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POINT PARTICLE KINEMATICS AND LIGHT PROPAGATION IN DE SITTER SPACETIME

BAKALÁŘSKÁ PRÁCE
BACHELOR'S THESIS

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KINEMATIKA BODOVÝCH ČÁSTIC A ŠÍŘENÍ SVĚTLA V DE SITTEROVĚ ČASOPROSTORU

BAKALÁŘSKÁ PRÁCE
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Akademický rok: 2014/2015

ZADÁNÍ BAKALÁŘSKÉ PRÁCE

student(ka): Tomáš Michalík

který/která studuje v **bakalářském studijním programu**

obor: **Fyzikální inženýrství a nanotechnologie (3901R043)**

Ředitel ústavu Vám v souladu se zákonem č.111/1998 o vysokých školách a se Studijním a zkušebním řádem VUT v Brně určuje následující téma bakalářské práce:

Kinematika bodových částic a šíření světla v de Sitterově časoprostoru

v anglickém jazyce:

Point particle kinematics and light propagation in de Sitter spacetime

Stručná charakteristika problematiky úkolu:

De Sitterův časoprostor je důležitý v moderné relativistické fyzice kvůli jeho roli oskulujícího časoprostoru, na kterém se odehrává lokální kinematika. Student prozkoumá rozdíly v kinematice v Minkowského a v de Sitterově časoprostoru, které vyplývají ze zakřivení a neexistence preferované soustavy souřadnic v de Sitterově časoprostoru. Student prozkoumá symetrie a tranzitivní vlastnosti de Sitterovho časoprostoru a odvodí změnu ve variacích časoprostorových proměnných, ze které vyplývá změna zákonu zachování odvozených z teorému Noetherové. Změna ve variacích taktéž vede ke změně funkcionální derivace, odlišným Eulerovým-Lagrangeovým rovnicím a modifikovaným rovnicím geodetických křivek. Student se taktéž zaměří na šíření elektromagnetických vln a odvodí rychlost jejich šíření.

Cíle bakalářské práce:

1. Parametrizovat grupu symetrie de Sitterova časoprostoru.
2. Najít tvar generátorů tranzitivních transformací de Sitterova časoprostoru.
3. Najít proud Noetherové generovaný tranzitivními transformacemi.
4. Získat modifikovanou funkcionální derivaci a odvodit rovnici geodetiky.
5. Vyšetřit šíření elektromagnetických vln v de Sitterově časoprostoru.

Seznam odborné literatury:

[1] S. W. Hawking, G. F. R. Ellis: The Large Scale Structure of Space-Time. ISBN 0-5210-9906-4.
[2] A. O. Barut, R. Raczka: Theory of Group Representations and Applications. ISBN 9-9715-0217-8.

Vedoucí bakalářské práce: Klaus Bering Larsen, Ph.D.

Termín odevzdání bakalářské práce je stanoven časovým plánem akademického roku 2014/2015.

V Brně, dne 21.11.2014

L.S.

prof. RNDr. Tomáš Šikola, CSc. Ředitel ústavu	doc. Ing. Jaroslav Katolický, Ph.D. Děkan fakulty

ABSTRACT
In this thesis we study de Sitterian special relativity, which takes place in de Sitter spacetime instead of Minkowski spacetime. We start with study of the symmetry group of de Sitter spacetime. We try to use these results to develop kinematics in this spacetime. We also review connection between electrodynamics on fixed background spacetime and electrodynamics in macroscopic media in flat spacetime. We apply this on de Sitter spacetime and find refractive index of associated macroscopic media.

KEYWORDS
de Sitter spacetime, kinematics, Lie groups, special relativity, electrodynamics , refractive index

ABSTRAKT
V této práci se zabýváme de Sitterovou speciální relativiou, která se odehrává v de Sitterově časoprostoru namísto Minkowského časoprostoru. Začínáme studií grupy symetrie de Sitterova časoprostoru. Snažíme se použít tyto poznatky k rozvinutí kinematiky v de Sitterově časoprostoru. Dále přezkoumáváme spojení mezi elektrodynamikou na pevně zvoleném časoprostoru a elektrodynamikou v makroskopickém médiu v plochém časoprostoru. Toto aplikujeme na de Sitterův časoprostor a hledáme index lomu sdruženého média.

KLÍČOVÁ SLOVA
de Sitterův časoprostor, kinematika, Lieovy grupy, speciální relativita, elektrodynamika, index lomu

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DECLARATION

I declare that I have elaborated my bachelor's thesis on the theme of "Point particle kinematics and light propagation in de Sitter spacetime" independently, under the supervision of the bachelor's thesis supervisor and with the use of technical literature and other sources of information which are all quoted in the thesis and detailed in the list of literature at the end of the thesis.

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Brno . . . .

(author's signature)

I wish to express my sincere thanks to my thesis advisor Klaus Bering Larsen, PhD. for his time, valuable advices and stimulating consultations. I am also grateful to doc. Franz Hinterleitner, PhD. for his time, valuable advices and his presence on the consultations with my advisor. I also thank my family and friends for the unceasing encouragement, support and attention.

CONTENTS

NOTATIONS AND CONVENTIONS

χ^A	Coordinates in 5-dimensional Minkowski spacetime
δ^0	Vertical variation
δ^0_ξ	Vertical variation with respect to ξ
δ^a	Horizontal variation
δ^a_ξ	Horizontal variation with respect to ξ
ℓ	Pseudoradius
ϵ_{ijk}	Totally antisymmetric tensor density, $\epsilon_{123} = 1$
$\eta_{\mu\nu}$	Metric of Minkowski spacetime
$\Gamma^\mu{}_{\nu\rho}$	Levi-Civita connection
γ^A	Basis of Clifford algebra
Λ	O(1,4) transformation
Λ_c	Cosmological constant
$\mathfrak{k}, \mathfrak{l}, \mathfrak{p}, \ldots$	Lie algebras
$\mathfrak{L}_m$	Matter lagrangian density
∇	Covariant derivative
$\Pi^{\mu\nu}$	de Sitterian energy-momentum tensor
π^μ	de Sitterian momentum
Π_μ	Generator of generalized translations
$\pounds$	Lie derivative
ρ	Charge density
Σ_μ	Spin matrices
$\Box$	Covariant d'Alembert operator
Θ	Cartan involution of Lie group
ϑ	Cartan involution of Lie algebra

A_μ	Four-potential
B^i	Magnetic induction
D^i	Electric induction
E_i	Electric intensity
$F^{\mu\nu}$	Field strength tensor
$G_{\mu\nu}$	Einstein tensor
$g_{\mu\nu}$	Metric tensor
H_i	Magnetic intensity
J^μ	Current four-vector
j^i	Electric current
$K^{\mu\nu}$	Proper conformal current
K_μ	Generator of proper conformal transformations
k_μ	Wave four-vector
n	Refractive index
P_μ	Generator of translations
R	Scalar curvature
$R_{\mu\nu}$	Ricci tensor
$T^{\mu\nu}$	Energy-momentum tensor
x	Coordinates on de Sitter spacetime

I will be using west coast metric $(+,-,-,...)$.

- 5-dimensional indices will be denoted by capital Latin letters $A, B, ...$ and will run from 0 to 4.
- 4-dimensional indices will be denoted by Greek letter $\alpha, \beta, ...$ and will run from 0 to 3.
- 3-dimensional indices will be denoted by small Latin letters $a, b, ...$ and will run from 1 to 3.

INTRODUCTION

Just like Newtonian mechanics was replaced by special relativity when the invariance of the speed of light appeared in Maxwell's equations, there exists a generalization of special relativity which also admits a invariant length parameter. Such a length parameter is for example the cosmological constant. In fact, an invariant length parameter is present in the special relativity as well, but it is easy to overlook, because it is infinite. If we want this parameter to be finite, we must replace Minkowski spacetime with de Sitter one. De Sitter spacetime is defined as a one sheeted hyperboloid embedded in 5-dimensional Minkowski spacetime and its invariant length parameter is its pseudoradius. In this thesis we will discuss changes which arise when we replace Minkowskian relativity with de Sitterian one. In the first chapter we will construct de Sitter spacetime and its metric and talk about its symmetries and symmetry group. We will find the explicit form of an arbitrary de Sitter transformation using the elegant isomorphism between quaternionic matrices and 5-dimensional Minkowski spacetime. In the second chapter we will discuss the notion of transitivity in de Sitter spacetime, which is changed significantly, as compared to the Minkowski spacetime. With this change comes also a change in the Euler-Lagrange equations and the Noether currents. The change in the Euler-Lagrange equations will modify every equation of motion obtained from the action principle, such as the geodesic equation. Modified Noether currents lead to the different conservation laws. We will observe that the correction term depends on the pseudoradius and we are able to recover the ordinary special relativity in the infinite pseudoradius limit. In the third chapter we will take a look at the propagation of electromagnetic waves. We will use the fact, that we are able to convert the problem of the electromagnetic wave propagation in a curved spacetime into propagation in a refractive media in the Minkowski spacetime, which will lead to an interesting results. The propagation speed of electromagnetic waves is actually different from the physical constant speed of light.

1 DE SITTER SPACETIME AND ITS SYMMETRIES

The aim of this chapter is to present de Sitter spacetime and explore its geometrical properties and symmetry group, which will help us to define kinematics in this spacetime later on.

1.1 De Sitter hyperboloid

The de Sitter hyperboloid $dS(1,3)$ is a hypersurface of 5-dimensional Minkowski spacetime given by the equation

$$\eta_{AB}\chi^A\chi^B = -\ell^2 \tag{1.1}$$

where l is the pseudoradius, a length parameter invariant under de Sitter transformations. Using basic differential geometry and the fact that de Sitter space is maximally symmetric, we are able to relate this pseudoradius to the cosmological constant using the Einstein field equations (EFE). The resulting relation is given by

$$\Lambda_c = \frac{3}{\ell^2} \tag{1.2}$$

This allows us to inspect our theory in various limits, mainly in the flat limit, which should lead to Einstein's special relativity.

The biggest difference between Minkowski and de Sitter space is that de Sitter space does not have a preferred coordinate system. The most important thing in de Sitter special relativity is therefore choice the of a coordinate system, in which we will be able to find similarities with Einstein's special relativity. I will use the coordinate system introduced in [1]. This coordinate system is given by

$$\left.\begin{aligned} \chi^\mu &= \Omega(x)x^\mu \\ \chi^4 &= \ell\Omega(x)\left(1+\frac{\sigma^2}{4\ell^2}\right) \end{aligned}\right\} \tag{1.3}$$

where

$$\Omega(x) := \frac{1}{1-\sigma^2/4\ell^2} \tag{1.4}$$

$$\sigma^2 := \eta_{\mu\nu}x^\mu x^\nu \tag{1.5}$$

The induced metric in this coordinate system is

$$g_{\mu\nu} = \Omega^2(x)\eta_{\mu\nu} \tag{1.6}$$

This coordinate patch is conformally equivalent to Minkowski spacetime and therefore we expect that kinematics in this coordinate system will have nice properties. This later turns out to be true.

Rewriting the equation for de Sitter spacetime in form

$$-\frac{1}{\ell^2}\Omega^2\left(x\right)\sigma^2+\left(\chi'^4\right)^2=1 \tag{1.7}$$

where χ'^4 is a dimensionless coordinate we can see that for a infinite pseudoradius, which corresponds to a vanishing cosmological constant, we obtain

$$\chi'^4=1 \tag{1.8}$$

which defines 4-dimensional Minkowski hyperspace as expected.

1.2 Symmetry group of dS(1,3)

The symmetry group of de Sitter spacetime is a group, which sends a point from de Sitter spacetime to another point in de Sitter spacetime. It clearly has to leave equation 1.1 invariant. In matrix notation, this equation is

$$\chi^\top\eta\chi=-\ell^2 \tag{1.9}$$

If we transform vectors by a linear transformation Λ, we the obtain vector

$$\chi'^A=\Lambda^A{}_B\chi^B \tag{1.10}$$

and we the get equation

$$\left(\Lambda\chi\right)^\top\eta\left(\Lambda\chi\right)=-\ell^2 \tag{1.11}$$

This gives us an equation for matrices belonging to the symmetry group of de Sitter spacetime

$$\Lambda^\top\eta\Lambda=\eta \tag{1.12}$$

This group is called de Sitter group and we will denote it $O(1,4)$. Let's take a look at these matrices.

By taking the determinant of the equation we find out, that

$$\det\left(\Lambda\right)^2=1 \tag{1.13}$$

When we take a look at the equation for the $\Lambda^0{}_0$, we get

$$\left(\Lambda^0{}_0\right)^2=1+\left(\Lambda^1{}_0\right)^2+\left(\Lambda^2{}_0\right)^2+\left(\Lambda^3{}_0\right)^2+\left(\Lambda^4{}_0\right)^2 \tag{1.14}$$

And therefore we can also choose the sign of $\Lambda^0{}_0$. Transformations with negative sign reverse the time flow. We can see that there is no continuous way to change the determinant of the matrix from -1 and 1 and sign of $\Lambda^0{}_0$ from $-$ to $+$. This tells us that the de Sitter group has (at least) four connected components. However, only one of these connected components is a subgroup, namely the one with positive determinant and sign of $\Lambda^0{}_0$. We will denote this subgroup $SO^+(1,4)$. It is called the restricted de Sitter group. The parametrization of this group would require solving a set of 10 quadratic equations for 20 unknowns. We are not able to do it directly, so we have to find smart way to do it.

1.2.1 De Sitter algebra

Let's take a look on infinitesimal de Sitter transformations. These transformations have the form

$$\Lambda = \mathbb{I} + \epsilon X + O\left(\epsilon^2\right) \tag{1.15}$$

By inserting this into the equation for de Sitter group we find out, that X has to satisfy the equation

$$X^\top = -\eta X \eta^{-1} \tag{1.16}$$

These transformations have one upper and one lower index. In indices, this equation has the form

$$\left(X^\top\right)_A{}^B = -\eta_{AC} X^C{}_D \eta^{DB} \tag{1.17}$$

These matrices form a 10-dimensional vector space denoted $\mathfrak{so}(1,4)$, hence every element of this space can be obtained as linear combination of basis elements. To find this subspace, it is convenient to define a basis of the linear space of matrices. This basis is given by

$$\left(e_C{}^D\right)^A{}_B \coloneqq \delta^A_C \delta^D_B \tag{1.18}$$

Multiplication is defined by

$$\left(e_C{}^D e_E{}^F\right)^A{}_B = \left(e_C{}^D\right)^A{}_G \left(e_E{}^F\right)^G{}_B = \delta^D_E \left(e_C{}^F\right)^A{}_B \tag{1.19}$$

Now we define matrices with lower indices as

$$\left(e_{CD}\right)^A{}_B \coloneqq \delta^A{}_C \eta_{DB} \tag{1.20}$$

It is easy to see that in this case the result of a matrix multiplication is

$$e_{CD}e_{EF} = \eta_{ED}e_{CF} \tag{1.21}$$

Using these relations, we find that the linear subspace is generated by antisymmetric matrices with lower indices

$$t_{AB} \coloneqq e_{AB} - e_{BA} \tag{1.22}$$

On this space, we are able to define another operation, called Lie bracket or commutator, defined as

$$[X, Y] = XY - YX \qquad X, Y \in \mathfrak{so}\,(1, 4) \tag{1.23}$$

By inserting this expression into the equation of our vector space we find out that this indeed lies inside and therefore corresponds to some other infinitesimal transformation Z. This promotes the space of infinitesimal transformations to a Lie algebra. An easy calculation for basis elements gives us the result

$$[t_{AB}, t_{CD}] = \eta_{BC}t_{AD} + \eta_{AD}t_{BC} - \eta_{AC}t_{BD} - \eta_{BD}t_{AC} \tag{1.24}$$

We are able to obtain group elements by exponentiating elements of the algebra.

1.2.2 De Sitter spacetime in terms of quaternionic matrices

To perform exponentiation, we will use a representation of 5-dimensional Minkowski spacetime using Clifford algebra. To a vector χ we assign the quaternionic matrix $\gamma_A\chi^A$ denoted $\not{\chi}$.

$$\not{\chi} \coloneqq \chi^A\gamma_A \qquad \chi^A \in \mathbb{R} \qquad \gamma_A \in Mat_{2\times 2}\,(\mathbb{H}) \tag{1.25}$$

The γ matrices satisfy an anticommutation relations

$$\{\gamma_A, \gamma_B\} = 2\eta_{AB}\mathbb{I}_{2\times 2} \qquad \gamma_A, \gamma_B \in Mat_{2\times 2}\,(\mathbb{H}) \tag{1.26}$$

We define "conjugated" γ matrices as

$$\widetilde{\gamma}_A = \gamma_0\gamma_A\gamma_0 \tag{1.27}$$

Indices on γ matrices are raised and lowered with η^{AB}. In this thesis, we will use an explicit representation in which γ matrices are [2]

$$\left.\begin{aligned}
\gamma_0 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
\gamma_1 &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \\
\gamma_2 &= \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix} \\
\gamma_3 &= \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix} \\
\gamma_4 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\end{aligned}\right\} \tag{1.28}$$

The squared length of χ is in this formulation is given as

$$\eta_{AB}\chi^A\chi^B\mathbb{I}_{2\times 2} = \not{\chi}^\dagger\widetilde{\not{\chi}} \tag{1.29}$$

and the coordinates χ^A are given by

$$\chi^A = \frac{1}{2}\mathrm{tr}_{\mathbb{H}}\left(\widetilde{\gamma}^A\not{\chi}\right) \tag{1.30}$$

where [1]

$$\mathrm{tr}_{\mathbb{H}}A \coloneqq \frac{1}{2}\mathrm{tr}\left(A + A^\dagger\right) \tag{1.31}$$

The symmetry group acts on these matrices by conjugation

$$\not{\chi}' = g\chi g^{-1} \tag{1.32}$$

Coordinates in this formulation are therefore transformed as

$$\chi'^A = \frac{1}{2}\mathrm{tr}_{\mathbb{H}}\left(\widetilde{\gamma}^A g\chi^B\gamma_B g^{-1}\right) \tag{1.33}$$

Let's find out which transformations leave the norm of a vector invariant.

First, we take a look at the squared length of the transformed vector, which is equivalent to

$$\left(g\not{\chi}g^{-1}\right)^\dagger\gamma_0 g\not{\chi}g^{-1}\gamma_0 \tag{1.34}$$

[1] $A^\dagger = \bar{A}^\top$, where $\bar{A}$ is quaternionic conjugate and $\top$ denotes transpose. We have $\bar{i} = -i$, $\bar{j} = -j$ and $\bar{k} = \overline{ij} = \bar{j}\,\bar{i} = ji = -k$.

Let's assume that

$$g^{\dagger}\gamma_0 g = \gamma_0 \tag{1.35}$$

We obtain

$$\left(g^{\dagger}\right)^{-1} \not{\chi}^{\dagger}\gamma_0\not{\chi} g^{-1}\gamma_0 \tag{1.36}$$

Using the expression for the norm of a vector we substitute $\eta_{AB}\chi^A\chi^B\gamma_0 = \not{\chi}^{\dagger}\gamma^0\not{\chi}$ and we have

$$\eta_{AB}\chi^A\chi^B \left(g^{\dagger}\right)^{-1} \gamma_0 g^{-1}\gamma_0 \tag{1.37}$$

From our assumption 1.35 we find out that

$$g^{\dagger} = \gamma_0 g^{-1}\gamma_0 \tag{1.38}$$

has to hold. Substituting this into our calculation gives us the squared length $\eta_{AB}\chi^A\chi^B$, the same squared length as before the transformation. The group 1.35 is called the pseudohyperunitary group and is denoted by $Sp(1,1)$ or $U(1,1,\mathbb{H})$. In this group, infinitesimal transformations X have to fulfill

$$X^{\dagger} = -\gamma_0 X\gamma_0 \tag{1.39}$$

Explicitly, these matrices are

$$X = \begin{pmatrix} k_1 & x \\ \bar{x} & k_2 \end{pmatrix} \qquad \bar{k}_1 = -k_1 \qquad \bar{k}_2 = -k_2 \qquad k_1, k_2, x \in \mathbb{H} \tag{1.40}$$

This is again a 10-dimensional real vector space. By the choice of a suitable basis, we are able to obtain the same Lie bracket as in the $SO^{+}(1,4)$ formulation and therefore these algebras are isomorphic. This algebra is, however, much easier to exponentiate and therefore we are able to parametrize our symmetry group.

1.2.3 Cartan decomposition of Spin(1,4) group

Now let us parametrize $Sp(1,1)$ using the Cartan decomposition. An introduction to various decompositions of Lie groups can be found in [3]. To perform the Cartan decomposition of a Lie group, we have to find the corresponding Cartan pair. We apply the Cartan involution

$$\Theta(g) = \left(g^{\dagger}\right)^{-1} \tag{1.41}$$

which will help us to identify the maximal compact subgroup K, which is invariant under this involution. Using the equation of the group and the invariance under the involution we find out that this subgroup consists of matrices with unit quaternions on the diagonal.

$$k = \mathrm{diag}\left(k_1, k_2\right) \qquad |k_1| = |k_2| = 1 \qquad k_1, k_2 \in \mathbb{H} \tag{1.42}$$

We can easily recognize the group

$$K = SU(2) \times SU(2) = Spin(4) \tag{1.43}$$

as the double cover of the rotational group in the 4-dimensional Euclidean space.

The next step is to induce an action of this involution on the Lie algebra and identify subspaces with eigenvalues 1 and -1 denoted $\mathfrak{k}$ and $\mathfrak{p}$ respectively. The action of this involution on the algebra is

$$\vartheta\left(X\right) = -X^{\dagger} \tag{1.44}$$

$\mathfrak{k}$ corresponds to the diagonal imaginary matrices, whereas $\mathfrak{p}$ consists of Hermitian quaternion 2×2 matrices with zeros on the diagonal.

Finally, any group element $g \in Sp(1,1)$ can be expressed as $k\mathrm{e}^X$, where $k \in K$ and $X \in \mathfrak{p}$. In our case, X is

$$X = \begin{pmatrix} 0 & x \\ \bar{x} & 0 \end{pmatrix} \qquad x \in \mathbb{H} \tag{1.45}$$

aand therefore e^X is

$$e^{\mathfrak{p}} \ni e^X = \begin{pmatrix} \cosh\frac{|x|}{2} & \frac{x}{|x|}\sinh\frac{|x|}{2} \\ \frac{\bar{x}}{|x|}\sinh\frac{|x|}{2} & \cosh\frac{|x|}{2} \end{pmatrix} \qquad x \in \mathbb{H} \tag{1.46}$$

1.2.4 Isomorphism between real and quaternionic formulation

We want to find a homomorphism between groups the $Sp(1,1)$ and $SO^+(1,4)$, that is a function $g \to \Lambda(g)$. We can do this by comparing the transformation laws for slashed and non-slashed quantities. Non-slashed vector transforms as 1.10, whereas vectors in quaternionic formulation are transformed by 1.33. The following equation has to hold.

$$\Lambda\left(g\right)^A{}_B \chi^B = \frac{1}{2}\mathrm{tr}_{\mathbb{H}}\left(\widetilde{\gamma}^A g \chi^B \gamma_B g^{-1}\right) \tag{1.47}$$

From this equation we can directly read off the relation between the elements of $Sp(1,1)$ and $SO^{+}(1,4)$

$$\Lambda\left(g\right)^{A}{}_{B} = \frac{1}{2}\mathrm{tr}_{\mathbb{H}}\left(\widetilde{\gamma}^{A} g \gamma_{B} g^{-1}\right) \tag{1.48}$$

Notice that g and $-g$ give the same image in $SO^{+}(1,4)$. The group $Sp(1,1)$ is therefore a double cover of the de Sitter group, called Spin group and denoted $Spin(1,4)$. This group is very important if we want to work with particles with half-integer spins - the fermions. In figure 1.1 we can see the action of a de Sitterian boost on our coordinate system with two of the spatial coordinates suppressed.

1.3 Construction of invariant metric on dS(1,3)

At the beginning of this chapter we defined the de Sitter spacetime as a hypersurface in the ambient Minkowski spacetime. However, there exists a more fundamental definition which allows us to define the de Sitter spacetime without EFE. It is a basic example of a homogeneous space. This space is defined as a quotient of the transitive symmetry group with respect to its subgroup, which leaves some point invariant, called the little group. If the little group fulfills some additional requirements [3], the homogeneous space becomes the symmetric space. Symmetric spaces are generalizations of flat Euclidean (or Minkowski, depending on signature of metric) space. We would like to the find metric, which is invariant under actions of the symmetry group on this space. In this section I follow Appendix A of [4].

A simple real Lie group G is naturally provided with a Killing metric. This metric is obtained from the Killing form on the Lie algebra $\mathfrak{l}$, defined

$$\langle\ ,\ \rangle : \mathfrak{l} \times \mathfrak{l} \to \mathbb{R}, (A, B) \mapsto \langle A, B\rangle = \mathrm{Tr}\left[(\mathrm{ad}A)\,(\mathrm{ad}B)\right] \tag{1.49}$$

where $A \to \mathrm{ad}A = [A, \cdot]$ is the adjoint representation. This bi-linear form is non-degenerate iff G is semi-simple.

Now we use left translations on a Lie group generated by multiplication to pull-back the Killing form the tangent space of identity to the tangent space at an arbitrary point. A left translation is a diffeomorphism

$$L_{g} : G \to G, h \mapsto L_{g}\left(h\right) = gh \tag{1.50}$$

Using this map, we can define a scalar product in arbitrary group element g by

$$\mathrm{d}s^{2} = \langle\ ,\ \rangle_{g} : T_{g}G \times T_{g}G \to \mathbb{R} \text{ as } \langle\ ,\ \rangle_{g} = L^{*}_{g^{-1}}\langle\ ,\ \rangle_{e} \tag{1.51}$$

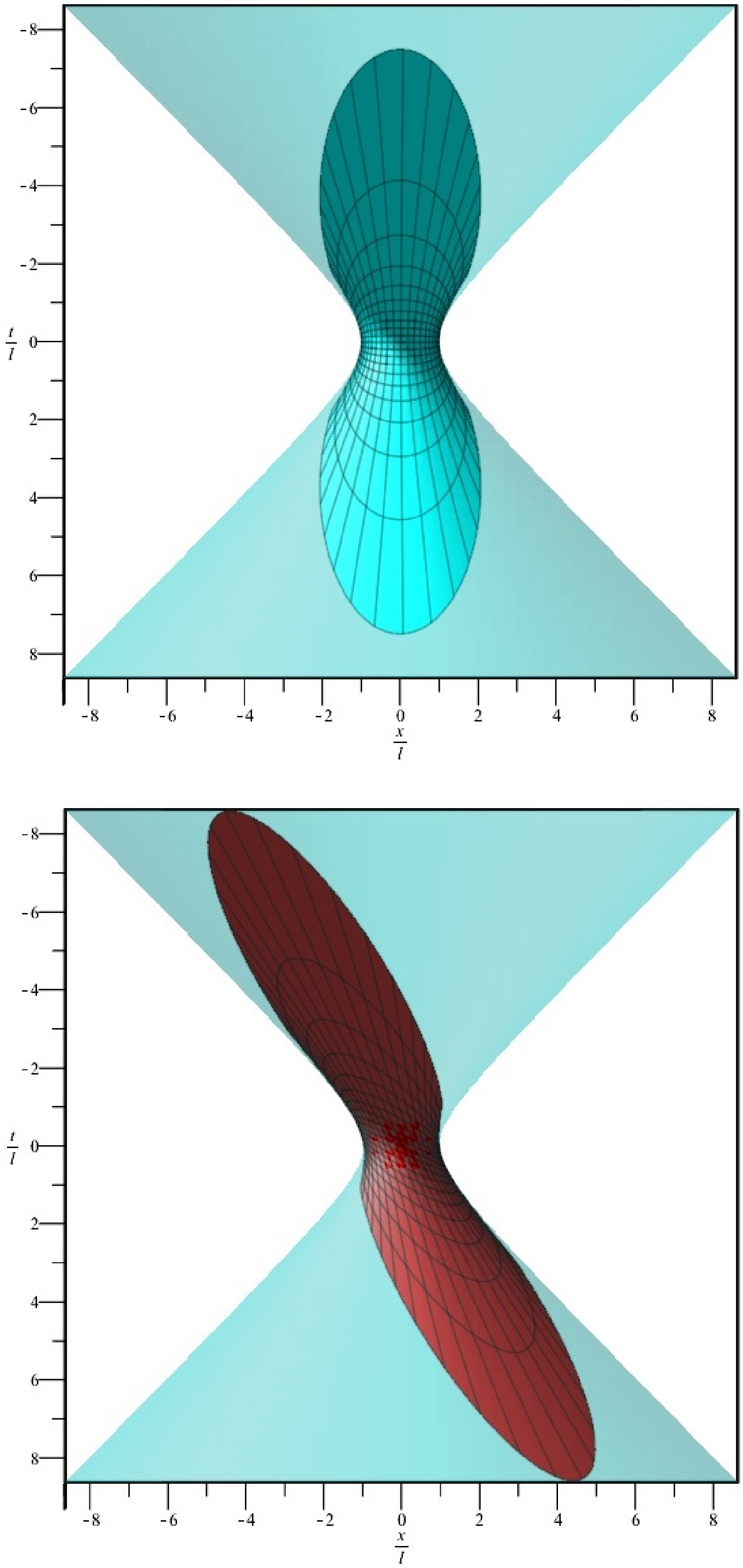

Fig. 1.1: Effects of boost on coordinate system 1.3

Now let us compute this pullback explicitly. Let $\varphi(\chi)$ be differentiable local coordinatization of G:

$$\varphi : \mathbb{R}^n \supset U \to G \tag{1.52}$$

An infinitesimal displacement $\mathrm{d}\chi$ in U determines an infinitesimal displacement $\mathrm{d}g = \mathrm{d}\varphi(\chi)$ is $T_g G$, where $g = \varphi(\chi)$. Using a left translation, we can push-forward such a displacement to $T_e G$ by

$$L_{g^{-1}*}\mathrm{d}g = g^{-1}\mathrm{d}g = \varphi(\chi)^{-1}\,\mathrm{d}\varphi\chi \tag{1.53}$$

It gives us a metric induced from the Killing metric.

$$\mathrm{d}s^2 = \langle \mathrm{d}g, \mathrm{d}g \rangle_g = \left\langle \varphi(\chi)^{-1}\,\mathrm{d}\varphi(\chi), \varphi(\chi)^{-1}\,\mathrm{d}\varphi(\chi) \right\rangle \tag{1.54}$$

If our group G is simple, an application of Schur's lemma tells us, that we can use any faithful representation instead of the adjoint representation. The resulting metric differs only by a multiplicative constant. To find a metric on $dS(1,3)$, we need to identify it with sub-manifold of the de Sitter group. Let us denote element of the restricted de Sitter group as Λ. Point in de Sitter spacetime embedded in the restricted de Sitter group is denoted as x.

A vector in de Sitter spacetime embedded in Minkowski spacetime transforms under the de Sitter transformation as

$$\chi' = \Lambda\,\chi \tag{1.55}$$

If we consider it as an element of the group, it transform under the most natural action of the group on itself - the adjoint action

$$x' = \Lambda x \Lambda^{-1} \tag{1.56}$$

Such actions correspond to isometries of the Killing metric. This fact is shown by simple calculation

$$x'^{-1}\mathrm{d}x = \left(\Lambda x \Lambda^{-1}\right)^{-1}\mathrm{d}\left(\Lambda x \Lambda^{-1}\right) = \Lambda x^{-1}\mathrm{d}x\Lambda^{-1} \tag{1.57}$$

And using the cyclic property of the trace we have

$$\begin{aligned}\mathrm{Tr}\left[\left(x'^{-1}\mathrm{d}x'\right)\left(x'^{-1}\mathrm{d}x'\right)\right] &= \mathrm{Tr}\left[\left(\Lambda x^{-1}\mathrm{d}x\Lambda^{-1}\right)\left(\Lambda x^{-1}\mathrm{d}x\Lambda^{-1}\right)\right] = \\ &= \mathrm{Tr}\left[\left(x^{-1}\mathrm{d}x\right)\left(x^{-1}\mathrm{d}x\right)\right]\end{aligned} \tag{1.58}$$

$dS(1,3)$ is identified with a sub-manifold $x(\chi)$ of $SO(1,4)$ which transforms as $x' = \Lambda x \Lambda^{-1}$ under the transformation $\chi' = \Lambda\chi$. x can be thought of as rank two tensor with one covariant and one contravariant index. The most general tensor of this type constructed from the vector χ is

$$x^A{}_B = \alpha\delta^A{}_B + \beta\chi^A\chi_B \tag{1.59}$$

This tensor belongs to $SO(1,4)$ iff

$$\eta_{AB} = x^C{}_A x^D{}_B \eta_{CD} \tag{1.60}$$

This gives us $\alpha = \pm 1$ and $\beta = \frac{2\alpha}{\ell^2}$. The condition $\det x = 1$ forces us to take the minus sign. Therefore we have

$$x^A{}_B = -\delta^A{}_B - \frac{2}{\ell^2}\chi^A\chi_B \tag{1.61}$$

We can note that $x^2 = \mathbb{I}$, so $x = x^{-1}$ and $x\mathrm{d}x + \mathrm{d}x x = 0$. Then

$$\mathrm{Tr}\left(x^{-1}\mathrm{d}x x^{-1}\mathrm{d}x\right) = \mathrm{Tr}\left(x\mathrm{d}x x^{-1}\mathrm{d}x\right) = -\mathrm{Tr}\left(\mathrm{d}x\mathrm{d}x\right) \tag{1.62}$$

Thus we find

$$\mathrm{d}s^2 = -\frac{4}{\ell^2}\mathrm{Tr}\left[\mathrm{d}\big(\chi^A\chi_C\big)\,\mathrm{d}\big(\chi^C\chi_B\big)\right] = \frac{8}{\ell^2}\eta_{AB}\mathrm{d}\chi^A\mathrm{d}\chi^B|_{\chi^2=-\ell^2} \tag{1.63}$$

because

$$d\left(\chi^C\chi_C\right) = d\left(\ell^2\right) = 0 \tag{1.64}$$

After dropping an unimportant constant $8/\ell^2$, we see that the invariant metric on $dS(1,3)$ is exactly the one inherited from the ambient Minkowski spacetime.

1.4 Killing vector fields as generators of symmetries

Killing vector fields are vector fields, along which the metric tensor is constant. They are defined as solutions to a set of differential equations given by

$$\pounds_\xi g = 0 \tag{1.65}$$

If we express the Lie derivative in coordinates, we obtain the equation

$$\nabla_{(\mu}\xi_{\nu)} = 0 \tag{1.66}$$

where ∇ is the Levi-Civita connection. These equations do not have solutions in general. However, if we know the action of the isometry group of our manifold, we can avoid solving this equation. We can push-forward left-invariant vector fields on the isometry group G to our manifold and the resulting vector fields are Killing vector fields. This approach is also used in the definition of generalized Killing vector fields on manifold with arbitrary connection. In most cases, the algebra of Killing vector fields is isomorphic to the Lie algebra $\mathfrak{g}$ of G.

In our case, Killing vectors are obtained by a restriction of the action of $SO(1,4)$ on the 5-dimensional Minkowski space $\mathbb{M}^5$. These are in fact pseudo-rotations given by

$$t_{AB} = \eta_{CA}\chi^C P_B - \eta_{CB}\chi^C P_A \tag{1.67}$$

where $P_A = \partial_A$. We can pullback these vector fields to de Sitter space coordinatized by 1.3. The resulting vector fields are

$$t_{\mu\nu} = \eta_{\mu\sigma}x^\sigma P_\nu - \eta_{\nu\sigma}x^\sigma P_\mu \tag{1.68}$$

$$t_{\mu 4} = \ell P_\mu - \frac{1}{4\ell}K_\mu \tag{1.69}$$

Transformations generated by $t_{\mu\nu}$ correspond to Lorentz transformation of de Sitter spacetime, whereas $t_{\mu 4}$ are de Sitterian equivalents of translations. The second term, K_μ, is a generator of proper conformal transformations

$$K_\mu = \left(2\eta_{\mu\nu}x^\nu x^\rho - \sigma^2\delta^\rho_\mu\right) P_\rho \tag{1.70}$$

While they lost their commutativity, they retained the most important property of translations - transitivity. The relation to ordinary translations is more obvious if we introduce normalized transitive vector fields defined by

$$\Pi_\mu = \frac{t_{\mu 4}}{\ell} = P_\mu - \frac{1}{4\ell^2}K_\mu \tag{1.71}$$

This can also be written as

$$\Pi_\mu = \xi^\rho_\mu P_\rho \tag{1.72}$$

where

$$\xi^\mu_\rho = \delta^\mu_\rho - \frac{1}{4\ell^2}\left(2\eta_{\rho\nu}x^\nu x^\mu - \sigma^2\delta^\mu_\rho\right) \tag{1.73}$$

We can notice, that these vector fields correspond to generators of translations in the infinite pseudoradius limit. Therefore this is the direct generalization of Einstein's special relativity.

2 KINEMATICS IN DS(1,3)

In this chapter I will derive a modification of variation suitable to use in de Sitter spacetime. This modification follows from the homogeneity and isotropy of de Sitter spacetime. I will use this modified variation to the find Euler-Lagrange equations of a free particle. In this chapter I closely follow the article [5].

2.1 Variation of metric tensor

Variation of a quantity describes how the quantity changes when we move around a little bit. Tensor fields can be viewed as fields of multilinear mappings, their variation is therefore a little bit more complicated than for ordinary functions. For example, variation of the metric tensor is given by

$$\delta g_{\mu\nu} = g'_{\mu\nu}\left(x'\right) - g_{\mu\nu}\left(x\right) \tag{2.1}$$

This can be rewritten as sum of a horizontal variation coming from a change of argument

$$\delta^a g_{\mu\nu} = g_{\mu\nu}\left(x'\right) - g_{\mu\nu}\left(x\right) \tag{2.2}$$

and a vertical variation, which is given by the Lie derivative of the metric

$$\delta^0 g_{\mu\nu} = g'_{\mu\nu}\left(x'\right) - g_{\mu\nu}\left(x'\right) = \pounds_\xi g_{\mu\nu} \tag{2.3}$$

We are interested in variations which arise if we are varying fields along transitive Killing vector fields. First, let's take a look at the vertical variation of a covector field. We have

$$\delta^0_\Pi \psi_\mu \equiv \left(\pounds_\Pi \psi\right)_\mu = -\epsilon^\alpha \xi^\gamma_\alpha \partial_\gamma \psi_\mu - \epsilon^\alpha \partial_\mu {\xi_\alpha}^\gamma \psi_\gamma \tag{2.4}$$

where ϵ^α is an infinitesimal, $x-$independent vector. This expression can be written as

$$\delta^0_\Pi \psi_\mu = -\epsilon^\alpha \Pi_\alpha \psi_\mu - \epsilon^\alpha \left(\Sigma_\alpha\right)_\mu{}^\gamma \psi_\gamma \tag{2.5}$$

with

$$\left(\Sigma_\alpha\right)_\mu{}^\gamma = \partial_\mu {\xi_\alpha}^\gamma \tag{2.6}$$

This reminds us of a Einsteins special relativity, where these matrices are representations of the Lorentz algebra. In case of ordinary special relativity these matrices vanish because the generators of translations are constant. The interpretation of these matrices in de Sitter relativity is not as obvious, because a simple calculation will show us that Σ_α does not satisfy the commutation relation of de Sitter algebra. To explore them deeper we take a look at the transformations of the metric that they generate.

A horizontal variation of the metric is given by

$$\delta^a_\Pi g_{\mu\nu} = \epsilon^\alpha \Pi_\alpha g_{\mu\nu}(x) \tag{2.7}$$

whereas the Lie derivative can be written as

$$\delta^0_\Pi g_{\mu\nu} = -\epsilon^\alpha \Pi_\alpha g_{\mu\nu} - \epsilon^\alpha (\Sigma_\alpha)_\mu{}^\gamma g_{\gamma\nu} - \epsilon^\alpha (\Sigma_\alpha)_\nu{}^\gamma g\gamma\mu \tag{2.8}$$

This of course vanishes, because we are varying along Killing vector. The total variation is therefore given by

$$\delta_\Pi g_{\mu\nu} = \delta^a_\Pi g_{\mu\nu} + \delta^0_\Pi g_{\mu\nu} = -\epsilon^\alpha (\Sigma_\alpha)_\mu{}^\gamma g_{\gamma\nu} - \epsilon^\alpha (\Sigma_\alpha)_\nu{}^\gamma g_{\gamma\mu} \tag{2.9}$$

We can substitute Σ_α from equation 2.6, which gives us the result of the variation of the metric tensor

$$g'_{\mu\nu}(x') = \omega^2 g_{\mu\nu}(x) \tag{2.10}$$

where [5]

$$\omega^2 := 1 + \frac{\epsilon_\alpha x^\alpha}{\ell^2} \tag{2.11}$$

For the metric tensor, the Σ_α matrices generate infinitesimal conformal rescalings with ω^2 as conformal factor.

This variation should, however, vanish, because of the symmetry of de Sitter spacetime. Such a transformation is just a redefinition of the origin of spacetime and should not affect physics. If we define the conformally compensated vertical variation as

$$\bar{\delta}^0_\Pi \psi_\mu = (\pounds_\Pi \psi)_\mu + \epsilon^\alpha (\Sigma_\alpha)_\mu{}^\gamma \psi_\gamma \tag{2.12}$$

we will obtain the vanishing variation. The second term in above definition is called the conformal compensator. It seems it is more convenient to use this modification variation in case of de Sitter spacetime. We can observe that we recover the original variations in the large pseudoradius limit. These variations will be used to obtain the conformally compensated geodesic equation, as well as conserved quantities.

2.2 Motion of a free particle

We will use the principle of stationary proper time to obtain trajectories of a free particle. We will use the unmodified vertical variation to show how the conformal compensator appears after several algebraic manipulations. The proper time functional is given by

$$S[\Gamma] = -mc \int_{\Gamma} \sqrt{g_{\mu\nu} u^{\mu} u^{\nu}} \mathrm{d}\lambda \tag{2.13}$$

where $u^{\mu} = \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\lambda}$ denotes the derivative with respect to λ. This calculation would, however, be very complicated and we can obtain equations of motions by minimizing the related, non-square root functional which yields the same equations of motion. This functional is given by

$$E[\Gamma] = \frac{mc^2}{2} \int_{\Gamma} g_{\mu\nu} u^{\mu} u^{\nu} \mathrm{d}\lambda \tag{2.14}$$

Its variation is

$$\delta E[\Gamma] = mc^2 \int_{\Gamma} \left[\frac{1}{2} \delta_{\Pi} \left(g_{\mu\nu} \right) u^{\mu} \mathrm{d}x^{\mu} + g_{\mu\nu} u^{\mu} \mathrm{d} \left(\delta_{\Pi} \left(x^{\nu} \right) \right) \right] \tag{2.15}$$

By using the fact that differentiation commutes with variation, the variation of the energy functional, up to a surface term which can be eliminated by fixed boundary conditions, is

$$\delta E[\Gamma] = mc^2 \int_{\Gamma} \left[\frac{1}{2} g_{\mu\nu,\gamma} u^{\mu} u^{\nu} - \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(g_{\mu\gamma} u^{\mu} \right) \right] \epsilon^{\alpha} \xi^{\gamma}_{\alpha} \mathrm{d}\lambda \tag{2.16}$$

This form, however does not contain much information because it is not covariant. To obtain a covariant form we will have to use some algebraic manipulations, which will give us the desired result

$$\delta E[\Gamma] = mc^2 \int_{\Gamma} u^{\gamma} \left(\nabla_{\gamma} u^{\beta} \right) \xi^{\rho}_{\beta} \epsilon_{\rho} \mathrm{d}\lambda \tag{2.17}$$

This gives us geodesic equation

$$u^{\gamma} \left(\nabla_{\gamma} u^{\beta} \right) = 0 \tag{2.18}$$

If we rewrite this variation in terms of an anholonomic four-velocity

$$U^{\mu} = \xi^{\mu}_{\rho} u^{\rho} \tag{2.19}$$

The conformal compensator will appear after several algebraic manipulations

$$\delta E[\Gamma] = mc^2 \int_{\Gamma} \left[u^{\gamma} \nabla_{\gamma} U^{\rho} - \frac{1}{2} u^{\beta} u^{\gamma} \left(\nabla_{\beta} \xi^{\rho}_{\gamma} + \nabla_{\gamma} \xi^{\rho}_{\beta} \right) \right] \epsilon_{\rho} \mathrm{d}\lambda \tag{2.20}$$

We can write down the covariant derivative of ξ in local coordinate system

$$\nabla_\beta \xi^\rho_\gamma = \partial_\beta \xi^\rho_\gamma + \Gamma^\rho{}_{\beta\alpha}\xi^\alpha_\gamma - \Gamma^\alpha{}_{\beta\gamma}\xi^\rho_\alpha = \left(\Sigma_\alpha\right)_\beta{}^\rho + \Gamma^\rho{}_{\beta\alpha}\xi^\alpha_\gamma - \Gamma^\alpha{}_{\beta\gamma}\xi^\rho_\alpha \tag{2.21}$$

It is obvious from 2.15 that terms containing Σ matrices correspond to conformal compensators and should be subtracted in conformally compensated variation. Thus we have finally obtained result

$$\bar{\delta}E\left[\Gamma\right] = mc^2 \int_\Gamma u^\gamma \left(\nabla_\gamma U^\rho - u^\beta \frac{2\Gamma^\alpha{}_{\beta\gamma}x_\alpha x^\rho - \Gamma^\rho{}_{\alpha\beta}x_\gamma x^\alpha - \Gamma^\rho{}_{\alpha\gamma}x_\beta x^\alpha}{4\ell^2}\right)\epsilon_\rho \mathrm{d}\lambda \tag{2.22}$$

If we use the conformally compensated variation, trajectories of free particles are given by equation

$$\frac{\mathrm{d}U^\rho}{\mathrm{d}\lambda} + \Gamma^\rho{}_{\alpha\gamma}U^\alpha u^\gamma = u^\beta u^\gamma \frac{2\Gamma^\alpha{}_{\beta\gamma}x_\alpha x^\rho - \Gamma^\rho{}_{\alpha\beta}x_\gamma x^\alpha - \Gamma^\rho{}_{\alpha\gamma}x_\beta x^\alpha}{4\ell^2} \tag{2.23}$$

This is equation of parallel transport of anholonomic four-velocity field along four-velocity field and in flat limit it gives us geodesic equation as expected. However we can feel that something is not right. The geodesic equation usually tells us that four-velocity is being conserved along trajectory, but this equation does not yield any conserved quantities. This is unacceptable and therefore we reject the conformally compensated variation approach to the de Sitterian relativity. The original idea was to find principle which allows us to modify the geodesic equation to the form

$$\frac{\mathrm{d}U^\rho}{\mathrm{d}\lambda} + \Gamma^\rho{}_{\alpha\gamma}U^\alpha u^\gamma = 0 \tag{2.24}$$

I tried to reproduce the results of the [5], however I have discovered that aforementioned article contains an error on page 8. They didn't take into account additional terms coming from covariant derivative, which led them to result which seemed plausible, because it implied the conservation of the anholonomic four-velocity. However, if we include these additional terms, the resulting equation does not yield any conserved quantities, which is physically unacceptable. Kinematics in the de Sitter spacetime is therefore governed by Einsteins original theory. If the equation 2.24 was the equation of motion of a free particle, apparent horizons would disappear from de Sitter spacetime and it would lead to the solution of the initial value problem in inflationary cosmology. For discussion on initial value problem in cosmology see [6].

3 PHOTON PROPAGATION IN DS(1,3)

In this final chapter I will investigate the behavior of the electromagnetic waves in the limit of geometrical optics. Propagation of light in a curved spacetime can be transformed into propagation of light in a medium with continuous refractive index. Our aim is to find the "refractive index" for de Sitter spacetime. The refractive index is, however, coordinate and observer dependent. We will consider an electromagnetic field with small energy density and we neglect its gravitational effects for simplicity. In this section, the Gaussian CGS unit system is used.

3.1 Electrodynamics in curved spacetime

In general spacetime, the Maxwell's equations can be compactly written as

$$\left.\begin{aligned} -\left[\sqrt{-g}F^{\mu\nu}\right]_{,\nu} &= \tfrac{4\pi}{c}\sqrt{-g}J^{\mu} \\ F_{[\mu\nu,\gamma]} &= 0 \end{aligned}\right\} \tag{3.1}$$

where J^μ are sources and $F_{\mu\nu}$ is the field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.2}$$

If we are interested in the coupling of the matter field to the gravity, we have to use the EFE, Coupling of the field to the gravity is obtained through energy-momentum tensor

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta \mathfrak{L}_m}{\delta g_{\mu\nu}} \tag{3.3}$$

where $\mathfrak{L}_m$ is the matter Lagrangian. The EFE in the presence of the matter fields therefore are

$$G^{\mu\nu} = \frac{8\pi G}{c^4}T^{\mu\nu} \tag{3.4}$$

We can note that the covariant divergence of left-hand side vanishes due to differential Bianchi identity. This implies that the covariant divergence of the right-hand side of 3.4 should also vanish. This is in fact true and proof can be found in [7], Appendix E. For an electromagnetic field we have energy-momentum tensor

$$T^{\mu\nu} = \frac{1}{4\pi c}\left(-F^{\mu\gamma}F^{\nu}{}_{\gamma} + \frac{1}{4}g^{\mu\nu}F^{\gamma\rho}F_{\gamma\rho}\right) + g^{\mu\nu}J^{\rho}A_{\rho} \tag{3.5}$$

This tensor is obtained from the Lagrangian density of an electromagnetic field

$$\mathfrak{L}_{EM} = \sqrt{-g}\left(-\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} - A_\mu J^\mu\right) \tag{3.6}$$

If we perform variation 3.3, the result is indeed 3.5. This tensor is therefore a suitable source term in the EFE. We can note that the Maxwell's equations 3.1 include the determinant of the metric tensor g and this problem is quite involved in general. However, we can consider small electromagnetic field $F^{\mu\nu}$, whose gravitational effects can be neglected. This leads to the separation of the Maxwell's equations from the EFE and leads to the problem of electromagnetic field on a fixed background spacetime. This approximation is used in this chapter.

Now, let's introduce electromagnetic 3-vectors

$$\left.\begin{aligned} E_i &:= F_{i0} \\ H_i &:= \tfrac{1}{2}\sqrt{-g}\epsilon_{ijk}F^{jk} \\ D^i &:= \sqrt{-g}F^{0i} \\ B^i &:= \tfrac{1}{2}\epsilon^{ijk}F_{jk} \end{aligned}\right\} \tag{3.7}$$

Maxwell's equations can be rewritten in terms of these vectors

$$\left.\begin{aligned} -D^i{}_{,0} + \epsilon^{ijk}H_{k,j} &= \tfrac{4\pi}{c}j^i \\ B^i{}_{,0} + \epsilon_{ijk}E_{k,j} &= 0 \\ D^i{}_i &= 4\pi\rho \\ B^i{}_i &= 0 \end{aligned}\right\} \tag{3.8}$$

where

$$\left.\begin{aligned} j^i &:= \sqrt{-g}J^i \\ \rho &:= \sqrt{-g}\tfrac{J^0}{c} \end{aligned}\right\} \tag{3.9}$$

The relations amongst E, D, H and B, called constitutive equations, are obtained from [8]

$$\left.\begin{aligned} \sqrt{-g}F^{\mu\nu} &= \sqrt{-g}g^{\mu\rho}g^{\nu\sigma}F_{\rho\sigma} \\ F_{\mu\nu} &= \tfrac{1}{\sqrt{-g}}g_{\mu\rho}g_{\nu\sigma}\sqrt{-g}F^{\rho\sigma} \end{aligned}\right\} \tag{3.10}$$

We define 3-dimensional metric used to lower and raise indices of 3-vectors as

$$e_{ij} = -g_{ij} + \frac{g_{i0}g_{j0}}{g_{00}} \tag{3.11}$$

Obtained constitutive equations are

$$\left.\begin{aligned} D^i &= E^i + \sigma^{ij}E_j + \epsilon^{ijk}g_jH_k \\ B^i &= H^i + \sigma^{ij}H_j - \epsilon^{ijk}g_jE_k \end{aligned}\right\} \tag{3.12}$$

where

$$\begin{aligned} \sigma_{ij} &:= -\frac{\sqrt{-g}}{g_{00}}g_{ij} - \delta_{ij} \\ g_i &:= \frac{g_{i0}}{g_{00}} \end{aligned} \tag{3.13}$$

Constitutive equations are usually additional information obtained either empirically or theoretically. In this case, constitutive equation are of purely geometrical origin, they encode information about four-dimensional metric. We can notice that due to the diffeomorphism invariance, we can always choose such a frame of reference that $g_i = 0$ and no mixing of electric and magnetic fields occur. We can note, that in the case of asymptotically flat spacetime $g_{\mu\nu}$ tends to $\eta_{\mu\nu}$ at infinity. ϵ_{ij} and g_i vanish in this case and the constitutive equations reduce to

$$\left.\begin{aligned} D^i &= E^i \\ B^i &= H^i \end{aligned}\right\} \tag{3.14}$$

Let's also express the components of the energy-momentum tensor density

$$\begin{aligned} 4\pi c\sqrt{-g}{T_0}^0 &= \frac{1}{2}\left(D^i E_i + B^i H_i\right) + 4\pi\left(J^0 A^0 - cJ^i A_i\right) \\ 4\pi c\sqrt{-g}{T_0}^i &= \epsilon^{ijk} E_j H_k \\ 4\pi c\sqrt{-g}T_i^{\ 0} &= \epsilon_{ijk} D^j B^k \\ 4\pi c\sqrt{-g}{T_j}^i &= -\frac{1}{2}{\delta_j}^i\left(D^k E_k + B^k H_k\right) + 4\pi{\delta_j}^i\left(J^0 A^0 - cJ^k A_k\right) + E_j D^i + H_j B^i \end{aligned} \tag{3.15}$$

3.2 Geometrical optics in de Sitter spacetime

If we want to apply the geometrical optics limit in flat spacetime, the following condition has to hold

$$\lambda \ll \ell \tag{3.16}$$

where ℓ is the characteristic size of the system and λ is the wavelength. If these conditions are fulfilled, any wave-optics quantity is given by the formula,

$$A = a\mathrm{e}^{i\phi} \tag{3.17}$$

where a is a slowly varying amplitude, which is a function of space and time coordinates (first and higher order derivatives can be neglected), whereas ϕ is the eikonal, an almost linear function of space and time coordinates (second and higher order derivatives can be neglected). The following equations hold

$$\left.\begin{aligned} \phi_{,0} &= \omega \\ \phi_{,i} &= -k_i \end{aligned}\right\} \tag{3.18}$$

where ω is the angular frequency and k_i is the wave vector. In this section, it is convenient to use a different coordinate system [9]

$$\begin{aligned} x^0 &= \ell \log \frac{\chi^0 + \chi^1}{\ell} \\ x^i &= \frac{\ell \chi^{i+1}}{\chi^0 + \chi^1} \end{aligned} \tag{3.19}$$

In these coordinates, the line element has a very useful form

$$\mathrm{d}s^2 = \left(\mathrm{d}x^0\right)^2 - n^2 \delta_{ij} \mathrm{d}x^i \mathrm{d}x^j \tag{3.20}$$

with

$$n := \exp\left(\sqrt{\frac{\Lambda_c}{3}} x^0\right) \tag{3.21}$$

We will assume the covariant Lorenz gauge[1] of the electromagnetic four-potential

$$\nabla_\mu A^\mu = 0 \tag{3.22}$$

The Maxwell's equation in this gauge is [10]

$$\Box A^\mu - R^\mu{}_\nu A^\nu = 0 \tag{3.23}$$

where $\Box$ is the covariant d'Alembert operator

$$\Box = g^{\mu\nu} \nabla_\mu \nabla_\nu \tag{3.24}$$

We can substitute the Ricci tensor of the de Sitter spacetime into Maxwell's equation

$$\Box A^\mu + \Lambda_c A^\mu = 0 \tag{3.25}$$

The term involving the cosmological constant looks like a mass term. The Maxwell's equations in four dimension are however conformally invariant and de Sitter spacetime is conformally flat. The photons will therefore move on the lightcone, which implies vanishing mass. This term therefore should not be considered as a background dependent mass of the photon. For an extensive discussion about the curvature-related mass term see [11].

Assuming a massless photon field in the geometrical optics limit 3.17, we are looking for a four-potential

$$A_\mu = a_\mu \exp\left(i k_\nu x^\nu\right) \tag{3.26}$$

[1]This gauge condition is named after a Danish physicist Ludvig Lorenz. In literature, it is often misspelled as "Lorentz" gauge, because it actually is Lorentz invariant. Hendrik Lorentz however did not introduce this gauge condition.

This leads us to the dispersion relation

$$\omega\left(k\right)=\frac{c}{n}\sqrt{k^{2}+n^{2}\Lambda_{c}} \tag{3.27}$$

Considering that

$$n\sqrt{\Lambda_{c}}\sim\ell^{-1} \tag{3.28}$$

and remembering that $k\sim\lambda^{-1}$, the condition 3.16 turns out to be

$$k\gg n\sqrt{\Lambda_{c}} \tag{3.29}$$

In this domain, the dispersion relation has a very simple form

$$\omega\left(k\right)=\frac{c}{n}k \tag{3.30}$$

The velocity of propagation of electromagnetic waves is given by the group velocity

$$v:=\frac{\mathrm{d}\omega\left(k\right)}{\mathrm{d}k}=\frac{c}{n} \tag{3.31}$$

We can note that the speed of light decreases with time, which leads to red shift which is often observed in light coming from very distant objects. In the limit of vanishing cosmological constant, we have $n\to1$ and the electromagnetic waves propagate at the speed of light.

4 CONCLUSION

In this thesis, I have studied the changes which are brought to kinematics and electrodynamics when curvature is introduced. I have successfully studied and parametrized the symmetry group of de Sitter spacetime and its action on a conformally flat coordinate system. In these coordinates, the action of the de Sitter algebra is very similar to the action of the Poincaré algebra. The only difference is that the transitive transformations contain an extra coordinate-dependent term depending on the inverse square of the pseudoradius. This coordinate dependence lures us to think that the notion of motion should be changed by addition of conformal compensator. This modification however does not give the desired results and I have to conclude, that the kinematics in de Sitter spacetime is ruled by Einsteins theory. In the last chapter I have studied the connection between electrodynamics in a curved spacetime and electrodynamics in a macroscopic material in flat spacetime. Using this connection, I have found the "refractive index" of de Sitter spacetime.

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