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EXISTENCE OF SOLUTIONS FOR A CLASS OF SECOND-ORDER BOUNDARY VALUE PROBLEMS

HADI HAGHSHENAS AND GHASEM A. AFROUZI

Abstract. We employ some known critical point theorems to establish results on the existence of weak solutions for an impulsive boundary value problem depending on two real parameters. One of the results ensures the existence of at least three weak solutions, while another one proves the existence of at least one.

1. Introduction

Many dynamical systems that describe models in applied sciences have an impulsive dynamical behavior due to abrupt changes at certain instants during the evolution process. The rigorous mathematical description of these phenomena leads to impulsive differential equations, which characterize various processes of the real world described by models that are subject to sudden changes in their states.

In the last few years, variational methods have been used to determine the existence of solutions for impulsive differential equations possessing variational structures under certain boundary conditions. See, for instance, [2, 3, 7, 9–11] and the references therein for detailed discussions.

In this paper, we study the second-order, impulsive boundary value problem

$$\begin{cases} -u''(t)+a(t)u'(t)+b(t)u(t)=\lambda g(t,u(t))+\mu k(t,u(t)),\ t\in[0,T],\ t\neq t_j,\\ \triangle(u'(t_j))=I_j(u(t_j)),\ j=1,2,\dots,m,\\ u(0)=u(T)=0, \end{cases} \tag{1.1}$$

where $\lambda\in]0,+\infty[$, $\mu\in[0,+\infty[$, and $a,b\in L^\infty\big([0,T]\big)$ satisfy the conditions

$$\operatorname*{ess\,inf}_{t\in[0,T]} a(t)\geq 0,\quad \operatorname*{ess\,inf}_{t\in[0,T]} b(t)\geq 0.$$

Also, the points $t_j,\ j=1,2,\dots,m$, are the instants where the impulses occur, $0=t_0<t_1<\cdots<t_m<t_{m+1}=T$, and $\triangle\big(u'(t_j)\big)=u'(t_j^+)-u'(t_j^-)=\lim_{t\to t_j^+}u'(t)-\lim_{t\to t_j^-}u'(t)$. Moreover, $I_j:\mathbb{R}\to\mathbb{R}$ is continuous for every $j\in\{1,2,\dots,m\}$, and $g,k:[0,T]\times\mathbb{R}\to\mathbb{R}$ are L^1-Carathéodory functions.

We emphasize that in [1], under assumptions similar to those of our results, the authors established the existence of infinitely many solutions for (1.1).

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Bonanno *et al.* proved in [5] the existence of at least three solutions for the problem

$$\begin{cases} -u''(t)+a(t)u'(t)+b(t)u(t)=\lambda g(t,u(t)),\ t\in[0,T],\ t\neq t_j,\\ \triangle(u'(t_j))=\mu I_j(u(t_j)),\ j=1,2,\dots,m,\\ u(0)=u(T)=0, \end{cases}$$

where $\lambda,\mu\in]0,+\infty[$, $g:[0,T]\times\mathbb{R}\to\mathbb{R}$, and $a,b\in L^\infty\big([0,T]\big)$. Also, $0=t_0<t_1<\dots<t_m<t_{m+1}=T$, $\triangle\big(u'(t_j)\big)=u'(t_j^+)-u'(t_j^-)$, and $I_j:\mathbb{R}\to\mathbb{R}$ is continuous for every $j\in\{1,2,\dots,m\}$.

In [6], the authors established the existence of at least one solution for the problem

$$\begin{cases} -u''(t)+a(t)u'(t)+b(t)u(t)=\lambda g(t,u(t))+k(u(t)),\ t\in[0,T],\ t\neq t_j,\\ \triangle(u'(t_j))=I_j(u(t_j)),\ j=1,2,\dots,m,\\ u(0)-u(T)=u'(0)-u'(T)=0, \end{cases}$$

where $\lambda\in]0,+\infty[$, $g:[0,T]\times\mathbb{R}\to\mathbb{R}$, and $a,b\in L^\infty\big([0,T]\big)$. Also, $0=t_0<t_1<\dots<t_m<t_{m+1}=T$, $\triangle\big(u'(t_j)\big)=u'(t_j^+)-u'(t_j^-)$, $I_j:\mathbb{R}\to\mathbb{R}$ is continuous for every $j\in\{1,2,\dots,m\}$, and $k:\mathbb{R}\to\mathbb{R}$ is a Lipschitz continuous function.

Compared to the results obtained in [5] and [6], we provide some new assumptions that ensure the existence of weak solutions for (1.1). More precisely, in one of our main results we establish the existence of at least three weak solutions for (1.1), while in the other theorem we prove the existence of at least one such solution.

The paper is organized as follows. In Section 2, we recall some basic concepts and results that constitute our main tools. Section 3 is devoted to our main results.

2. Preliminaries

To ensure the existence of weak solutions for problem (1.1), we use the following critical point theorems as our main tools. The proofs of these results can be found in [4] and [8], respectively.

Theorem 2.1. ([4]) *Let X be a real, reflexive Banach space, $\phi:X\to\mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, and $\psi:X\to\mathbb{R}$ be a sequentially weakly upper semicontinuous, continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that*

$$\phi(0)=\psi(0)=0.$$

Assume that there exist $r>0$ and $\bar{x}\in X$, with $r<\phi(\bar{x})$, such that

(i) $\sup_{\phi(x)\le r}\psi(x)<r\frac{\psi(\bar{x})}{\phi(\bar{x})}$, *and*

(ii) *for each λ in*

$$\Lambda_r=\left]\frac{\phi(\bar{x})}{\psi(\bar{x})},\frac{r}{\sup_{\phi(x)\le r}\psi(x)}\right[,$$

the functional $\phi - \lambda\psi$ is coercive. Then for each $\lambda \in \Lambda_r$, the functional $\phi - \lambda\psi$ has at least three distinct critical points in X.

Theorem 2.2. ([8]) *Let X be a real, reflexive Banach space. Also, let $\phi, \psi : X \to \mathbb{R}$ be Gâteaux differentiable functionals such that ϕ is sequentially weakly lower semicontinuous, strongly continuous and coercive in X, and ψ is sequentially weakly upper semicontinuous in X. Let $r > \inf_X \phi$ and φ be the function defined by*

$$\varphi(r) = \inf_{u\in\phi^{-1}(]-\infty,r[)} \frac{\sup_{v\in\phi^{-1}(]-\infty,r[)} \psi(v) - \psi(v)}{r - \phi(u)}. \tag{2.1}$$

Then, for any $\lambda \in]0, 1/\varphi(r)[$, the restriction of functional $\phi - \lambda\psi$ to $\phi^{-1}\big(]-\infty, r[\big)$ admits a global minimum, which is a critical point of $\phi - \lambda\psi$ in X.

We consider the following problem, which is slightly different from problem (1.1).

$$\begin{cases} -(p(t)u'(t))' + q(t)u(t) = \lambda f(t, u(t)) + \mu h(t, u(t)),\ t \in [0, T],\ t \neq t_j, \\ \triangle(u'(t_j)) = I_j(u(t_j)),\ j = 1, 2, \dots, m, \\ u(0) = u(T) = 0. \end{cases} \tag{2.2}$$

Herein, $\lambda \in\ ]0, +\infty[$, $\mu \in [0, +\infty[$, $p \in C^1\big([0, T], [0, +\infty[\big)$, and $q \in L^\infty\big([0, T]\big)$ satisfies $\operatorname{ess\,inf}_{t\in[0,T]} q(t) \geq 0$. Also, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\triangle\big(u'(t_j)\big) = u'(t_j^+) - u'(t_j^-) = \lim_{t\to t_j^+} u'(t) - \lim_{t\to t_j^-} u'(t)$, $I_j : \mathbb{R} \to \mathbb{R}$ is continuous for every $j \in \{1, 2, \dots, m\}$, and $f, h : [0, T] \times \mathbb{R} \to \mathbb{R}$ are L^1-Carathéodry functions.

It is easy to see that the solutions of problem (2.2) are those of problem (1.1) if

$$p(t) = e^{-\int_0^t a(s)ds},\ q(t) = b(t)e^{-\int_0^t a(s)ds},$$

$$f(t, u) = g(t, u)e^{-\int_0^t a(s)ds},\ h(t, u) = k(t, u)e^{-\int_0^t a(s)ds}.$$

By analogy to f and h, we introduce the potential functions F and H from $[0, T]\times\mathbb{R}$ into $\mathbb{R}$ by

$$F(t, x) = \int_0^x f(t, \xi)d\xi,\ H(t, x) = \int_0^x h(t, \xi)d\xi,$$

for all $(t, x) \in [0, T] \times \mathbb{R}$. In the Sobolev space $H_0^1\big(]0, T[\big)$, consider the inner product

$$(u, v) = \int_0^T p(t)u'(t)v'(t)dt + \int_0^T q(t)u(t)v(t)dt,$$

which induces the norm

$$||u|| = \left(\int_0^T p(t)(u^{'})^2(t)dt + \int_0^T q(t)u^2(t)dt\right)^{\frac{1}{2}}.$$

Then, the following Poincaré-type inequality holds.

$$\left[\int_0^T u^2(t)dt\right]^{\frac{1}{2}} \leq \frac{T}{\pi}\left[\int_0^T (u')^2(t)dt\right]^{\frac{1}{2}}. \tag{2.3}$$

Proposition 2.3. *If* $u \in H_0^1\big(]0,T[\big)$, *then*

$$||u||_\infty \leq \frac{1}{2}\sqrt{\frac{T}{p^*}}||u||, \tag{2.4}$$

where $||u||_\infty = \max_{t\in[0,T]} |u(t)|$ *and* $p^* = \min_{t\in[0,T]} p(t)$.

Proof. In view of Hölder's inequality,

$$||u||_\infty \leq \frac{\sqrt{T}}{2}||u'||_{L^2\big([0,T]\big)} \leq \frac{1}{2}\sqrt{\frac{T}{p^*}}||u||.$$

□

Definition 2.4. A function $u \in H_0^1\big(]0,T[\big)$ is said to be a weak solution of problem (2.2) if it satisfies

$$\int_0^T p(t)u'(t)v'(t)dt + \int_0^T q(t)u(t)v(t)dt - \lambda\int_0^T f(t,u(t))v(t)dt$$
$$- \mu\int_0^T h(t,u(t))v(t)dt + \sum_{j=1}^m p(t_j)I_j(u(t_j))v(t_j) = 0,$$

for any $v \in H_0^1\big(]0,T[\big)$.

Here, we define the functionals ϕ and ψ from $H_0^1\big(]0,T[\big)$ into $\mathbb{R}$ by

$$\begin{aligned}\phi(u) &= \frac{1}{2}||u||^2,\\ \psi(u) &= \int_0^T J(t,u(t))dt - \frac{1}{\lambda}\sum_{j=1}^m p(t_j)\int_0^{u(t_j)} I_j(x)dx,\end{aligned} \tag{2.5}$$

where $J(t,x) = F(t,x) + \frac{\mu}{\lambda}H(t,x)$, for each $(t,x) \in [0,T]\times\mathbb{R}$. Using the properties of f and h, and the continuity of I_j for $j = 1,2,\dots,m$, we find that $\phi,\psi \in C^1\big(H_0^1(]0,T[),\mathbb{R}\big)$, and that for every $v \in H_0^1\big(]0,T[\big)$,

$$\phi'(u)(v) = \int_0^T p(t)u'(t)v'(t)dt + \int_0^T q(t)u(t)v(t)dt,$$

and

$$\psi'(u)(v) = \int_0^T f(t,u(t))v(t)dt + \frac{\mu}{\lambda}\int_0^T h(t,u(t))v(t)dt - \frac{1}{\lambda}\sum_{j=1}^m p(t_j)I_j(u(t_j))v(t_j).$$

So, using standard arguments, we deduce that the critical points of the functional

$$E_{\lambda,\mu}(u) = \phi(u) - \lambda\psi(u),\ u \in H_0^1\big(]0,T[\big),$$

are the weak solutions of problem (2.2).

3. The main results

In this section, we formulate and prove our main results.

Lemma 3.1. ([5, Lemma 3.1]) *Assume that*

(H1) *there exist constants* $\alpha, \beta > 0$ *and* $\sigma \in [0, 1[$ *such that* $|I_j(x)| \leq \alpha + \beta |x|^\sigma$ *for all* $x \in \mathbb{R}$, $j = 1, 2, \ldots, m$.

Then, for any $u \in H_0^1\big(]0, T[\big)$,

$$\Big|\sum_{j=1}^{m} p(t_j) \int_0^{u(t_j)} I_j(x)dx\Big| \leq \sum_{j=1}^{m} p(t_j)\Big(\alpha ||u||_\infty + \frac{\beta}{\sigma+1}||u||_\infty^{\sigma+1}\Big).$$

Now, let

$$\tilde{p} = \sum_{j=1}^{m} p(t_j),\ s = \frac{6p^*}{12||p||_\infty + T^2||q||_\infty},\ \Gamma_c = \frac{\alpha}{c} + (\frac{\beta}{\sigma+1})c^{\sigma-1},$$

where α, β, σ are given by (H1), and c is a positive constant.

Theorem 3.2. *Suppose that* (H1) *holds. Furthermore, assume that there exist positive constants* c, d, *with* $c < d$, *such that*

(H2) $F(t, \xi) \geq 0$ *for all* $(t, \xi) \in \big([0, \frac{T}{4}] \cup [\frac{3T}{4}, T]\big) \times [0, d]$;

(H3) $H(t, \xi) \geq 0$ *for all* $(t, \xi) \in [0, T] \times \mathbb{R}^+$;

(H4) $M < \left(\frac{2sp^*}{2p^* + \tilde{p}T\Gamma_{(\frac{d}{\sqrt{s}})}}\right)B$, *where*

$$M = \frac{\int_0^T \max_{|\xi| \leq c} F(t, \xi)dt}{c^2},\ B = \frac{\int_{\frac{T}{4}}^{\frac{3T}{4}} F(t, d)dt}{d^2};$$

(H5) $\limsup_{|\xi| \to +\infty} \frac{\sup_{t \in [0,T]} F(t,\xi)}{\xi^2} \leq \frac{\pi^2}{4T} \frac{\int_0^T \max_{|\xi| \leq c} F(t,\xi)dt}{c^2}$;

(H6) $N = \sup_{t \in [0,T], \xi \in \mathbb{R}^+} H(t, \xi) < +\infty$.

If $D = \frac{\int_0^T \max_{|\xi| \leq c} H(t,\xi)dt}{c^2}$ *is a positive real number, then for every*

$$\lambda \in \Lambda = \Big]\frac{2p^* + \tilde{p}T\Gamma_{(\frac{d}{\sqrt{s}})}}{sTB}, \frac{2p^*}{TM}\Big[,$$

and for each $\mu \in [0, \delta[$, *where*

$$\delta = \frac{1}{TD}\Big(2p^* - \lambda TM - \tilde{p}T\Gamma_c\Big),$$

problem (2.2) *has at least three distinct, weak solutions in* $H_0^1\big(]0, T[\big)$.

Proof. First, we observe that, due to (H4), the interval Λ is non-empty. Now, let λ and μ be as in the conclusion. Our aim is to apply Theorem 2.1 to problem (2.2). To do so, let $X = H_0^1\big(]0, T[\big)$ and assume that ϕ and ψ are as in (2.5). We first observe that the functionals ϕ and ψ satisfy the regularity assumptions of Theorem 2.1. Note that $\phi(0) = \psi(0) = 0$. Let $r = \frac{2c^2p^*}{T}$ and define

$$\bar{v}(t) = \begin{cases} \frac{4d}{T}t, & t \in \left[0, \frac{T}{4}\right], \\ d, & t \in \left]\frac{T}{4}, \frac{3T}{4}\right], \\ \frac{4d}{T}(T-t), & t \in \left]\frac{3T}{4}, T\right]. \end{cases}$$

Clearly, $\bar{v} \in X$ and

$$\phi(\bar{v}) \geq \frac{p^*}{2}\int_0^T (\bar{v}')^2(t)dt = \frac{4p^*d^2}{T}.$$

So, from $c < \sqrt{2}d$, we obtain $r < \phi(\bar{v})$. Next, we show that

$$\frac{\sup_{\phi(u)\leq r}\psi(u)}{r} < \frac{1}{\lambda} < \frac{\psi(\bar{v})}{\phi(\bar{v})}.$$

Taking (2.4) into account, for every $u \in X$ such that $\phi(u) \leq r$ one obtains $\max_{t\in[0,T]}|u(t)| \leq c$. Consequently, from Lemma 3.1 it follows that

$$\sup_{\phi(u)\leq r}\psi(u) \leq \int_0^T \max_{|x|\leq c} J(t,x)dt + \frac{1}{\lambda}\tilde{p}\left(\alpha c + \frac{\beta}{\sigma+1}c^{\sigma+1}\right),$$

that is,

$$\frac{\sup_{\phi(u)\leq r}\psi(u)}{r} < \frac{T}{2p^*}\left(M + \frac{\mu}{\lambda}D + \frac{1}{\lambda}\tilde{p}\Gamma_c\right).$$

Hence, having $\mu < \delta$ in mind, we obtain

$$\frac{\sup_{\phi(u)\leq r}\psi(u)}{r} < \frac{1}{\lambda}. \tag{3.1}$$

On the other hand,

$$\begin{aligned} ||\bar{v}||^2 &= \int_0^T p(t)(\bar{v}')^2(t)dt + \int_0^T q(t)(\bar{v})^2(t)dt \\ &\leq \frac{2d^2(12||p||_\infty + T^2||q||_\infty)}{3T} \\ &= \frac{4p^*d^2}{sT}. \end{aligned} \tag{3.2}$$

So, $\phi(\bar{v}) \leq \frac{2p^*d^2}{sT}$. Now, Lemma 3.1, (H2), (H3), (2.4) and (3.2) allow us to write

$$\begin{aligned} \psi(\bar{v}) &\geq \int_{\frac{T}{4}}^{\frac{3T}{4}} F(t,d)dt - \frac{1}{\lambda}\tilde{p}\big(\alpha||\bar{v}||_\infty + \frac{\beta}{\sigma+1}||\bar{v}||_\infty^{\sigma+1}\big) \\ &\geq \int_{\frac{T}{4}}^{\frac{3T}{4}} F(t,d)dt - \frac{1}{\lambda}\frac{\tilde{p}d^2}{s}\Gamma_{(\frac{d}{\sqrt{s}})}. \end{aligned}$$

Thus, we obtain

$$\frac{\psi(\bar{v})}{\phi(\bar{v})} \geq \frac{sT\int_{\frac{T}{4}}^{\frac{3T}{4}} F(t,d)dt - \frac{1}{\lambda}\tilde{p}Td^2\Gamma_{(\frac{d}{\sqrt{s}})}}{2p^*d^2}.$$

Since $\lambda > \frac{2p^* + \tilde{p}T\Gamma_{(\frac{d}{\sqrt{s}})}}{sTB}$,

$$\frac{\psi(\bar{v})}{\phi(\bar{v})} > \frac{1}{\lambda}. \tag{3.3}$$

Therefore, it follows from (3.1) and (3.3) that condition (i) of Theorem 2.1 is satisfied. To prove the coercivity of the functional $E_{\lambda,\mu}$, we note that by (H5) and the inequality $\lambda < \frac{2p^*}{TM}$,

$$\limsup_{|\xi|\to+\infty} \frac{\sup_{t\in[0,T]} F(t,\xi)}{\xi^2} < \left(\frac{\pi^2 p^*}{2T^2}\right)\frac{1}{\lambda}.$$

So, we can fix $\epsilon > 0$ satisfying

$$\limsup_{|\xi|\to+\infty} \frac{\sup_{t\in[0,T]} F(t,\xi)}{\xi^2} < \epsilon < \left(\frac{\pi^2 p^*}{2T^2}\right)\frac{1}{\lambda}.$$

Then, there exists a positive constant w such that

$$F(t,\xi) \le \epsilon|\xi|^2 + w, \quad \forall t \in [0,T],\ \forall \xi \in \mathbb{R}. \tag{3.4}$$

It follows from Lemma 3.1, (2.3), (2.4), (3.4) and (H6) that

$$\begin{aligned}
\phi(u) - \lambda\psi(u) &= \frac{1}{2}||u||^2 - \lambda\int_0^T F(t,u(t))dt \\
&\quad -\mu\int_0^T H(t,u(t))dt + \sum_{j=1}^m p(t_j)\int_0^{u(t_j)} I_j(x)dx \\
&\ge \frac{1}{2}||u||^2 - \lambda\epsilon||u||^2_{L^2([0,T])} - \lambda wT - \mu NT \\
&\quad - \tilde{p}\left[\alpha\frac{1}{2}\sqrt{\frac{T}{p^*}}||u|| + \frac{\beta}{\sigma+1}\left(\frac{1}{2}\sqrt{\frac{T}{p^*}}\right)^{\sigma+1}||u||^{\sigma+1}\right] \\
&\ge \left(\frac{1}{2} - \lambda\epsilon\frac{T^2}{\pi^2 p^*}\right)||u||^2 - \lambda wT - \mu NT \\
&\quad - \tilde{p}\left[\alpha\frac{1}{2}\sqrt{\frac{T}{p^*}}||u|| + \frac{\beta}{\sigma+1}\left(\frac{1}{2}\sqrt{\frac{T}{p^*}}\right)^{\sigma+1}||u||^{\sigma+1}\right],
\end{aligned}$$

for all $u \in H_0^1(]0,T[)$. So, the functional $E_{\lambda,\mu}$ is coercive. Therefore, by Theorem 2.1, the functional $E_{\lambda,\mu}$ has at least three distinct critical points in X. This completes the proof. □

Let $A(t)$ be a primitive of $a(t)$, $\tilde{s} = \frac{6e^{-A(T)}}{12+T^2||be^{-A}||_\infty}$, and

$$G(t,\xi) = \int_0^\xi g(t,x)dx, \quad K(t,\xi) = \int_0^\xi k(t,x)dx.$$

Now, we obtain the following multiplicity property for problem (1.1).

Theorem 3.3. *Suppose that* (H1) *holds. Furthermore, assume that there exist positive constants* c *and* d*, with* $c < d$*, such that*

(H2') $G(t,\xi) \ge 0$ *for all* $(t,\xi) \in \left([0,\frac{T}{4}] \cup [\frac{3T}{4},T]\right) \times [0,d]$;

(H3') $K(t,\xi) \geq 0$ *for all* $(t,\xi) \in [0,T] \times \mathbb{R}^+$*;*
(H4') $M' < \left(\frac{2\tilde{s}e^{-||a||_1}}{2e^{-||a||_1}+T\Gamma_{(\frac{d}{\sqrt{\tilde{s}}})}}\right)B'$*, where*

$$M' = \frac{\int_0^T e^{-A(t)} \max_{|\xi|\leq c} G(t,\xi)dt}{c^2}, \quad B' = \frac{\int_{\frac{T}{4}}^{\frac{3T}{4}} e^{-A(t)}G(t,d)dt}{d^2};$$

(H5') $\limsup_{|\xi|\to+\infty} \frac{\sup_{t\in[0,T]} e^{-A(t)}G(t,\xi)}{\xi^2} \leq \frac{\pi^2}{4T} \frac{\int_0^T e^{-A(t)} \max_{|\xi|\leq c} G(t,\xi)dt}{c^2}$*;*
(H6') $N' = \sup_{t\in[0,T],\xi\in\mathbb{R}^+} \ e^{-A(t)}K(t,\xi) < +\infty$.

If $D' = \frac{\int_0^T e^{-A(t)} \max_{|\xi|\leq c} K(t,\xi)dt}{c^2}$ *is a positive real number, then for every*

$$\lambda \in \left]\frac{2e^{||a||_1} + T\Gamma_{(\frac{d}{\sqrt{\tilde{s}}})}}{\tilde{s}TB'}, \frac{2e^{-||a||_1}}{TM'}\right[,$$

and for every $\mu \in [0,\delta[$*, where*

$$\delta = \frac{1}{TD'}\left[2e^{-||a||_1} - \lambda TM' - T\Gamma_c\right],$$

problem (1.1) *has at least three distinct weak solutions in* $H_0^1\big(]0,T[\big)$*.*

The proof of the above theorem follows from Theorem 3.2 by choosing

$$p(t) = e^{-\int_0^t a(s)ds}, \quad q(t) = b(t)e^{-\int_0^t a(s)ds},$$

and also

$$f(t,u) = g(t,u)e^{-\int_0^t a(s)ds}, \quad h(t,u) = k(t,u)e^{-\int_0^t a(s)ds}.$$

Here, we employ Theorem 2.2 to ensure the existence of at least one non-trivial solution to problem (2.2).

Theorem 3.4. *Suppose that* (H1) *holds. Furthermore, assume that*
(M2) $F(t,x) \geq 0$ *for all* $(t,x) \in [0,T] \times \mathbb{R}^+$*;*
(M3) $H(t,x) \geq 0$ *for all* $(t,x) \in [0,T] \times \mathbb{R}^+$*;*
(M4) V *and* W *are non-negative real numbers, where*

$$V = \frac{\int_0^T \max_{|x|\leq\pi} F(t,x)dt}{\pi^2}, \quad W = \frac{\int_0^T \max_{|x|\leq\pi} H(t,x)dt}{\pi^2}.$$

If η *and* γ *are positive real numbers and*

$$\gamma < \xi = \frac{2p^*}{T}\left[\frac{1}{V + \frac{\eta}{\gamma}W + \frac{1}{\gamma}\tilde{p}\Gamma_\pi}\right], \tag{3.5}$$

then for every $\mu \in]0,\eta[$ *and for each* $\lambda \in \Lambda' =]\gamma,\xi[$*, problem* (2.2) *has at least one weak solution.*

Proof. First, we observe that due to (3.5), the interval Λ' is non-empty. Now, let μ and λ be as in the conclusion. Our aim is to apply Theorem 2.2 to problem (2.2). To do so, let $X = H_0^1\big(]0,T[\big)$ and assume that ϕ and ψ are as in (2.5). We first observe that the functionals ϕ and ψ satisfy the regularity assumptions of Theorem 2.2. Note that $\inf_X \phi = 0$. For every $r > 0$, define $\varphi(r)$ as in (2.1). Since $0 \in \phi^{-1}\big(]-\infty, r[\big)$ and $\phi(0) = \psi(0) = 0$,

$$\varphi(r) = \inf_{u \in \phi^{-1}\left(]-\infty,r[\right)} \frac{\sup_{v \in \phi^{-1}\left(]-\infty,r[\right)} \psi(v) - \psi(u)}{r - \phi(u)} \leq \frac{\sup_{v \in \phi^{-1}\left(]-\infty,r[\right)} \psi(v)}{r}. \tag{3.6}$$

Let $r' = \dfrac{2\pi^2 p^*}{T}$. Taking (2.4) into account, $||u||_\infty \leq \pi$ for every $v \in X$ such that $\phi(v) < r'$. Lemma 3.1 and a simple calculation show that

$$\frac{\sup_{\phi(v)<r'} \psi(v)}{r'} \leq \frac{T}{2p^*}\left[V + \frac{\eta}{\gamma}W + \frac{1}{\gamma}\tilde{p}\Gamma_\pi\right].$$

Consequently, by (3.6),

$$\varphi(r) \leq \frac{T}{2p^*}\left[V + \frac{\eta}{\gamma}W + \frac{1}{\gamma}\tilde{p}\Gamma_\pi\right]$$

for all $r > 0$. Hence, Theorem 2.2 implies that, for every $\mu \in \big]0, \eta\big[$ and for each $\lambda \in]\gamma, \xi[\subseteq]\, 0, 1/\varphi(r)[$, the functional $E_{\lambda,\mu}$ admits at least one critical point in $\phi^{-1}\big(]-\infty, r[\big)$. This completes the proof. □

Finally, we obtain the following corollary concerning problem (1.1).

Theorem 3.5. *Suppose that (H1) holds. Furthermore, assume that*

(M2') $G(t,x) \geq 0$ *for all* $(t,x) \in [0,T] \times \mathbb{R}^+$*;*
(M3') $K(t,x) \geq 0$ *for all* $(t,x) \in [0,T] \times \mathbb{R}^+$*;*
(M4') V' *and* W' *are non-negative real numbers, where*

$$V' = \frac{\int_0^T e^{-A(t)} \max_{|x|\leq\pi} G(t,x)dt}{\pi^2}, \quad W' = \frac{\int_0^T e^{-A(t)} \max_{|x|\leq\pi} K(t,x)dt}{\pi^2}.$$

If η and γ are positive real numbers and

$$\gamma < \xi = \frac{2e^{-||a||_1}}{T}\left[\frac{1}{V' + \frac{\eta}{\gamma}W' + \frac{1}{\gamma}\Gamma_\pi}\right],$$

then for every $\mu \in]0, \eta[$ and for each $\lambda \in]\gamma, \xi[$, problem (1.1) *has at least one weak solution.*

It is mentioned that the proof of the above theorem follows from Theorem 3.4 by choosing

$$p(t) = e^{-\int_0^t a(s)ds}, \quad q(t) = b(t)e^{-\int_0^t a(s)ds},$$

and also

$$f(t,u) = g(t,u)e^{-\int_0^t a(s)ds}, \quad h(t,u) = k(t,u)e^{-\int_0^t a(s)ds}.$$

We end this paper by giving the following application to illustrate Theorem 3.2.

Example 3.6. Let $f : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ be two continuous functions defined as follows

$$f(x) = \begin{cases} 15, & \text{if } x \leq 1, \\ 15(400x - 399), & \text{if } 1 < x \leq 2, \\ 6015, & \text{if } x > 2, \end{cases}$$

and

$$h(x) = \begin{cases} 7x^6, & \text{if } x \leq 1, \\ 7x, & \text{if } 1 < x \leq 4, \\ 0, & \text{if } x > 4. \end{cases}$$

Consider the autonomous problem

$$\begin{cases} -u''(t) + (t-\pi)u(t) = \lambda f(u(t)) + \mu h(u(t)), \ a.e. \ in \ [0,1], \\ \triangle(u'(t_1)) = u'(t_1^+) - u'(t_1^-) = 1 - \sqrt[3]{u(t_1)}, \\ u(0) = u(1) = 0. \end{cases} \tag{3.7}$$

We observe that condition (H1) holds. In this case, one has $\alpha = 1$, $\beta = 1$ and $\sigma = \frac{1}{3}$. Our aim is to apply Theorem 3.2. To this end, we have $T = 1$, $p^* = 1$, $\tilde{p} = 1$, $\|p\|_\infty = 1$, $\|q\|_\infty = \pi$ and $s = \frac{6}{12+\pi} \simeq 0.39$. So, by choosing $c = 1$ and $d = 2$, one has $\Gamma_{(\frac{d}{\sqrt{s}})} \simeq 0.65$, $M = (\frac{T}{c^2}) \max_{|\xi| \leq c} F(\xi) = F(1) = 15$, $B = (\frac{T}{2}) \frac{F(d)}{d^2} = \frac{1}{8} F(2) = \frac{3030}{8} = 378.75$, and so (H4) is satisfied. Moreover, (H5) is clearly true, since $\lim_{|\xi| \to +\infty} \frac{F(\xi)}{\xi^2} = 0$. Finally, since $D = (\frac{T}{c^2}) \max_{|\xi| \leq c} H(\xi) = H(1) = 1$ and $\Gamma_c = 1.75$, owing to Theorem 3.2, for every $\lambda \in]0.01, 0.13[$ and for each $\mu \in [0, 0.25 - 15\lambda[$, the problem (3.7) possesses at least three distinct weak solutions.

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Hadi Haghshenas, Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran
e-mail: haghshenas60@gmail.com

Ghasem A. Afrouzi, Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran
e-mail: afrouzi@umz.ac.ir