ASYMPTOTIC CHARACTERIZATION OF SOLUTIONS OF EMDEN-FOWLER TYPE DIFFERENCE EQUATION

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Abstract: The paper derives an asymptotic formula describing the long-time behaviour of a solution of a nonlinear Emden-Fowler type difference equation.

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1 INTRODUCTION

We discuss the asymptotic behaviour of a solution of Emden-Fowler type difference equation

\[ \Delta^2 u(k) \pm k^\alpha u^m(k) = 0, \]  \hspace{1cm} (1)

where \( k \in \mathbb{N}(k_0) := \{k_0, k_0+1, \ldots\}, k_0 \in \mathbb{N}, k_0 > 0, u: \mathbb{N}(k_0) \to \mathbb{R}, \Delta u(k) = u(k+1) - u(k) \) is the first and \( \Delta^2 u(k) \) the second difference with \( \alpha, m \in \mathbb{R}, m > 0, m \neq 1 \). When the + sign is assumed in (1), the below result holds only if \( m = p/q \) where \( p \) and \( q \) are integer numbers such that \( p - q \) is odd. Equation (1) is a difference analogue of the Emden-Fowler differential second-order equation, known from the theory of ordinary differential equations (we refer to [4]), being significant in astrophysics, cosmology, atomic physics, and other areas.

2 MAIN RESULT

We will prove that there exists a solution \( u = u(k) \) of (1) such that

\[ u(k) = ak^{-s} + bk^{-(s+1)} + O(k^{-(s+\gamma+1)}) \]  \hspace{1cm} (2)

when \( k \to \infty \) where

\[ a = \left\lceil s(s+1) \right\rceil^{1/(m-1)}, \quad b = as(s+2)/(s+2-ms), \quad s = (\alpha + 2)/(m-1), \]  \hspace{1cm} (3)

\( \gamma \in (0, 1) \) is a fixed number and \( O \) is the Landau order symbol big “O”. The equation (1) was investigated in [3] where the existence of a solution with asymptotic behaviour determined by formula (2) was proved under the assumption that \( s \in (-2, -1) \) and \( m < 0 \). Our main result below establishes the validity of formula (2) under a different set of conditions.

Theorem 1. Let \( s > 0 \). If

\[ m < \frac{(s+2)(s+3)}{s(s+1)} \]  \hspace{1cm} (4)

then there exists a solution \( u = u(k) \) of equation (1) defined on \( \mathbb{N}(k_0) \), where \( k_0 \) is sufficiently large, with asymptotic behaviour determined by the formula (2).
Proof. In the proof we will refer to some parts of the paper [3]. Let
\[ u(k) = a/k^s + b/(k^{s+1})(1 + Y_0(k)), \]
\[ \Delta u(k) = \Delta(a/k^s) + \Delta(b/k^{s+1})(1 + Y_1(k)), \]
\[ \Delta^2 u(k) = \Delta^2(a/k^s) + \Delta^2(b/k^{s+1})(1 + Y_2(k)) \]
where \( Y_i; \mathbb{N}(k_0) \to \mathbb{R}, i = 0, 1, 2 \) are new unknown functions. Then, equation (1) can be converted (for details we refer to [3, Part 3]) into a system
\[ \Delta Y_0(k) = F_1(k,Y_0,Y_1) := (-s+1)k^{-1} + O(k^{-2}) + Y_0(k) + Y_1(k), \]
\[ \Delta Y_1(k) = F_2(k,Y_0,Y_1) := (-s+2)k^{-1} + O(k^{-2})(ms + 2)^{-1}Y_0(k) - Y_1(k) + O(k^{-1}). \]

This will allow us to use [3, Lemma 2] (being a modification of [1, Theorem 8]) to prove Theorem 1. As the scheme of the proof is the same as that of Theorem 1 in [3], we will only emphasize in detail the parts that are different, referring the remaining parts to this source. Let \( \epsilon_i > 0, i = 1, \ldots, 4, \gamma > 0 \) and \( \beta > 0 \) be fixed. Define, as in [3, Part 4], auxiliary functions \( b_1(k) := -\epsilon_1/k^{\gamma}, c_1(k) := \epsilon_2/k^{\gamma}, b_2(k) := -\epsilon_3/k^{\beta} \) and \( c_2(k) := \epsilon_4/k^{\beta} \). To apply [3, Lemma 2], the following inequalities
\[ F_1(k,b_1(k),Y_1) < b_2(k+1) - b_1(k), \]
\[ F_1(k,c_1(k),Y_1) > c_2(k+1) - c_1(k), \]
\[ F_2(k,Y_0,b_2(k)) < b_2(k+1) - b_2(k), \]
\[ F_2(k,Y_0,c_2(k)) > c_2(k+1) - c_2(k) \]
must hold whenever \( -\epsilon_3 k^{-\beta} \leq Y_1 \leq \epsilon_4 k^{-\beta} \) in (8), (9) and \( -\epsilon_1 k^{-\gamma} \leq Y_0 \leq \epsilon_2 k^{-\gamma} \) in (10), (11). Let us find the conditions for validity of inequalities (8) – (11). As we assume \( m > 0 \) and \( s > 0 \), we have \( ms > 0, s + 1 > 0 \) obtaining:
\[ F_1(k,b_1(k),Y_1) = \left( -\frac{s+1}{k} + O\left( \frac{1}{k^2} \right) \right) \cdot \left( \frac{\epsilon_1}{k^{\gamma}} + Y_1(k) \right) < \left( -\frac{s+1}{k} + O\left( \frac{1}{k^2} \right) \right) \cdot \left( \frac{\epsilon_1}{k^{\gamma}} - \frac{\epsilon_3}{k^{\beta}} \right) \]
\[ < b_1(k+1) - b_1(k) = -\frac{\epsilon_1}{(k+1)^{\gamma}} + \frac{\epsilon_1}{k^{\gamma}} = \frac{\epsilon_1}{(k+1)^{\gamma}} \left( 1 + O\left( \frac{1}{k} \right) \right). \]

Then we have the following
\[ -\frac{s+1}{k^{\gamma+1}} \epsilon_1 + \frac{s+1}{k^{\beta+1}} \epsilon_3 + O\left( \frac{1}{k^{\beta+2}} \right) < \frac{\epsilon_1}{k^{\gamma+1}} \left( 1 + O\left( \frac{1}{k^{\gamma+2}} \right) \right) \]
or, after simplifying,
\[ \frac{s+1}{k^{\beta+1}} \epsilon_3 + O\left( \frac{1}{k^{\beta+2}} \right) < \frac{\epsilon_1}{k^{\gamma+1}} \left( 1 + O\left( \frac{1}{k^{\gamma+2}} \right) \right). \]

If \( k \to \infty \), then the last inequality hold either if \( \gamma < \beta \), or if \( \gamma = \beta \) and \( \epsilon_3 < \epsilon_1(\gamma + s + 1)/(s + 1) \). Similarly, inequality
\[ F_1(k,c_1(k),Y_1) = \left( -\frac{s+1}{k} + O\left( \frac{1}{k^2} \right) \right) \cdot \left( -\frac{\epsilon_2}{k^{\gamma}} + Y_1(k) \right) > \left( -\frac{s+1}{k} + O\left( \frac{1}{k^2} \right) \right) \cdot \left( -\frac{\epsilon_2}{k^{\gamma}} + \frac{\epsilon_4}{k^{\beta}} \right) \]
\[ > c_1(k+1) - c_1(k) = \frac{\epsilon_2}{(k+1)^{\gamma}} - \frac{\epsilon_2}{k^{\gamma}} = -\frac{\epsilon_2}{k^{\gamma}} \left( 1 + O\left( \frac{1}{k} \right) \right) \]

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implies either \( \gamma < \beta \), or \( \gamma = \beta \) and \( \varepsilon_4 < \varepsilon_2 (\gamma + s + 1)/(s + 1) \). Hence, both the above inequalities hold if either \( \gamma < \beta \), or if
\[
\gamma = \beta, \quad \varepsilon_3 < \varepsilon_1 \frac{s + 1}{s + 1}, \quad \varepsilon_4 < \varepsilon_2 \frac{s + 1}{s + 1}.
\]
In much the same way we estimate (recall that \( ms > 0 \)):
\[
F_2(k, Y_0, b_2(k)) = \left( -\frac{s + 2}{k} + O \left( \frac{1}{k^2} \right) \right) \cdot \left( \frac{ms}{s + 2} Y_0(k) + \frac{\varepsilon_3}{k^2} + O \left( \frac{1}{k} \right) \right) < \frac{ms}{k} \frac{s + 2}{k} \cdot \frac{\varepsilon_3}{k^2} + O \left( \frac{1}{k^2} \right).
\]
Then, inequality
\[
\frac{ms}{k^{r+1}} \varepsilon_1 - \frac{s + 2}{k^{r+1}} \varepsilon_3 + O \left( \frac{1}{k^{r+2}} \right) < \frac{\varepsilon_4}{k^{r+1}} \quad \text{and the system of inequalities}
\]
holds if either \( \gamma > \beta \) (this case is excluded), or if \( \gamma = \beta < 1 \) and \( \varepsilon_2 < \varepsilon_4 (\gamma + s + 2)/ms \). Summing up all the conditions, we get (except for \( m > 0, s > 0 \)) the system of inequalities
\[
0 < \gamma = \beta < 1, \quad \frac{\gamma + s + 1}{s + 1}, \quad \frac{\gamma + s + 1}{s + 1} < \frac{\gamma + s + 2}{ms}, \quad \frac{\gamma + s + 2}{ms}.
\]
Then, the necessary conditions for the solvability of this system are expressed by the below inequalities
\[
0 < \gamma = \beta < 1, \quad 1 < \frac{(\gamma + s + 1)(\gamma + s + 2)}{ms(s + 1)}.
\]
As \( ms(s + 1) > 0 \), it is easy to see that the last inequality is equivalent to the following one
\[
\gamma^2 + \gamma(2s + 3) + (s + 1)(s + 2 - ms) > 0.
\]
Consider the equation
\[
\gamma^2 + \gamma(2s + 3) + (s + 1)(s + 2 - ms) = 0.
\]
Its discriminant \( D \) is positive since \( D = (2s + 3)^2 - 4(s + 1)(s + 2 - ms) = 4ms(s + 1) + 1 > 0 \) and the roots \( \gamma_\pm \) of (15) are
\[
\gamma_\pm = \frac{1}{2} \left[ -(2s + 3) \pm \sqrt{4ms(s + 1) + 1} \right].
\]
Obviously \( \gamma_- < 0 \) and
\[
\gamma_+ < 1 \implies \sqrt{4ms(s + 1) + 1} < 2s + 5 \implies m < \frac{(s + 2)(s + 3)}{s(s + 1)}.
\]
The last inequality guarantees the existence of a \( \gamma \in (0, 1) \) such that inequalities (13) hold. Moreover, if a \( \gamma \in (0, 1) \) is fixed, then it is easy to show that there exist an \( \varepsilon_i > 0, i = 1, \ldots, 4 \) such that the system of inequalities (12) is satisfied. Then, by the above-mentioned results, system (6), (7) has a solution \((k, Y_1(k), Y_2(k)), k \in \mathbb{N}(k_0)\), where \( k_0 \) is sufficiently large, such that \( -\varepsilon_1 k^{-\gamma} \leq Y_0(k) \leq \varepsilon_2 k^{-\gamma}, -\varepsilon_3 k^{-\gamma} \leq Y_1(k) \leq \varepsilon_4 k^{-\gamma} \). Formula (2) follows from (5).
3 COROLLARY TO MAIN RESULT

In the following corollary, values of $\alpha$ and $m$ are specified such that Theorem 1 holds.

**Corollary 1.** If $s > 0$ and either
i) $0 < m < 1$ and $\alpha < -2$
or
ii) $m > 1$ and
\[-2 < \alpha < \frac{1}{2} \left[-(m-1) + \sqrt{(m-1)^2 + 16m}\right],\]
then the conclusion of Theorem 1 holds.

**Proof.** Assumption (4) of Theorem 1 holds if
\[ms(s+1) < s^2 + 5s + 6.\]
According to formula (3), this inequality will be valid if
\[m(\alpha+2)(\alpha+m+1) < (\alpha+2)^2 + 5(\alpha+2)(m-1) + 6(m-1)^2.\]
This inequality, after some computations, turns into
\[(m-1) \left[\alpha^2 + \alpha(m-1) - 4m\right] < 0. \quad (16)\]
As $m > 0$, inequality (16) will hold if either
\[0 < m < 1, \; \alpha^2 + \alpha(m-1) - 4m > 0 \quad (17)\]
or
\[m > 1, \; \alpha^2 + \alpha(m-1) - 4m < 0. \quad (18)\]
First, analyze the inequality (17). Since $s > 0$, (3) implies $\alpha < -2$ and
\[\alpha^2 + \alpha(m-1) - 4m = (\alpha-m)(\alpha+2m) + 2(m-1)^2 - (2+\alpha) > 0.\]
That is, in the case i), inequality (16) holds and, consequently, assumption (4) of Theorem 1 is valid.

Next, analyze the inequality (18). In this case, $m > 1$ and (3) implies $\alpha > -2$. Considering the equation
\[\alpha^2 + \alpha(m-1) - 4m = 0\]
we find its roots
\[\alpha_{\pm} = \frac{1}{2} \left[-(m-1) \pm \sqrt{(m-1)^2 + 16m}\right].\]
Let us show that $\alpha_- < -2$. Obviously, $(m-1)^2 + 16m > 16$ and
\[-\sqrt{(m-1)^2 + 16m} < -4.\]
Therefore,
\[\alpha_- = \frac{1}{2} \left[-(m-1) - \sqrt{(m-1)^2 + 16m}\right] < -2.\]
Now, show that $\alpha_+ > -2$. This inequality is equivalent with
\[\sqrt{(m-1)^2 + 16m} > (m-1) - 4,\]
which holds for $1 < m \leq 5$ and, taking the second power, for $m > 5$, is equivalent with
\[(m-1)^2 + 16m > (m-1)^2 - 8(m-1) + 16,\]
which obviously holds as well. We conclude that the second inequality in (18) holds if $-2 < \alpha < \alpha_+$. Thus, the case ii) holds as well.

\[\square\]
4 EXAMPLE

Let $m = 2$, $\alpha = 1$. Then, equation (1) takes the form

$$\Delta^2 u(k) \pm ku^2(k) = 0. \quad (19)$$

By (3), we obtain $s = 3$, $a = \mp 12$, $b = \pm 80$. Condition (4) is satisfied since

$$m < (s+2)(s+3)/s(s+1) = 2.5$$

and Theorem 1 is applicable. Therefore, by formula (2), there exist two solutions to equation (1) with the asymptotic behaviour

$$u(k) = \mp 12k^{-3} \pm 80k^{-4} + O(k^{-(4+n)})$$

when $k \to \infty$ where $\gamma \in (0, 1)$ is a fixed number.

5 CONCLUSION

The results obtained generalize those published in [2] where only one case of equation (19) with the term $-ku^2(k)$ was considered. The method used seems to be efficient for further investigation of the behaviour of solutions to Emden-Fowler types of discrete equations. For example, it can be expected that new results will be achieved if the constants $\alpha$ and $m$ satisfy sets of assumptions different from those described by Theorem 1. Another challenge for further investigation is the following one. Using a discretization suggested by the forward difference formula

$$\Delta f(k) = \frac{f(k+h) - f(k)}{h},$$

where $f$ is a well-defined function and $h$ is a step of discretization, the Emden-Fowler differential second-order equation can be transformed into a difference one. Analyze the asymptotic behaviour of the solutions to a derived difference equation and show that, for $h \to 0$, formulas describing the asymptotic behaviour give formulas known for differential Emden-Fowler equation.

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