COMPARISON OF DISCRETIZATION METHODS

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Abstract: This paper deals with discretization of the continuous systems. There will be presented two common methods how to do this job and one uncommon. The uncommon method is to look at the system as filter. So the system could be implemented as a FIR filter. In the end of this paper these methods will be compared.

Keywords: Convolution, Discretization, Impulse response, FIR filtering, Equivalent Z-transfer function

1 INTRODUCTION

In this time we often need to implement continuous systems into microcontrollers. Microcontrollers unfortunately can only work with discrete systems, so we need to calculate discrete model of the continuous system which we will implement into microcontroller. There are many methods ho to do this job, but much of these methods work directly with transfer functions. In this paper we will show method which is based on discretization of the impulse response of the continuous system and implement it as a FIR (finite impulse response) filter.

2 MOST COMMON METHODS

In this section we will deal with bilinear transform (Tusntin's method) and with method which uses step-response of continuous system to obtain equivalent discrete transfer function. These methods consume very low computation power.

2.1 BILINEAR TRANSFORM

The simplest method of discretization of the continuous system is to use bilinear (Tustin's) transform [1] according to:

$$\mathrm{p} = \frac{2}{T_{vz}} \frac{\mathrm{z}-1}{\mathrm{z}+1}, \tag{1}$$

where T_{vz} is sampling period, p is continuous p operator and z is discrete z operator. From this discrete transfer function we could compute a difference equation which is possible to implement into a microcontroller.

2.2 EQUIVALENT Z-TRANSFER FUNCTION

This method of discretization is based on discretization of the step-response of the continuous system. First step is to compute impulse response

$$h(t) = \mathscr{L}^{-1}\left\{\frac{1}{\mathrm{p}}F(\mathrm{p})\right\}, \tag{2}$$

where $h(t)$ is step-response, $\mathscr{L}^{-1}$ means inverse Laplace transform and $F(\mathrm{p})$ is our continuous transfer function. Second step id to sample this step-response

$$h(k) = h(t)|_{t=k\cdot T_{vz}}, \tag{3}$$

where T_{vz} is sampling period. In the next step we could obtain discrete transfer function

$$F(\mathrm{z}) = \frac{\mathrm{z}-1}{\mathrm{z}} \mathscr{Z}\{h(k)\}, \tag{4}$$

where $\mathscr{Z}$ means Z-transform. This final discrete transfer function can be transformed into difference equation which could be implemented into a microcontroller.

3 SYSTEM IMPLEMENTATION AS FIR FILTER

In this section we will look at the system as on the FIR filter. So we need to obtain impulse response, sample it and then finally compute convolution in microcontroller. Main disadvantage of this method is that this method consumes quite a lot of computation power.

First step in this method is to obtain impulse response

$$g(t) = \mathscr{L}^{-1}\{F(\mathrm{p})\}, \tag{5}$$

so we need to calculate inverse Laplace transform of our continuous transfer function. Second step is to sample our impulse response

$$g(k) = g(t)|_{t=k\cdot T_{vz}} \tag{6}$$

And finally we can filter input signal according to

$$y(k) = T_{vz} \sum_{i=0}^{N} g(k-i)u(i), \tag{7}$$

where N is length of $g(k)$ and $u(i)$ is input signal. As you can see, this method consumes quite a lot of computation power. On the other hand, this method could be used for implementation of the fractional order transfer functions, where another methods doest work well. How was written in [2, 3, 4] it is possible to obtain impulse response of fractional order transfer function.

4 EXAMPLE

Lets assume we have continuous system described

$$F(\mathrm{p}) = \frac{1}{\mathrm{p}+2}, \tag{8}$$

and we want to implement it into microcontroller. We will compare previous three method.

4.1 BILINEAR TRANSFORM

Lets substitute p in our transfer function (8):

$$F(z) = \frac{1}{\frac{2}{T_{vz}}\cdot\frac{\mathrm{z}-1}{\mathrm{z}+1}+2} = \frac{\mathrm{z}+1}{\left(\frac{2}{T_{vz}}+2\right)\mathrm{z}+2-\frac{2}{T_{vz}}} = \frac{1+\mathrm{z}^{-1}}{\left(\frac{2}{T_{vz}}+2\right)+\left(2-\frac{2}{T_{vz}}\right)\mathrm{z}^{-1}}. \tag{9}$$

From this discrete transfer function we can get difference equation

$$u(k)+u(k-1) = \left(\frac{2}{T_{vz}+2}\right)y(k) + \left(2-\frac{2}{T_{vz}}\right)y(k-1), \tag{10}$$

where T_{vz} is sampling period, $u(k)$ is input signal and $y(k)$ is output signal. This equation can be easily implemented into microcontroller.

4.2 EQUIVALENT Z-TRANSFER FUNCTION

For the first step we need to get step response of our transfer function (8):

$$h(t) = \mathscr{L}^{-1}\left\{\frac{1}{\mathrm{p}}\frac{1}{\mathrm{p}+2}\right\} = \frac{1}{2}\left(1-\mathrm{e}^{-2t}\right). \tag{11}$$

Now we can discretizate this step response

$$h(k) = h(t)|_{t=k\cdot T_{vz}} = \frac{1}{2}\left(1-\mathrm{e}^{-2t}\right)|_{t=k\cdot T_{vz}} = \frac{1}{2}\left(1-\mathrm{e}^{-2kT_{vz}}\right). \tag{12}$$

Now we can compute Z-transform of $h(k)$ and obtain discrete transfer function

$$F(\mathrm{z}) = \frac{\mathrm{z}-1}{\mathrm{z}}\mathscr{Z}\{h(k)\} = \frac{1}{2}\frac{\mathrm{z}-1}{\mathrm{z}}\left(\frac{\mathrm{z}}{\mathrm{z}-1} - \frac{\mathrm{z}}{\mathrm{z}-\mathrm{e}^{-2T_{vz}}}\right) = \frac{1}{2}\frac{\left(1-\mathrm{e}^{-2T_{vz}}\right)\mathrm{z}^{-1}}{1-\mathrm{e}^{-2T_{vz}}\mathrm{z}^{-1}}. \tag{13}$$

From this discrete transfer function we can calculate difference equation

$$y(k) = \frac{1}{2}\left(1-\mathrm{e}^{-2T_{vz}}\right)u(k-1) + \mathrm{e}^{-2T_{vz}}y(k-1), \tag{14}$$

where T_{vz} is sampling period, $u(k)$ is input signal and $y(k)$ is output signal. This equation can be easily implemented into microcontroller.

4.3 SYSTEM AS A FIR FILTER

In this subsection we compute impulse response, which will be used as a FIR filter. In this process we get

$$g(t) = g(t) = \mathscr{L}^{-1}\{F(\mathrm{p})\} = \mathscr{L}^{-1}\left\{\frac{1}{\mathrm{p}+2}\right\} = \mathrm{e}^{-2t}. \tag{15}$$

Now we can discretizate this impulse response.

$$g(k) = g(t)|_{t=k\cdot T_{vz}} = \mathrm{e}^{-2t}|_{t=k\cdot T_{vz}} = e^{-2kT_{vz}}. \tag{16}$$

So in this point we have coefficients of the FIR filter an now we can implement our discretized system into microcontroller. Final sampled (sampling period $T_{vz} = 10^{-3}s$) impulse response you can is in figure 1.

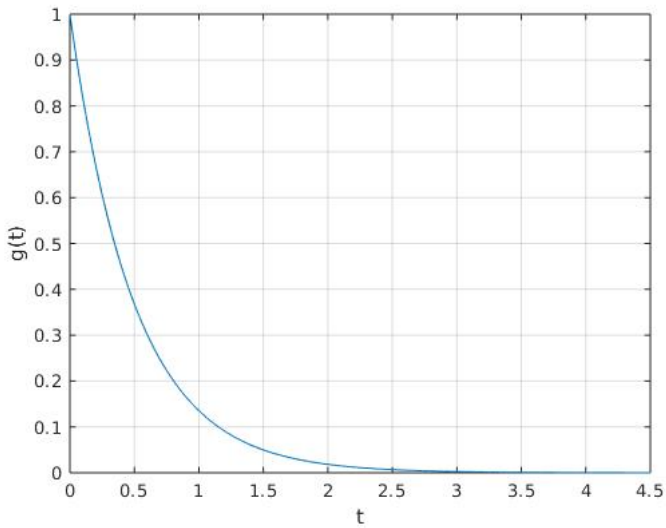

Figure 1: Impulse response

4.4 METHODS COMPARISON

To compare results of previously mentioned methods lets show response to unity step function

$$\sigma(k) = \begin{cases} 0 & \text{if } k < 0 \\ 1 & \text{if } k \geq 0 \end{cases} \tag{17}$$

which is used as an input signal to our systems. The sampling period T_{vz} for all systems was chosen to $T_{vz} = 10^{-3}s$. So the final difference equation for system discretized by bilinear transform is

$$y(k) = \frac{u(k) + u(k-1) + 1998y(k-1)}{2002}. \tag{18}$$

Final difference equation for system discretized using equivalent Z-transfer function is

$$y(k) = 9,99 \cdot 10^{-4} u(k-1) + 0.998y(k-1) \tag{19}$$

The response to unity step signal is showed in figure 2. As you can see the results are the same, so all of these methods work really well.

5 CONCLUSION

In this paper were shown methods to discretizate continuous system. The classical methods are easy to implement and they consume really low of computation power. On the other hand implement discretized continuous system as a FIR filter and compute discrete convolution consumes much more computation power but it allows us to implement not only integer order systems but also fractional order systems.

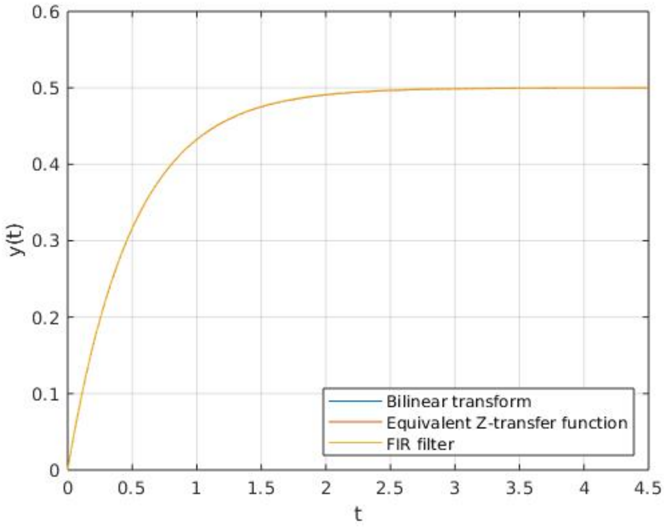

Figure 2: Response to input signal

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