STABILITY OF THE ZERO SOLUTION OF STOCHASTIC DIFFERENTIAL SYSTEM

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Abstract: Stability of stochastic differential equations (SDEs) has become a very popular theme of recent research in mathematics and its applications. The method of Lyapunov functions for the analysis of qualitative behavior of SDEs provide some very powerful instruments in the study of stability properties for concrete stochastic dynamical systems, conditions of existence the stationary solutions of SDEs and related problems.

Keywords: Brownian motion, stochastic differential equation, Lyapunov function, stability.

1 INTRODUCTION

Stochastic modeling has an important role in many branches of science and industry. Stochastic model can be used to solve problem which evinces by accident, noise, etc. Definition of probability spaces, Brownian motion, SDE and an existence and uniqueness of solution of these equations were mentioned in Student EEICT 2014 [4]. It was taken from B. Øksendal [8]. The stability theory was introduced by R. Z. Khasminskii [3]. The basic principles of various types of stochastic systems are described by X.Mao [7]. In this paper we will follow up on previous proofs in Student EEICT 2015 [5] and we derived sufficient conditions for general system of the zero solution of the stochastic differential system using Lyapunov function. The results are verified on examples.

2 MAIN RESULTS - SYSTEM WITH TWO-DIMENSIONAL BROWNIAN MOTION

We have a homogenous linear stochastic differential equation

$$\mathrm{d}X_t = AX_t\mathrm{d}t + G\mathrm{d}B_t, \tag{1}$$

where $X_t = \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix}$, $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$, $G = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix}$, $B_t = \begin{pmatrix} B_1(t) \\ B_2(t) \end{pmatrix}$,

$a_1, a_2, a_3, a_4, g_1, g_2, g_3, g_4$ are constants.

Definition 2.1 *Lyapunov quadratic function V is given*

$$V(X_t) = X_t^T Q\, X_t,$$

where $Q = \begin{pmatrix} q_1 & q_2 \\ q_2 & q_1 \end{pmatrix}$ is a symmetric positive-definite matrix, i.e. $q_1 > 0, q_1^2 - q_2^2 > 0$.

Theorem 2.1 *Zero solution of equation* (1) *is stochastically stable if holds $LV < 0$, where*

$$LV = 2\left[a_1X_1^2(t) + a_4X_2^2(t) + (a_2+a_3)X_1(t)X_2(t) + g_1^2 + g_2^2 + g_3^2 + g_4^2\right].$$

Proof: [6], pp. 12.

Now we can do a discussion under which conditions the system will be stable.
The Euclidean matrix norm on the space R^n can be define as

$$\|A\|_E := \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}^2},$$

where a_{ij} is a matrix element of the i-th line and of the j-th column of the matrix, n is number of matrix raws, m is number of matrix columns. We denote $g_1^2+g_2^2+g_3^2+g_4^2=\|G\|^2$ and give

$$LV = 2\left[a_1X_1^2(t)+a_4X_2^2(t)+(a_2+a_3)X_1(t)X_2(t)+\|G\|^2\right]. \tag{2}$$

The Lyapunov function LV will be negative definite if and only if

$$a_1X_1^2(t)+a_4X_2^2(t)+(a_2+a_3)X_1(t)X_2(t)+\|G\|^2 \leq 0,$$

because $\|G\|^2 \geq 0$, therefore the matrix A must be sufficiently negative, to obtain a negative definite function. We use the **Sylvester's criterion** which is a necessary and sufficient criterion to determine whether a matrix is positive-definite. [2] In the following consequences we construct solutions to better imagine the stochastic stability.

Example 2.1 *First, we consider a diagonal matrix A and G of equation* (1) *in the form*

$$A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, G = \begin{pmatrix} \frac{a}{10} & 0 \\ 0 & \frac{a}{10} \end{pmatrix}.$$

Solution: The matrix A will be negative definite under following conditions:

$$\begin{aligned} D_1 &= |a_{11}| = a < 0, \\ D_2 &= a^2 > 0 \end{aligned}$$

if holds D_1 then the condition D_2 is obvious.

Then from (2) follows

$$\begin{aligned} aX_1^2(t)+aX_2^2(t) &\leq -\|G\|^2, \\ a\|X_t\|^2 &\leq -\|G\|^2. \end{aligned}$$

If the variable a is negative and also inequality $a\|X_t\|^2 \leq -\|G\|^2$ is valid, then the system is stochastically stable.
We find a solution of the stochastic system based on eigenvalues. If $a_{12}=a_{21}=0$, then $\lambda_1=a_{11}$, $\lambda_2=a_{22} \Rightarrow \lambda_{1,2}=a$. Because a is negative we make substitution $a=-\alpha, \alpha>0$. We give a solution of the system

$$\begin{aligned} X_1(t) &= C_1e^{-\alpha t}, \\ X_2(t) &= C_2te^{-\alpha t}, \end{aligned}$$

when C_1, C_2 are constants.
Zero solution of equation (1) with a matrix A is stochastically stable if holds the inequality $a\|X_t\|^2 \leq -\|G\|^2$. We determine stability of solution for $Q=E$

$$\begin{aligned} \mathrm{d}V(X_t) &= 2\left[aX_1^2(t)+aX_2^2(t)+2\left(\frac{a}{10}\right)^2\right]\mathrm{d}t+\frac{aX_1(t)}{5}\mathrm{d}B_1(t)+\frac{aX_2(t)}{5}\mathrm{d}B_2(t), \\ E\{\mathrm{d}V(X_t)\} &= 2\left[aX_1^2(t)+aX_2^2(t)+2\left(\frac{a}{10}\right)^2\right]\mathrm{d}t = LV\mathrm{d}t. \end{aligned}$$

There has to hold the inequality $a\|X_t\|^2 \leq -\|G\|^2$, so

$$a^2+50a\|X_t\|^2<0 \Leftrightarrow a<0 \vee a<-50\|X_t\|^2.$$

For $X_1(t)=C_1e^{-\alpha t}, X_2(t)=C_2te^{-\alpha t}$ we get

$$a<-50\left(C_1^2e^{-2\alpha t}+C_2^2t^2e^{-2\alpha t}\right).$$

Stochastic differential system is stable for $a<0$ or $a<-50\left(C_1^2e^{-2\alpha t}+C_2^2t^2e^{-2\alpha t}\right)$.

Example 2.2 *We consider a diagonal matrix A and G of equation* (1) *in the form*

$$A=\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, G=\begin{pmatrix} \frac{a}{10} & 0 \\ 0 & \frac{b}{10} \end{pmatrix}.$$

Solution: The matrix A will be negative definite under following conditions:

$$\begin{aligned} D_1 &= a<0, \\ D_2 &= ab>0 \Rightarrow b<0. \end{aligned}$$

Then from (2) follows

$$aX_1^2(t)+bX_2^2(t) \quad \leq \quad -\|G\|^2.$$

We find a solution of the stochastic system based on eigenvalues. $\lambda_1=a, \lambda_2=b$. We substitute $a=-\alpha, \alpha>0, b=-\beta, \beta>0$. We give a solution of the system

$$\begin{aligned} X_1(t) &= C_1e^{-\alpha t}, \\ X_2(t) &= C_2te^{-\beta t}, \end{aligned}$$

C_1, C_2 are constants.
Zero solution of equation (1) with a matrix A is stochastically stable if holds the inequality $aX_1^2(t)+bX_2^2(t) \leq -\|G\|^2$.
We determine stability of solution for $Q=E$

$$\begin{aligned} \mathrm{d}V(X_t) &= 2\left[aX_1^2(t)+bX_2^2(t)+\left(\frac{a}{10}\right)^2+\left(\frac{b}{10}\right)^2\right]\mathrm{d}t+\frac{aX_1(t)}{5}\mathrm{d}B_1(t)+\frac{bX_2(t)}{5}\mathrm{d}B_2(t), \\ E\left\{\mathrm{d}V(X_t)\right\} &= 2\left[aX_1^2(t)+bX_2^2(t)+\left(\frac{a}{10}\right)^2+\left(\frac{b}{10}\right)^2\right]\mathrm{d}t = LV\mathrm{d}t. \end{aligned}$$

There has to hold the inequality $aX_1^2(t)+bX_2^2(t) \leq -\|G\|^2$, so if for $X_1(t)=C_1e^{-\alpha t}, X_2(t)=C_2te^{-\beta t}$ holds the inequality

$$aC_1^2e^{-2\alpha t}+bC_2^2t^2e^{-2\beta t} \leq -\frac{a^2+b^2}{100},$$

then the system is stable.

Example 2.3 *We consider a symmetric matrix A and G of equation* (1) *in the form*

$$A=\begin{pmatrix} a & b \\ b & a \end{pmatrix}, G=\begin{pmatrix} \frac{a}{10} & \frac{b}{10} \\ \frac{b}{10} & \frac{a}{10} \end{pmatrix}.$$

Solution: The matrix A will be negative definite under following conditions:

$$\left.\begin{array}{l} D_1 = a < 0, \\ D_2 = a^2 - b^2 > 0 \Rightarrow |a| > |b| \end{array}\right\} \text{ i.e. must be valid } |a| > |b| > 0.$$

Then from (2) follows

$$aX_1^2(t) + aX_2^2(t) + 2bX_1(t)X_2(t) \quad \leq \quad -\|G\|^2.$$

The variable a must be sufficiently negative and also inequality

$$a\|X(t)\|^2 + 2bX_1(t)X_2(t) \leq -\|G\|^2$$

must be valid, then we can say that the system is stochastically stable.
We find eigenvalues of matrix A as the solution of the characteristic equation

$$det(A - \lambda E) = 0,$$

where E is the unit matrix.

$$\begin{aligned} |A - \lambda E| = (a-\lambda)^2 - b^2 &= 0, \\ (a-\lambda)^2 &= b^2, \\ |a-\lambda| &= |b|. \end{aligned}$$

Eigenvalues are

$$\begin{aligned} -a + \lambda_1 = |b| \Rightarrow \lambda_1 = a + |b|, \\ a - \lambda_2 = |b| \Rightarrow \lambda_2 = a - |b|. \end{aligned}$$

We substitute $a = -\alpha, \alpha > 0, |b| > 0, \alpha < |b|$, i.e. $\lambda_1 = -\alpha + |b|, \lambda_2 = -\alpha - |b|$. For the eigenvalue $\lambda_1 = -\alpha + |b|$ we find the eigenvector $v_1 = (v_{11}, v_{12})$. There is any nonzero vector which fulfills a following relation

$$\begin{aligned} (A - \lambda_1 E)\, v_1 &= 0, \\ \begin{pmatrix} a - (a+|b|) & b \\ b & a - (a+|b|) \end{pmatrix} v_1 &= 0. \end{aligned}$$

For $b > 0$ we choose an arbitrary vector $v_1 = (1,1)^T$, for $b < 0$ we choose $v_1 = (-1,1)^T$. Then

$$\begin{aligned} &\text{for} \quad b > 0 \quad \text{is } X_1(t) = (1,1)^T e^{(-\alpha+b)t}, \\ &\text{for} \quad b < 0 \quad \text{is } X_1(t) = (-1,1)^T e^{(-\alpha+b)t}. \end{aligned}$$

For the eigenvalue $\lambda_1 = -\alpha - |b|$ we find an eigenvector $v_2 = (v_{21}, v_{22})$

$$\begin{aligned} (A - \lambda_1 E)\, v_2 = 0, \\ \begin{pmatrix} a - (a-|b|) & b \\ b & a - (a-|b|) \end{pmatrix} v_2 = 0. \end{aligned}$$

For $b > 0$ we choose an arbitrary vector $v_2 = (1,-1)^T$, for $b < 0$ we choose $v_2 = (1,1)^T$. Then

$$\begin{aligned} &\text{for} \quad b < 0 \quad \text{is } X_2(t) = (1,1)^T e^{-(\alpha+b)t}, \\ &\text{for} \quad b > 0 \quad \text{is } X_2(t) = (1,-1)^T e^{-(\alpha+b)t}. \end{aligned}$$

The general solution is given by a linear combination $X_t = C_1X_1(t) + C_2X_2(t)$, with arbitrary constants C_1, C_2.
Zero solution of equation (1) with a matrix A is stochastically stable if holds the inequality
$a\|X(t)\|^2 + 2bX_1(t)X_2(t) \leq -\|G\|^2$.
We determine stability of solution for $Q = E$

$$\begin{aligned}
\mathrm{d}V(X_t) &= 2\left[a\left(X_1^2(t) + X_2^2(t)\right) + 2bX_1(t)X_2(t) + \frac{a^2}{50} + \frac{b^2}{50}\right]\mathrm{d}t + \frac{aX_1(t) + bX_2(t)}{5}\mathrm{d}B_1(t) \\
&+ \frac{bX_1(t) + aX_2(t)}{5}\mathrm{d}B_2(t), \\
E\{\mathrm{d}V(X_t)\} &= 2\left[a\left(X_1^2(t) + X_2^2(t)\right) + 2bX_1(t)X_2(t) + \frac{a^2 + b^2}{50}\right]\mathrm{d}t = LV\mathrm{d}t.
\end{aligned}$$

There has to hold the inequality $a\|X(t)\|^2 + 2bX_1(t)X_2(t) \leq -\|G\|^2$, so if holds the inequality

$$a\|X(t)\|^2 + 2bX_1(t)X_2(t) \leq -\frac{a^2 + b^2}{50},$$

for $b > 0, X_1(t) = (1,1)^T e^{(-\alpha+b)t}, X_2(t) = (1,-1)^T e^{-(\alpha+b)t}$; for $b < 0, X_1(t) = (-1,1)^T e^{(-\alpha+b)t}$, $X_2(t) = (1,1)^T e^{-(\alpha+b)t}$, then the system is stable.

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