COMPARISON OF METHODS FOR IMPULSE RESPONSE COMPUTATION

Vilem Karsky
Doctoral Degree Programme (4), FEEC BUT
E-mail: xkarsk010@stud.feec.vutbr.cz

Supervised by: Pavel Jura
E-mail: jura@feec.vutbr.cz

Abstract: This paper deals with obtaining impulse responses from integer order and fractional order transfer functions. There are shown three method how to compute inverse Laplace transform. The first method is based on Mittag-Leffler functions, the second method is formed on generalized Laguerre functions and the third method lays on Fourier transform. These methods are also compared on two examples.

Keywords: Mittag-Leffler functions, Generalized Laguerre functions, Fourierre transform, Fractional order transfer function, Inverse Laplace transform

1 INTRODUCTION

When we deal with systems we often need to obtain impulse response. This isn't huge problem for integer order systems but for fractional order systems it could be quite difficult. In this paper three methods for impulse response computation will be compared.

2 MATHEMATICAL BACKGROUND

In this section some basic concepts will be described. Also in this section methods for computation impulse response will be rely briefly outlined. In following section we will assume that our system is described with transfer function in form

$$F(s) = \frac{\sum_{m=0}^{M} b_m s^{qm}}{\sum_{n=0}^{N} a_n s^{qn}}, \tag{1}$$

where $q \in \mathbb{R}$.

2.1 Impulse Response

Impulse response is system response to dirac impulse. When we have system describer with transfer function, the impulse response could by obtained by calculating inverese Laplace transform of our transfer function. We can write

$$g(t) = \mathscr{L}^{-1}\left\{F(s)\right\}. \tag{2}$$

2.2 Direct Computation of Impulse Response

When we have system described with transfer function in form (1) we could split this transfer function into sum of partial fractions and transform each partial fraction according to formula [1]

$$\mathscr{L}\left\{t^{\alpha k+\beta-1} E^{(k)}_{\alpha,\beta}(\pm a t^{\alpha})\right\} = \frac{k! s^{\alpha-\beta}}{(s^{\alpha} \mp a)^{k+1}}, \tag{3}$$

where $E_{\alpha;\beta}(z)$ are Mittag-Leffler functions defined by

$$E_{\alpha;\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad [1]. \tag{4}$$

This method lead to solution in form of quite slow convergent infinite series.

2.3 Employing Generalized Laguerre Functions for Computation of Impulse Response

Generalized Laguerre functions we can define, according to [2], as

$$l_n^{\alpha}(t) = \sqrt{2\lambda}\mathrm{e}^{-\lambda t} L_n^{\alpha}(2\lambda t), \tag{5}$$

where $L_i^{\alpha}(t)$ are generalized Laguerre polynomials. Generalized Laguerre polynomials are defined as

$$L_n^{\alpha}(x) = \frac{\mathrm{e}^x x^{-\alpha}}{n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left(x^{n+\alpha} \mathrm{e}^{-x} \right) \quad [3]. \tag{6}$$

Generalized Laguerre functions generates orthogonal base in time space and also in operator space and spectrum coefficients of function $f(t)$ in generalized Laguerre functions base could be computed according to formula

$$c_n = \langle l_n^{\alpha}(t), f(t) \rangle = \langle \mathscr{L}\{l_n^{\alpha}(t)\}, \mathscr{L}\{f(t)\} \rangle, \tag{7}$$

where $\langle f(t), g(t) \rangle$ means scalar product of functions $f(t)$ and $g(t)$.

This property of generalized Laguerre functions could be used for computation inverse Laplace transform. As was shown in [2, 3] we could compute spectrum coefficients using formula

$$c_n^1 = \frac{-\sqrt{2\lambda}}{(n+1)!} \left[\frac{\mathrm{d}^{n+1}}{\mathrm{d}z^{n+1}} F(z) \right]_{z=0}, \tag{8}$$

where $F(z)$ is our transfer function after applying bilinear transform in form

$$s = \lambda \frac{1+z}{1-z} \tag{9}$$

on it. So final impulse response would be in form

$$g(t) = \sum_{n=0}^{\infty} c_n^1 l_n^1 = \sqrt{2\lambda}\mathrm{e}^{-\lambda t} \sum_{n=0}^{\infty} c_n^1 L_n^1(2\lambda t). \tag{10}$$

2.4 Fourier Method for Computation of Impulse Response

In paper [4] was shown method how to compute impulse response from transfer functions using Fourier transform. If we substitute s by $\mathrm{j}\omega$ in our transfer function we could rewrite our transfer function into

$$F(\mathrm{j}\omega) = \frac{M(\omega) + \mathrm{j}N(\omega)}{Q(\omega) + \mathrm{j}Z(\omega)}. \tag{11}$$

After that the impulse response could be calculated according to

$$g(t) = \frac{2}{\pi} \int_0^{\infty} \frac{M(\omega)Q(\omega) + N(\omega)Z(\omega)}{Q^2(\omega) + Z^2(\omega)} \cos(\omega t) \mathrm{d}t. \tag{12}$$

3 COMPARISON OF BOTH METHODS

For comparison of all three methods was chosen two transfer functions (first is integer order transfer function and second is fractional order transfer function).

3.1 INTEGER ORDER SYSTEM

Firs system is defined by this transfer function

$$F(s) = \frac{10}{s^2 + 4s + 8}. \tag{13}$$

Analytical solution of this transfer function is

$$g_a(t) = 5\mathrm{e}^{-2t}\sin(2t). \tag{14}$$

This impulse response is plotted in Figure 1 with blue line.

When we use formula (4) we get impulse response in form

$$g_m(t) = \frac{5}{2}\mathrm{j}\left[E_{1;1}(-(2+2\mathrm{j})t) - E_{1;1}(-(2-2\mathrm{j})t)\right]. \tag{15}$$

This impulse response can be modified to equation (14), but it is possible only for integer order system. In Figure 1 the impulse response, which was calculated for first 100 terms, is drawn with red dashed line.

Third way to get impulse response is by using Generalized Laguerre functions as mentioned earlier. This impulse response will be in form (10). For this system was employed only first 7 generalized Laguerre functions with $\lambda = 3.7465$. In Table 1 you can see spectrum coefficients. This impulse response $g_g(t)$ is plotted in Figure 1 with green dash-dotted line.

Table 1: The coefficients' spectrum: integer order system

i	0	1	2	3	4	5	6
c_i^1	1.7199	-1.1594	0.0505	0.2043	-0.0762	-0.0140	0.0199

And finally the fourth way is to employ Fourier transform as was said previously. Final result $g_w(t)$ is shown in Figure 1 with black dotted line.

For comparison of the approximations was calculated three differences $g_a(t) - g_m(t)$, $g_a(t) - g_g(t)$ and $g_a(t) - g_w(t)$. These differences are shown in Figure 2. You can see that MLF better approximate impulse response in first part but, then they diverge. In opposite GLF have some approximation error in the beginning, and then they are converging to $g(t)$. It is worth mention that for MLF was used 100 terms and for GLF was used only 7 terms. But the best solution is obtained by employing Fourier method.

3.2 FRACTIONAL ORDER SYSTEM

For fractional order transfer function was chosen quite similar system. This system is described

$$F(s) = \frac{10}{s^{1.2} + 4s^{0.6} + 8}. \tag{16}$$

If we use formula (4) to get impulse response, we get

$$g_m(t) = \frac{5}{2}\mathrm{j}t^{-0.4}\left[E_{0.6;0.6}(-(2+2j)t^{0.6} - E_{0.6;0.6}(-(2-2j)t^{0.6}\right]. \tag{17}$$

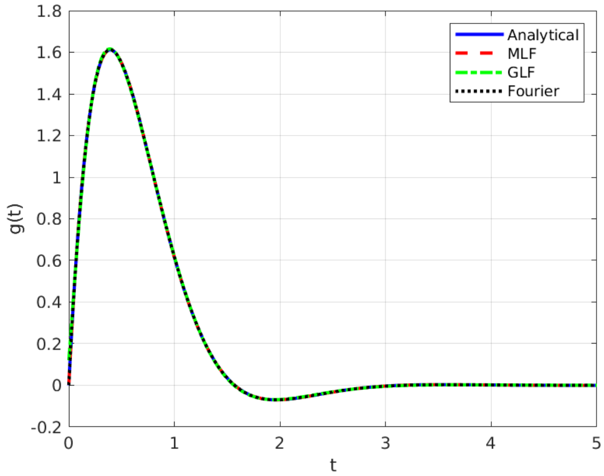

Figure 1: Impulse responses of the first system

You can see it in Figure 3 with blue line. For computation was used first 200 terms.

But when we employ first 7 GLF with $\lambda = 4.8697$ we get spectrum coefficients, which are in Table 2. It's plotted in Figure 3 with red dashed line.

Table 2: The coefficients' spectrum: fractional order system

i	0	1	2	3	4	5	6
c_i^1	1.4178	-0.2816	0.1962	-0.1093	0.0773	-0.0586	0.0430

And finally solution obtained using Fourier transform is shown in Figure 3 with green dotted line. As you can see in Figure 3 all methods give similar result, but MLF diverge.

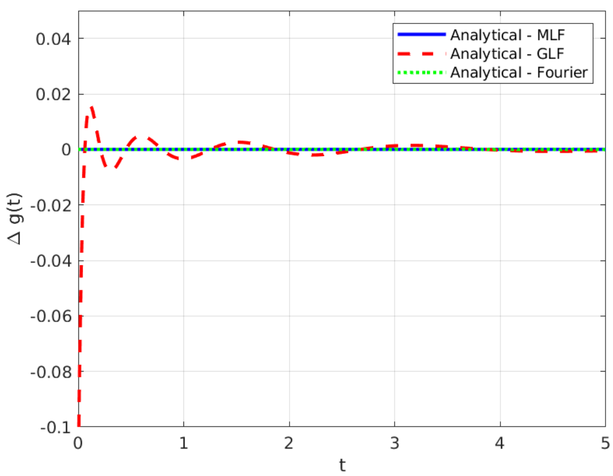

Figure 2: Differences from the $g_a(t)$

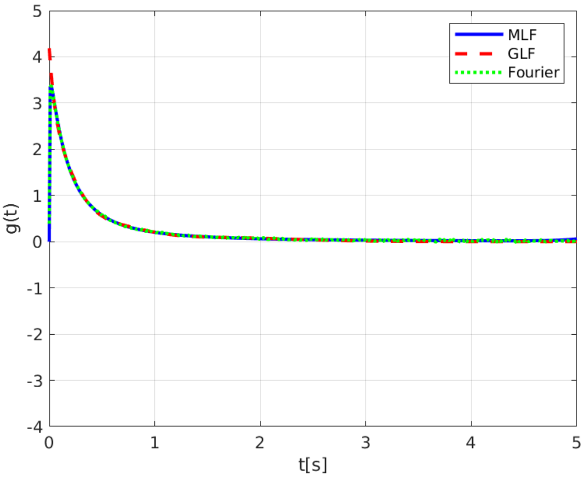

Figure 3: Impulse response of the second system

4 CONCLUSION

In this paper was shown that all methods are suitable for approximation impulse response. Solution obtained with MLF offers better results in the beginning of the impulse response but it diverges and needs a lot of terms. On the other hand solution using GLF converges and needs only a few terms but the approximation of the beginning of the impulse response is little worse. And solution using Fourier transform offers great results but this method could consume a lot of computation power to compute integrals numerically.

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