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FAKULTA STROJNÍHO INŽENÝRSTVÍ

INSTITUTE OF MATHEMATICS

ÚSTAV MATEMATIKY

LERCH'S THEOREM IN THE TIME-SCALES THEORY AND ITS CONSEQUENCES FOR FRACTIONAL CALCULUS

LERCHOVA VĚTA V TEORII ČASOVÝCH ŠKÁL A JEJÍ DŮSLEDKY PRO ZLOMKOVÝ KALKULUS

MASTER'S THESIS
DIPLOMOVÁ PRÁCE

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Department:	Institute of Mathematics
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Pursuant to Act no. 111/1998 concerning universities and the BUT study and examination rules, you have been assigned the following topic by the institute director Master's Thesis:

Lerch's theorem in the time–scales theory and its consequences for fractional calculus

Concise characteristic of the task:

Fractional calculus and time–scales theory represent disciplines dealing with generalization of classical analysis. Lerch's theorem is fundamental assertion of Laplace transform theory, however, for the nabla time–scales calculus it was not proven so far. Main goal of this thesis is to study nabla Laplace transform on time scales, especially to formulate and prove the corresponding Lerch's theorem. Further, its consequences for fractional calculus in time–scales theory will be discussed.

Goals Master's Thesis:

1. To formulate and prove Lerch's Theorem for the nabla Laplace transform on time scales.
2. To discuss its consequences for introduction of fractional calculus to the time scales theory.

Recommended bibliography:

DOETSCH, Gustav. Introduction to the Theory and Application of the Laplace Transformation. ISBN 978-3-642-65692-7.

KISELA, Tomáš. Power functions and essentials of fractional calculus on isolated time scales, Advances in Difference Equations, Vol.2013, (2013), No.8, pp.1-18.

BOHNER, Martin a Allan C. PETERSON. Advances in Dynamic Equations on Time Scales. ISBN 0-8176-4293-5.

Deadline for submission Master's Thesis is given by the Schedule of the Academic year 2016/17

In Brno,

L. S.

prof. RNDr. Josef Šlapal, CSc.
Director of the Institute

doc. Ing. Jaroslav Katolický, Ph.D.
FME dean

Summary
Main concern of the diploma thesis is the study of the generalized nabla time scale Laplace transform and its uniqueness, including the proof of uniqueness and the application of uniqueness to fractional calculus on time scales.

Abstrakt
Hlavným zájmem diplomové práce je studium zobecněné nabla Laplaceové transformace na časových škálach a její jednoznačnosti, včetně důkazu jednoznačnosti a aplikace jednoznačnosti v zlomkovém kalkulu na časových škálach.

Keywords
time scale, fractional calculus, Laplace transform, Lerch's theorem, uniqueness, unicity, power function, monomial

Klíčová slova
časová škála, zlomkový kalkulus, Laplaceova transformace, Lerchova věta, jednoznačnost, mocninná funkce, monomiál

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I declare that I have written this master's thesis all by myself under the direction of my supervisor Ing. Tomáš Kisela, Ph.D. using the literature listed in the bibliography.

Bc. Matej Dolník

My most sincere thanks go to my advisor and mentor Ing. Tomáš Kisela, Ph.D. for guidance, support and patience. Secondly, I would like to thank my brother Viktor for support and doc. Ing. Luděk Nechvátal, Ph.D. for promptness.

Bc. Matej Dolník

Contents

1. Introduction

"The art of doing mathematics consists in finding that special case which contains all the germs of generality."
David Hilbert

The generalization of previously obtained results into one theory is the grand goal of mathematics. The purpose of this thesis is to enhance the results of two theories that unify the results of calculus, namely:

- the time scale calculus
- the fractional calculus.

Continuous fractional calculus is as old as the continuous calculus itself. Even though the first mention of continuous fractional calculus was at the end of the 17th century, in the Leibniz's letter to l'Hospital and the first applications lead to Heaviside - continuous fractional calculus did not meet a lot of popularity. Throughout the history, many authors contributed to the continuous fractional calculus, but since it had only a handful of applications, it did not become popular. Not until the first international conference in 1974, which was specialised on this subject. At the same time, the comprehensive survey of fractional calculus [1] was published. After 1974, the continuous fractional calculus became a fast-growing and respected branch of mathematics. With this development, numerous applications of fractional calculus were discovered, for instance in: transmission line theory, chemical analysis of aqueous solutions, design of heat-flux meters, rheology of soils, growth of intergranular grooves at metal surfaces, quantum mechanical calculations, and dissemination of atmospheric pollutants.

However, continuous fractional calculus is by no means the final generalization of continuous calculus. In 1969 ([2] and [3]), motivated by numerical methods for solving continuous fractional differential equations, the theory of discrete fractional calculus was founded (a q-calculus or a quantum calculus deserved a special attention due to its applications, but we do not deal with q-calculus in this thesis).

In the past decade, the development of discrete fractional calculus became more popular however, with the increased number of scientists interested in this subject, large number of an unexpected difficulties were uncovered.

Time scale calculus is a theory, which was introduced in Stefan Hilger's PhD thesis [4] in 1988, later on published [5]. (A *time scale* is an arbitrary non-empty closed subset of real numbers.) The theory unifies the discrete and continuous calculus into one, under one framework independent of set of points, that we deal with, which has a tremendous potential for applications.

The applications, which prospered from the employment of time scale calculus are, for instance, already mentioned discrete and continuous fractional calculus. One of major achievements of time scale calculus is the possibility to unite the discrete and the continuous fractional calculus (all the special cases such as q-calculus) into a single mathematical discipline the *fractional calculus on time scales*.

The fundamental question in the fractional calculus on time scales is the question:
How to define fractional power functions on an arbitrary time scale?
The case for monomials of natural orders in real numbers is answered naturally - the monomial of the n-th order is the n-th integral of one. The time scale framework provided the answer for discrete or hybrid (continuous/discrete) cases of natural orders.

The answers for non-natural orders (which are fundamental for development of fractional calculus) were proposed and discussed in, for example [6], [7], [8], [9], [10], but it is yet to be proved, or disproved, whether the proposed definitions are well-defined on an arbitrary time scale.

The pursue of answers to the uniqueness of one of these definitions required one of the most well-known tool of mathematics - the Laplace transform or, more precisely speaking - its time scale generalization. The main purpose of this work is to study the uniqueness of such transform and to extend the results of the generalized Lerch's theorem (theorem stating the uniqueness of the generalized nabla time scale Laplace transform).

The thesis is organized as follows:
The second chapter (the first chapter being the introduction) presents elementary results in time scale calculus and a brief survey of time scale functions.
The third chapter introduces the generalized nabla time scale Laplace transform, investigates the region of convergence of mentioned transform and the uniqueness property of the transform.
The last chapter provides the introduction to fractional calculus on time scales and the application of previously acquired results of the generalized nabla time scale Laplace transform.

2. Time scale calculus

In his PhD thesis [4] from 1988, Stefan Hilger introduced calculus on *measure chains*, in order to unify discrete and continuous analysis. Eventually, a special case of measure chains gained on popularity - the so-called *time scale*.

2.1. Elementary definitions of time scale calculus

In this section, we introduce the basic notions of time scale calculus. The general idea of time scale calculus is to prove the result on *time scale* $\mathbb{T}$, which is an arbitrary closed non-empty subset of real numbers.

Examples of such sets are $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$ and also the more complicated examples such as Cantor set. We might notice that $\mathbb{Q}$ is not a time scale, since it is not closed set.

Following basic definitions and theorems can be found in [11].

Let us put $\sup\{\mathbb{T}\} = \inf\{\emptyset\}$ and $\inf\{\mathbb{T}\} = \sup\{\emptyset\}$ in the following definitions.

Definition 2.1.1. *Forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by:

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}.$$

Similarly, *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by:

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

Forward graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by:

$$\mu(t) := \sigma(t) - t$$

and *backward graininess function* $\nu : \mathbb{T} \to [0, \infty)$:

$$\nu(t) := t - \rho(t).$$

Informally speaking, when t denotes the 'current point' on time scale, $\sigma(t)$ denotes the 'next point' and $\rho(t)$ the 'point before'.

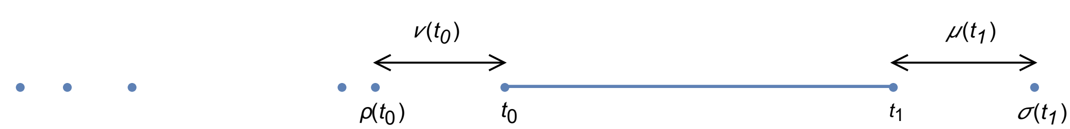

Figure 2.1: An example of a time scale with a demonstration of graininess functions and jump operators.

We may observe that in the case of real numbers ($\mathbb{T} = \mathbb{R}$), both graininess functions are $\mu(t) = \nu(t) = 0$ for all $t \in \mathbb{T}$.
In the case of $\mathbb{T} = h\mathbb{Z}$, both graininess functions are $\mu(t) = \nu(t) = h$.

Let us define for $n \in \mathbb{N}$:

$$\begin{aligned} \nu^0(t) &:= 0, \qquad & \nu^n(t) &:= \nu^{n-1}(t) + \nu(t - \nu^{n-1}(t)), \\ \rho^0(t) &:= 0, \qquad & \rho^n(t) &:= \rho(\rho^{n-1}(t)). \end{aligned}$$

Similarly, we define for $n \in \mathbb{N}$:

$$\begin{aligned} \mu^0(t) &:= 0, \qquad & \mu^n(t) &:= \mu^{n-1}(t) + \mu(t + \mu^{n-1}(t)), \\ \sigma^0(t) &:= 0, \qquad & \sigma^n(t) &:= \sigma(\sigma^{n-1}(t)). \end{aligned}$$

We may notice that $\rho^n(t) = t - \nu^n(t)$ and $\sigma^n(t) = t + \mu^n(t)$.

Therefore, isolated time scale $\mathbb{T}$ can be written in the form:

$$\mathbb{T} = \{...\rho^2(t), \rho^1(t), t, \sigma^1(t), \sigma^2(t), ...\}.$$

To clarify, whether the examined point is isolated or a part of a real interval, we define the following properties:

Definition 2.1.2. a) If $\sigma(t) > t$, we say that t is a *right-scattered point.*

b) If $\rho(t) < t$, we say that t is a *left-scattered point.*

c) If $\sigma(t) > t$ and $\rho(t) < t$, we say that t is an *isolated point.*

d) If $t < \sup\{\mathbb{T}\}$ and $\sigma(t) = t$, we say that t is a *right-dense point.*

e) If $t > \inf\{\mathbb{T}\}$ and $\rho(t) = t$, we say that t is a *left-dense point.*

f) If $\sup\{\mathbb{T}\} > t > \inf\{\mathbb{T}\}$ and $\rho(t) = t = \sigma(t)$, we say that t is a *dense point.*

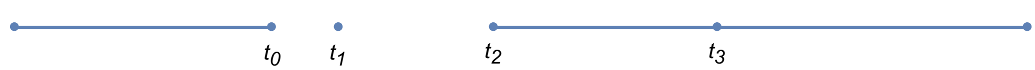

Figure 2.2: An example of a time scale with a demonstration of different points. In particular - point t_0 is left-dense and right scattered, point t_1 is an isolated point, point t_2 is left-scattered and right-dense, point t_3 is a dense point.

If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_\kappa := \mathbb{T} \setminus \{m\}$, otherwise $\mathbb{T}_\kappa := \mathbb{T}$. We call $\mathbb{T}_\kappa$ the *truncated time scale.*
Similarly, if $\mathbb{T}$ has a left-scattered maximum M, then $\mathbb{T}^\kappa := \mathbb{T} \setminus \{M\}$, otherwise $\mathbb{T}^\kappa := \mathbb{T}$.

Now we are in a position to define a generalized notion of continuity:

Definition 2.1.3. A function $f : \mathbb{T} \to \mathbb{R}$ is *right-dense continuous (rd-continuous)*, if it is continuous at each right-dense point in $\mathbb{T}$ and if it has finite left sided limits at left dense points in $\mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is *left-dense continuous (ld-continuous)*, if it is continuous at each left-dense point in $\mathbb{T}$ and if it has finite right sided limits at right dense points in $\mathbb{T}$.

Theorem 2.1.1. Assume $f : \mathbb{T} \to \mathbb{R}$. Then, the following statements are true:

a) If f is continuous, then f is rd-continuous and ld-continuous.

b) Forward jump operator σ is rd-continuous.

c) Backward jump operator ρ is ld-continuous.

2.2. Time scale derivatives and integrals

This section provides the fundamentals of generalized time scale derivatives and integrals. Used definitions and theorems can be found in [11], [12]. Since there are two possible definitions of derivatives, time scale calculus is divided into Δ-calculus (forward derivative) and ∇-calculus (backward derivative). Throughout this work, we deal with ∇-calculus.

Definition 2.2.1. Let $f : \mathbb{T} \to \mathbb{R}$ be a function and let $t \in \mathbb{T}_\kappa$. Then, if it exists, ∇-*derivative* $\nabla f(t)$ (or $f^\nabla(t)$) is defined as a number with the property that for any given number $\epsilon > 0$, there is a delta neighbourhood U_δ of t (i.e., $U_\delta = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$), such that:

$$|f(s) - f(\rho(t)) - f^\nabla(t)(s - \rho(t))| \leq \epsilon |s - \rho(t)| \qquad \forall s \in U_\delta.$$

Theorem 2.2.1. Let $f : \mathbb{T} \to \mathbb{R}$ be a function and let $t \in \mathbb{T}_\kappa$. Then, the following statements are true.

a) If f is differentiable at t, then f is continuous at t.

b) If f is continuous at t and t is left-scattered, then f is ∇-differentiable at t with:

$$f^\nabla(t) = \frac{f(t) - f(\rho(t))}{\nu(t)}.$$

c) If t is left-dense, then f is ∇-differentiable at t iff the limit:

$$\lim_{s \to t} \frac{f(s) - f(t)}{s - t}$$

exists and is finite. Then:

$$f^\nabla(t) = \lim_{s \to t} \frac{f(s) - f(t)}{s - t}.$$

d) If t is ∇-differentiable at t, then:

$$f(\rho(t)) = \nu(t)f^{\nabla}(t) + f(t).$$

Theorem 2.2.2. Assume that $f, g : \mathbb{T} \to \mathbb{R}$ are ∇-differentiable at $t \in \mathbb{T}_{\kappa}$. Then, the following statements hold:

a) $(fg)^{\nabla}(t) = f^{\nabla}(t)g(t) + f(\rho(t))g^{\nabla}(t) = f(t)g^{\nabla}(t) + f^{\nabla}(t)g(\rho(t))$,

b) $(f+g)^{\nabla}(t) = f^{\nabla}(t) + g^{\nabla}(t)$,

c) $\left(\frac{f}{g}\right)^{\nabla}(t) = \frac{f^{\nabla}(t)g(t) - f(t)g^{\nabla}(t)}{g(t)g(\rho(t))}$.

Definition 2.2.2. Let $a, b \in \mathbb{T}$, such that $a < b$. Let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ and $F : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be functions, such that $F^{\nabla}(t) = f(t)$ for all $t \in \mathbb{T}^{\kappa}$. Then, the function $F(t)$ is called the *antiderivative* of function $f(t)$ and we define the *definite nabla integral over* $[a,b]_{\mathbb{T}}$ as $\int_a^b f(t)\nabla t := F(b) - F(a)$.

We may also note that $\int_b^a f(t)\nabla t = -\int_a^b f(t)\nabla t$ and $\int_a^a f(t)\nabla t = 0$.

Theorem 2.2.3. Every ld-continuous function has a ∇-antiderivative.

The following theorem allows the representation of integral by sums, if the time scale is isolated. Throughout this work, we utilize this theorem multiple times.

Theorem 2.2.4. Let $\mathbb{T}$ be an isolated time scale. Then, the ∇-integral can be calculated as:

$$\int_a^b f(t)\nabla t = \sum_{t \in (a,b]_{\mathbb{T}}} \nu(t)f(t).$$

Definition 2.2.3. Let $a \in \mathbb{T}$, $\sup\{\mathbb{T}\} = \infty$, $f : [a,\infty)_{\mathbb{T}} \to \mathbb{R}$ be ld-continuous. Then, the *improper integral of first kind* over $f(t)$ over $[a,\infty)_{\mathbb{T}}$ is defined by:

$$\int_a^{\infty} f(t)\nabla t := \lim_{b\to\infty} \int_a^b f(t)\nabla t.$$

Definition 2.2.4. Let $a, b, c \in \mathbb{T}$ be such that $a < b < c$ and let $f : (a,c]_{\mathbb{T}} \to \mathbb{R}$ be ld-continuous on any interval $[b,c]_{\mathbb{T}}$. Then, the *improper integral of second kind* of $f(t)$ over $[a,c]_{\mathbb{T}}$ is defined as:

$$\int_a^c f(t)\nabla t := \begin{cases} \lim_{b\to a+}(\int_b^c f(t)\nabla t), & \text{if } a \text{ is right-dense,} \\ \int_a^c f(t)\nabla t, & \text{if } a \text{ is right-scattered.} \end{cases}$$

From the Theorem 2.2.2 a) we obtain the following *integral by parts formula*:

Theorem 2.2.5. Assume that $f, g : \mathbb{T} \to \mathbb{R}$ are ∇-differentiable at $t \in \mathbb{T}_{\kappa}$. Then:

$$\int f^{\nabla}(t)g(t)\nabla t = f(t)g(t) - \int f(\rho(t))g^{\nabla}(t)\nabla t.$$

Proof of the next theorem is a modified version of the delta version of this proof presented in [12].

Theorem 2.2.6. Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose that $g : \mathbb{T} \to \mathbb{R}$ is ∇-differentiable. Then, $f \circ g : \mathbb{T} \to \mathbb{R}$ is ∇-differentiable and the following formula holds:

$$(f \circ g)^{\nabla}(t) = g^{\nabla}(t) \int_0^1 f'(g(t) - u\nu(t)g^{\nabla}(t))du.$$

2.3. Time scale exponential

In this section, we present the generalized nabla time scale exponential and some of its properties needed for a generalized nabla time scale Laplace transform examination.
The following definitions may be found at [12].

Definition 2.3.1. A function $f : \mathbb{T} \to \mathbb{R}$ is ν-regressive if:

$$1 - \nu(t)f(t) \neq 0 \quad \text{for all } t \in \mathbb{T}_{\kappa}.$$

Definition 2.3.2. The class of all scalar ld-continuous and ν-regressive functions on $\mathbb{T}$ is denoted by $\mathcal{R}_{\nu}$, i.e. $\mathcal{R}_{\nu} := \{f : \mathbb{T} \to \mathbb{R}; f(t) \text{ is ld-continuous and } \nu\text{-regressive}\}$.
Furthermore, $\mathcal{R}_{\nu}^{+} := \{f \in \mathcal{R}_{\nu}; 1 - f(t)\nu(t) > 0\}$ for all $t \in \mathbb{T}_{\kappa}$.

The following two definitions define the generalized addition and subtraction for time scale exponentials.

Definition 2.3.3. We define a circle plus addition for $p, q \in \mathcal{R}_{\nu}$ by:

$$(p \oplus_{\nu} q)(t) := p(t) + q(t) - p(t)q(t)\nu(t)$$

for all $t \in \mathbb{T}_{\kappa}$.

Definition 2.3.4. We define a circle minus subtraction for $p \in \mathcal{R}_{\nu}$ by:

$$\ominus_{\nu} p(t) := -\frac{p(t)}{1 - p(t)\nu(t)}$$

for all $t \in \mathbb{T}_{\kappa}$.

Throughout this work, we use $\oplus$,$\ominus$ instead of $\oplus_{\nu}$,$\ominus_{\nu}$.

Theorem 2.3.1. $(\mathcal{R}_{\nu}, \oplus)$ is an Abelian group and $(\mathcal{R}_{\nu}^{+}, \oplus)$ is a subgroup of $(\mathcal{R}_{\nu}, \oplus)$.

The following results are dual to the delta results presented in [12]. We utilize the same methods in ∇-calculus to determine them.

Using the chain rule provided by the Theorem 2.2.6, we derive the following formula for $(f \circ g)(x) = x^{\alpha}$, $\alpha \in \mathbb{R}$:

$$(x^{\alpha})^{\nabla}(t) = x^{\nabla}(t) \int_0^1 \alpha(x(t) - u\nu(t)x^{\nabla}(t))^{\alpha - 1} dh.$$

If $x(t) \neq 0$, then:

$$(x^\alpha)^\nabla(t) = x^\alpha(t)\frac{x^\nabla(t)}{x(t)}\alpha\int_0^1(1-u\nu(t)\frac{x^\nabla(t)}{x(t)})^{\alpha-1}du. \tag{2.1}$$

In order to have everything well-defined, we want to assume for $\alpha \in \mathbb{R}\setminus\mathbb{N}$ that:

$$(1-\nu(t)\frac{x^\nabla(t)}{x(t)}u)^{\alpha-1} > 0,$$

for $u \in [0,1]$ and $t \in \mathbb{T}$. A sufficient condition for this is:

$$\mathcal{R}(\alpha) := \begin{cases} \mathcal{R}^+ & \text{for} \quad \alpha \in \mathbb{R}\setminus\mathbb{N}, \\ \mathcal{R} & \text{for} \quad \alpha \in \mathbb{N}. \end{cases}$$

From the computation above, we obtain the following definition:

Definition 2.3.5. For $\alpha \in \mathbb{R}$ and for $p \in \mathcal{R}(\alpha)$, we define the dot multiplication:

$$(\alpha \odot p)(t) := p(t)\alpha\int_0^1(1-u\nu(t)p(t))^{\alpha-1}du.$$

Theorem 2.3.2. Let $\alpha \in \mathbb{R}$. If $\alpha \in \mathbb{N}$, suppose that $\alpha \neq 0$ for all $t \in \mathbb{T}$. If $\alpha \notin \mathbb{N}$, suppose that $x(t)x(\rho(t)) > 0$ for all $t \in \mathbb{T}$. Then:

$$\frac{(x^\alpha)^\nabla}{x^\alpha} = (\alpha \odot \frac{x^\nabla}{x}).$$

Proof.

$$1-\nu\frac{(x(t))^\nabla}{x(t)} = \frac{x(\rho(t))}{x(t)}$$

Then, the theorem follows directly from the expression 2.1. □

Theorem 2.3.3. Let $\alpha \in \mathbb{R}$. If $p \in \mathcal{R}(\alpha)$, then:

$$1-\nu(\alpha \odot p) = (1-\nu p)^\alpha.$$

Proof.

$$\begin{aligned} 1-\nu(\alpha \odot p) &= 1-\nu p\alpha\int_0^1(1-u\nu p))^{\alpha-1}du = 1-\int_0^1 \nu p\alpha(1-u\nu p))^{\alpha-1}du = \\ &= 1-\int_1^{1-\nu p}\alpha s^{\alpha-1}ds = 1+(1-\nu p)^\alpha - 1^\alpha = (1-\nu p)^\alpha. \end{aligned}$$

□

Theorems and definitions comprising of a generalized nabla time scale exponential function can be found at [11], [12].

Definition 2.3.6. For any $f \in \mathcal{R}_\nu$ and fixed $s \in \mathbb{T}$, the exponential function $\hat{e}_f(\cdot, s)$ is defined as a unique solution of the initial value problem:

$$\nabla y(t) = f(t)y(t) \qquad y(s) = 1.$$

Definition 2.3.7. For $h > 0$, the *Hilger complex numbers* are defined as:

$$\mathbb{C}_h := \{z \in \mathbb{C} : z \neq \frac{1}{h}\}$$

and the *strip* $\mathbb{Z}_h$ is defined as:

$$\mathbb{Z}_h := \{z \in \mathbb{C} : -\frac{\pi}{h} < \mathrm{Im}(z) \leq \frac{\pi}{h}\}.$$

The *h-cylinder transformation* $\hat{\xi}_h : \mathbb{C}_h \to \mathbb{Z}_h$ is defined by:

$$\hat{\xi}_h(z) := -\frac{1}{h}\mathrm{Log}(1 - zh)$$

for $h > 0$ and where Log is the principal logarithm function.
For $h = 0$ we define $\hat{\xi}_0(z) := z$ for all $z \in \mathbb{C}_0 := \mathbb{C}$.

The following theorem may be used as an equivalent definition of ∇-exponential function:

Theorem 2.3.4. Solution of the initial value problem $\nabla y(t) = f(t)y(t)$, $y(s) = 1$ can be written as:

$$\hat{e}_f(t, s) = \exp\{\int_s^t \hat{\xi}_{\nu(\tau)}(f(\tau))\nabla\tau\} \qquad s, t \in \mathbb{T}$$

where $\hat{\xi}_h(z)$ is the ν-cylinder transformation.

Theorem 2.3.5. If $p \in \mathcal{R}_\nu$, then the *semi-group property* for $s, t, r \in \mathbb{T}$ is satisfied:

$$\hat{e}_f(t, r)\hat{e}_f(r, s) = \hat{e}_f(t, s).$$

Theorem 2.3.6. Let $p, q \in \mathcal{R}_\nu$ and $s, t \in \mathbb{T}$. Then, the following statements are valid:

a) $\hat{e}_0(t, s) = 1$,

b) $\hat{e}_p(t, t) = 1$,

c) $\hat{e}_p(t, s)\hat{e}_q(t, s) = \hat{e}_{p \oplus q}(t, s)$,

d) $\frac{1}{\hat{e}_p(t,s)} = \hat{e}_p(s, t) = \hat{e}_{\ominus p}(t, s)$,

e) $\hat{e}_p(\rho(t), s) = (1 - \nu(t)p(t))\hat{e}_p(t, s)$.

Theorem 2.3.7. Let $p \in \mathcal{R}_\nu$ and $t_0 \in \mathbb{T}$. Then, the following statements are true:

a) If $p \in \mathcal{R}_\nu^+$, then $\hat{e}_p(t, t_0) > 0$ for all $t \in \mathbb{T}$.

b) If $1 - \nu(t)p(t) < 0$ for some $t \in \mathbb{T}_\kappa$, then $\hat{e}_p(\rho(t), t_0)\hat{e}_p(t, t_0) < 0$.

c) If $1 - \nu(t)p(t) < 0$ for all $t \in \mathbb{T}_\kappa$, then $\hat{e}_p(t, t_0)$ changes its sign at every point $t \in \mathbb{T}$.

Theorem 2.3.8. Let $z \in \mathcal{R}_\nu$. Then:

$$\hat{e}_{\ominus z}(\rho(t), s) = \frac{- \ominus z}{z} \hat{e}_{\ominus z}(t, s).$$

Proof. Utilizing the Theorem 2.3.6 e):

$$\hat{e}_{\ominus z}(\rho(t), s) = (1 - \nu(t) \ominus z)\hat{e}_{\ominus z}(t, s) = \frac{\hat{e}_{\ominus z}(t, s)}{1 - \nu(t)z} = \frac{- \ominus z}{z} \hat{e}_{\ominus z}(t, s).$$

□

The following representation theorem is essential for further computations.

Theorem 2.3.9. Let $\mathbb{T}$ be discrete and $t > s$, $z \in \mathbb{C}$. Then:

$$\hat{e}_{\ominus z}(t, s) = \prod_{\tau \in (s,t]_{\mathbb{T}}} (1 - z\nu(\tau)).$$

Proof. By the Theorem 2.3.4 we may write for discrete time scales:

$$\hat{e}_{\ominus z}(t, s) = \exp\{\int_s^t \hat{\xi}_{\nu(\tau)}(z) \nabla \tau\},$$

where $\hat{\xi}_{\nu(\tau)}(z)$ is ν-cylinder transformation. Therefore:

$$\exp\{\int_s^t \hat{\xi}_{\nu(\tau)}(z) \nabla \tau\} = \exp\{\int_s^t -\frac{1}{\nu(\tau)} \mathrm{Log}(1 - \ominus z\nu(\tau)) \nabla \tau\}.$$

Utilizing the Theorem 2.2.4, we rewrite the integral:

$$\exp\{\int_s^t -\frac{1}{\nu(\tau)} \mathrm{Log}(1 - \ominus z\nu(\tau)) \nabla \tau\} = \exp\{\sum_{(s,t]_{\mathbb{T}}} \mathrm{Log}(1 - \ominus z\nu(\tau)) \nabla \tau\}^{-1}.$$

Using the property of Principal Logarithm and exponential function:

$$\exp\{\sum_{(s,t]_{\mathbb{T}}} \mathrm{Log}(1 - \ominus z\nu(\tau)) \nabla \tau\}^{-1} = \prod_{(s,t]_{\mathbb{T}}} (1 - \ominus z\nu(\tau)) \nabla \tau\}^{-1}.$$

Finally, by the Definition 2.3.4, we prove the result. □

As for the delta case - [13], we define:

Definition 2.3.8. Let $\mathbb{T}$ be a time scale, such that $sup\{\mathbb{T}\} = \infty$. We call $\mathbb{T}$ a *periodic time scale with period* p, if there is a $p \in \mathbb{R}$, such that $\nu(t) = \nu(t + p)$ for all $t \in \mathbb{T}$.

Example 2.3.1. We present some examples of a nabla time scale exponential functions.
If $\mathbb{T} = h\mathbb{Z}$:

$$\hat{e}_c(t,s) = \frac{1}{(1-ch)^{\frac{t-s}{h}}},$$
$$\hat{e}_{\ominus c}(t,s) = (1-ch)^{\frac{t-s}{h}}.$$

If $\mathbb{T} = q^{\mathbb{Z}}$:

$$\hat{e}_{f(t)}(\sigma^a(s),s) = \prod_{k=1}^{a} \frac{1}{1-(1-\frac{1}{q})\sigma^k(s)(f(\sigma^k(s))},$$
$$\hat{e}_{f(t)}(\rho^a(s),s) = \prod_{k=1}^{a} 1-(1-\frac{1}{q})\rho^k(s)(f(\rho^k(s)).$$

If $\mathbb{T} = \mathbb{R}$:

$$\hat{e}_c(\sigma(t),s) = \hat{e}_c(\rho(t),s) = e^{c(t-s)}.$$

If $\mathbb{T} = \bigcup_{k=0}^{\infty}\{[3k,3k+1] \cup \{3k+2\}\}$:

$$\begin{aligned} \hat{e}_c(t,0) &= e^{ck}(1-c)^k e^{c(t-3k)} && \text{for } t \in [3k,3k+1],\ k \in \mathbb{N}, \\ &= e^{c(k+1)}(1-c)^{k+1} && \text{for } t = 3k+2,\ k \in \mathbb{N}. \end{aligned}$$

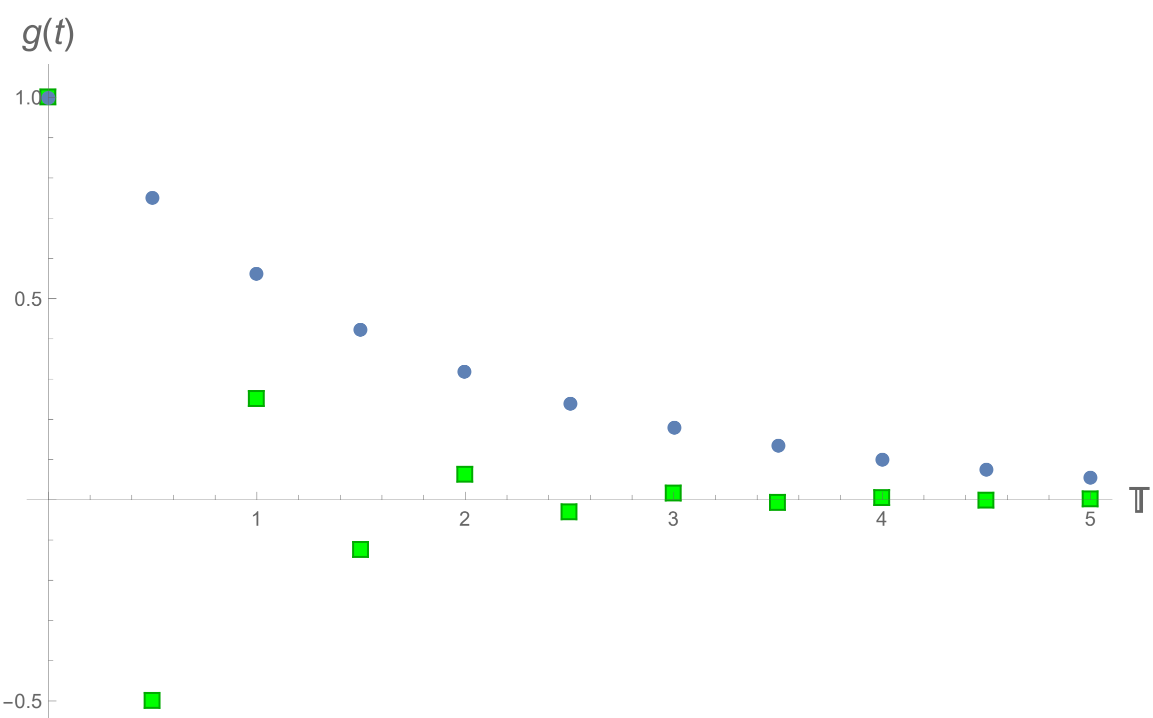

Figure 2.3: An example of a time scale exponential functions on $\mathbb{T} = \frac{1}{2}\mathbb{Z}$. The function $\hat{e}_{\ominus 0,5}(t,0)$ is represented by the blue dots, while $\hat{e}_{\ominus 3}(t,0)$ is represented by the green squares. We may notice that $3 \notin \mathcal{R}_\nu^+$, whereas $1 \in \mathcal{R}_\nu^+$.

The following theorem is dual to delta versions presented in [12].

Theorem 2.3.10. If $\alpha \in \mathbb{R}$ and $p \in \mathcal{R}(\alpha)$, then:

$$\hat{e}_{\alpha \odot p} = \hat{e}_p^{\alpha}.$$

Proof. Let $t_0 \in \mathbb{T}$ and put:

$$y = \hat{e}_p^{\alpha}(\cdot, t_0).$$

Then, $y(t_0) = 0$ and by the Theorem 2.3.2:

$$y^{\nabla} = (\hat{e}_p^{\alpha})^{\nabla} = (\alpha \odot \frac{\hat{e}_p^{\nabla}}{\hat{e}_p})\hat{e}_p^{\alpha} = (\alpha \odot p)y.$$

We know that y solves the initial value problem:

$$\nabla y(t) = (\alpha \odot p)y(t) \qquad y(t_0) = 1.$$

Therefore,

$$\hat{e}_p^{\alpha} = y = \hat{e}_{\alpha \odot p}.$$

$\square$

2.4. Time scale monomials

This section provides a brief survey of nabla time scale monomials. The following results are adapted from [6].

Definition 2.4.1. Monomials $\hat{h}_n : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ for $n \in \mathbb{N}$ are defined via recursion:

$$\hat{h}_0(t,s) := 1,$$
$$\hat{h}_n(t,s) := \int_s^t \hat{h}_{n-1}(\tau, s)\nabla\tau.$$

Theorem 2.4.1. Let $n \in \mathbb{Z}^+$ and $s, t \in \mathbb{T}$. Then:

a) $\hat{h}_n(t,t) = 0$,

b) $\nabla \hat{h}_n(t,s) = \hat{h}_{n-1}(t,s)$ for $t \in \mathbb{T}_\kappa$,

c) $\nabla \hat{h}_1(t,s) = (t - s)$.

Theorem 2.4.2. Let $s, t \in \mathbb{T}$, $n \in \mathbb{Z}_0^+$. Then, the following statements are true:

a) If $\mathbb{T} = \mathbb{R}$, then $\hat{h}_n(t,s) = \frac{(t-s)^m}{m!}$.

b) If $\mathbb{T} = h\mathbb{Z}$ and $\sigma^k(s) = t$, then:

$$\hat{h}_n(t,s) = h^n \binom{n+k-1}{k-1} = (-1)^{k-1} h^n \binom{-n-1}{k-1}.$$

3. Generalized nabla time scale Laplace transform

Laplace transform plays a crucial role in real and complex analysis and is commonly appliedn i other fields, such as solving differential equations or signal processing. In this section, we examine the notion of a generalized nabla time scale Laplace transform and its properties - some of them very similar to or the same as in real Laplace transform. We focus our examination on the topic of the uniqueness property of the generalized nabla time scale Laplace transform, which is neglected by many authors. In this work, the presented application of the generalized time scale Laplace transform is in the fractional calculus on time scales.

3.1. Basic definitions

In this section, we provide the most commonly used definition of the generalized nabla time scale Laplace transform. This definition was introduced in [14] and utilized on numerous occasions - for example: [6], [7], [9], [10], [15], [16], [17].

Definition 3.1.1. Let $\sup\{\mathbb{T}\} = \infty$, $s \in \mathbb{T}$ and let $f(t)$ be a real function defined at least on $(s, \infty)_{\mathbb{T}}$. The generalized nabla time scale Laplace transform of $f(t)$ is defined by:

$$\mathcal{L}_s^{\mathbb{T}}\{f\}(z) := \int_s^{\infty} f(t)\hat{e}_{\ominus z}(\rho(t), s)\nabla t \qquad for \quad z \in \mathcal{D}(f), \tag{3.1}$$

where $\mathcal{D}(f)$ consist of all complex numbers z, such that $\mathrm{Re}(z) \in \mathcal{R}_\nu$ for which the improper integral exists.

Throughout this work, we call $\mathcal{D}(f)$ the *domain of convergence of the generalized nabla time scale Laplace transform*. Note that since the points $z = \frac{1}{\nu(t)}$ for any $\nu(t) > 0$ do not lie in $\mathcal{D}(f)$, we put them to $\mathcal{D}(f)$ via definition.

Theorem 3.1.1. The necessary condition for the existence of the generalized nabla time scale Laplace transform $\mathcal{L}_s\{f\}(z)$ is:

$$\lim_{t\to\infty} f(t)\hat{e}_{\ominus z}(\rho(t), s) = 0.$$

Definition 3.1.2. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be of *exponential type I*, if there exist constants $M, c > 0$ s.t. $|f(t)| \leq Me^{ct}$ for every t. Furthermore, a function f is set to be of *exponential type II*, if there exist constants $N > 0$, $\alpha \in {\mathcal{R}_\nu}^+$ s.t. $|f(t)| \leq N\hat{e}_\alpha(t, 0)$.

The proof of the following theorem may be found in [16].

Theorem 3.1.2. Let $s \in \mathbb{T}$, $f : [s, \infty)_{\mathbb{T}} \to \mathbb{R}$ be a function of exponential type II - $|f(t)| \leq N\hat{e}_\alpha(t, 0)$, where $\alpha \in {\mathcal{R}_\nu}^+$. Then:

$$\lim_{t\to\infty} f(t)\hat{e}_{\ominus z}(t, s) = 0,$$

for $z \in \mathbb{C}_{\inf_{\tau\in[s,t)_{\mathbb{T}}}\{\nu(\tau)\}}$, s.t. $Re(z) > \alpha$.

From the linearity property of $\nabla - integral$, we may easily deduce a *linearity property* of the generalized nabla time scale Laplace transform.

The generalized transform also has the well-known *transform of the derivative property*, ([14],[16],[18]).

Theorem 3.1.3. For $m \in \mathbb{N} \cup \{0\}$:

$$\mathcal{L}_s^{\mathbb{T}}\{\nabla^k f\}(z) = z^k \mathcal{L}_s^{\mathbb{T}}\{f\}(z) - \sum_{j=0}^{k-1} z^j \nabla^{m-j-1} f(t)|_{t=a}.$$

The generalized notion of convolution was resolved in the delta case [19] via dynamic version of transport equation. Later on, it was adapted for the nabla case in [7].

Definition 3.1.3. The shift of $f : (a, \infty)_{\mathbb{T}} \to \mathbb{R}$ is defined as a solution of the shifting problem:

$$\begin{aligned} \nabla_t u(t, \rho(s)) &= -\nabla_s u(t, s) & s, t \in \mathbb{T} \quad t > s > a, \\ u(t, a) &= f(t) & t \in \mathbb{T} \quad t > a. \end{aligned}$$

Note that for the constant graininess function $u(t, s) = f(t - s)$.

Definition 3.1.4. Let $s, t \in \mathbb{T}$, $t \geq s$. The convolution of $f, g : \mathbb{T} \to \mathbb{R}$ is defined as:

$$(f * g)(t, s) := \int_s^t \hat{f}(t, \rho(\tau)) g(\tau) \nabla\tau,$$

where $\hat{f}$ is the shift of f.

The following *Convolution theorem* was proved in [7].

Theorem 3.1.4. Let $f, g : \mathbb{T} \to \mathbb{R}$ be functions, s.t. $\mathcal{L}_s^{\mathbb{T}}(f)$,$\mathcal{L}_s^{\mathbb{T}}(g)$ exist. Then:

$$\mathcal{L}_s^{\mathbb{T}}\{(f * g)(\cdot, s)\}(z) = \mathcal{L}_s^{\mathbb{T}}(f) \cdot \mathcal{L}_s^{\mathbb{T}}(g).$$

Utilizing the definition of the generalized nabla time scale Laplace transform, we may calculate the following transform of generalized nabla exponential functions (see [14]).

Example 3.1.1. The generalized nabla time scale Laplace transform of nabla exponential function $\hat{e}_c(t, s)$ via Theorems 2.3.6 e), 2.3.4, 2.3.3:

$$\begin{aligned} \mathcal{L}_s^{\mathbb{T}}\{\hat{e}_c(\rho(t), s)\}(z) &= \int_s^\infty \hat{e}_c(t, s)\hat{e}_{\ominus z}(\rho(t), s)\nabla t \\ &= \int_s^\infty \frac{1}{1 - \nu(t)z} \hat{e}_{c\ominus z}(t, s)\nabla t \\ &= \int_s^\infty \frac{c \ominus z}{c - z} \hat{e}_{c\ominus z}(t, s)\nabla t \\ &= \frac{1}{c - z} \int_s^\infty (c \ominus z)\hat{e}_{c\ominus z}(., s)\nabla t \\ &= \frac{1}{c - z}. \end{aligned}$$

Theorem 3.1.5. For $n \in \mathbb{Z}$:

$$\mathcal{L}_s^{\mathbb{T}}\{\hat{h}_n(\cdot, 0)\}(z) = z^{-n-1}.$$

The following lemma utilizes the periodicity of a time scale to represent the generalized nabla time scale Laplace transform as a sum.

Lemma 3.1.1. Let $\mathbb{T}$ be a periodic time scale with period p. Let $f : \mathbb{T} \to \mathbb{R}$, and let the generalized nabla time scale Laplace transform of f exist. Then:

$$\mathcal{L}_s^{\mathbb{T}}\{f\}(z) = \sum_{n=0}^{\infty} (\hat{e}_{\ominus z}(s+p, s))^n \Big(\int_{s+np}^{s+(n+1)p} f(t)\hat{e}_{\ominus z}(\rho(t), s+np)\nabla t\Big).$$

Proof. Let us use the linearity property of time scale integral:

$$\mathcal{L}_s^{\mathbb{T}}\{f\}(z) = \sum_{n=0}^{\infty} \int_{s+np}^{s+(n+1)p} f(t)\hat{e}_{\ominus z}(\rho(t), s)\nabla t.$$

We utilize the semi-group property of time scale exponential 2.3.5:

$$\mathcal{L}_s^{\mathbb{T}}\{f\}(z) = \sum_{n=0}^{\infty} \int_{s+np}^{s+(n+1)p} f(t)\hat{e}_{\ominus z}(s+np, s)\hat{e}_{\ominus z}(\rho(t), s+np)\nabla t.$$

Since $\mathbb{T}$ is periodic, we know that $\hat{e}_{\ominus z}(s+np, s+(n-1)p) = \hat{e}_{\ominus z}(s+p, s)$ for any $n \in \mathbb{N}$. Thus, $\hat{e}_{\ominus z}(s+np, s) = (\hat{e}_{\ominus z}(s+p, s))^n$. Therefore:

$$\mathcal{L}_s^{\mathbb{T}}\{f\}(z) = \sum_{n=0}^{\infty} (\hat{e}_{\ominus z}(s+p, s))^n \int_{s+np}^{s+(n+1)p} f(t)\hat{e}_{\ominus z}(\rho(t), s+np)\nabla t.$$

□

3.2. Domain of convergence

An important issue in an investigation of the generalized nabla time scale Laplace transform is the so called domain of convergence $\mathcal{D}$, which we defined in 3.1.1. In the following examples, we provide basic definitions regarding the domain of convergence of the generalized nabla time scale Laplace trnasfom, estimations of the domain of convergence and some examples to demostrate some chosen properties of the domain of convergence. Some of the work presented in this section was inspired by the corresponding delta case presented in [13].

We define similarly as in [20]:

Definition 3.2.1. The *fundamental region* of the generalized nabla time scale Laplace transform is a Hilger circle with radius $R = \frac{1}{\min(\nu(k))}$, centred at $\frac{1}{\min(\nu(k))}$. In the case of $\nu(t) = 0$ for some $t \in \mathbb{T}$, the fundamental region of convergence is the set of all points $z \in \mathbb{C}$, s.t. $\mathrm{Re}(z) > 0$.

The main reason to define fundamental region of convergence is the fact that it contains all zeros of the generalized time scale exponential function.

Example 3.2.1. Let us consider the periodic time scale $\mathbb{T} = \{0, 1, 7, 8, 14, 15, ...\}$ with the backward graininess function $\nu(\sigma^{2k+1}(t)) = 1$ and $\nu(\sigma^{2k}(t)) = 6$ for $k \in \mathbb{N}$.
It is clear that the fundamental region of the generalized nabla time scale Laplace transform on $\mathbb{T}$ is the Hilger circle centered at 1 with radius $R = 1$.

For another examination of the domain of convergence, we use the the generalized nabla time scale Laplace transform of 1. Utilizing theorem 2.2.4 and representation theorem for periodic time scales 3.1.1, we obtain:

$$\begin{aligned}\mathcal{L}_0^{\mathbb{T}}\{1\}(z) &= \sum_{n=0}^{\infty}(\hat{e}_{\ominus z}(s+p,s))^n\left(\int_{s+np}^{s+(n+1)p} \hat{e}_{\ominus z}(\rho(t), s+np)\nabla t\right) \\ &= \sum_{n=0}^{\infty}(1-z)^n(1-6z)^n(1+6(1-z)) \\ &= 1 + 6(1-z) + (1-z)(1-6z) + 6(1-z)^2(1-6z) + ...\end{aligned}$$

A necessary condition for this series to converge is $|(1-z)(1-6z)| < 1$. From this property, we may observe that the domain of convergence is not connected, as shown in the following figure.

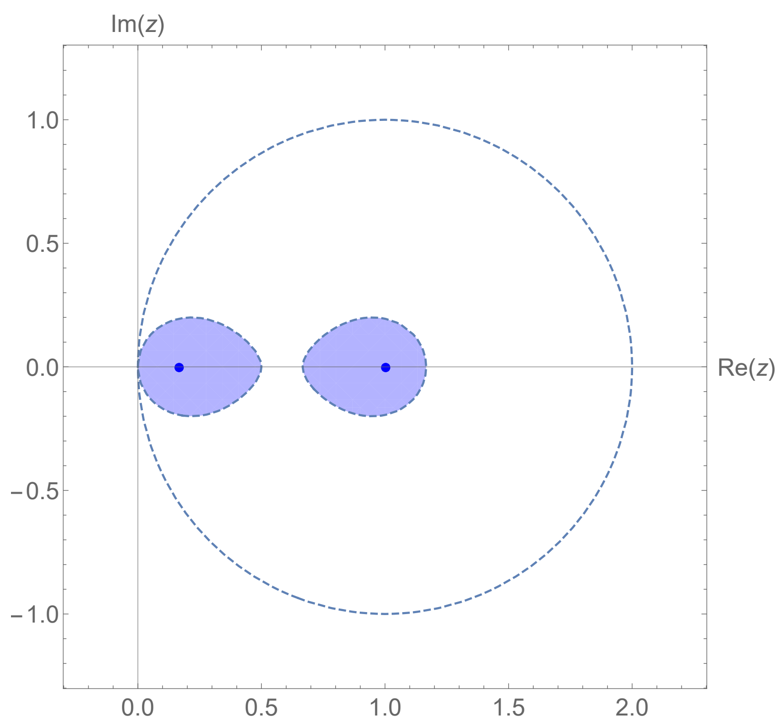

Figure 3.1: The blue area represents the domain of convergence of unitary function on $\mathbb{T} = \{0, 1, 7, 8, 14, 15, ...\}$. We observe that we may place two balls of a sufficiently small radius ϵ centred at $\frac{1}{6}$ and 1 that both lie in domain of convergence. However, the domain of convergence is not connected. Secondly, we observe that the fundamental region of convergence - boundary represented by the dashed blue circle - is not equal to the domain of convergence.

Example 3.2.2. Let us consider the time scale $\mathbb{T} = \mathbb{N} \cup \{0, \frac{1}{n}\}$, where $0 < \frac{1}{n} < 1$. It is clear that the fundamental region of the generalized nabla time scale Laplace transform on $\mathbb{T}$ is the Hilger circle centred at 1 with radius $R = max(n, 1)$.

For examination of the domain of convergence, we again use the generalized nabla time scale Laplace transform of the unitary function. We obtain:

$$\mathcal{L}_0^{\mathbb{T}}\{1\}(z) = \sum_{t\in(0,\infty)_{\mathbb{T}}} \nu(t) \prod_{x\in(0,\rho(t))} (1-\nu(x)z) =$$

$$= \frac{1}{n} + \frac{n-1}{n}(1-\frac{1}{n}z) + (1-\frac{1}{n}z)(1-\frac{n-1}{n}z)\sum_{m=0}^{\infty}(1-z)^m.$$

A necessary condition for this series to converge is $|1-z| < 1$. Note that the value of z in first two terms has no impact on the convergence of whole series.

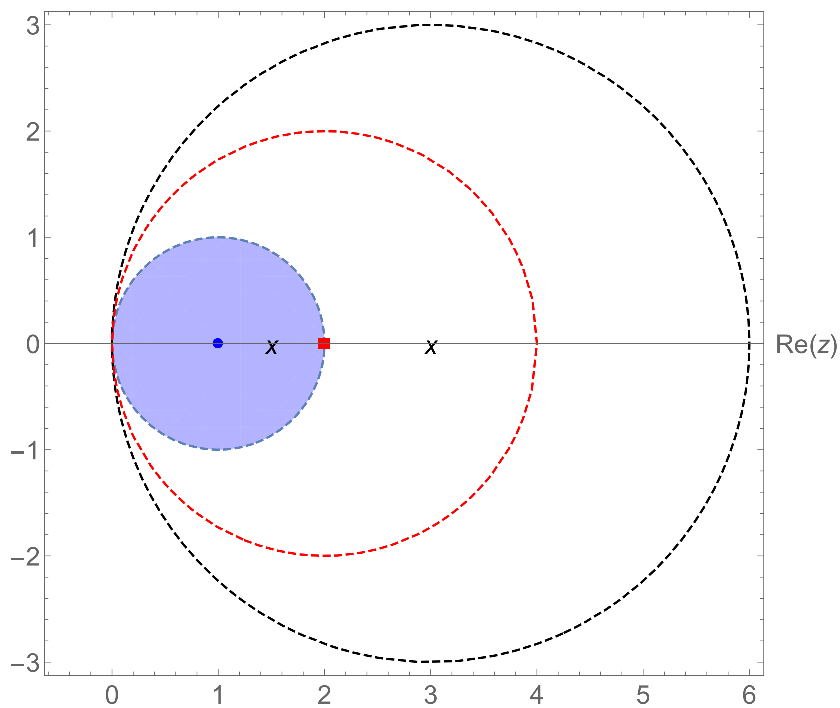

Figure 3.2: The blue area represents the domain of convergence of 1 on $\mathbb{T} = \mathbb{N} \cup \{0, \frac{1}{n}\}$ where $0 < \frac{1}{n} < 1$ for chosen values $n = 2, 3$. For $n = 2$ the red cube is the point of convergence of series of the generalized nabla time scale Laplace transform, red circle is the boundary of the fundamental region of convergence. For $n = 3$ - black crosses are points of convergence, the black circle is the boundary of the fundamental region of convergence.

In the picture above, we may notice that there is a point of convergence for the case $n = 3$, namely the point $z = 3$, that lies outside of the domain of convergence $\mathcal{D}(1)$, but lies inside the fundamental region of convergence. However, the point $z = \frac{3}{2}$ lies inside the domain of convergence.
In the case $n = 2$, there is a point, namely the point $z = 2$, that lies on the boundary of $\mathcal{D}(1)$.

In the following example, we present the domain of convergence of the exponential function. Since in any form of Laplace transform, we only deal with functions of exponential order, it is convenient to examine this special case.

Example 3.2.3. Let $\mathbb{T}$ be a periodic time scale s.t. $\mathbb{T} = \mathbb{N} \cup \bigcup_{k \in \mathbb{N}\cup\{0\}}[3k, 3k+1]$. By the Theorem 3.1.1 the generalized nabla time scale Laplace transform of $f(t)$ can be written as:

$$\begin{aligned}\mathcal{L}_0^{\mathbb{T}}\{f(t)\}(z) &= \sum_{n=0}^{\infty}((1-z)e^{-z})^n \int_{3n}^{3n+2} f(t)\hat{e}_{\ominus z}(\rho(t), 3n)\nabla t = \\ &= \sum_{n=0}^{\infty}((1-z)e^{-z})^n (\int_{3n}^{3n+1} f(t)e^{-z(t-3n)}dt + e^{-z}(1-z)f(3n+2)).\end{aligned}$$

Consider the function:

$$\begin{aligned} f(t) &= e^{ct} & t &\in [3n, 3n+1] \\ f(3n+2) &= e^{c(3n)} & t &= 3n+2. \end{aligned}$$

Then, using substitution we obtain:

$$\begin{aligned}\mathcal{L}_0^{\mathbb{T}}\{f(t)\}(z) &= \sum_{n=0}^{\infty}((1-z)e^{-z})^n (\int_0^1 e^{c(x+3n)}e^{-z(x)}dx + e^{-z}(1-z)e^{c(3n)}) = \\ &= \sum_{n=0}^{\infty}((1-z)e^{-z}e^{3c})^n (\int_0^1 e^{c(x)}e^{-z(x)}dx + e^{-z}(1-z)).\end{aligned}$$

The sum converges if:

$$|(1-z)e^{-z}e^{3c})| < 1.$$

The following figures show the domain of convergence of $f(t)$ on time scale $\mathbb{T} = \mathbb{N} \cup \bigcup_{k \in \mathbb{N}\cup\{0\}}[3k, 3k+1]$ for different parameters:

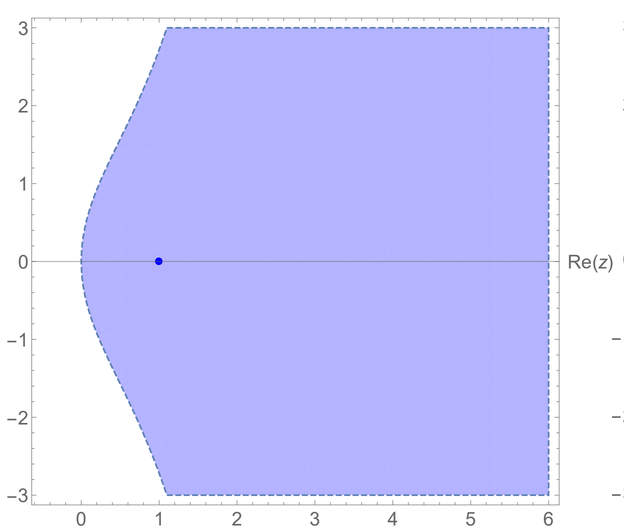

Figure 3.3: parameter $c = 0$.

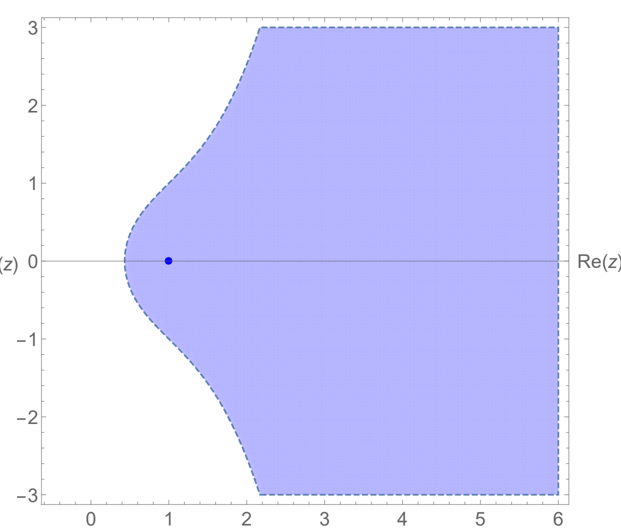

Figure 3.4: parameter $c = \frac{1}{3}$.

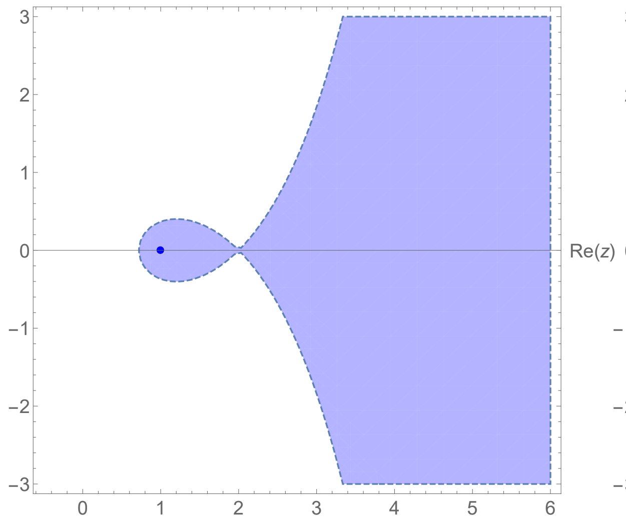

Figure 3.5: parameter $c = \frac{2}{3}$.

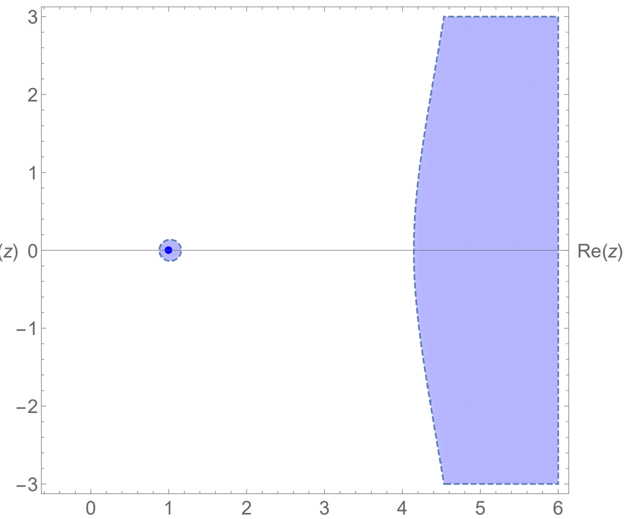

Figure 3.6: parameter $c = \frac{3}{4}$.

What we observe is again the possibility that the domain of convergence is not connected. Also, note that in every figure from some value α, complex numbers z, s.t. $Re(z) > \alpha$, belong to the domain of convergence. Thus, the domain of convergence is unbounded. This property could be expected, since the set with zero graininess function is of infinite measure. Secondly, we might notice that in every picture, the domain of convergence contains open subset containing point $z = 1$.

Previous observations lead to the next lemma, which is the *estimation of domain of convergence of periodic time scale.*

Lemma 3.2.1. Let $\mathbb{T}$ be a time scale, such that the generalized nabla time scale Laplace transform exists. Let $\mathbb{T}$ be the periodic time scale. Let $f : \mathbb{T} \to \mathbb{R}$ be the exponential function of type II.

Then, the domain of convergence $\mathcal{D}(f)$ contains the union of balls of radius $\epsilon > 0$ centred at $\frac{1}{\nu_i}$, where ν_i are values of the backward graininess function ν.

If $\nu = 0$ in any interval, then there exists a value α, s.t. the whole half plane $z > \alpha$ lies in the domain of convergence $\mathcal{D}(f)$.

Proof. Using the lemma 3.1.1, we write:

$$\mathcal{L}_s^{\mathbb{T}}\{f\}(z) = \sum_{n=0}^{\infty} (\hat{e}_{\ominus z}(s+p, s))^n \Big(\int_{s+np}^{s+(n+1)p} f(t)\hat{e}_{\ominus z}(\rho(t), s+np)\nabla t\Big).$$

Since f is a function of exponential order II, there is some constant $M > 0$, $c > 0$:

$$\mathcal{L}_s^{\mathbb{T}}\{f\}(z) \le \sum_{n=0}^{\infty} (\hat{e}_{\ominus z}(s+p, s))^n \Big(\int_{s+np}^{s+(n+1)p} M\hat{e}_c(\rho(t), s)\hat{e}_{\ominus z}(\rho(t), s+np)\nabla t\Big).$$

Utilizing the semi-group property of the time scale exponential function 2.3.5:

$$\sum_{n=0}^{\infty}(\hat{e}_{\ominus z}(s+p,s))^n(\int_{s+np}^{s+(n+1)p} M\hat{e}_c(\rho(t),s)\hat{e}_{\ominus z}(\rho(t),s+np)\nabla t) =$$
$$= M\sum_{n=0}^{\infty}(\hat{e}_{\ominus z}(s+p,s))^n(\int_{s+np}^{s+(n+1)p} \hat{e}_c(s+np,s)\hat{e}_c(\rho(t),s+np)\hat{e}_{\ominus z}(\rho(t),s+np)\nabla t).$$

Since $\mathbb{T}$ is periodic, $\hat{e}_c(s+np,s) = \hat{e}_c^n(p,s)$. Thus:

$$M\sum_{n=0}^{\infty}(\hat{e}_{\ominus z}(s+p,s))^n(\int_{s+np}^{s+(n+1)p} \hat{e}_c(s+np,s)\hat{e}_c(\rho(t),s+np)\hat{e}_{\ominus z}(\rho(t),s+np)\nabla t) =$$
$$= M\sum_{n=0}^{\infty}(\hat{e}_{\ominus z}(s+p,s)\hat{e}_c(s+p,s))^n(\int_{s+np}^{s+(n+1)p} \hat{e}_{c\ominus z}(\rho(t),s+np)\nabla t) =$$
$$= M\sum_{n=0}^{\infty}(\hat{e}_{c\ominus z}(s+p,s))^n(\int_{s+np}^{s+(n+1)p} \hat{e}_{\ominus z}(\rho(t),s+np)\nabla t).$$

By the integration we obtain:

$$M\sum_{n=0}^{\infty}(\hat{e}_{c\ominus z}(s+p,s))^n(\int_{s+np}^{s+(n+1)p} \hat{e}_{c\ominus z}(\rho(t),s+np)\nabla t) =$$
$$= M\sum_{n=0}^{\infty}(\hat{e}_{c\ominus z}(s+p,s))^n(\hat{e}_{c\ominus z}(s+(n+1)p,s+np)-1) =$$
$$= M\sum_{n=0}^{\infty}(\hat{e}_{c\ominus z}(s+p,s))^n(\hat{e}_{c\ominus z}(s+p,s)-1).$$

Therefore, the generalized nabla time scale Laplace transform converges whenever:

$$|\hat{e}_{c\ominus z}(s+p,s)| < 1,$$

or equivalently, if:

$$|\hat{e}_{\ominus z}(s+p,s)| < |\hat{e}_{\ominus c}(s+p,s)|.$$

We aim to show that around any point $\frac{1}{\nu(x)}$ for $x \in \mathbb{T}$, there is a ball centred around $\frac{1}{\nu(x)}$ that lies in the domain of convergence.
Let us choose $z_x = \frac{1}{\nu(x)} + \epsilon e^{i\phi}$, where $\phi \in [0, 2\pi)$ and $x \in \mathbb{T}$, $x \neq 0$. The exponential may be rewritten in the following fashion:

$$|\hat{e}_{\ominus z}(s+p,s)| = |e^{-z_x L} \prod_{\tau \in (s,s+p]_{\mathbb{T}_d}} (1-\nu(\tau)z_x)|,$$

where $L = \sum_j L_j$, such that L_j are lengths of continuous intervals of time scale $\mathbb{T}$, and $\mathbb{T}_d$ is part of the time scale, where $\nu(t) \neq 0$ (discrete). Thus:

$$|e^{-z_x L} \prod_{\tau \in (s,s+p]_{\mathbb{T}_d}} (1-\nu(\tau)z_x)| = |e^{-(\frac{1}{\nu(x)}+\epsilon e^{i\phi})L} \prod_{\tau \in (s,s+p]_{\mathbb{T}_d}} (1-\nu(\tau)(\frac{1}{\nu(x)}+\epsilon e^{i\phi}))|.$$

There is at least one value τ, s.t. $\tau = x$, so the following holds:

$$|e^{-(\frac{1}{\nu(x)}+\epsilon e^{i\phi})L} \prod_{\tau \in (s,s+p]_{\mathbb{T}_d}} (1 - \nu(\tau)(\frac{1}{\nu(x)} + \epsilon e^{i\phi}))| =$$
$$= \epsilon e^{-(\frac{1}{\nu(x)}+\epsilon e^{i\phi})L} | \prod_{\tau \in (s,s+p]_{\mathbb{T}_d \setminus \{x\}}} (1 - \nu(\tau)(\frac{1}{\nu(x)} + \epsilon e^{i\phi}))|.$$

We conclude that if there is a finite number of values $\tau \in (s, s+p]_{\mathbb{T}_d \setminus \{x\}}$, this quantity can be arbitrarily small and, therefore, the ball of radius ϵ around $\frac{1}{\nu(x)}$ lies in the domain of convergence $\mathcal{D}(f)$ for any $x \in \mathbb{T}_d$.

On the other hand, if there is an infinite number of points, it must be true that $\nu(\tau)$, for $\tau \in (s, s+p]_{\mathbb{T}_d \setminus \{x\}}$, tends to zero on some subset of $\mathbb{T}_d$ (otherwise, there is no such p that $\mathbb{T}$ is periodic) and since the generalized nabla time scale Laplace transform exists, the infinite product converges. From these assumptions we arrive to the same conclusion: the ball of radius ϵ around $\frac{1}{\nu(x)}$ lies in the domain of convergence $\mathcal{D}(f)$.

Since the choice of x is arbitrary (but discrete), we may conclude that around any point $\frac{1}{\nu(x)}$, there is a ball of radius ϵ lying in the domain of convergence $\mathcal{D}(f)$.

It remains to prove that there is a value α, for which for any z, s.t. $Re(z) > \alpha$, the generalized nabla time scale Laplace transform converges. This is obvious, since the real exponential is a function that grows faster than any polynomial. Therefore, the quantity $|e^{-z_x L} \prod_{\tau \in (s,s+p]_{\mathbb{T}_d}} (1 - \nu(\tau) z_x)|$ tends to zero as z_x goes to infinity.

This proves our assertion. □

Motivated by the proof of Lemma 3.2.1, we define:

Definition 3.2.2. Let $\mathbb{T}$ be a time scale. The union of balls B_{ϵ_i} of radius $\epsilon_i > 0$ in complex plane, each of them centred around the points of $\frac{1}{\nu(t)}$ for all $t \in \mathbb{T}$, possibly in union with the plane $Re(z) > \alpha$, $\alpha \in \mathbb{R}$ (if $\mathbb{T}$ contains the interval), such that the whole union lies in the domain of convergence of the generalized nabla time scale Laplace transform is called a *guaranteed domain of convergence* of the generalized time scale Laplace transform.

Theorem 3.2.1. The generalized nabla time scale Laplace transform converges absolutely inside the guaranteed domain of convergence.

Proof. The proof of this theorem is a direct consequence of the proof of Lemma 3.2.1. □

The estimation of the guaranteed domain of convergence lemma can be extended to a slightly more general time scale:

Lemma 3.2.2. Let $\mathbb{T}$ be a time scale, such that the generalized nabla time scale Laplace transform exists and $\mathbb{T} = \mathbb{T}_1 \cup \mathbb{T}_p$, where T_1 is an arbitrary time scale bounded by $r \in \mathbb{R}$ and T_p is a periodic time scale s.t. $inf\{\mathbb{T}_p\} = r$. Let $f : \mathbb{T} \to \mathbb{R}$ be the exponential function of type II.

Then, the domain of convergence $\mathcal{D}(f)$ contains the union of balls of radius $\epsilon > 0$ centred at $\frac{1}{\nu_i}$, where ν_i are the values of the backward graininess function $\nu(t)$ for $t > r$.

If $\nu(t) = 0$, $t > r$ in any interval, instead of ball of radius ϵ centred at $\frac{1}{\nu_i}$, then, there exists a value α, s.t. the whole half plane $z > \alpha$ lies in the domain of convergence $\mathcal{D}(f)$.

Proof. Let us use the linearity of the generalized nabla time scale Laplace transform:

$$\mathcal{L}_s^{\mathbb{T}}\{f\}(z) = \int_s^r f(t)\hat{e}_{\ominus z}(\rho(t), s)\nabla t + \int_r^\infty f(t)\hat{e}_{\ominus z}(\rho(t), s)\nabla t.$$

For the estimation of the guaranteed domain of convergence, we may neglect the first integral, since it has no impact on the convergence of the second integral. Then, the proof is a trivial consequence of the previous theorem, since:

$$\int_r^\infty f(t)\hat{e}_{\ominus z}(\rho(t), s)\nabla t = \hat{e}_{\ominus z}(s, r)\mathcal{L}_r^{\mathbb{T}}\{f\}(z).$$

□

One might think that the domain of convergence $\mathcal{D}(f)$ is always a subset of the fundamental region of convergence. But this is not the case.

Example 3.2.4. Let $\mathbb{T} = \mathbb{Z}$. Let $f(t) = \hat{e}_{\ominus 0.9}(\rho(t), 0)$. Then:

$$\mathcal{L}_s^{\mathbb{T}}\{\hat{e}_{\ominus 0.9}(\rho(t), 0)\}(z) = \sum_{n=0}^{\infty}(1 - z)^n(0,1)^n.$$

Clearly, the domain of convergence is the circle centred in 1 with radius 10, while the fundamental region of convergence is also centred in 1 but of radius 1.

3.3. Generalized Lerch's theorem on uniform time scales

To better understand the nature of the generalized nabla time scale Laplace transform, it is convenient to examine the special cases on uniform time scales. This chapter is dedicated to the examination of such cases - namely, the case of real numbers and integers (we also examine the case $\mathbb{T} = h\mathbb{Z}$, but it is only a trivial generalization of $\mathbb{T} = \mathbb{Z}$). The real case is well-documented over the course of history, however, we need to investigate the proof methods used to be able to find possible generalizations to an arbitrary time scale.

Definition 3.3.1. We call time scale $\mathbb{T}$ to be a *uniform time scale*, if $\nu(t) = c$ for all $t \in \mathbb{T}$ and for some $c \in \mathbb{R}$.

3.3.1. Real Laplace transform

The most well-known case of the generalized nabla time scale Laplace transform is the real Laplace transform - $\mathbb{T} = \mathbb{R}$. We investigate the proof methods found in literature and discuss applicability of these methods in proofs for more general time scales. Two different proof methods are presented in [21],[22].

It is impossible to show for the real Laplace transform that the mapping from the image function to the original function is uniquely determined. However, this fact holds for two functions that differ only in a finite number of points - or in the words of Lebesgue measure - that differ only by a set of zero Lebesgue measure. For this purpose, we need the following definition.

Definition 3.3.2. $f : \mathbb{R} \to \mathbb{R}$ is a *null-function* if and only if:

$$\int_0^t f(\tau)d\tau := 0$$

for all $t \geq 0$.

The proof of the following lemma can be found at [21]:

Lemma 3.3.1. If the Laplace transform of f:

$$\mathcal{L}_0^{\mathbb{R}}\{f\}(z)$$

converges for $z = z_0$, then it converges in an open half plane $Re(z) > Re(z_0)$, where it can be expressed by the absolutely converging integral:

$$(z - z_0)\int_0^\infty e^{-(z-z_0)t}\phi(t)dt$$

with

$$\phi(t) = \int_0^t e^{-z_0\tau} f(\tau)d\tau.$$

The proof of the following lemma can be found at [21]:

Lemma 3.3.2. Let ψ be a continuous function, $a, b \in \mathbb{R}$, $a > b$ and suppose:

$$\int_a^b x^n \psi(x)dx = 0$$

holds on (a, b) and for $n = 0, 1, ...$
Then, $\psi(x) = 0$ for all $x \in (a, b)$.

We continue by stating Lerch's theorem, after which we discuss the two proof methods found in literature.

Theorem 3.3.1. If Laplace transform $\mathcal{L}_0^{\mathbb{R}}\{f\}$ vanishes on an infinite sequence of points that are located at equal intervals along a line parallel to the real axis:

$$\mathcal{L}_0^{\mathbb{R}}\{f\}(z_0 + n\xi) = 0 \qquad (\xi > 0, n = 1, 2, ...) \tag{3.2}$$

with z_0 being the point of convergence of $\mathcal{L}_0^{\mathbb{R}}\{f\}$; then, it follows that $f(t)$ is a null-function.

The following proof can be found at [21]:

Proof. Invoking the lemma 3.3.1, for $Re(z) > Re(z_0)$:

$$\mathcal{L}_0^{\mathbb{R}}\{f\}(z) = (z - z_0)\int_0^\infty e^{-(z-z_0)t}\phi(t)dt$$

with

$$\phi(t) = \int_0^t e^{-z_0\tau} f(\tau)d\tau.$$

Hence:

$$\mathcal{L}_0^{\mathbb{R}}\{f\}(z_0 + n\xi) = (n\xi)\int_0^{\infty} e^{-n\xi t}\phi(t)dt.$$

By hypothesis 3.2:

$$\int_0^{\infty} e^{-n\sigma t}\phi(t)dt = 0$$

for all $n = 1, 2, \ldots.$ Employing the substitution $e^{-\xi t} = x$, $t = -\frac{\log(x)}{\xi}$, $\phi(-\frac{\log(x)}{\xi}) = \psi(x)$, we rewrite the last equation:

$$\frac{1}{\xi}\int_0^{1} x^{n-1}\psi(x)dx = 0$$

for $n = 1, 2, \ldots$ or $\int_0^1 x^n\psi(x)dx = 0$ for $n = 0, 1, \ldots.$
We define:

$$\psi(0) = \lim_{t\to\infty}\phi(t) = \mathcal{L}_0\{f\}(z_0)$$

and

$$\psi(1) = \phi(0) = 0$$

to make ψ continuous. Thus, by applying Lemma 3.3.2 we get:

$$\psi(x) = 0$$

for all $x \in (0, 1)$, that is:

$$\phi(t) = \int_0^{t} e^{-z_0\tau} f(\tau)d\tau = 0 \qquad \text{for all } t \geq 0.$$

Per partes integration yields:

$$e^{-z_0 t}\int_0^{t} f(\tau)d\tau + z_0\int_0^{t} e^{-z_0\tau}d\tau\int_0^{\tau} f(u)du = 0 \qquad \text{for all } t \geq 0.$$

Differentiation produces:

$$-z_0 e^{-z_0 t}\int_0^{t} f(\tau)d\tau + e^{-z_0 t}\frac{d}{dt}\int_0^{t} f(\tau)d\tau + z_0 e^{-z_0 t}\int_0^{t} f(u)du = 0 \qquad \text{for all } t \geq 0.$$

Therefore:

$$\frac{d}{dt}\int_0^{t} f(\tau)d\tau = 0 \qquad \text{for all } t \geq 0.$$

With $\int_0^t f(\tau)d\tau$ for $t = 0$, it follows that:

$$\int_0^{t} f(\tau)d\tau = 0 \qquad \text{for all } t \geq 0.$$

That is, f is a null-function. □

We state the consequence of this theorem - the uniqueness theorem, found at [21]:

Theorem 3.3.2. Two original functions, whose image functions assume equal values on an infinite sequence of points that are located along a line parallel to the real axis, differ at most by a null-function.

When trying to generalize this proof to an arbitrary time scales, the following questions arise:

- Which assumptions are the most important?
- What can and what cannot be generalized on an arbitrary time scale?
- Is it possible to simplify such a proof?

In the real Laplace transform, the domain of convergence plays a crucial role, but is often ignored - since it is easy to show that for some values z, s.t. $Re(z) > \alpha$, $\alpha \in \mathbb{R}$, the Laplace transform converges. In this case, we extended the domain of convergence from a point to the half-plane of convergence by Lemma 3.3.1. Moreover, the integral converges absolutely on this set. Therefore, we do not need to go into further specifications of the domain of convergence, since the half-plane provides everything necessary for the proof.

In the real Lerch's theorem, we assume that there exists an infinite sequence of equidistant points parallel to the real line. The following theorem - (The Principle of Analytic continuation - Identity theorem found at [24]) shows that if Laplace transform converges to zero on this sequence of numbers, then it converges in the whole domain of convergence.

Theorem 3.3.3. Let f and g be analytic in a region A. Suppose that there is a sequence $z_1, z_2, ...$ of distinct points of A converging to $z_0 \in A$, such that $f(z_n) = g(z_n)$ for all $n \in \mathbb{N}$. Then, $f = g$ on all of A. The conclusion is valid, in particular, if $f = g$ on some neighbourhood of some point in A.

There also exists a relaxed assumption for this proof, that is - let the assumption 3.2 hold on half-plane for some $Re(z) > \alpha$, $\alpha \in \mathbb{R}$.
It is not an easy task to generalize such an assumption to an arbitrary time scale - firstly, we have to consider the changes to the domain of convergence based on the time scale that we are examining. Then, we have to make a reasonable assumption based on the character of the given domain of convergence. As we may observe in the next chapters - by making unreasonable assumptions, we obtain unreasonable results. As we have seen in the previous chapter, the domain of convergence might be concentrated around some point in $\mathbb{C}$ and it might be a bounded set. In that case, we are unable to make a sequence of equidistant points. However, we may assume that the condition holds on some neighbourhood of such point (and possibly relax the assumption to the sequence of points). Secondly, the domain of convergence might not be connected - thus, we need to make assumptions for every region contained in the domain of convergence.

Since the domain of convergence is in historical cases a region, it might be tempting to call it the 'region of convergence'. However, it is not a region. That is the reason, why we call it in this work a domain of convergence instead of a region of convergence..

Second idea of this proof, which in our opinion may not be neglected in the generalization attempts, is an attribute of the real exponential function:

$$e^{-(z_0+\sigma k)t} = e^{-z_0 t}e^{-\sigma k t}.$$

This operation looks trivial on real numbers, yet the generalization of such result is not trivial at all.
In time scale calculus, we defined in 2.3.3, 2.3.4 the generalized plus and generalized minus, which at a first glance works fine for this purpose and possibly may be utilized in proving results on more general time scales than $\mathbb{R}$:

$$\hat{e}_{\ominus(z_0\oplus\sigma k)}(\rho(t), s) = \hat{e}_{\ominus z_0}(\rho(t), s)\hat{e}_{\ominus\sigma k}(\rho(t), s). \tag{3.3}$$

However, there is a serious issue with this operation in the proof of generalized Lerch's theorem, which is the fact that $z_0 \oplus \sigma k$ is a function of t instead of a point in the domain of convergence. Therefore, we are unable to utilize similar results on the sequence of points in the domain of convergence - points of the form $z_0 \oplus \sigma k$, since this quantity is a function of t, not a constant point.
Moreover, trying to imitate the proof methods requires the definition of generalized multiplication 2.3.10, which prevents us in utilization of same proof methods.

For the simplified version of the proof of Lerch's theorem, we need Fatou's lemma found in [23]:

Theorem 3.3.4. Let $\{f_n\}_n^\infty$ be a sequence of non-negative measurable functions. Define $f = \liminf_{n\to\infty} f_n$ almost everywhere on a set E. Then, f is measurable and:

$$\int_E f \leq \liminf_{n\to\infty} \int_E f_n.$$

Let us present the simplified version of the proof of Lerch's theorem 3.3.1:

Proof. Let $z_k = z + nk$. From the assumption of the theorem:

$$\mathcal{L}_0^{\mathbb{R}}\{f\}(z_k) = \int_0^\infty f(t)e^{-z_k t}dt = 0.$$

Choose $d > 0$. Without loss of generality, we may choose $\tau \in [0, d]$ s.t. $f(t) \geq 0$ on $[0, \tau]$. Let us multiply the quantity with $e^{\tau z_k}$:

$$0 = \int_0^\infty f(t)e^{-z_k(t-\tau)}dt.$$

From the linearity property of the real Laplace transform:

$$0 = \int_0^\tau f(t)e^{-z_k(t-\tau)}dt + \int_\tau^\infty f(t)e^{-z_k(t-\tau)} = \int_0^\tau f(t)e^{-z_k(t-\tau)}dt + \mathcal{L}_\tau^{\mathbb{R}}\{f\}(z_k).$$

Since the $\lim_{z_k\to\infty} \mathcal{L}_\tau^{\mathbb{R}}\{f\}(z_k) \to 0$, we may take the limit $z_k \to \infty$ and split the limits:

$$0 = \lim_{z_k\to\infty} \int_0^\tau f(t)e^{-z_k(t-\tau)}dt + \lim_{z_k\to\infty} \mathcal{L}_\tau^{\mathbb{R}}\{f\}(z_k).$$

Therefore:

$$0 = \lim_{z_k \to \infty} \int_0^\tau f(t) e^{-z_k(t-\tau)} dt.$$

Since the sequence of functions $f(t)e^{-z_k(t-\tau)}$ is the sequence of positive measurable functions on $[0, \tau]$ we may conclude by Fatou's lemma 3.3.4:

$$0 = \lim_{z_k \to \infty} \int_0^\tau f(t) e^{-z_k(t-\tau)} dt \geq \int_0^\tau \liminf_{z_k \to \infty} f(t) e^{-z_k(t-\tau)} dt.$$

Let us suppose by contradiction that f is not a null function on $[0, \tau]$. Then, it must be true that:

$$\liminf_{z_k \to \infty} f(t) e^{-z_k(t-\tau)} = \infty$$

for some t, since f is non-negative. This produces a contradiction, so our claim is satisfied - f is the null function on $[0, \tau]$. We may repeat the process for any non-positive or non-negative part of f and conclude that f is a null function on $[0, d]$. Since the choice of d is arbitrary, it must be true that f is the null function on $\mathbb{R}$. $\square$

Note, that the choice of z_k might be different (but $\lim_{k\to\infty} z_k$ must still diverge, and all of z_k must lie in the domain of convergence).

3.3.2. Laplace transform on integers

Another case of the generalized nabla time scale Laplace transform is the case $\mathbb{T} = \mathbb{Z}$. In this subsection, we present the proof of Lerch's theorem for integers.

From the definition of the time scale exponential 2.3.6 and the generalized nabla time scale Laplace transform 3.1.1, we obtain:

$$\begin{aligned}
\mathcal{L}_s^{\mathbb{Z}}\{f\}(z) &= \int_s^\infty f(t) \hat{e}_{\ominus z}(\rho(t), s) \nabla t \\
&= \sum_{t=s+1}^{\infty} \nu(t) f(t) (1-z)^{\rho(t)-s} \\
&= \sum_{t=s+1}^{\infty} f(t)(1-z)^{t-s-1} \\
&= \sum_{t=0}^{\infty} f(t+s+1)(1-z)^t.
\end{aligned}$$

Firstly, we may notice that this series is in fact Laurent (or possibly Taylor for $s > 0$) expansion with coefficients $f(t+s+1)(-1)^t$.
Secondly, it is useful to look into the lemma 3.2.1 to realize that the domain of convergence of the generalized nabla time scale Laplace transform on $\mathbb{T} = \mathbb{Z}$ will always be centred around 1, so we are always able to find a ball $B_\epsilon(1)$, centred at 1 of radius ϵ, such that inside the ball, the Laplace integral converges.

Now, we have enough background to study the uniqueness of such a transform.

Theorem 3.3.5. Let $\mathbb{T} = \mathbb{Z}$. Suppose:

$$\int_s^\infty f(t)\hat{e}_{\ominus z}(\rho(t), s)\nabla t = 0$$

for all $z \in B_\epsilon(1)$.
Then, $f(t) = 0$ for all $t \in \mathbb{T}$, $t > s$.

The following proof is a slightly modified proof of the uniqueness of Laurent series stated in [24]:

Proof. From the assumption, we know that inside $B_\epsilon(1)$:

$$\int_s^\infty f(t)\hat{e}_{\ominus z}(\rho(t), s)\nabla t = \sum_{t=0}^{\infty} f(t+s+1)(1-z)^t = 0$$

Multiplying the expression by $(1-z)^{-n-1}$ yields:

$$\sum_{t=0}^{\infty} f(t+s+1)(1-z)^{t-n-1} = 0.$$

Let γ be the path inside $B_\epsilon(1)$. Since the series converges uniformly inside $B_\epsilon(1)$, we may integrate over γ and interchange the sum with the integral.

$$\sum_{t=0}^{\infty} f(t+s+1)\int_\gamma (1-z)^{t-n-1} dz = 0$$

From the Cauchy integral formula, we know that:

$$\int_\gamma (1-z)^{t-n-1} dz = 0$$

for $t \neq n$, and for $t = n$:

$$\int_\gamma (1-z)^{t-n-1} dz = -2\pi i$$

Therefore:

$$0 = \sum_{t=0}^{\infty} f(t+s+1)\int_\gamma (1-z)^{t-n-1} dz = -2\pi i \delta_{nk} f(n+s+1)$$

Finally we obtain the result that f vanishes in every point $t \in [s, \infty]_{\mathbb{T}}$. □

Note that since we are on a discrete time scale, we do not need the definition of a null function.

3.3.3. Laplace transform on $h\mathbb{Z}_a$

This is a slightly more general case than $\mathbb{T} = \mathbb{Z}$, even though the proof method stays the same. We define the set $h\mathbb{Z}_a = \{..., a-2h, a-h, a, a+h, a+2h, ...\}$.

Theorem 3.3.6. Let $\mathbb{T} = h\mathbb{Z}_a$. Suppose:

$$\int_s^\infty f(t)\hat{e}_{\ominus z}(\rho(t), s)\nabla t = 0$$

for all $z \in B_\epsilon(\frac{1}{h})$.
Then, $f(t) = 0$ for all $t \in \mathbb{T}$.

Proof. From the assumption, we know that inside $B_\epsilon(1)$:

$$\int_s^\infty f(t)\hat{e}_{\ominus z}(\rho(t), s)\nabla t = \sum_{t=0}^{\infty} f(t+s+1)(1-hz)^t = 0$$

Multiply the expression by $(1-hz)^{-n-1}$.

$$\sum_{t=0}^{\infty} f(t+s+1)(1-hz)^{t-n-1} = 0$$

Let γ be the path inside $B_\epsilon(\frac{1}{h})$. Since the series converges uniformly inside $B_\epsilon(\frac{1}{h})$, we might integrate over γ and interchange the sum and the integral.

$$\sum_{t=0}^{\infty} f(t+s+1)\int_\gamma (1-hz)^{t-n-1}dz = 0$$

From the Cauchy integral formula, we know that:

$$\int_\gamma (1-hz)^{t-n-1}dz = 0$$

for $t \neq n$, and for $t = n$:

$$\int_\gamma (1-hz)^{t-n-1}dz = -\frac{1}{h}2\pi i$$

Therefore:

$$0 = \sum_{t=0}^{\infty} f(t+s+1)\int_\gamma (1-hz)^{t-n-1}dz = -2\pi i\delta_{nk}\frac{1}{h}f(n+s+1)$$

Finally, we obtain the result that f vanishes in every point $t \in [s, \infty]_\mathbb{T}$. □

3.4. Generalized Lerch's theorem on non-uniform time scales

In this subsection, we provide counterexamples to the uniqueness of the generalized nabla time scale Laplace transform on some specific time scales.

3.4.1. Counterexamples

Let us start with a simple example of discrete time scales:

Example 3.4.1. Let $\mathbb{T} = \{0, 2, 3, 4, ...\} = (\mathbb{N} \cup \{0\}) \setminus \{1\}$. Suppose for $z \in B_\epsilon(1)$:

$$\mathcal{L}_0^{\mathbb{T}}\{f\}(z) = \int_0^\infty f(t)\hat{e}_{\ominus z}(\rho(t), 0)\nabla t = 0.$$

Then, by the Definition 3.1.1:

$$\mathcal{L}_0^{\mathbb{T}}\{f\}(z) = 2f(2) + (1-2z)\sum_{t=0}^{\infty} f(t+3)(1-z)^t.$$

Let us choose $f(t+3) = f(2)2^{t+1}$ for $t \geq 0$:

$$\begin{aligned}
\mathcal{L}_0^{\mathbb{T}}\{f\}(z) &= 2f(2) + 2(1-2z)\sum_{t=0}^{\infty} f(2)2^t(1-z)^t = \\
&= 2f(2) + 2(1-2z)\sum_{t=0}^{\infty} f(2)2^t(1-z)^t = \\
&= 2f(2) + 2f(2)(1-2z)\frac{1}{1-2(1-z)} = \\
&= 2f(2) + 2f(2)(1-2z)\frac{1}{-1+2z} = 0.
\end{aligned}$$

Therefore, we have found a function whose transform vanishes and is non-zero on a set of non-zero measure in complex plane, namely $2|1-z| < 1$. Thus, the generalized nabla time scale Laplace transform on this time scale is not unique, since we may add this function to any other function and obtain the same generalized nabla time scale Laplace transform.

We may look at the same problem from a different perspective. The expression:

$$2f(2) + (1-2z)\sum_{t=0}^{\infty} f(t+3)(1-z)^t = 0$$

may be rewritten as:

$$\frac{2f(2)}{1-2z} + \sum_{t=0}^{\infty} f(t+3)(-1)^t(z-1)^t = 0.$$

This means that the we are looking for the Taylor expansion of the function $-\frac{2f(2)}{1-2z}$ around 1 with coefficients $f(t+3)(-1)^t$. If such an expansion exists, the generalized nabla time scale Laplace transform on this time scale is not unique.

Moreover, from the uniqueness of Taylor series, we may notice that there is only one function f satisfying $\mathcal{L}_0^{\mathbb{T}}\{f\}(z) = 0$ with a domain of convergence of non-zero measure up to a multiplicative constant. This is true only for some examples.

Let us consider the following example of a discrete time scale:

Example 3.4.2. Let $\mathbb{T} = \mathbb{Z} \cup \{\frac{1}{2}\}$. Suppose for $z \in B_\epsilon(1)$:

$$\mathcal{L}_0^{\mathbb{T}}\{f\}(z) = \int_0^\infty f(t)\hat{e}_{\ominus z}(\rho(t),0)\nabla t = 0.$$

Then:

$$\int_0^\infty f(t)\hat{e}_{\ominus z}(\rho(t),0)\nabla t = \int_0^1 f(t)\hat{e}_{\ominus z}(\rho(t),0)\nabla t + \hat{e}_{\ominus z}(1,0)\int_1^\infty f(t)\hat{e}_{\ominus z}(\rho(t),1)\nabla t =$$

$$= f\Big(\frac{1}{2}\Big)\nu\Big(\frac{1}{2}\Big)\hat{e}_{\ominus z}(\rho\Big(\frac{1}{2}\Big),0) + f(1)\nu(1)\hat{e}_{\ominus z}(\rho(1),0) + \hat{e}_{\ominus z}(1,0)\sum_{t=0}^{\infty} f(t+1)(1-z)^t =$$

$$= f\Big(\frac{1}{2}\Big)\frac{1}{2} + f(1)\frac{1}{2}(1-\frac{1}{2}z) + (1-\frac{1}{2}z)^2\sum_{t=0}^{\infty} f(t+1)(1-z)^t = 0$$

Let $f(0,5) = 0$ and divide by $(1-0,5z)^2$:

$$\frac{f(1)\frac{1}{2}}{1-\frac{1}{2}z} + \sum_{t=0}^{\infty} f(t+1)(1-z)^t =$$

$$= \frac{f(1)}{1-(z-1)} + \sum_{t=0}^{\infty} f(t+1)(1-z)^t =$$

$$= f(1)\sum_{t=0}^{\infty}(z-1)^t + \sum_{t=0}^{\infty} f(t+1)(1-z)^t = 0$$

Let $f(t+1) = (-1)^t f(1)$ and let $f(1) = c \neq 0$.

We found a function f that does not vanish for every value of t and still satisfies the equation. Therefore, the generalized nabla time scale Laplace transform on $\mathbb{T} = \mathbb{Z} \cup \{\frac{1}{2}\}$ is not unique.

Let us consider the following example of a hybrid - discrete/continuous time scale:

Example 3.4.3. Let $\mathbb{T} = \mathbb{Z} \cup [0,1]$. Suppose for $z \in B_\epsilon(1)$:

$$\int_0^\infty f(t)\hat{e}_{\ominus z}(\rho(t),0)\nabla t = 0$$

Then:

$$\int_0^\infty f(t)\hat{e}_{\ominus z}(\rho(t),0)\nabla t = \int_0^1 f(t)\hat{e}_{\ominus z}(\rho(t),0)\nabla t + \hat{e}_{\ominus z}(1,0)\int_1^\infty f(t)\hat{e}_{\ominus z}(\rho(t),1)\nabla t =$$

$$= \int_0^1 f(t)e^{-zt} + e^{-z}\sum_{t=0}^{\infty} f(t+1)(1-z)^t$$

Let $f(t) = 1$ on $[0, 1]$ and multiply by e^z.

$$\int_0^1 e^{z(1-t)} + \sum_{t=0}^{\infty} f(t+1)(1-z)^t =$$

$$= \frac{1-e^z}{z} + \sum_{t=0}^{\infty} f(t+1)(-1)^t(z-1)^t$$

Note that the function $\frac{1-e^z}{z}$ is an analytic function around 1 (also everywhere). Without further computations, we might conclude that there exist coefficients $f(t+1) \neq 0$, such that the whole expression gives us 0. Therefore, the generalized nabla time scale Laplace Transform on this time scale is not unique.

Let us consider another example of a discrete/hybrid time scale:

Example 3.4.4. Let $\mathbb{T} = [1, \infty] \cup \{0\}$. Suppose for $|z| > \alpha$:

$$\int_0^{\infty} f(t)\hat{e}_{\ominus z}(\rho(t), 0)\nabla t = 0$$

Then:

$$\int_0^{\infty} f(t)\hat{e}_{\ominus z}(\rho(t), 0)\nabla t = \int_0^1 f(t)\hat{e}_{\ominus z}(\rho(t), 0)\nabla t + \hat{e}_{\ominus z}(1, 0)\int_1^{\infty} f(t)\hat{e}_{\ominus z}(\rho(t), 1)\nabla t =$$

$$= f(1) + (1-z)\int_1^{\infty} f(t)e^{-z(t-1)}dt =$$

$$= f(1) + (1-z)\int_0^{\infty} f(x+1)e^{-zx}dx =$$

$$= f(1) + (1-z)\mathcal{L}_1^{\mathbb{T}}\{f(t)\}\{z\}$$

Let us choose a function $f(x+1) = -e^x$, $f(1) = 1$. Then, the expression vanishes.

From our investigation, it is clear that there are multiple examples of time scales on which the generalized nabla time scale Laplace transform is not unique. With this discovery, a natural question arises: How to evaluate, whether the studied time scale has the unique generalized nabla time scale Laplace transform? How can we construct a counterexample?

Construction of counterexamples

To find a counterexample of uniqueness, we might consider only an example of a function that is not a null function and has zero generalized nabla time scale Laplace transform. If we are able to construct such an example, then, we may conclude that the transform may not be unique on such time scale.

Let $\mathbb{T} = \mathbb{T}_a \cup \mathbb{T}_p$, where $\mathbb{T}_a$ is an arbitrary time scale, s.t. $\sup\{T_a\} = a$, $\mathbb{T}_p$ is an arbitrary periodic time scale, s.t. $\inf\{\mathbb{T}_p\} = \infty$. For simplicity, suppose that $\mathbb{T}$ is discrete. Let $s \in \mathbb{T}_a$. Consider the generalized nabla time scale Laplace transform:

$$\mathcal{L}_s^{\mathbb{T}}\{f\}(z).$$

By Lemma 3.2.2, we know that there is a guaranteed domain of convergence - union of balls of radius ϵ around $\frac{1}{\nu(x)}$, where $x \in \mathbb{T}_p$. Let us suppose for z in the guaranteed domain of convergence that:

$$\mathcal{L}_s^{\mathbb{T}}\{f\}(z) = 0.$$

To show that the generalized nabla time scale Laplace transform on the time scale of this form is not unique, we need to find such a function f that the generalized nabla time scale Laplace transform of f is zero, while f is non-zero on some points. By linearity property:

$$0 = \int_s^a f(t)\hat{e}_{\ominus z}(\rho(t), s)\nabla t + \int_a^\infty f(t)\hat{e}_{\ominus z}(\rho(t), s)\nabla t.$$

Let us multiply the equation by $\hat{e}_{\ominus z}(s, a)$:

$$0 = \int_s^a f(t)\hat{e}_{\ominus z}(\rho(t), a)\nabla t + \int_a^\infty f(t)\hat{e}_{\ominus z}(\rho(t), a)\nabla t.$$

We observe that:

$$\hat{e}_{\ominus z}(\rho(t), a) = \prod_{\tau \in (a,t)} (1 - z\nu(\tau)) \qquad a > \rho(t)$$
$$\hat{e}_{\ominus z}(\rho(t), a) = \prod_{\tau \in (a,t)} (1 - z\nu(\tau))^{-1} \qquad a < \rho(t).$$

Thus, the time scale exponential function in the former integral has singularities in points $z = \frac{1}{\nu(x)}$, $x \in \mathbb{T}_a$, whereas the time scale exponential function in second integral has zeros in points $z = \frac{1}{\nu(x)}$, $x \in \mathbb{T}_p$ and is entire (analytic in whole $\mathbb{C}$).

Since around any point $\frac{1}{\nu(x)} \in \mathbb{T}_p$ there is a small ball, we may consider the curve γ around the first point $\frac{1}{\nu(x)} \in \mathbb{T}_p$ that is contained inside the ball and integrate around γ in the complex plane:

$$0 = \int_\gamma (\int_s^a f(t)\hat{e}_{\ominus z}(\rho(t), a)\nabla t + \int_a^\infty f(t)\hat{e}_{\ominus z}(\rho(t), a)\nabla t) dz.$$

Interchanging integrals, we obtain:

$$0 = \int_s^a f(t)(\int_\gamma \hat{e}_{\ominus z}(\rho(t), a) dz)\nabla t + \int_a^\infty f(t)(\int_\gamma \hat{e}_{\ominus z}(\rho(t), a) dz)\nabla t.$$

Since the exponential in the second integral is entire, the second integral vanishes. However, in a small neighbourhood of $\frac{1}{\nu(x)}$, the exponential in the first integral is analytic, therefore, both of the integrals vanish.
Let us then go back:

$$0 = \int_s^a f(t)\hat{e}_{\ominus z}(\rho(t), a)\nabla t + \int_a^\infty f(t)\hat{e}_{\ominus z}(\rho(t), a)\nabla t,$$

and multiply the equation by $\frac{1}{1-\nu(x)z}$, where x is the first point in $\mathbb{T}_p$, as previously, and then integrate around γ and interchange the integrals:

$$0 = \int_s^a f(t)(\int_\gamma \frac{\hat{e}_{\ominus z}(\rho(t),a)}{1-\nu(x)z}dz)\nabla t + \int_a^\infty f(t)(\int_\gamma \frac{\hat{e}_{\ominus z}(\rho(t),a)}{1-\nu(x)z}dz)\nabla t.$$

Since the exponential in the second integral is:

$$\hat{e}_{\ominus z}(\rho(t),a) = \prod_{\tau\in(a,t)} (1 - z\nu(\tau))$$

and we divided it by $\frac{1}{1-z\nu(x)}$, we obtain:

$$0 = \int_s^a f(t)(\int_\gamma \frac{\hat{e}_{\ominus z}(\rho(t),a)}{1-\nu(x)z}dz)\nabla t - f(x)2\pi i.$$

Also, by the Cauchy integral formula:

$$0 = \int_s^a f(t)(\frac{-2\pi i}{\nu(x)}\frac{d}{dz}\hat{e}_{\ominus z}(\rho(t),a)|_{z=\frac{1}{\nu(x)}})\nabla t - f(x)2\pi i.$$

To prove that f is zero everywhere, we need:

$$\int_\gamma \frac{\hat{e}_{\ominus z}(\rho(t),a)}{1-\nu(x)z}dz = \frac{-2\pi i}{\nu(x)}\frac{d}{dz}\hat{e}_{\ominus z}(\rho(t),a)|_{z=\frac{1}{\nu(x)}} = 0.$$

However, that is untrue in any case.
Finally, to construct the counterexample, we go back:

$$0 = \int_s^a f(t)\hat{e}_{\ominus z}(\rho(t),a)\nabla t + \int_a^\infty f(t)\hat{e}_{\ominus z}(\rho(t),a)\nabla t.$$

We may choose f arbitrarily, so let $f(t) = 0$ for every $t \in \mathbb{T}_a$, except for $f(a)$. Then, by Theorem 2.2.4:

$$0 = f(a)\hat{e}_{\ominus z}(\rho(a),a)\nu(a) + \int_a^\infty f(t)\hat{e}_{\ominus z}(\rho(t),a)\nabla t$$

$$0 = f(a)\frac{\nu(a)}{1-\nu(a)z} + \int_a^\infty f(t)\hat{e}_{\ominus z}(\rho(t),a)\nabla t.$$

If $|\nu(a)z| < 1$, $\frac{1}{1-\nu(a)z}$ is the geometric series:

$$0 = f(a)\nu(a)(\sum_{k=0}^\infty z^k\nu(a)^k) + \int_a^\infty f(t)\hat{e}_{\ominus z}(\rho(t),a)\nabla t.$$

Since both terms are in fact Taylor series in variable z, we may conclude that there may exist a non-zero function f, s.t. the series are equal and we obtain the counterexample.

If $|\nu(a)z| > 1$, $\frac{1}{1-\nu(a)z} = \frac{-\frac{1}{\nu(a)z}}{1-\frac{1}{\nu(a)z}}$ and we use the same principle:

$$0 = f(a)\nu(a)(\sum_{k=1}^\infty -\frac{1}{z^k\nu(a)^k}) + \int_a^\infty f(t)\hat{e}_{\ominus z}(\rho(t),a)\nabla t.$$

Note that since the function $\frac{1}{1-\nu(a)z}$ is analytic everywhere except the point $\frac{1}{\nu(a)}$, our computations are correct from the complex analysis point of view.

Thus, we have found some possible counterexamples on the time scale $\mathbb{T}$.

3.4.2. Generalized Lerch's theorem for periodic time scales

We recall the definition of periodic time scale 2.3.8.

Theorem 3.4.1. Let $\mathbb{T}$ be a discrete periodic time scale, such that the generalized nabla time scale Laplace transform exists and $sup\{\mathbb{T}\} = \infty$. Assume that the domain of convergence $\mathcal{D}(f)$ has a non-empty interior. Assume for every z in the guaranteed domain of convergence:

$$\mathcal{L}_s^{\mathbb{T}}\{f\}(z) = 0.$$

Then, $f(t) = 0$ for every $t \in \mathbb{T}$.

Proof. Utilizing the lemma 3.2.1 and the fact that the domain of convergence has non-empty interior, we may conclude that there exists a guaranteed domain of convergence. Direct calculation yields (z being in the guaranteed domain of convergence):

$$\begin{aligned} 0 = \mathcal{L}_s^{\mathbb{T}}\{f\}(z) &= \sum_{t \in (s,\infty)_{\mathbb{T}}} f(t)\nu(t) \prod_{x \in (s,\rho(t))_{\mathbb{T}}} (1 - \nu(x)z) = \\ &= f(t_1)\nu(t_1) + f(t_2)\nu(t_2)(1-\nu(t_1)z) + f(t_3)\nu(t_3)(1-\nu(t_1)z)(1-\nu(t_2)z) + ... \end{aligned}$$

Let us divide the following expression by $(1 - \nu(t_1)z)$ and integrate along a positively oriented circle γ inside the ball $B_\epsilon(\frac{1}{\nu(t_1)})$, containing the point $\frac{1}{\nu(t_1)}$. By the assumption, such a ball exists and is fully contained in the domain of convergence.

We carefully observe that we may interchange the integral and the sum, since it converges absolutely and uniformly inside the guaranteed domain of convergence.
Since the integral of an analytic function vanishes, we may conclude that:

$$\int_\gamma f(t_i)\nu(t_i) \prod_{x \in (s,\rho(t_i))_{\mathbb{T}}} (1 - \nu(x)z) \frac{1}{(1-\nu(t_1)z)} dz = 0$$

for all $i > 1$ and

$$\int_\gamma f(t_1)\nu(t_1) \frac{1}{(1-\nu(t_1)z)} dz = 2\pi i f(t_1).$$

Therefore, $f(t_1) = 0$.

Now, we proceed by dividing the original expression by $(1-\nu(t_1)z)(1-\nu(t_2)z)$ and utilizing the fact that $f(t_1) = 0$. Using the same process for every i, we obtain the desired result: $f(t) = 0$ for all $t \in \mathbb{T}$. □

Theorem 3.4.2. Let $\mathbb{T}$ be a periodic time scale, such that the generalized nabla time scale Laplace transform exists and $sup\{\mathbb{T}\} = \infty$. Assume that the domain of convergence $\mathcal{D}(f)$ has a non-empty interior. Assume for every z in the guaranteed domain of convergence:

$$\mathcal{L}_s\{f\}(z) = 0.$$

Then, $f(t) = 0$ for every $t \in \mathbb{T}$.

Proof. Since we have already proved the discrete case, we may assume that $\nu(t) = 0$ for some values $t \in \mathbb{T}$.
Utilizing the lemma 3.2.1 and the fact that the domain of convergence has a non-empty interior, we may conclude that the guaranteed domain of convergence exists. Utilizing the lemma 3.1.1 (z being in the guaranteed domain of convergence):

$$0 = \mathcal{L}_s^{\mathbb{T}}\{f\}(z) = \sum_{n=0}^{\infty} \hat{e}^n_{\ominus z}(\rho(s+p), s)(\int_{s+np}^{s+(n+1)p} f(t)\hat{e}_{\ominus z}(\rho(t), s+np)\nabla t).$$

Without loss of generality, we may assume that the interval $[s, d] \subset \mathbb{T}$ (that the first interval is continuous). We aim to show that $f(t)$ is a null function on interval $[s, d]$.

Let us multiply the whole expression by $e^{z\tau}$, where $\tau \in [s, d]$, s.t. $f(t) \geq 0$ (or possibly $f(t) \leq 0$), when $t \leq \tau$. Let us rewrite the generalized nabla time scale Laplace transform in the following fashion:

$$0 = \mathcal{L}_s^{\mathbb{T}}\{f\}(z) = \int_s^{\tau} f(t)e^{-z(t-s-\tau)}dt + \mathcal{L}_{\tau}^{\mathbb{T}}\{f\}(z).$$

Now, we use our assumption that the generalized nabla time scale Laplace transform is equal to 0 for any point of form $z_k = z_0 + \sigma k$. Note that we do not need the whole assumption that the generalized nabla time scale Laplace transform converges in the half plane $Re(z) > Re(z_0)$, but only on a sequence of equidistant points. Therefore, let $z_k \to \infty$:

$$0 = \lim_{z_k \to \infty} (\int_s^{\tau} f(t)e^{-z_k(t-s-\tau)}dt + \mathcal{L}_{\tau}\{f\}(z_k)).$$

At this point, caution is advised. Since the second part of the limit vanishes, we may conclude:

$$0 = \lim_{z_k \to \infty} \int_s^{\tau} f(t)e^{-z_k(t-s-\tau)}dt + \lim_{z_k \to \infty} \mathcal{L}_{\tau}\{f\}(z_k).$$

And therefore:

$$0 = \lim_{z_k \to \infty} \int_s^{\tau} f(t)e^{-z_k(t-s-\tau)}dt.$$

Realizing that on $[s, \tau]$ the sequence of functions $f(t)e^{-z_k(t-s-\tau)}$ is a sequence of positive measurable functions, we apply Fatou's lemma 3.3.4:

$$0 = \lim_{z_k \to \infty} \int_s^{\tau} f(t)e^{-z_k(t-s-\tau)}dt \geq \int_s^{\tau} \liminf_{z_k \to \infty} f(t)e^{-z_k(t-s-\tau)}dt.$$

Let us assume, by contradiction, that f is non-zero on some sub-interval of $[s, \tau]$ with non-zero measure. Then, $\liminf_{z_k \to \infty} f(t)e^{-z_k(t-s-\tau)} = \infty$ (or $-\infty$) at points of chosen subinterval, and since f is a non-negative function (or a non-positive), the whole integral cannot be zero. This produces a contradiction, so our claim that f is zero on $[s, \tau]$ is satisfied. We may repeat the process for any non-negative (or non-positive) part of f so that we obtain the conclusion that f is the null function on $[s, d]$.

With a simple application of this method and the method of Lerch's theorem for the discrete periodic time scale, we iteratively complete our proof. □

3.4.3. Generalized Lerch's theorem for arbitrary discrete time scales

Even though we proposed several counterexamples to the uniqueness of the generalized nabla time scale Lapalce transform on some time scales, we are able to prove some results (for an arbitrary discrete time scales), with the effect of relaxing the assumptions of the generalized Lerch's theorem. That occasionally poses a problem, as in some cases, the theorem cannot be utilized due to the impossibility to satisfy the assumptions.

Theorem 3.4.3. Let $\mathbb{T}$ be an arbitrary discrete time scale, such that the generalized nabla time scale Laplace transform exists and $sup\{\mathbb{T}\} = \infty$. Assume, that the domain of convergence $\mathcal{D}(f)$ contains the set Ω - the union of open balls centred at $\frac{1}{\nu_i}$, where ν_i are all values of the backward graininess function. If $\nu(t) = 0$ for some $t \in \mathbb{T}$, assume that there is a half-plane $Re(z) > Re(\alpha)$ contained in Ω. Assume that in Ω, the generalized nabla time scale Laplace transform converges absolutely and uniformly. Assume for every z in Ω:

$$\mathcal{L}_s\{f\}(z) = 0$$

Then, $f(t) = 0$ for every $t \in \mathbb{T}$.

Proof. The proof method is the same as proof methods in previous chapters. □

Even though we propose the proof of the uniqueness of the generalized nabla time scale Laplace transform on any non-uniform time scales, it might not be unique. The proof technique holds, but the assumptions cannot be always met, since the guaranteed domain of convergence does not contain balls in every point $\frac{1}{\nu(x_i)}$. This might be true only for points, which are repeated infinitely many times. Therefore, in cases of time scales that differ from periodic time scales proposed in the previous chapter, we must proceed carefully and we must not expect the uniqueness automatically, as many authors do.

3.4.4. Summary

In this section, we proved Lerch's theorem for uniform time scales and periodic time scales; showed by counterexamples that the uniqueness of the generalized nabla time scale Laplace transform might not hold and covered the possible proof methods for other time scales, for which reasonable assumptions are met. The main result of our investigation is the following corollary.

Definition 3.4.1. Let $f : \mathbb{T} \to \mathbb{R}$, s.t.:

$$\int_s^x f(t)\nabla t = 0$$

for $s \in \mathbb{T}$ and for all $x \in \mathbb{T}$. Then f is called the *generalized null function*.

Clearly, the generalized null function is a null function on continuous integrals and vanishes on discrete parts of chosen time scale $\mathbb{T}$.

Theorem 3.4.4. Let $\mathbb{T}$ be the periodic time scale, such that the generalized nabla time scale Laplace transform exists. Two original functions, whose image functions assume equal values in the guaranteed domain of convergence of the generalized nabla time scale Laplace transform, differ at most by a generalized null function.

In further examination, studying the properties of the domain of convergence might be useful in proving the uniqueness of sets broader than periodic time scales - the perspective regions might be on time scales, where any value of backward graininess function is repeated infinitely many times.

Furthermore, after proving the uniqueness of the generalized nabla time scale Laplace transform, we are able to define the Inverse generalized nabla time scale Laplace transform. This cannot be done the other way around.

The groundwork in the delta case of the inverse transform can be found in [25].

4. Fractional calculus on time scales

4.1. Continuous fractional calculus

Continuous fractional calculus is a branch of mathematics that studies generalizations of continuous differential and integral calculus to non-integer or possibly even complex orders - in other words, continuous fractional calculus studies powers of operator:

$$D = \frac{d}{dx}.$$

A critical role in achieving a generalization of such an operator is played by the well-known Cauchy integral formula:

$${}_aD^m f(t) = \int_a^t \int_a^{\tau_{m-1}} ... \int_a^{\tau_1} f(\tau_0)d\tau_0...d\tau_{m-1} = \int_a^t \frac{(t-\tau)}{(m-1)!} f(\tau)d\tau,$$

where $m \in \mathbb{N} \cup \{0\}$, $a \in \mathbb{R}$, $t > a$ and $f : \mathbb{R} \to \mathbb{R}$ is integrable on $[a,t]$. To obtain the generalization of this result, we use the *Euler* Γ*-function*:

$$\Gamma(z) := \int_0^\infty x^{z-1}e^{-x}dx = \lim_{n\to\infty} \frac{n!n^{z-1}}{z(z+1)...(z+n)}, \tag{4.1}$$

that is defined for all complex numbers $z \in \mathbb{C} \setminus \{\mathbb{Z}^- \cup \{0\}\}$.

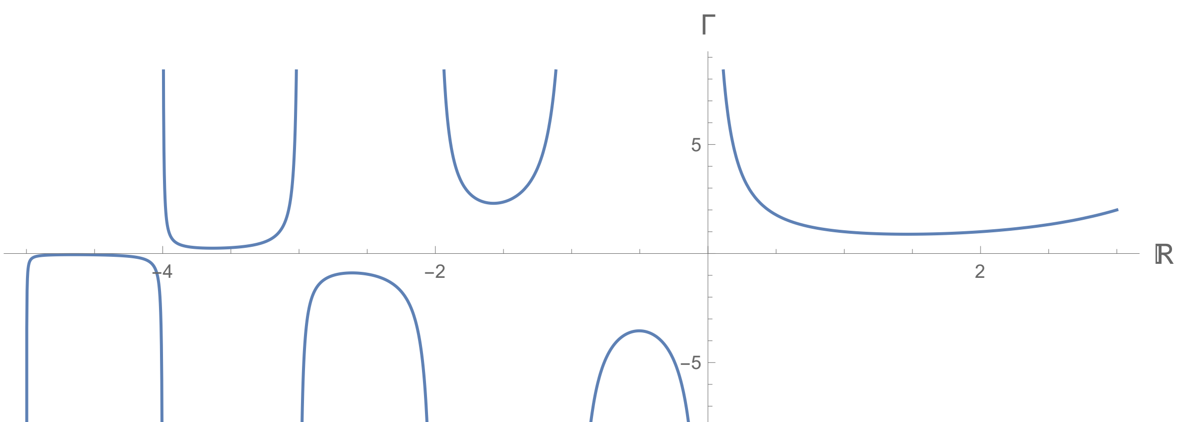

Figure 4.1: The Euler Γ-function plot.

It is easy to show that:

$$\Gamma(z+1) = z\Gamma(z),$$

therefore the Γ-function represents the generalization of factorial function. Thus, we can define the generalized integral.
Following definitions can be found in [26], [1] or [6]:

Definition 4.1.1. Let $\gamma \in \mathbb{R}$, $\gamma \geq 0$, $a, b, c \in \mathbb{R}$, $c > b \geq a$ and let $f : (a, c] \to \mathbb{R}$. We define *Riemann-Liouville fractional integral of order* $\gamma > 0$ with the lower limit a as:

$${}_aD^{-\gamma} f(t) := \int_a^t \frac{(t-\tau)^{\gamma-1}}{\Gamma(\gamma)} f(\tau)d\tau$$

for $t \in [b, c]$. For $\gamma = 0$ we define ${}_0D^{-\gamma} f(t) := f(t)$ for $t \in [b, c]$.

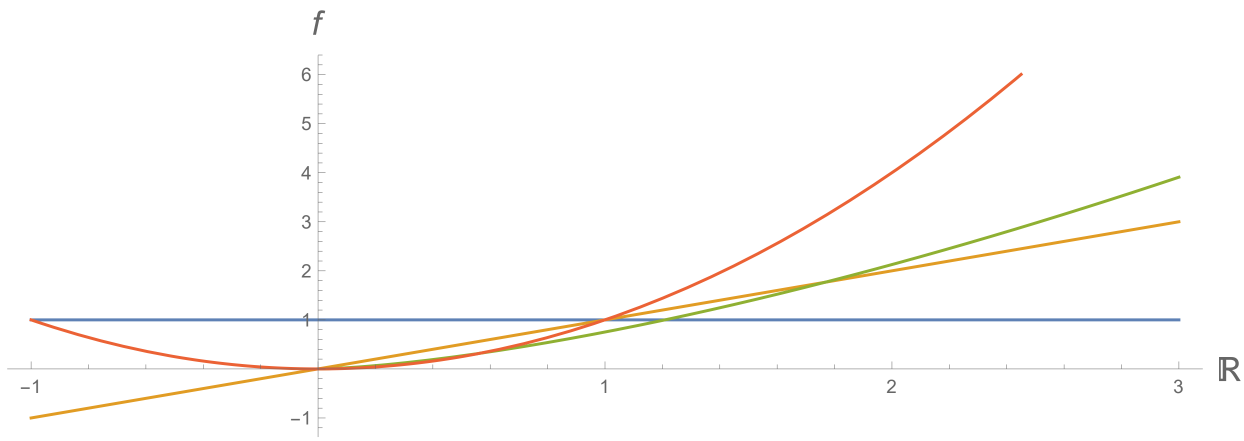

Figure 4.2: The fractional integral of 1 with chosen $\gamma = 0, \frac{1}{2}, 1, 2$.

It is possible to generalize fractional integral to negative orders using the generalized functions, but since it also implies a loss of integrability, other definitions were proposed.

Definition 4.1.2. Let $\gamma \in \mathbb{R}$, $\gamma \geq 0$, $a, b, c \in \mathbb{R}$, $c > b \geq a$ and let $f : (a, c] \to \mathbb{R}$. We define *Riemann-Liouville fractional derivative of order* γ with the lower limit a as:

$$ {}_aD^{\gamma} f(t) :=_a D_a^{\lceil\alpha\rceil} D^{-(-\lceil\alpha\rceil-\alpha)} f(t) $$

for $t \in [b, c]$.

We may notice that, choosing $\alpha \in \mathbb{N}$, we obtain the regular derivative.

If we change the order of operations, we obtain the following definition.

Definition 4.1.3. Let $\gamma \in \mathbb{R}$, $\gamma \geq 0$, $a, b, c \in \mathbb{R}$, $c > b \geq a$ and let $f : (a, c] \to \mathbb{R}$. We define *Caputo fractional derivative of order* γ with the lower limit a as:

$$ {}_a^C D^{\gamma} f(t) :=_a D_a^{-(-\lceil\alpha\rceil-\alpha)} D^{\lceil\alpha\rceil} f(t) $$

for $t \in [b, c]$.

4.2. Introduction to fractional calculus on time scales

In this section, we examine the possible natural question - how to generalize continuous fractional calculus in a time scale framework.

Naturally, we introduce the generalization of continuous Cauchy formula, as in [6]:

Definition 4.2.1. Let $\mathbb{T}$ be an arbitrary time scale. Let $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$ be ld-continuous. We define the *nabla Cauchy formula*:

$$ {}_a\nabla^{-1} f(t) := \int_a^t f(\tau) \nabla\tau $$

and recursively:

$$ {}_a\nabla^{-m} f(t) := \int_a^t {}_a\nabla^{-m+1} f(\tau) \nabla\tau $$

for $m \in \mathbb{N}$.

Proof of the following theorem can be found in [6].

Theorem 4.2.1. Let $m \in \mathbb{Z}$, $m \geq 0$, $a, b \in \mathbb{T}$, let $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$ be ld-continuous. Then:

$$_a\nabla^{-m} f(t) = \int_a^t \hat{h}_{m-1}(t, \rho(\tau)) f(\tau) \nabla\tau$$

for $t \in [a, b]_{\mathbb{T}}$.

Hence, generalization of fractional calculus to time scales is:

$$_a\nabla^{-\gamma} f(t) = \int_a^t \hat{h}_{\gamma-1}(t, \rho(\tau)) f(\tau) \nabla\tau \tag{4.2}$$

for fractional orders γ.
However, the expression above is only symbolical expression. In order to obtain meaning behind this formula, we need to give meaning to power functions $\hat{h}_\gamma$, which is the main topic of the following section.

4.2.1. Fractional calculus on uniform time scales

Before defining power functions on an arbitrary time scale, we might look into some specific results of nabla fractional calculus on time scales. This subsection is a survey of some proven results in discrete fractional calculus presented in [17].

We define the sets $\mathbb{N}_a := \{a, a+1, a+2, ...\}$.

Definition 4.2.2. Assume n is a positive integer and $t \in \mathbb{R}$. Then, we define the *rising function* $t^{\overline{n}}$ by:

$$t^{\overline{n}} := t(t+1)...(t+n-1).$$

Theorem 4.2.2. The following statement is true:
For $n \in \mathbb{N}$ and $t \in \mathbb{N}_a$:

$$\hat{h}_0(t, a) = 1$$

and for $t \in \mathbb{N}_{a-n+1}$, $n \in \mathbb{N}_1$:

$$\hat{h}_n(t, a) = \frac{(t-a)^{\overline{n}}}{n!}.$$

Note that the theorem above coincides with the definition 2.4.1.

The following theorem is a nabla equivalent of Taylor's formula for $\mathbb{N}_a$:

Theorem 4.2.3. Assume $f : \mathbb{N}_{a-n} \to \mathbb{R}$, where $n \in \mathbb{N}_0$. Then:

$$f(t) = p_n(t) + R_n(t)$$

for $t \in \mathbb{N}_{a-n}$, where $n-th$ degree Taylor polynomial $p_n(t)$ is given by:

$$p_n(t) := \sum_{k=0}^{n} \nabla^k f(a) \frac{(t-a)^{\overline{k}}}{k!} = \sum_{k=0}^{n} \nabla^k f(a) \hat{h}_k(t,a)$$

and the Taylor's remainder $R_n(t)$ is given by:

$$R_n(t) = \int_a^t \frac{(t-\rho(s))^{\overline{n}}}{n!} \nabla^{n+1} f(s) \nabla s = \int_a^t \hat{h}_n(t,\rho(s)) \nabla^{n+1} f(s) \nabla s$$

for $t \in \mathbb{N}_{a-n}$.

With the Taylor's formula, we may expand our "toolbox" for solving initial value problems (Integer Order Variation of Constants Formula), define formal Taylor series and study the expansions of some generalized functions. For more information on these topics, see [17].

We define *nabla integral order sum*:

Definition 4.2.3. Let $f : \mathbb{N}_{a+1} \to \mathbb{R}$ be given and $n \in \mathbb{N}_1$. Then:

$$\nabla_a^{-n} f(t) := \int_a^t \hat{h}_{n-1}(t,\rho(s)) f(s) \nabla s$$

for $t \in \mathbb{N}_a$. Also, we define:

$$\nabla_a^0 f(t) := f(s).$$

In this case, since we know the formula for nabla monomials, we easily generalize them by utilizing the Euler Gamma function 4.1 in the following definition:

Definition 4.2.4. Let $\alpha \neq -1, -2, ...$ We define α-th order nabla power functions $\hat{h}_\alpha(t,a)$ by:

$$\hat{h}_\alpha(t,a) := \frac{(t-a)^{\overline{\alpha}}}{\Gamma(\alpha+1)}.$$

We may observe that the regular properties of power functions hold, which is shown in the following theorem.

Theorem 4.2.4. The following statements are true, provided that the expressions in this theorem are well-defined:

- $\hat{h}_\alpha(a,a) = 0$
- $\nabla \hat{h}_\alpha(t,a) = \hat{h}_{\alpha-1}(t,a)$
- $\int_a^t \hat{h}_\alpha(s,a) \nabla s = \hat{h}_{\alpha+1}(t,a)$
- $\int_a^t \hat{h}_\alpha(t,\rho(s)) \nabla s = \hat{h}_{\alpha+1}(t,a)$
- for $k \in \mathbb{N}_1$, $\hat{h}_{-k}(t,a) = 0$, $t \in \mathbb{N}_a$.

We define *nabla fractional sum* in terms of nabla power functions:

Definition 4.2.5. Let $f : \mathbb{N}_{a+1} \to \mathbb{R}$ be given, assume $\alpha > 0$. Then:

$$\nabla_a^{-\alpha} f(t) := \int_a^t \hat{h}_{\alpha-1}(t, \rho(s)) f(s) \nabla s,$$

for $t \in \mathbb{N}_a$. We also define:

$$\nabla_a^{-\alpha} f(a) := 0.$$

We define *nabla Riemann-Liouville fractional difference*:

Definition 4.2.6. Let $f : \mathbb{N}_{a+1} \to \mathbb{R}$, $\alpha \in \mathbb{R}^+$ and choose n s.t. $n - 1 < \alpha \leq n$. Then, we define α-th order *nabla Riemann-Liouville fractional difference*, $\nabla_a^\alpha f(t)$, by:

$$\nabla_a^\alpha f(t) := \nabla^N \nabla_a^{-N-\alpha} f(t)$$

for $t \in \mathbb{N}_{a+N}$.

4.3. Power functions

In the previous section, we arrived at the expression 4.2 that might be used for generalization of fractional calculus. By giving meaning to generalized nabla time scale power functions on time scales of form $\mathbb{T} = \mathbb{N}_a$, we defined some elementary definitions of fractional calulus. However, we also found that in order to do this, we need explicit formulas for the power functions on time scales. We could continue with this approach and generalize some other time scales, for example, the well-known case of *q-calculus*. However, since the explicit formulas of power functions are not known for an arbitrary time scale, the axiomatic definition is inevitable. Attempts to generalize power functions by various definitions may be found, for example, in [6],[7], [9], [10], [7], [8]. With the axiomatic definition of power function, we are able to achieve generalizations of fractional operators for almost any time scale.

4.3.1. Axiomatic definition of power functions

The following axiomatic definition of power functions can be found in [7].

Definition 4.3.1. Let $s, t \in \mathbb{T}$ and let $\beta, \alpha \in (-1, \infty)$. The time scale power functions $\hat{h}_\beta(t, s)$ are defined by a family of non-negative functions satisfying:

i) $\int_s^t \hat{h}_\beta(t, \rho(\tau)) \hat{h}_\gamma(\tau, s) \nabla\tau = \hat{h}_{\beta+\gamma+1}(t, s)$ for $t \geq s$,

ii) $\hat{h}_0(t, s) = 1$ for $t \geq s$,

iii) $\hat{h}_\beta(t, t) = 1$ for $\beta \in (0, 1)$.

Further, the parameter β in $\hat{h}_\beta(t, s)$ is called the order of the function $\hat{h}_\beta(t, s)$.

Theorem 4.3.1. Let $m \in \mathbb{N}_0$, $\beta \in (-1, \infty)$, $s, t \in \mathbb{T}$ such that $t > s$. Then:

$$\nabla^m \hat{h}_\beta(t,s) = \begin{cases} \hat{h}_{\beta-m}(t,s) & \beta > m-1 \\ 0 & \beta \in 0, 1, ..., m-1. \end{cases}$$

The theorem 4.3.1 does not discuss the case $\beta \in (-1, m-1]\big|_{\{0,1,...m-1\}}$ due to an occurrence of a power function of order less than -1. Since such functions cannot be included in the Definition 4.3.1, we define them by the Theorem 4.3.1.

Definition 4.3.2.

$$\hat{h}_\beta(t,s) := \nabla^{-\lceil\beta\rceil} \hat{h}_{\beta-\lceil\beta\rceil}(t,s)$$

for $\beta \in (-\infty, -1)|\mathbb{Z}$, $s, t \in \mathbb{T}$, $t \geq \sigma^{-\beta}(s)$, where $\lceil\beta\rceil$ is the ceiling function $\lceil\beta\rceil = min\{m \in \mathbb{Z}; m \geq \beta\}$.

Theorem 4.3.2. Let $\beta \in (-1, \infty)$, $s, t \in \mathbb{T}_\kappa$ such that $t > s$. Then, $\hat{h}_\beta$ solves the shifting problem, i.e.:

$$\nabla_t \hat{h}(t, \rho(s)) = -\nabla_s \hat{h}_\beta(t,s)$$

Theorem 4.3.2 enables us to rewrite the Definition 4.3.1 i) via the convolution:

$$(\hat{h}_\beta * \hat{h}_\gamma)(t,s) = \hat{h}_{\beta+\gamma+1}(t,s) \quad t \geq s, \beta, \gamma > -1.$$

To show that the axiomatic definition is well-defined, we need to prove the existence and the uniqueness property of such definition. The following results on isolated time scales are provided in [7].

Theorem 4.3.3. Let $\mathbb{T}$ be an isolated time scale, and let $r \in (-1, \infty)_Q$. Then, the Definition 4.3.1 uniquely determines the power function $\hat{h}_r(t,s)$ for all $s, t \in T$, such that $t > s$.

Theorem 4.3.4. Let $\mathbb{T}$ be an isolated time scale and let $r \in (-1, \infty)_Q$, $s, t \in \mathbb{T}$, $t > s$, $\nu(t) \neq \nu(s)$ for all $t, s \in \mathbb{T}$. Then, the following formula holds:

$$\hat{h}_r(t,s) = \frac{\nu(t)\hat{h}_r(t, \sigma(s)) - \nu(\sigma(s))\hat{h}_r(\rho(t), s)}{\nu(t) - \nu(\sigma(s))}.$$

Theorem 4.3.5. Let $\mathbb{T}$ be an isolated time scale, $t \in \mathbb{T}$ and let $r \in (-1, \infty)_\mathbb{Q}$. Then, the following statements are true:

a) $\hat{h}_r(t,t) = 0$ for $r > 0$

b) $\hat{h}_r(t,t) = 0$ for $r = 0$

c) the value of $\hat{h}_r(t,t)$ for $-1 < r < 0$ is unbounded.

Definition 4.3.3. Let $\mathbb{T}_\mathcal{L}$ be an isolated time scale, such that $sup\{\mathbb{T}_\mathcal{L}\} = \infty$ and $sup\{\nu(t), t \in \mathbb{T}_\mathcal{L}\} < \infty$.

Theorem 4.3.6. Let $a \in \mathbb{T}_{\mathcal{L}}$, and let $r \in (-1, \infty)_{\mathbb{Q}}$. Then, the following expression holds:

$$\mathcal{L}_a^{\mathbb{T}_{\mathcal{L}}}\{\hat{h}_r(.,a)\}(z) = z^{-r-1}.$$

Theorem 4.3.7. Let $\mathbb{T}$ be an arbitrary time scale. Then:

$$\mathcal{L}_s^{\mathbb{T}}\{1\}(z) = \frac{1}{z}.$$

Proof.

$$\mathcal{L}_s^{\mathbb{T}}\{1\}(z) = \int_s^{\infty} \hat{e}_{\ominus z}(\rho(\eta), s)\nabla\eta = -\frac{1}{z}\hat{e}_{\ominus z}(\eta, s)|_{\eta\to s}^{\eta\to\infty} = \frac{1}{z}.$$

□

Theorem 4.3.8. Let $\mathbb{T}$ be an arbitrary time scale such that the generalized nabla time scale Laplace transform exists, $\alpha \in \mathbb{Q}$. Then, the generalized nabla time scale Laplace transform of $\hat{h}_\alpha$ is $\frac{1}{z^{\alpha+1}}$.

Proof. We know that $\mathcal{L}_s^{\mathbb{T}}\{\hat{h}_k(t,s)\}(z) = \frac{1}{z^{k+1}}$ for $n \in \mathbb{N}$. Let the generalized nabla time scale Laplace transform of the convolution of m times $\hat{h}_\beta$ be:

$$\mathcal{L}_s^{\mathbb{T}}\{\hat{h}_\beta * \hat{h}_\beta ... * \hat{h}_\beta * \hat{h}_\beta\}(z) = \frac{1}{z^k}.$$

Via 4.3.2:

$$\mathcal{L}_s^{\mathbb{T}}\{\hat{h}_{m\beta+m-1}\}(z) = \frac{1}{z^k}$$
$$\mathcal{L}_s^{\mathbb{T}}\{\hat{h}_{m\beta}\}(z) = \frac{1}{z^{k+1-m}}.$$

We may assume that $k = m\beta + m$, so $\beta = \frac{k}{m} - 1$. Then:

$$\mathcal{L}_s^{\mathbb{T}}\{\hat{h}_\beta\}(z) = \frac{1}{z^{\frac{k}{m}}} = \frac{1}{z^{\beta+1}}.$$

□

Theorem 4.3.9. Power functions on $\mathbb{T}$, where the generalized nabla time scale Laplace transform exists and is unique, are defined correctly by the Definition 4.3.1.

Proof. Proof is the direct consequence of Theorem 4.3.8. Calculating the generalized nabla time scale Laplace transform using properties of the Definition 4.3.1, we are able to obtain only one result. Then, the uniqueness of generalized time scale nabla Laplace transform provides the uniqueness of the Definition 4.3.1, except for a generalized null function. □

By the Theorem 4.3.9 we have shown that the axiomatic definition 4.3.1 is defined correctly for those time scales, where the generalized nabla time scale Laplace transform exists and is unique. However as we have shown previously, this does not necessarily mean that the Definition 4.3.1 defines the power functions uniquely on every time scale, since there are examples of time scales, where the uniqueness might not hold. Therefore, in future studies, it has to be proven, whether the axiomatic definition of power function is unique on those time scales, where the generalized nabla time scale Laplace transform is not unique and, moreover, other proof methods have to be developed to do that.

4.3.2. Power functions as an inverse Laplace transform

In delta calculus there are attempts to define power functions of fractional orders as an inverse delta Laplace transform, for instance [27]. This approach, however, fails in the nabla case due to the possibility of mapping multiple functions to a single Laplace transform.

Let us consider the following definition of nabla time scale power functions, found in [10],[9] (to differentiate between the previous definition and this one, we denote power function by capital $\hat{H}$ instead of $\hat{h}$):

Definition 4.3.4. Let $z \in \mathbb{C} \setminus \{0\}$, $\alpha > -1$. For $t \geq s$, we define:

$$\hat{H}_\alpha(t, t_0) := \mathcal{I}_s^{\mathbb{T}}\{\frac{1}{z^{\alpha+1}}\}(t),$$

where $\mathcal{I}$ denotes the inverse of $\mathcal{L}$. For $t < s$ we put:

$$\hat{H}_\alpha(t, t_0) := 0.$$

As we have shown in examples in subsection 3.4.1, the unicity property of the generalized nabla time scale Laplace transform does not hold on any time scale. Therefore, it is possible on some time scales to arrive at more than one result when we perform the inverse operation. It may be demonstrated by the following example.

Example 4.3.1. Let us consider the time scale $\mathbb{T} = \{0, 2, 3, 4, ...\} = (\mathbb{N} \cup \{0\}) \setminus \{1\}$ defined in the same manner as in 3.4.1. Let us consider the function $f : \mathbb{T} \to \mathbb{R}$, s.t. $f(2) = 1$, $f(t + 3) = 2^{t+1}$ for $t > 2$. Utilizing the definition 4.3.4, we obtain:

$$\hat{H}_0(t, t_0) = \mathcal{I}_s^{\mathbb{T}}\{\frac{1}{z}\}(t).$$

Since $\mathcal{I}$ is the inverse generalized nabla time scale Laplace transform, we are looking for a function g satisfying $\mathcal{L}_s^{\mathbb{T}}\{g\}(z) = \frac{1}{z}$. The possible options are:

$$g(t) = 1 + cf(t),$$

where $c \in \mathbb{R}$ (or possibly $c \in \mathbb{C}$). Thus, the definition 4.3.4 defines a whole class of functions that may be classified as a nabla time scale power functions.

In the theorem 4.3.9, we proved the uniqueness of the axiomatic definition 4.3.1 by the uniqueness property of generalized nabla time scale Laplace transform. One might think that since the definition 4.3.4 also depends on the same uniqueness property, the set of time scales, on which the definition is unique, is the same. However, this is not true, since there are other proof methods to show uniqueness of power function. Namely, the theorem 4.3.3 (which does not use the generalized nabla time scale Laplace transform at all) extends the set of time scales on which the definition is unique to all isolated time scales, while the generalized nabla time scale Laplace transform is not unique on any isolated time scale.

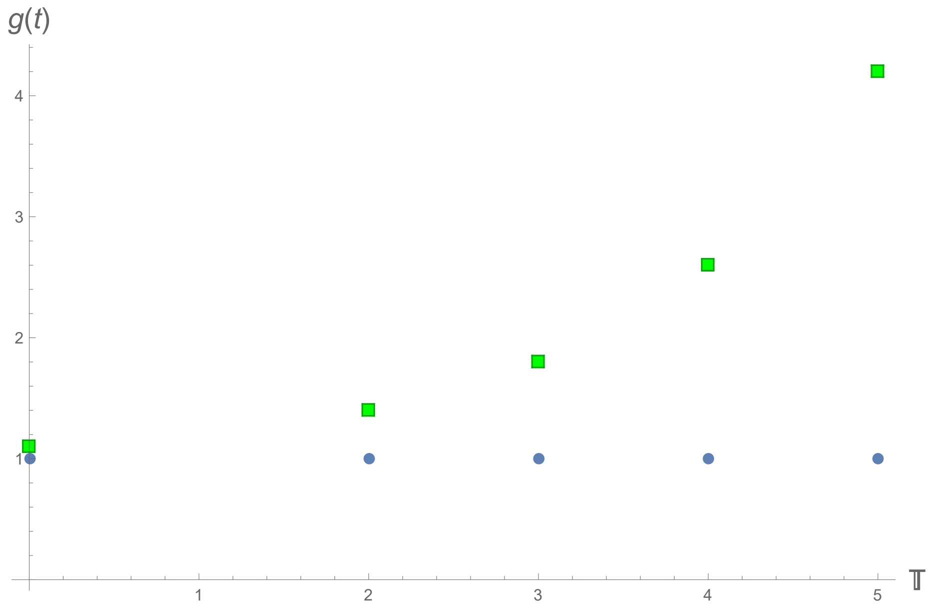

Figure 4.3: In the picture, we may observe plots of two functions with the same generalized nabla time scale Laplace transform, namely $\frac{1}{z}$.

4.4. Fractional operators on time scales

In this section, we arrive at some generalizations of fractional operators on time scales, utilizing the generalizations of time scale power functions.

Definition 4.4.1. Let $\tilde{\mathbb{T}}$ be a time scale, such that the time scale monomials of any rational order are defined uniquely.

Example 4.4.1. All isolated, uniform or periodic time scales satisfy the Definition 4.4.1.

Previously, in definition 4.2.3, utilizing generalized power functions to time scales of the form $\mathbb{T} = \mathbb{N}_a$, we were able to define Integral order sum. Now, since we successfully defined (by the Definition 4.3.1, or possibly on some time scales 4.3.4) the power functions for more general time scales, we are in a position to define the *fractional integral of order* α on general time scale.

Definition 4.4.2. Let $\alpha > 0$, $\tilde{a}, a, b \in \tilde{\mathbb{T}}$ be such that $\tilde{a} \leq a < b$. Then, for the function $f : (\tilde{a}, b]_{\mathbb{T}} \to \mathbb{R}$, we define *the fractional integral of order* $\alpha > 0$ with the lower limit a as:

$$_a\nabla^{-\alpha} f(t) := \int_a^t \hat{h}_{\alpha-1}(t, \rho(\tau)) f(\tau) \nabla\tau$$

for $t \in [a, b]_{\mathbb{T}} \cap (\tilde{a}, b]_{\mathbb{T}}$.

Theorem 4.4.1. Let $\gamma \geq 0$, $\alpha > 0$, $\tilde{a}, a, b \in \tilde{\mathbb{T}}$ be such that $\tilde{a} \leq a < b$. Then, for any function $f : (\tilde{a}, b]_{\mathbb{T}} \to \mathbb{R}$, the fractional integral with the lower limit a is linear.

Proof. Linearity of fractional integral follows directly from the linearity of the time scale integral. $\square$

For $\gamma = 1$, Definition 4.4.2 is reduced to a formula for the anti-derivative:

$$_a\nabla^{-1}f(t) = \int_a^t f(\tau)\nabla\tau$$

known from the time scales theory. We also may note that in Definition 4.4.2, if $a > \tilde{a}$, we get the usual definitions of difference calculus.

In similar fashion, we are able to define the generalized fractional derivative operators:

Definition 4.4.3. Let $\alpha > 0$, $\tilde{a}, a, b \in \tilde{\mathbb{T}}$ be such that $\tilde{a} \leq a < b$. Then, for the function $f : (\tilde{a}, b]_{\mathbb{T}} \to \mathbb{R}$, we define *the Riemann-Liouville fractional derivative of order* α with the lower limit a as:

$$_a\nabla^{\alpha}f(t) := \nabla^{\lceil\alpha\rceil}{}_a\nabla^{\alpha-\lceil\alpha\rceil}f(t), \quad t \in [\sigma(a), b]_{\mathbb{T}} \cap (\sigma(\tilde{a}), b]_{\mathbb{T}}$$

Definition 4.4.4. Let $\alpha > 0$, $\tilde{a}, a, b \in \tilde{\mathbb{T}}$ be such that $\tilde{a} \leq a < b$. Then, for the function $f : (\tilde{a}, b]_{\mathbb{T}} \to \mathbb{R}$, we define *the Caputo fractional derivative of order* α with the lower limit a $(a > \tilde{a})$ as:

$$^C_a\nabla^{\alpha}f(t) := {}_a\nabla^{\alpha-\lceil\alpha\rceil}\nabla^{\lceil\alpha\rceil}f(t), \quad t \in [\sigma(a), b]_{\mathbb{T}}$$

Finally, we are able to prove some basic properties of fractional derivatives in the following theorems.

Theorem 4.4.2. Let $\alpha \in \mathbb{R}$, $\beta \in (-1, \infty)$ and $s, t \in \tilde{\mathbb{T}}$ such that $s > t$. Then, it holds:

$$_a\nabla^{\alpha}\hat{h}_{\beta}(t, a) = \begin{cases} \hat{h}_{\beta-\alpha}(t, s) & \text{for} \quad \beta > \alpha - 1 \\ 0 & \text{for} \quad \beta \in \{\alpha - \lceil\alpha\rceil, \alpha - \lceil\alpha\rceil + 1, ..., \alpha - 1\}. \end{cases}$$

Theorem 4.4.3. Let $\alpha > 0$, $\beta \in (-1, \infty)$ and $s, t \in \tilde{\mathbb{T}}$ be such that $t > s$. Then, it holds:

$$^C_a\nabla^{\alpha}\hat{h}_{\beta}(t, a) = \begin{cases} \hat{h}_{\beta-\alpha}(t, s) & \text{for} \quad \beta > \lceil\alpha\rceil - 1 \\ 0 & \text{for} \quad \beta \in \{\alpha - \lceil\alpha\rceil, \alpha - \lceil\alpha\rceil + 1, ..., \alpha - 1\}. \end{cases}$$

5. Conclusions and Future Work

This master thesis covers the topic of uniqueness of the generalized nabla time scale Laplace transform, as well as some consequences of the uniqueness in fractional calculus. The first chapter introduces the theoretical basis of the time scale framework required to understand further studies in this work.
The main results of this thesis are contained in the second and the third chapter and they improve the results of [28].
The second chapter contains results regarding the generalized nabla time scale Laplace transform, namely:

- An introduction of the most commonly used definition of the generalized nabla time scale Laplace transform, as well as the basic properties of such definition.
- An examination of the domain of convergence of the generalized nabla time scale Laplace transform and a discussion regarding the importance of this set to generalized Lerch's theorem.
- An investigation of proof methods required to show uniqueness of the generalized nabla time scale Laplace transform on uniform time scales (time scales with constant backward graininess function) and a discussion about possible generalizations of such methods.
- A counterexamples of uniqueness of the generalized nabla time scale Laplace transform as well as the method of generating additional counterexamples, if possible.
- A proof of uniqueness of the generalized nabla time scale Laplace transform (generalized Lerch's theorem) of all periodic time scales.
- A general proof of uniqueness of the generalized nabla time scale Laplace transform (generalized Lerch's theorem) of all time scales that are able to satisfy the given assumptions.

The third chapter contains the results of fractional calculus, namely:

- An introduction to continuous fractional calculus and discrete calculus, with emphasis on the definition of fractional power functions.
- The axiomatic definition of fractional power functions and comparison of various definitions of fractional power functions.
- Extended results of the uniqueness of axiomatic definition of fractional power functions by application of generalized Lerch's theorem.

We believe that the main results of this master thesis enhanced the present knowledge of time scale calculus as well as the fractional calculus on time scales and thus contributed to its further development. Additionally, since some questions remained unanswered, the thesis provided some objects of future studies, namely:

- An investigation of counterexamples of uniqueness of the generalized nabla time scale Laplace transform, a determination of the number of possible counterexamples (possibly in regard to a multiplicative constant).

- An examination of implications of counterexamples of uniqueness on applications of time scale theory.
- An extension of generalized Lerch's theorem to all possible time scales and determining which they are.
- A definition of the Inverse generalized nabla time scale Laplace transform and identification of its properties and applications.
- An application of generalized Lerch's theorem to fractional calculus.
- A demonstration of the uniqueness of fractional power functions defined by axiomatic definition 4.3.1 on time scales, where generalized Lerch's theorem does not hold.

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6. The list of symbols

Set notation

$\emptyset$	empty set
$\mathbb{N}$	set of natural numbers
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers
$\mathbb{T}$	time scale, 2.1
$\mathbb{T}^{\kappa}$	truncated time scale (left scattered minimum), 2.1
$\mathbb{T}_{\kappa}$	truncated time scale (right scattered minimum), 2.1
$\mathbb{Z}$	set of whole numbers
$\mathbb{N}_a$	set $\{a, a+1, a+2, ...\}$, 4.2.1
$h\mathbb{Z}_a$	set $\{..., a-2h, a-h, a, a+ha, a+2h, ...\}$ 3.3.3
$\mathbb{C}_h$	Hilger complex numbers, 2.3.7
$\mathbb{Z}_h$	strip of complex numbers, 2.3.7
$\mathcal{R}_{\nu}$	class of ν-regressive functions, 2.3.1
$\mathcal{R}_{\nu}^{+}$	class of positively ν-regressive functions, 2.3.2
$\mathcal{D}(f)$	domain of convergence of the generalized nabla time Laplace transform of a function f, 3.1.1

Operator and function notation

$\min\{\cdot\}$	minimum
$\sup\{\cdot\}$	supremum
$\inf\{\cdot\}$	infimum
$f \circ g$	composition of functions f, g
$\sigma(\cdot)$	forward jump operator, 2.1.1
$\rho(\cdot)$	backward jump operator, 2.1.1
$\mu(\cdot)$	forward graininess function, 2.1.1
$\nu(\cdot)$	backward graininess function, 2.1.1

$\nabla f(\cdot)$ or $f^{\nabla}(\cdot)$	nabla derivative of function f, 2.2.1
$\hat{e}_c(\cdot,\cdot)$	nabla time scale exponential function, 2.3.6
$\hat{h}_n(\cdot,\cdot)$	nabla time scale power function (also nabla time scale monomial), 2.4.1, 4.3.1
$\mathcal{L}_s^{\mathbb{T}}\{f\}(\cdot)$	generalized nabla time scale Laplace transform, 3.1.1
$\Gamma(\cdot)$	Euler Gamma function, 4.1
${}_aD^{-\gamma}f$	Riemann-Liouville fractional integral of order γ, 4.1.1
${}_aD^{\gamma}f$	Riemann-Liouville fractional derivative of order γ, 4.1.2
${}_a^CD^{\gamma}f$	Caputo fractional derivative of order γ, 4.1.3
${}_a\nabla^{-m}f$	nabla Cauchy integral, 4.2.1
$\nabla_a^{\alpha}f$	nabla Riemann-Liouville fractional difference, 4.2.6
$\nabla_a^{-n}f$	nabla integral order sum, 4.2.3
$\hat{H}_n(\cdot,\cdot)$	nabla time scale power function, 4.3.4
$\mathcal{I}^{\mathbb{T}}{}_s\{f\}$	inverse generalized nabla time scale Laplace transform, 4.3.4
$\oplus_\nu$ or $\oplus$	generalized addition, 2.3.3
$\ominus_\nu$ or $\ominus$	generalized subtraction, 2.3.4