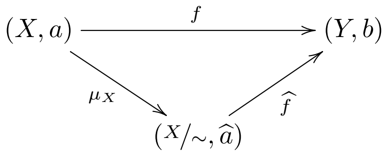
Let us show that $\mu_X : X \longrightarrow {}^{X}/\!\sim$ is a V-continuous map. For this, let $\mu_X(A)$ be a V-closed subset of $VC\,({}^{X}/\!\sim)$, where A is a subset of X.

For every $x \in a\left(\mu_X^{-1}\,(\mu_X\,(A))\right)$, we have:

$$\mu_X\,(x) \in \mu_X\left(a\left(\mu_X^{-1}\,(\mu_X\,(A))\right)\right) = \widehat{a}(\mu_X\,(A)) = \mu_X(A).$$

Thus, $a\left(\mu_X^{-1}\,(\mu_X\,(A))\right) \subseteq \mu_X^{-1}\,(\mu_X\,(A))$. The inverse inclusion is immediate and, consequently, $\mu_X^{-1}\,(\mu_X\,(A))$ is a V-closed subset of (X, a).

To finish, we will prove that $({}^{X}/\!\sim, \widehat{a})$ is the T_0-reflection of (X, a). So, let (Y, b) be a T_0-space and $f : (X, a) \longrightarrow (Y, b)$ be a V-continuous map. It is sufficient to prove that there exists a unique V-continuous map $\widehat{f}$ rending the following diagram commutative



Uniqueness: Straightforward since $\widehat{f}\left(\mu_X(x)\right) = f(x)$.

Existence: Let $x, y \in X$ be such that $\mu_X(x) = \mu_X(y)$. Suppose that $f(x) \neq f(y)$. Since (Y, b) is T_0, there exists a $B \in VC(Y)$ such that, for example, $f(x) \in B$ and $f(y) \notin B$. Then, $x \in f^{-1}(B)$ and $y \notin f^{-1}(B)$. By the V-continuity of f, we have $f^{-1}(B) \in VC(X)$ and then $\mu_X(x) \neq \mu_X(y)$, which is a contradiction.

$\widehat{f}$ is V-continuous: Let $B \in VC(Y)$. We need to show that

$$\widehat{f}^{-1}(B) \in VC\left(X/\sim\right).$$

Since f is V-continuous, $a\left(f^{-1}(B)\right) = f^{-1}(B)$ and

$$a\left(\mu_X^{-1}\left(\widehat{f}^{-1}(B)\right)\right) = \mu_X^{-1}\left(\widehat{f}^{-1}(B)\right)$$

so that

$$\mu_X\left(a\left(\mu_X^{-1}\left(\widehat{f}^{-1}(B)\right)\right)\right) = \mu_X\left(\mu_X^{-1}\left(\widehat{f}^{-1}(B)\right)\right),$$

which implies $\widehat{a}\left(\widehat{f}^{-1}(B)\right) = \widehat{f}^{-1}(B)$. This fact completes the proof. □

In the rest of this paper we denote the T_0-reflection of a V-pretopological space (X, a) by $(T_0(X), \widehat{a})$.

Example 2.3. Let $X = \{x, y, z, t\}$ and (X, a) be the V-pretopological space such that:

$$a(\{x\}) = \{x\},\ a(\{y\}) = \{x, y\},\ a(\{z\}) = \{z, t\} \text{ and } a(\{t\}) = X.$$

Then, $(T_0(X), \widehat{a})$, is given by: $T_0(X) = \{u, v, w\}$ and:

$$\widehat{a}(\{u\}) = \{u\},\ \widehat{a}(\{v\}) = \{u, v\} \text{ and } \widehat{a}(\{w\}) = T_0(X).$$

3. New separation axioms in V-PreTop

By the Definition given in [3], we introduce the following definition particularly in the case of T_0-reflection.

Definition 3.1. A topological space X is said to be a $T_{(0,i)}$*-space* if $\mathbf{T}_0(X)$ is a T_i-space (thus there are two new types of separation axioms namely; $T_{(0,1)}$, $T_{(0,2)}$.

The goal of this section is to characterize $T_{(0,j)}$-spaces for $j \in \{1, 2\}$. Let (X, a) be a V-pretopological space and $x \in X$. We start by proving the following Lemma.

Lemma 3.2. Let (X, a) be a V-pretopological space. Then, the following statements are equivalent.

1. $T_0(X)$ is a T_1-space;
2. For each $x \in X$, we have $\mu_X^{-1}\left(\{\mu_X(x)\}\right) = vc(x)$.

Proof. 1. $\Longrightarrow$ 2. Suppose $T_0(X)$ is a T_1-space. If $y \in \mu_X^{-1}(\{\mu_X(x)\})$, then $vc(x) = vc(y)$ so that $y \in vc(x)$. Conversely, it is clear that $\mu_X^{-1}(\{\mu_X(x)\})$ is a V-closed subset of X containing x. Now, since $vc(x)$ is the smallest V-closed subset of X containing x, $vc(x) \subseteq \mu_X^{-1}(\{\mu_X(x)\})$.

2. $\Longrightarrow$ 1. Let $x, y \in X$ be such that $\mu_X(x) \neq \mu_X(y)$. Then, by the hypothesis, $y \notin vc(x)$. So, there exists an $A \in VC(X)$ containing x but not y. Finally, $\mu_X(A) \in VC(T_0(X))$ contains $\mu_X(x)$ but not $\mu_X(y)$. □

The following theorem characterizes $T_{(0,1)}$-spaces in **V-PreTop**.

Theorem 3.3. *Let (X, a) be a V-pretopological space. Then, the following statements are equivalent:*

(1) *X is $T_{(0,1)}$.*
(2) *For each $x, y \in X$ such that $vc(x) \neq vc(y)$, there is $A \in VC(X)$ containing x but not y.*
(3) *For each $x, y \in X$, $x \in vc(y) \Longrightarrow y \in vc(x)$.*
(4) *For each $x \in X$ and each $A \in VC(X)$ such that $vc(x) \cap A \neq \emptyset$, we have $x \in A$.*
(5) *For each $U \in VO(X)$ and $x \in U$ we have $vc(x) \subseteq U$.*

Proof. (1) $\Rightarrow$ (2) Let $x, y \in X$ be such that $vc(x) \neq vc(y)$. Then, $\mu_X(x) \neq \mu_X(y)$. Since $T_0(X)$ is T_1, there exists a $B \in VC(T_0(X))$ such that $\mu_X(x) \in B$ and $\mu_X(y) \notin B$. Hence, $\mu_X^{-1}(B) \in VC(X)$ which contains x but not y.

(2) $\Rightarrow$ (3) Suppose that $y \notin vc(x)$. Then, $vc(x) \neq vc(y)$ and thus, by hypothesis, there exists an $A \in VC(X)$ which contains y but not x. Therefore, $x \notin vc(y)$.

(3) $\Rightarrow$ (4) Let $A \in VC(X)$ be such that $vc(x) \cap A \neq \emptyset$ and $t \in vc(x) \cap A$. Then, (using 3) $x \in vc(t)$ so that $x \in A$.

(4) $\Rightarrow$ (5) Let $U \in VO(X)$ be such that $x \in U$. Then, $vc(x) \cap U^c = \emptyset$ and consequently, $vc(x) \subseteq U$.

(5) $\Rightarrow$ (1) Clearly, $\mu_X^{-1}(\{\mu_X(x)\}) \subseteq vc(x)$. Conversely, let $y \in vc(x)$. Suppose that $\mu_X(x) \neq \mu_X(y)$. Then, there exists an $A \in VC(X)$ which contains y but not x. So $A^c \in VO(X)$ and $x \in A^c$, which implies that $vc(x) \subseteq A^c$ and then $y \in A^c$, which is a contradiction. We deduce that $\mu_X^{-1}(\{\mu_X(x)\}) = vc(x)$. By Lemma 3.2 we have X is $T_{(0,1)}$. □

The following theorem characterizes $T_{(0,2)}$-spaces in **V-PreTop**.

Theorem 3.4. *Let (X, a) be a V-pretopological space. Then, the following statements are equivalent:*

(1) *(X, a) is $T_{(0,2)}$.*
(2) *For each $x, y \in X$ such that $vc(x) \neq vc(y)$, there exist two disjoint subsets $B_1, B_2 \in VO(X)$ such that $x \in B_1$ and $y \in B_2$.*

Proof. (1) $\Rightarrow$ (2) Let $x, y \in X$ be such that $vc(x) \neq vc(y)$. Since $\mu_X(x) \neq \mu_X(y)$ and $T_0(X)$ is T_2, there exist disjoint $A_1, A_2 \in VO(T_0(X))$ such that $\mu_X(x) \in A_1$ and $\mu_X(y) \in A_2$. Then, $x \in B_1 = \mu_X^{-1}(A_1)$ and $y \in B_2 = \mu_X^{-1}(A_2)$. It is clear that B_1, B_2 are disjoint V-open subsets in X.

(2) $\Rightarrow$ (1) Suppose $\mu_X(x) \neq \mu_X(y) \in T_0(X)$. Let $B_1, B_2 \in VO(X)$ be such that B_1 and B_2 are disjoint, $x \in B_1$ and $y \in B_2$. $\mu_X(B_1), \mu_X(B_2) \in VO(T_0(X))$ are disjoint and contain separately $\mu_X(x)$ and $\mu_X(y)$. So, (X, a) is $T_{(0,2)}$. □

4. Orthogonality

A morphism $f : A \longrightarrow B$ and an object X in a category C are called orthogonal [5], if the mapping $hom_C(f; X) : hom_C(B; X) \longrightarrow hom_C(A; X)$ that takes g to gf is bijective. For a class of morphisms Σ (a class of objects D), we denote by $\Sigma^{\perp}$ the class of objects orthogonal to every f in Σ (by $D^{\perp}$ the class of morphisms orthogonal to all X in D) [5].

The orthogonality class of morphisms $D^{\perp}$ associated with a reflective subcategory D of a category C satisfies the following identity $D^{\perp\perp} = D$ [2, Proposition 2.6]. Thus, it is of interest to give explicitly the class $D^{\perp}$. Note also that, if $I : D \longrightarrow C$ is the inclusion functor and $F : C \longrightarrow D$ is a left adjoint functor of I, then the class $D^{\perp}$ is the collection of all morphisms of C rendered invertible by the functor F (i.e. $D^{\perp} = \{f \in hom_C : F(f)$ is an isomorphism of $D\}$) [2, Proposition 2.3].

This section is devoted to the study of the orthogonal class $\mathbf{V\text{-}PreTop}^{\perp}_{\mathbf{S0}}$; hence, we will give a characterization of morphisms rendered invertible by the functor T_0 in the category **V-PreTop**.

4.1. Top viewed as a reflective subcategory of V-PreTop

It is well known that the category **Top** is a reflective subcategory in **V-PreTop**. The left adjoint functor of the inclusion functor I from **Top** to **V-PreTop** is defined as follows:

Given a V-pretopological space (X, a), we define the topological space $(X, \widehat{a})$, where

$$\widehat{a}(A) = \bigcap[B : \ B \supseteq A, B = a(B)], \ \forall A \subseteq X,$$

and $\mu_X : (X, a) \longrightarrow (X, \widehat{a})$ is the identity map.

Now, in order to characterize the class $\mathbf{Top}^{\perp}$ in the category **V-PreTop**, we introduce the following definition and proposition.

Definition 4.1. Let $f : (X, a) \longrightarrow (Y, b)$ be a V-continuous map. f is said to be V-closed if:

$$\forall A \in VC(X), \quad f(A) \in VC(Y).$$

Proposition 4.2. Let (X, a) be a V-pretopological space and $(X, \widehat{a})$ its associated topological space. Then, for any subset A of X, we have $\widehat{a}(A) = A$ if and only if $a(A) = A$.

Proof. If $a(A) = A$, then it is clear that $\widehat{a}(A) = A$. Conversely, suppose that $\widehat{a}(A) = A$ and consider $x \in a(A)$. Then, for every B such that $A \subseteq B = a(B)$, we have $x \in a(A) \subseteq a(B) = B$. Thus, $x \in \widehat{a}(A) = A$. □

Theorem 4.3. $\mathbf{Top}^{\perp}$ *is the family of all bijective V-closed V-continuous maps in* **V-PreTop**.

Proof. Let us consider the following commutative diagram:

$$\begin{array}{ccc} (X,a) & \xrightarrow{f} & (Y,b) \\ \downarrow{id_X} & \circlearrowleft & \downarrow{id_Y} \\ (X,\widehat{a}) & \xrightarrow[\widehat{f}]{} & \left(Y,\widehat{b}\right) \end{array}$$

Clearly, it is enough to show that $\widehat{f}$ is closed if and only if f is V-closed. So,

$$\begin{aligned} \widehat{f} \text{ is closed} &\iff \forall A \subseteq X: \ \widehat{a}(A) = A, \ \widehat{b}(\widehat{f}(A)) = \widehat{f}(A) \\ &\iff \forall A \subseteq X: \ \widehat{a}(A) = A, \ \widehat{b}(f(A)) = f(A) \\ &\iff \forall A \subseteq X: \ a(A) = A, \ b(f(A)) = f(A) \quad \text{(see Proposition 4.2)} \\ &\iff \forall A \in VC(X), \ f(A) \in VC(Y) \\ &\iff f \text{ is V} - \text{closed.} \end{aligned}$$

□

The following result is an immediate consequence of Theorem 4.3.

Corollary 4.4. *The orthogonal class of* **$\mathbf{Top_0}$** *viewed as a reflective subcategory of* **V-PreTop$_0$** *is exactly the family of all bijective V-closed V-continuous maps in* **V-PreTop$_0$**.

4.2. The orthogonal of V-PreTop$_0$ in the category V-PreTop

By Grothendiek's definition of quasihomeomorphisms we introduce the following definition:

Definition 4.5. Let $f : (X,a) \longrightarrow (Y,b)$ be a V-continuous map between V-pretopological spaces. f is said to be a V-quasihomeomorphism if the map

$$\begin{array}{cccc} \varphi_f : & VC\,(Y) & \longrightarrow & VC\,(X) \\ & A & \longmapsto & f^{-1}\,(A) \end{array}$$

is bijective.

Remark 4.6. Using notations in the Definition 4.5, we have

1. In the particular case of topological spaces f is a V-quasihomeomorphism if and only if it is a quasihomeomorphism.
2. Let $f : (X,a) \longrightarrow (Y,b)$ and $g : (Y,b) \longrightarrow (Z,c)$ be two V-continuous maps. Then, if two from $\{f, g, g \circ f\}$ are V-quasihomeomorphisms, then the third one is too.

Proposition 4.7. *Let $f : (X,a) \longrightarrow (Y,b)$ be a V-continuous map between V-pretopological spaces. If f is a V-quasihomeomorphism, then for all A in $VC\,(X)$ we have*

$$f^{-1}\,(b\,(f\,(A))) = A.$$

Proof. It is clear that $A \subseteq f^{-1}\,(b\,(f\,(A)))$. Conversely, let $B \in VC\,(Y)$ be such that $A = f^{-1}\,(B)$. Then, $f\,(A) = f\left(f^{-1}\,(B)\right) \subseteq B$ and then $b\,(f\,(A)) \subseteq b\,(B) = B$. So, $f^{-1}\,(b\,(f\,(A))) \subseteq f^{-1}\,(B) = A$. □

Proposition 4.8. *If f satisfies the condition $b\big(f(f^{-1}(B))\big) = B$ for all $B \in VC(Y)$, then φ_f is one to one.*

Proof. Let $B_1, B_2 \in VC(Y)$ be such that $\varphi_f(B_1) = \varphi_f(B_2)$, which is equivalent to $f^{-1}(B_1) = f^{-1}(B_2)$. Then, $b\big(f(f^{-1}(B_1))\big) = b\big(f(f^{-1}(B_2))\big)$ so that $B_1 = B_2$, which implies the injectivity of φ_f. □

Theorem 4.9. *Let f be a V-closed V-continuous map. Then, f is a V-quasihomeomorphism if and only if f satisfies the following statements:*

1. *For all A in $VC(X)$ we have $f^{-1}\big(b(f(A))\big) = A$;*
2. *For all $B \in VC(Y)$ we have $b\big(f(f^{-1}(B))\big) = B$.*

Proof. It is sufficient to use Proposition 4.7 and Proposition 4.8. □

Definition 4.10. Let $f : (X, a) \longrightarrow (Y, b)$ be a V-continuous map:

(a) f is called V-onto if for all $y \in Y$ there exists an $x \in X$ such that $vc(f(x)) = vc(y)$;
(b) f is called V-one to one if $vc(f(x)) = vc(f(y))$ implies $vc(x) = vc(y)$, for every $x, y \in X$.

Remark 4.11. In the particular case of topological spaces, f is V-onto (V-one to one) if and only if f is topologically onto (topologically one to one) (see [3, Definition 4.10]).

Theorem 4.12. *$\mu_X : X \longrightarrow T_0(X)$ is a V-quasihomeomorphism.*

Proof. μ_X is V-closed: Let $A \in VC(X)$. Then,

$$\begin{aligned}\widehat{a}(\mu_X(A)) &= \mu_X\left(a\left(\mu_X^{-1}(\mu_X(A))\right)\right) \\ &= \mu_X(a(A)) \\ &= \mu_X(A)\end{aligned}$$

so that $\mu_X(A) \in VC(T_0(X))$.

Let $A \in VC(X)$. Then,

$$\mu_X^{-1}(\widehat{a}(\mu_X(A))) = \mu_X^{-1}(\mu_X(A)) = A.$$

If $B \in VC(T_0(X))$, then

$$\begin{aligned}\widehat{a}\left(\mu_X\left(\mu_X^{-1}(B)\right)\right) &= \mu_X\left(a\left(\mu_X^{-1}\left(\mu_X\left(\mu_X^{-1}(B)\right)\right)\right)\right) \\ &= \mu_X\left(a\left(\mu_X^{-1}(B)\right)\right) \\ &= \widehat{a}(B) \\ &= B.\end{aligned}$$

Finally, we can use Theorem 4.9 to complete the proof. □

Lemma 4.13. Let $f : (X, a) \longrightarrow (Y, b)$ be a V-continuous map and $\widehat{f}$ the map defined by the following diagram:

$$\begin{array}{ccc} (X,a) & \xrightarrow{\ f\ } & (Y,b) \\ {\scriptstyle \mu_X}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \mu_Y} \\ (T_0(X),\widehat{a}) & \xrightarrow[\widehat{f}]{} & \left(T_0(Y),\widehat{b}\right) \end{array}$$

Then, we have the following statements:

(1) f is V-onto if and only if $\widehat{f}$ is onto;

(2) f is V-one to one if and only if $\widehat{f}$ is one to one.

Proof. (1) Suppose f is V-onto. Let $\mu_Y(y) \in T_0(Y)$ and $x \in X$ be such that $vc(f(x)) = vc(y)$. Then, $\mu_Y(y) = \mu_Y(f(x))$, which implies $\mu_Y(y) = \mu_Y(f(x)) = \mu_Y \circ f(x) = \widehat{f} \circ \mu_X(x) = \widehat{f}(\mu_X(x))$. Hence, $\mu_Y(y) = \widehat{f}(\mu_X(x))$ and $\widehat{f}$ is onto.

Conversely, if $\widehat{f}$ is onto, $y \in Y$ and $x \in X$ such that $\widehat{f}(\mu_X(x)) = \mu_Y(y)$, then $vc(f(x)) = vc(y)$.

(2) Suppose $\widehat{f}$ is one to one. Let $x, y \in X$ be such that $vc(f(x)) = vc(f(y))$. Then, $\mu_Y(f(x)) = \mu_Y(f(y))$ and $\widehat{f}(\mu_X(x)) = \widehat{f}(\mu_X(y))$. Since $\widehat{f}$ is one to one, $\mu_X(x) = \mu_X(y)$ and then $vc(x) = vc(y)$, which implies that f is V-one to one.

Conversely, suppose f be V-one to one. Let $x, y \in X$ be such that $\widehat{f}(\mu_X(x)) = \widehat{f}(\mu_X(y))$. Then, $\mu_Y(f(x)) = \mu_Y(f(y))$ and $vc(f(x)) = vc(f(y))$. Since f is V-one to one, $vc(x) = vc(y)$. So, $\mu_X(x) = \mu_X(y)$, which proves that $\widehat{f}$ is one to one. □

Now, we are in position to give the second main result of this paper.

Theorem 4.14. *Let $f : (X,a) \longrightarrow (Y,b)$ be a V-continuous map and $\widehat{f}$ the map defined by the following diagram:*

$$\begin{array}{ccc} (X,a) & \xrightarrow{\ f\ } & (Y,b) \\ {\scriptstyle \mu_X}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \mu_Y} \\ (T_0(X),\widehat{a}) & \xrightarrow[\widehat{f}]{} & \left(T_0(Y),\widehat{b}\right) \end{array}$$

Then, the following statements are equivalent:

1. *$\widehat{f}$ is an isomorphism;*
2. *f is a V-onto V-quasihomeomorphism.*

Proof. 1. ⇒ 2. Using Lemma 4.13, f is V-onto. Since $\widehat{f}, \mu_X$ and μ_Y are V-quasihomeomorphisms, f is also a V-quasihomeomorphism.

2. ⇒ 1. Since f is V-onto, $\widehat{f}$ is onto.

Let us prove that f is V-one to one. For this let $x, y \in X$ be such that $vc(f(x)) = vc(f(y))$. Suppose $vc(x) \neq vc(y)$. Then, there exists an $A \in VC(X)$ which contains, for example, x, but not y. Let $B \in VC(Y)$ be such that $f^{-1}(B) = A$. Then, $f(x) \in B$ and $f(y) \notin B$, which is a contradiction. We conclude that f is V-one to one so that $\widehat{f}$ is one to one.

μ_X, f and μ_Y are V-continuous so $\widehat{f}$ is V-continuous.

$\widehat{f}^{-1}$ is V-continuous: Let $A \in VC(T_0(X))$. Since μ_X is V-continuous, $\mu_X^{-1}(A) \in VC(X)$. f is a V-quasihomeomorphism, then there exists a $B \in VC(Y)$ such that $f^{-1}(B) = \mu_X^{-1}(A)$, which implies $f^{-1}\left(\mu_Y^{-1}(\mu_Y(B))\right) = \mu_X^{-1}(A)$ and then

$$\mu_X^{-1}(A) = (\mu_Y \circ f)^{-1}(\mu_Y(B)) = \left(\widehat{f} \circ \mu_X\right)^{-1}\big(\mu_Y(B)\big) = \mu_X^{-1}\Big(\widehat{f}^{-1}\big(\mu_Y(B)\big)\Big).$$

So, $\widehat{f}(A) = \mu_Y(B)$. Or, $\mu_Y(B) \in VC(T_0(Y))$ then $\widehat{b}\big(\widehat{f}(A)\big) = \widehat{f}(A)$, which implies that $\widehat{f}^{-1}$ is V-continuous. □

Corollary 4.15. [3, Theorem 2.4] *The orthogonal class of* $\mathbf{Top_0}$ *viewed as a reflective subcategory of* **Top** *is exactly the family of all topologically onto quasi-homeomorphisms.*

Proof. It is an immediate consequence of Theorem 4.14 and Remarks 4.6, 4.11. □

References

[1] M. Abbassi, S. Lazaar and A. Mhemdi, *T_0-reflection and some separation axioms in PreTop*, Filomat **32** (2018), 3289–3296.

[2] A. Ayache and O. Echi, *The envelope of a subcategory in Topology and group theory*, International Journal of Mathematics and Mathematical Sciences **21** (2005), 3787–3404.

[3] K. Belaid, O. Echi and S. Lazaar, *$T_{(\alpha,\beta)}$-spaces and the Wallman compactification*, International Journal of Mathematics and Mathematical Sciences **68** (2004), 3717–3735.

[4] E. Čech, *Topological Spaces*, Academia, Publishing House of the Czechoslovak Academy of Sciences, Prague, 1966.

[5] P. J. Freyd and G. M. Kelly, *Categories of continuous functors, I*, Journal of Pure and Applied Algebra **2** (1972), 169–191.

[6] H. Herrlich and G. Strecker, *H-closed spaces and reflective subcategories*, Mathematische Annalen **177** (1968), 302–309.

[7] H.-P. A. Künzi, T. A. Richmond, *T_i-ordered reflections*, Applied General Topology **6** (2005), 207–216.

[8] S. Mac Lane, *Categories for the Working Mathematician*, Graduate Texts in Mathematics **5**, Springer-Verlag, New York, 1971.

[9] Gh. Mirhosseinkhani, *Quasihomeomorphisms and meet-semilattice equivalence of generalaized topological spaces*, Acta Mathematica Hungarica **147** (2015), 272–285.

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