STABILITY OF THE STOCHASTIC DIFFERENTIAL EQUATIONS

Marie Klimešová
Doctoral Degree Programme (2.), FEEC BUT
E-mail: xklime01@stud.feec.vutbr.cz

Supervised by: Jaromír Baštinec
E-mail: bastinec@feec.vutbr.cz

Abstract: Stability of stochastic differential equations (SDEs) has become a very popular theme of recent research in mathematics and its applications. The method of Lyapunov functions for the analysis of qualitative behavior of SDEs provide some very powerful instruments in the study of stability properties for concrete stochastic dynamical systems, conditions of existence the stationary solutions of SDEs and related problems. The study of exponential stability of the moments makes natural the consideration of certain properties of the moment Lyapunov exponents. Another important characteristic for stability (or instability) of the stochastic systems is the stability index.

Keywords: Brownian motion, stochastic differential equation, Lyapunov function, stability.

1 INTRODUCTION

Stochastic modeling has come to play an important role in many branches of science and industry where more and more people have encountered stochastic differential equations. Stochastic model can be used to solve problem which evinces by accident, noise, etc. Definition of probability spaces, stochastic process (Brownian motion and his basic properties), stochastic differential equation and an existence and uniqueness of solution of these equations, were mentioned in Student EEICT 2014 [8]. It was taken from B. Øksendal [7] and E. Kolářová [5]. In this paper we focus on the description of the stochastic stability. The stability theory was introduced by R. Z. Khasminskii [4]. The basic principles of various types of stochastic systems are described by X.Mao [6]. In the paper we derived sufficient conditions for general system of the zero solution of the stochastic differential equation using Lyapunov function. The results are verified on several examples.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. On the probability space is defined stochastic differential equation.

Definition 1.1 *Let $B_t = (B_1(t), ..., B_m(t))$ be m-dimensional Brownian motion and $b : [0,T] \times R^n \to R^n$, $\sigma : [0,T] \times R^n \to R^{n\times m}$ be measurable functions. Then the process $X_t = (X_1(t), ..., X_m(t))$, $t \in [0,T]$ is the solution of the stochastic differential equation*

$$dX_t = b(t,X_t)\mathrm{d}t + \sigma(t,X_t)\mathrm{d}B_t, \tag{1}$$

$b(t,X_t) \in R$, $\sigma(t,X_t)W_t \in R$. After the integration of equation (1) we give the solution of the SDE

$$X_t = X_0 + \int_0^t b(s,X_s)ds + \int_0^t \sigma(s,X_s)dB_s. \tag{2}$$

Assume that for every initial value $X_t(0) = X_0 \in R^n$, there exists a unique global solution which is denoted by $X(t;t_0,X_0)$. So equation (1) has the solution $X_t(0) \equiv 0$ corresponding to the initial value $X_t(0) = 0$. This solution is called the **trivial solution** or equilibrium position.

2 STABILITY OF STOCHASTIC DIFFERENTIAL EQUATIONS

In 1892, A.M. Lyapunov introduced the concept of stability of a dynamic system. The stability means insensitivity of the state of the system to small changes in the initial state or the parameters of the system. For a stable system, the trajectories which are close to each other at a specific instant should therefore remain close to each other at all subsequent instants.

Lyapunov developed a methods for determining stability without solving the equation. We are used the second Lyapunov method: Let K denote the family of all continuous nondecreasing functions $\mu : R_+ \to R_+$ such that $\mu(0) = 0$ and $\mu(r) > 0$ if $r > 0$. For $h > 0$, let $S_h = \{x \in R^n : |x| < h\}$. A continuous function $V(x,t)$ defined on $S_h \times [t_0,\infty)$ is said to be **positive-definite** (in the sense of Lyapunov) if $V(0,t) \equiv 0$ and, for some $\mu \in K$,

$$V(x,t) \geq \mu(|x|)$$

for all $(x,t) \in S_h \times [t_0,\infty)$.
A function $V(x,t)$ is said to be **negative-definite** if $-V$ is positive-definite. A continuous non-negative function $V(x,t)$ is said to be **decrescent** (i.e. to have an arbitrarily small upper bound) if for some $\mu \in K$,

$$V(x,t) \leq \mu(|x|)$$

for all $(x,t) \in S_h \times [t_0,\infty)$.
A function $V(x,t)$ defined on $R^n \times [t_0,\infty)$ is said to be **radially unbounded** if

$$\lim_{|x|\to\infty} \left(\inf_{t \geq t_0} V(x,t) \right) = \infty$$

.
Let $C^{1,1}(S_h \times [t_0,\infty), R_+)$ denote the family of all continuous functions $V(x,t)$ from $S_h \times [t_0,\infty)$ to R_+ with continuous first partial derivatives with respect to every component of x and to t. Then $v(t) = V(t,X_t)$ represents a function of t with the derivative

$$\dot{v}(t) = V_t(t,X_t) + V_x(t,X_t)b(t,X_t) = \frac{\partial V}{\partial t}(t,X_t) + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(t,X_t)b_i(t,X_t).$$

If $\dot{v}(t) \leq 0$, then $v(t)$ will not increase so the distance of X_t from the equilibrium point measured by $V(t,X_t)$ does not increase. If $\dot{v}(t) < 0$, then $v(t)$ will decrease to zero so the distance will decrease to zero, that is $X_t \to 0$.

Theorem 2.1 ***(Lyapunov theorem)*** *If there exists a positive-definite function $V(x,t) \in C^{1,1}(S_h \times [t_0,\infty), R_+)$ such that*

$$\dot{V}(x,t) := V_t(t,X_t) + V_x(t,X_t)b(t,X_t) \leq 0$$

for all $(x,t) \in S_h \times [t_0,\infty)$, then the trivial solution is stable. If there exists a positive-definite decrescent function $V(x,t) \in C^{1,1}(S_h \times [t_0,\infty), R_+)$ such that $\dot{V}(x,t)$ is negative-definite, then the trivial solution is asymptotically stable.

2.1 STABILITY IN PROBABILITY

Definition 2.1 *The trivial solution of equation (1) is said to be*

*(i) stochastically **stable** or **stable in probability** if for every pair of $\varepsilon \in (0,1)$ and $r > 0$, there exists $\delta = \delta(\varepsilon, r, t_0) > 0$ such that*

$$P\{|x(t,t_0,x_0)| < r\} \geq 1 - \varepsilon \tag{3}$$

*for all $t \geq t_0$, whenever $|x_0| < \delta$. Otherwise, it is said to be stochastically **unstable**.*

*(ii) stochastically **asymptotically stable** if it is stochastically stable and, moreover, for every* $\varepsilon \in (0,1)$, *there exists* $\delta_0 = \delta_0(\varepsilon, t_0) > 0$ *such that*

$$P\left\{\lim_{t\to\infty} x(t,t_0,x_0) = 0\right\} \geq 1-\varepsilon \tag{4}$$

whenever $|x_0| < \delta_0$.

*(iii) stochastically **asymptotically stable in the large** if it is stochastically stable and, moreover, for all* $x_0 \in R^n$

$$P\left\{\lim_{t\to\infty} x(t,t_0,x_0) = 0\right\} = 1. \tag{5}$$

Suppose one would like to let the initial value be a random variable. It should also be pointed out that when $\sigma^(x,t) = 0$, these definitions reduce to the corresponding deterministic ones. We now extend the Lyapunov Theorem (2.1) to the stochastic case. Let $0 < h \leq \infty$. Denote by $C^{2,1}(S_h \times R_+, R_+)$ the family of all nonnegative functions $V(x,t)$ defined on $S_h \times R_+$ such that they are continuously twice differentiable in x and once in t. Define the differential operator L associated with equation (1) by

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(t,X_t) b_i(x,t) + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}\left[\sigma(x,t)\sigma^T(x,t)\right]_{ij}.$$

The inequality $\dot{V}(x,t) \leq 0$ will be replaced by $LV(x,t) \leq 0$ in order to get the stochastic stability assertions.

Theorem 2.2 *If there exists a positive-definite*

(i) function $V(x,t) \in C^{2,1}(S_h \times [t_0,\infty), R_+)$ *such that* $LV(x,t) \leq 0$ *for all* $(x,t) \in S_h \times [t_0,\infty)$, *then the trivial solution of equation (1) is stochastically **stable**.*

(ii) decrescent function $V(x,t) \in C^{2,1}(S_h \times [t_0,\infty), R_+)$ *such that* $LV(x,t)$ *is negative-definite, then the trivial solution of equation (1) is stochastically **asymptotically stable**.*

(iii) decrescent radially unbounded function $V(x,t) \in C^{2,1}(R^n \times [t_0,\infty), R_+)$ *such that* $LV(x,t)$ *is negative-definite, then the trivial solution of equation (1) is stochastically **asymptotically stable in the large.***

Proof: [6], pp. 111.

3 MAIN RESULTS

Definition 3.1 *Lyapunov quadratic function V is given*

$$V(X_t) = X_t^T Q\, X_t, \tag{6}$$

where Q is a symmetric positive-definite matrix.

Theorem 3.1 *The function LV*

$$LV(X_t) = X_t^T Q\, b(t,X_t) + b(t,X_t)^T Q\, X_t + \sigma(t,X_t)^T Q\, \sigma(t,X_t), \tag{7}$$

is negative-definite in some neighbourhood of $X_t = 0$ *for* $t \geq t_0$, *with respect to system (1). Then the trivial solution of equation (1) is stochastically asymptotically stable in the large according to Theorem (2.2).*

Proof. First we compute the Lyapunov function according to (6) of system (1).

$$\begin{aligned}
dV(X_t) &= V(X_t+dX_t)-V(X_t) = (X_t^T+dX_t^T)Q\,(X_t+dX_t)-X_t^TQ\,X_t \\
&= (X_t^T+b(t,X_t)^Tdt+\sigma(t,X_t)^TdB_t)Q\,(X_t+b(t,X_t)dt+\sigma(t,X_t)dB_t)-X_t^TQ\,X_t \\
&= X_t^TQ\,X_t+X_t^TQ\,b(t,X_t)dt+X_t^TQ\,\sigma(t,X_t)dB_t+b(t,X_t)^TdtQ\,X_t+b(t,X_t)^TdtQ\,b(t,X_t)dt \\
&+ b(t,X_t)^TdtQ\,\sigma(t,X_t)dB_t+\sigma(t,X_t)^TdB_tQ\,X_t+\sigma(t,X_t)^TdB_tQ\,b(t,X_t)dt \\
&+ \sigma(t,X_t)^TdB_tQ\,\sigma(t,X_t)dB_t-X_t^TQ\,X_t
\end{aligned}$$

We use the rules $dt\cdot dt = dt\cdot dB_t = dB_t\cdot dt = 0, dB_t\cdot dB_t = dt$ and we have

$$dV(X_t)=X_t^TQ\,b(t,X_t)dt+X_t^TQ\,\sigma(t,X_t)dB_t+b(t,X_t)^TdtQ\,X_t+\sigma(t,X_t)^TdB_tQ\,X_t+\sigma(t,X_t)^TQ\,\sigma(t,X_t)dt$$

We apply expectation $E\{dV(X_t)\}$ and we get

$$E\{dV(X_t)\}=X_t^TQ\,b(t,X_t)dt+b(t,X_t)^TQ\,X_tdt+\sigma(t,X_t)^TQ\,\sigma(t,X_t)dt=LV(X_t)dt,$$
$$-LV(X_t)\geq kV(X_t),\quad k=const.$$

$$\frac{d}{dt}E\{V(X_t)\}\leq -kE\{V(X_t)\},$$
$$E\{V(X_t)\}\leq exp(-kt).$$

Therefore,

$$\lim_{t\to\infty}E^2\{X_t\}=\lim_{t\to\infty}E\left\{X_tX_t^T\right\}=\Theta,$$

which implies asymptotically stable in the large. If $LV(X_t)$ is positive-definite in some neighbourhood of $X_t=0$ with respect to system (1). Then the trivial solution of equation (1) is unstable according to Theorem (2.2).

In the last part of this section, we derive the linear stochastic system of differential equations,

$$dX_t=\alpha X_tdt+\beta X_tdB_t,\ \ t\geq 0, \tag{8}$$

where α,β are $m\times m$ constant matrices.

Theorem 3.2 *We define*

$$LV(X_t)=X_t^T\alpha^TQ\,X_t+X_t^TQ\,\alpha,X_t+X_t^T\beta^TQ\,\beta X_t, \tag{9}$$

is negative-definite in neighbourhood of $X_t=0$ for $t\geq t_0$, with respect to system (8). Then the solution of equation (8) is stochastically asymptotically stable in the large according to Theorem (2.2).

Proof.

$$\begin{aligned}
dV(X_t) &= V(X_t+dX_t)-V(X_t)=(X_t^T+dX_t^T)Q\,(X_t+dX_t)-X_t^TQ\,X_t \\
&= (X_t^T+(\alpha X_t)^Tdt+(\beta X_t)^TdB_t)Q\,(X_t+\alpha X_tdt+\beta X_tdB_t)-X_t^TQ\,X_t \\
&= X_t^TQ\,X_t+X_t^TQ\,\alpha X_tdt+X_t^TQ\,\beta X_tdB_t+(\alpha X_t)^TdtQ\,X_t+(\alpha X_t)^TdtQ\,\alpha X_tdt \\
&+ (\alpha X_t)^TdtQ\,\beta X_tdB_t+(\beta X_t)^TdB_tQ\,X_t+(\beta X_t)^TdB_tQ\,\alpha X_tdt+(\beta X_t)^TdB_tQ\,\beta X_tdB_t \\
&- X_t^TQX_t \\
&= X_t^TQ\,\alpha X_tdt+X_t^TQ\,\beta X_tdB_t+(\alpha X_t)^TdtQ\,X_t+(\beta X_t)^TdB_tQ\,X_t+(\beta X_t)^TQ\,\beta X_tdt,
\end{aligned}$$

$$E\{dV(X_t)\}=X_t^TQ\,\alpha X_tdt+(\alpha X_t)^TQ\,X_tdt+(\beta X_t)^TQ\,\beta X_tdt=LV(X_t)dt.$$

4 EXAMPLES

Example 4.1 *We have stochastic differential equation in the form*

$$dX_t = -X_t dt + exp(-t)dB_t$$

Stability can be determine on the basis of the Theorem (3.1). We define Lyapunov function in the form (7), with $Q = 1$,

$$dV(X_t) = X_t^T(-X_t)dt + X_t^T exp(-t)dB_t + (-X_t)^T X_t dt + (exp(-t))^T X_t dB_t + (exp(-t))^T exp(-t)dt.$$

Then, we compute expectation $E\{dV(X_t)\}$

$$E\{dV(X_t)\} = -X_t^T X_t dt - X_t^T X_t dt + (exp(-t))^T exp(-t)dt = (-2X_t^2 + exp(-2t))dt$$

Function $LV(X_t) = -2X_t^2 + exp(-2t)$ *is negative definite. The following inequality is satisfied* $-2X_t^2 + exp(-2t) < 0$. *The trivial solution is stable. Now we determine a limit*

$$\lim_{t\to\infty} E^2\{X_t\} = \lim_{t\to\infty} E\left\{(-2X_t^2 + exp(-2t))^2\right\} \neq \Theta.$$

So, the solution is stable, but not asymptotically stable.

Example 4.2 *We have stochastic differential equation in the form*

$$dX_t = X_t dt + dB_t$$

We define Lyapunov function in the form (7), with $Q = 1$,

$$dV(X_t) = X_t^T X_t dt + X_t^T dB_t + X_t^T X_t dt + X_t dB_t + dt.$$

$$E\{dV(X_t)\} = X_t^T X_t dt + X_t^T X_t dt + dt = (2X_t^2 + 1)dt.$$

The funtion $LV(X_t) = 2X_t^2 + 1 > 0$ *is positive definite. The trivial solution is unstable.*

ACKNOWLEDGEMENT

This research was supported by Grant FEKT-S-11-2-921 of Faculty of Electrical Engineering and Communication, BUT.

REFERENCE

[1] DITLEVSEN, S.; SAMSON, A: *Stochastic Biomathematical Models*, Softcover, 2013.

[2] DURRETT, R.: *Probability: theory and examples*, Duxbury Press, 1996.

[3] DZHALLADOVA, I.A.: *Optimization of stochastic systems*, Kiev, KNEU Press, 2005. (in Ukrainien)

[4] KHASMINSKII, R. *Stochastic stability of differential equations.* 2nd ed. Berlin: Springer, 2012. ISBN 9783642232800.

[5] KOLÁŘOVÁ, E.: *Stochastické diferenciální rovnice v elektrotechnice*, VUT, Brno, 2005.

[6] MAO, X. *Stochastic differential equations and applications.* 2nd ed. Chichester: Horwood, 2008, xviii, 422 s. ISBN 978-1-904275-34-3.

[7] ØKSENDAL, B.: Stochastic Differential Equations, *An Introduction with Applications*, Springer-Verlag, 1995.

[8] KLIMEŠOVÁ, M.: *Stochastic Differential Equations*, Student EEICT. Tábor 43a, 61200 Brno: LITERA, 2014. s. 150-154. ISBN: 978-80-214-4924- 4.