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Fredholm alternative for the second-order singular Dirichlet problem

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Abstract

Consider the singular Dirichlet problem

$$u'' = p(t)u + q(t); \qquad u(a) = 0, \qquad u(b) = 0,$$

where $p, q :]a, b[\to \mathbb{R}$ are locally Lebesgue integrable functions. It is proved that if

$$\int_a^b (s-a)(b-s)\big[p(s)\big]_- \, ds < +\infty,$$

then the Fredholm alternative remains true.

MSC: 34B05

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1 Introduction

Consider the boundary value problem

$$u'' = p(t)u + q(t), \tag{1}$$

$$u(a) = 0, \qquad u(b) = 0, \tag{2}$$

where $p, q \in L_{\text{loc}}(]a, b[)$. We are mainly interested in the case when the functions p and q are not (in general) integrable on $[a, b]$. In this case, equation (1) as well as problem (1), (2) are said to be singular. It is well known that for singular problem (1), (2), the condition

$$\int_a^b (s-a)(b-s)\big|p(s)\big| \, ds < +\infty \tag{3}$$

guarantees the validity of the Fredholm alternative. More precisely, if (3) holds, then problem (1), (2) is uniquely solvable for any q satisfying

$$\int_a^b (s-a)(b-s)\big|q(s)\big| \, ds < +\infty \tag{4}$$

iff the corresponding homogeneous equation

$$u'' = p(t)u \tag{1_0}$$

has no nontrivial solution satisfying (2). The above statement plays an important role in the theory of singular problems; however, it does not cover many interesting, even rather simple, equations. For example, consider the Dirichlet problem for the Euler equation

$$u'' = \frac{\alpha}{(t-a)^2}u + \beta; \qquad u(a) = 0, \qquad u(b) = 0, \tag{5}$$

where α and β are real constants. By direct calculations, one can easily verify that if $\alpha > 0$, then the homogeneous problem

$$u'' = \frac{\alpha}{(t-a)^2}u; \qquad u(a) = 0, \qquad u(b) = 0$$

has only the trivial solution, while problem (5) is uniquely solvable. However, in this case $p(t) = \frac{\alpha}{(t-a)^2}$ and therefore condition (3) is not satisfied.

The aim of this paper is to show that the Fredholm alternative remains true even in the case when instead of (3) only the condition

$$\int_a^b (s-a)(b-s)\big[p(s)\big]_-\,ds < +\infty \tag{6}$$

holds. The paper is organized as follows. At the end of this section, we state our main results, the proofs of which one can find in Section 4. In Section 2, we recall some known results in a suitable for us form. Section 3 is devoted to *a priori* estimates and plays a crucial role in the proofs of the main results.

Throughout the paper we use the following notation.

$\mathbb{R}$ is the set of real numbers.

For $x \in \mathbb{R}$, we put $[x]_- = \frac{1}{2}(|x| - x)$.

$C(I)$, where $I \subset \mathbb{R}$ is the set of continuous functions $u : I \to \mathbb{R}$.

For $u \in C([\alpha,\beta])$, we put $\|u\|_{[\alpha,\beta]} = \max\{|u(t)| : t \in [\alpha,\beta]\}$.

$AC'_{\mathrm{loc}}(]\alpha,\beta[)$ is the set of functions $u :]\alpha,\beta[\to \mathbb{R}$, which are absolutely continuous together with their first derivative on every closed subinterval of $]\alpha,\beta[$.

$L_{\mathrm{loc}}(]\alpha,\beta[)$ is the set of functions $p :]\alpha,\beta[\to \mathbb{R}$, which are Lebesgue integrable on every closed subinterval of $]\alpha,\beta[$.

By $f(a)$ (resp., $f(b)$) we denote the right (resp., left) limit of the function $f :]a,b[\to \mathbb{R}$ at the point a (resp., b).

Under a solution of equation (1) we understand a function $u \in AC'_{\mathrm{loc}}(]a,b[)$ which satisfies it almost everywhere in $]a,b[$. A solution of equation (1) satisfying (2) is said to be a solution of problem (1), (2).

We say that a certain property holds in $]\alpha,\beta[$ if it takes place on every closed subinterval of $]\alpha,\beta[$.

Recall that we consider problem (1), (2), where $p, q \in L_{\mathrm{loc}}(]a,b[)$. Our main results are the following.

Theorem 1.1 *Let condition* (6) *hold. Then problem* (1), (2) *is uniquely solvable for any* q *satisfying* (4) *iff homogeneous problem* (1_0), (2) *has no nontrivial solution.*

Remark 1.1 In Theorem 1.1, condition (4) is essential and cannot be omitted. Indeed, let $p \equiv 0$, $q \in L_{\mathrm{loc}}(]a,b[)$, $q(t) \geq 0$ for $t \in]a,b[$, and

$$\int_a^{\frac{a+b}{2}} (s-a)q(s)\,ds = +\infty. \tag{7}$$

Evidently, (6) holds and problem (1_0), (2) has no nontrivial solution. On the other hand, a general solution of (1) is of the form

$$u(t) = \alpha + \beta t + \int_t^{\frac{a+b}{2}} (s-a)q(s)\,ds - (t-a)\int_t^{\frac{a+b}{2}} q(s)\,ds \quad \text{for } t \in]a,b[.$$

However, for $a < t < x < \frac{a+b}{2}$, we have

$$u(t) \geq \int_x^{\frac{a+b}{2}} (s-a)q(s)\,ds - (t-a)\int_x^{\frac{a+b}{2}} q(s)\,ds + \alpha + \beta t.$$

Hence,

$$\liminf_{t\to a+} u(t) \geq \alpha + \beta a + \int_x^{\frac{a+b}{2}} (s-a)q(s)\,ds.$$

Therefore, in view of (7), we get $\lim_{t\to a+} u(t) = +\infty$ and, consequently, problem (1), (2) has no solution.

Remark 1.2 Theorem 1.1 concerns half homogeneous problem (1), (2) and does not remain true for the fully nonhomogeneous problem

$$u'' = p(t)u + q(t); \qquad u(a) = c_1, \qquad u(b) = c_2. \tag{8}$$

Let, for example, $p(t) = \frac{2}{(t-a)^2}$, $q \equiv 0$, $c_1 \neq 0$, and $c_2 = 0$. It is clear that (6) holds and the corresponding homogeneous problem (1_0), (2) has no nontrivial solution. On the other hand, a general solution of (1) is of the form $u(t) = \frac{\alpha}{t-a} + \beta(t-a)^2$ for $t \in]a,b[$ and, therefore, (8) has no solution.

Theorem 1.2 *Let* (6) *hold and problem* (1_0), (2) *have no nontrivial solution. Then there exists* $r > 0$ *such that for any* q *satisfying* (4), *the solution* u *of problem* (1), (2) *admits the estimate*

$$\left|u(t)\right| + (t-a)(b-t)\left|u'(t)\right| \leq r\int_a^b (s-a)(b-s)\left|q(s)\right|ds \quad \textit{for } t \in]a,b[. \tag{9}$$

Consider now a sequence of equations

$$u'' = p(t)u + q_n(t), \tag{10_n}$$

where $q_n \in L_{\mathrm{loc}}(]a,b[)$ are such that

$$\int_a^b (s-a)(b-s)\left|q_n(s)\right|ds < +\infty \quad \text{for } n = 1,2,\ldots. \tag{11}$$

Let, moreover, $q \in L_{\mathrm{loc}}(]a,b[)$ satisfy (4) and

$$\lim_{n\to+\infty} \int_a^b (s-a)(b-s)\big|q_n(s)-q(s)\big|\,ds = 0. \tag{12}$$

Corollary 1.1 *Let* (4), (6) *hold and problem* (1_0), (2) *have no nontrivial solution. Let, moreover,* (11) *and* (12) *be fulfilled. Then the problems* (1), (2) *and* (10_n), (2) *have unique solutions* u *and* u_n, *respectively,*

$$\lim_{n\to+\infty} u_n(t) = u(t) \quad \textit{uniformly on } [a,b] \tag{13}$$

and

$$\lim_{n\to+\infty} u_n'(t) = u'(t) \quad \textit{uniformly in }]a,b[. \tag{14}$$

2 Auxiliary statements

In this section, we consider the equation

$$v'' = h(t)v + q(t),$$

where $h, q \in L_{\mathrm{loc}}(]a,b[)$, q satisfies (4), and

$$\int_a^b (s-a)(b-s)\big|h(s)\big|\,ds < +\infty. \tag{15}$$

Below we state some known results in a suitable for us form.

Proposition 2.1 *Let* (15) *hold. Then the problem*

$$v'' = h(t)v + q(t); \qquad v(a) = c_1, \qquad v(b) = c_2$$

is uniquely solvable for any $c_1, c_2 \in \mathbb{R}$ *and* q *satisfying* (4) *iff the homogeneous problem*

$$v'' = h(t)v; \qquad v(a) = 0, \qquad v(b) = 0$$

has no nontrivial solution.

Proof See, *e.g.*, [1, Theorem 3.1] or [2, Theorem 1.1]. □

Proposition 2.2 *Let* (15) *hold. Then there exist* $a_0 \in]a,b[$ *and* $b_0 \in]a_0,b[$ *such that, for any* $t_1 < t_2$ *satisfying either* $t_1, t_2 \in [a,a_0]$ *or* $t_1, t_2 \in [b_0,b]$, *the homogeneous problem*

$$v'' = h(t)v; \qquad v(t_1) = 0, \qquad v(t_2) = 0 \tag{16}$$

has no nontrivial solution. Moreover, for any $w \in C'_{\mathrm{loc}}(]t_1,t_2[)$ (*where* $t_1 < t_2$ *are the same as above*) *satisfying*

$$w''(t) \ge h(t)w(t) \quad \textit{for } t \in]t_1,t_2[; \qquad w(t_1) = 0, \qquad w(t_2) = 0,$$

the inequality

$$w(t) \le 0 \quad \textit{for } t \in [t_1, t_2]$$

holds.

Proof In view of (15), there exist $a_0 \in]a, b[$ and $b_0 \in]a_0, b[$ such that

$$\int_a^{a_0} (s-a)\big|h(s)\big|\,ds < 1, \qquad \int_{b_0}^{b} (b-s)\big|h(s)\big|\,ds < 1.$$

Hence, the inequalities

$$\int_a^{a_0} (s-a)(a_0-s)\big|h(s)\big|\,ds < a_0 - a, \qquad \int_{b_0}^{b} (s-b_0)(b-s)\big|h(s)\big|\,ds < b - b_0$$

hold as well. The latter inequalities, by virtue of [2, Lemma 4.1], imply that for any $t_1 < t_2$ satisfying either $t_1, t_2 \in [a, a_0]$ or $t_1, t_2 \in [b_0, b]$, homogeneous problem (16) has no nontrivial solution.

The second part of the proposition follows easily from the above-proved part and [2, Lemma 1.3]. □

Proposition 2.3 *Let* (15) *hold. Let, moreover,* $a_0 \in]a, b[$ *and* $b_0 \in]a_0, b[$ *be from the assertion of Proposition* 2.2. *Then there exists* $\varrho > 0$ *such that for any* $c \in \mathbb{R}$ *and any* q *satisfying* (4), *the solution* v *of the problem*

$$v'' = h(t)v + q(t); \qquad v(a) = 0, \qquad v(a_0) = c \tag{17}$$

admits the estimate

$$\big|v(t)\big| \le \varrho\left(|c|(t-a) + \int_a^t (s-a)\big|q(s)\big|\,ds + (t-a)\int_t^{a_0}\big|q(s)\big|\,ds\right) \tag{18}$$

for $t \in]a, a_0]$, *while the solution* v *of the problem*

$$v'' = h(t)v + q(t); \qquad v(b_0) = c, \qquad v(b) = 0 \tag{19}$$

admits the estimate

$$\big|v(t)\big| \le \varrho\left(|c|(b-t) + \int_t^b (b-s)\big|q(s)\big|\,ds + (b-t)\int_{b_0}^{t}\big|q(s)\big|\,ds\right) \tag{20}$$

for $t \in [b_0, b[$.

Proof By virtue of (15) and [1, Lemma 2.2], the initial value problems

$$v_1'' = h(t)v_1; \qquad v_1(a) = 0, \qquad v_1'(a) = 1$$

and

$$v_2'' = h(t)v_2; \qquad v_2(a_0) = 0, \qquad v_2'(a_0) = -1$$

have unique solutions v_1 and v_2, respectively, and the estimates

$$\left|v_1(t)\right| \le \varrho_0(t-a), \qquad \left|v_2(t)\right| \le \varrho_0(a_0-t) \quad \text{for } t \in [a,a_0] \tag{21}$$

are fulfilled, where

$$\varrho_0 = \exp\left(2\int_a^{a_0}(s-a)\left|h(s)\right|ds\right).$$

On the other hand, by virtue of Proposition 2.2,

$$v_1(a_0) \ne 0 \quad \text{and} \quad v_2(a) \ne 0.$$

In view of Propositions 2.1 and 2.2, problem (17) has a unique solution v. By direct calculations, one can easily verify that

$$v(t) = \frac{c}{v_1(a_0)}v_1(t) - \frac{1}{v_2(a)}\left(v_2(t)\int_a^t v_1(s)q(s)\,ds + v_1(t)\int_t^{a_0} v_2(s)q(s)\,ds\right) \tag{22}$$

for $t \in [a,a_0]$. Analogously, the (unique) solution v of problem (19) is of the form

$$v(t) = \frac{c}{v_4(b_0)}v_4(t) - \frac{1}{v_3(b)}\left(v_4(t)\int_{b_0}^t v_3(s)q(s)\,ds + v_3(t)\int_t^{b} v_4(s)q(s)\,ds\right) \tag{23}$$

for $t \in [b_0,b]$, where v_3 and v_4 are solutions of the problems

$$v_3'' = h(t)v_3; \qquad v_3(b_0) = 0, \qquad v_3'(b_0) = 1$$

and

$$v_4'' = h(t)v_4; \qquad v_4(b) = 0, \qquad v_4'(b) = -1,$$

respectively, $v_3(b) \ne 0$, $v_4(b_0) \ne 0$, and the estimates

$$\left|v_3(t)\right| \le \varrho_1(t-b_0), \qquad \left|v_4(t)\right| \le \varrho_1(b-t) \quad \text{for } t \in [b_0,b] \tag{24}$$

are fulfilled with

$$\varrho_1 = \exp\left(2\int_{b_0}^{b}(b-s)\left|h(s)\right|ds\right).$$

Now, it follows from (22) and (23), in view of (21) and (24), that the estimates (18) and (20) hold with

$$\varrho = \frac{\rho_0}{|v_1(a_0)|} + \frac{\rho_1}{|v_4(b_0)|} + \frac{a_0-a}{|v_2(a)|}\varrho_0^2 + \frac{b-b_0}{|v_3(b)|}\varrho_1^2. \qquad \square$$

3 Lemmas on *a priori* estimates

Lemma 3.1 *Let* (4) *and* (6) *hold. Then, for any* $\alpha \in [a,b[$ *and* $\beta \in]\alpha,b]$, *every solution* u *of equation* (1) *satisfying*

$$u(\alpha)=0, \qquad u(\beta)=0 \tag{25}$$

admits the estimate

$$(t-a)(b-t)\left|u'(t)\right| \le \|u\|_{[\alpha,\beta]}\left(b-a+\int_a^b (s-a)(b-s)\left[p(s)\right]_-\,ds\right) + \int_a^b (s-a)(b-s)\left|q(s)\right|\,ds \quad \text{for } t \in]\alpha,\beta[. \tag{26}$$

Proof Let $t_0 \in]\alpha,\beta[$. Then it is clear that either

$$u(t_0)u'(t_0)>0, \tag{27}$$

or

$$u(t_0)u'(t_0)<0, \tag{28}$$

or

$$u(t_0)u'(t_0)=0. \tag{29}$$

Assume that (27) (resp., (28)) holds. Then, in view of (25), there is $t^* \in]t_0,\beta[$ (resp., $t_* \in]\alpha,t_0[$) such that

$$\begin{gathered} u(t)\operatorname{sgn} u'(t_0)>0 \quad \text{for } t \in \left[t_0,t^*\right] \quad \text{and} \quad u'\left(t^*\right)=0 \\ \left(\text{resp., } u(t)\operatorname{sgn} u'(t_0)<0 \quad \text{for } t \in [t_*,t_0] \quad \text{and} \quad u'(t_*)=0\right). \end{gathered} \tag{30}$$

Multiplying both sides of (1) by $b-t$ (resp., by $t-a$) and integrating it from t_0 to t^* (resp., from t_* to t_0), we get

$$(b-t_0)u'(t_0)=u\left(t^*\right)-u(t_0)-\int_{t_0}^{t^*}(b-s)\left(p(s)u(s)+q(s)\right)ds$$
$$\left(\text{resp., } (t_0-a)u'(t_0)=u(t_0)-u(t_*)+\int_{t_*}^{t_0}(s-a)\left(p(s)u(s)+q(s)\right)ds\right).$$

Hence, in view of (30), we obtain

$$(b-t_0)\left|u'(t_0)\right| \le \|u\|_{[\alpha,\beta]}\left(1+\int_{t_0}^b (b-s)\left[p(s)\right]_-\,ds\right)+\int_{t_0}^b (b-s)\left|q(s)\right|ds$$
$$\left((t_0-a)\left|u'(t_0)\right| \le \|u\|_{[\alpha,\beta]}\left(1+\int_a^{t_0}(s-a)\left[p(s)\right]_-\,ds\right)+\int_a^{t_0}(s-a)\left|q(s)\right|ds\right).$$

Multiplying both parts of the latter inequality by $t_0 - a$ (resp., by $b - t_0$), we get

$$(t_0 - a)(b - t_0)\big|u'(t_0)\big| \le \|u\|_{[\alpha,\beta]}\left(b - a + \int_a^b (s-a)(b-s)\big[p(s)\big]_-\,ds\right) + \int_a^b (s-a)(b-s)\big|q(s)\big|\,ds. \tag{31}$$

Suppose now that (29) holds. Then either there is $\beta_0 \in]t_0, \beta[$ such that

$$u(t)u'(t) = 0 \quad \text{for } t \in [t_0, \beta_0], \tag{32}$$

or there is a sequence $\{t_n\}_{n=1}^{+\infty} \subset]t_0, \beta[$ such that

$$\lim_{n\to+\infty} t_n = t_0, \tag{33}$$

$$u(t_n)u'(t_n) \neq 0 \quad \text{for } n \in \mathbb{N}. \tag{34}$$

If (32) holds, then evidently $u(t) = u(t_0)$ for $t \in [t_0, \beta_0]$ and, consequently, (31) is fulfilled. On the other hand, if (34) holds, then, by virtue of the above-proved, the inequalities

$$(t_n - a)(b - t_n)\big|u'(t_n)\big| \le \|u\|_{[\alpha,\beta]}\left(b - a + \int_a^b (s-a)(b-s)\big[p(s)\big]_-\,ds\right) + \int_a^b (s-a)(b-s)\big|q(s)\big|\,ds \quad \text{for } n = 1, 2, \ldots$$

are fulfilled, and therefore, in view of (33), inequality (31) holds as well. Thus, estimate (26) is fulfilled. □

Lemma 3.2 *Let* (6) *hold. Then there exist* $a_0 \in]a, b[$, $b_0 \in]a_0, b[$, *and* $\varrho > 0$ *such that for any* $\alpha \in [a, a_0[$, $\beta \in]b_0, b]$ *and any* q *satisfying* (4), *every solution* u *of equation* (1) *satisfying*

$$u(\alpha) = 0 \tag{35}$$

admits the estimate

$$\big|u(t)\big| \le \varrho\left((t-a)\|u\|_{[\alpha,a_0]} + \int_a^t (s-a)\big|q(s)\big|\,ds + (t-a)\int_t^{a_0}\big|q(s)\big|\,ds\right) \quad \text{for } t \in]\alpha, a_0], \tag{36}$$

while every solution u *of equation* (1) *satisfying*

$$u(\beta) = 0 \tag{37}$$

admits the estimate

$$\big|u(t)\big| \le \varrho\left((b-t)\|u\|_{[b_0,\beta]} + \int_t^b (b-s)\big|q(s)\big|\,ds + (b-t)\int_{b_0}^t\big|q(s)\big|\,ds\right) \quad \text{for } t \in [b_0, \beta[. \tag{38}$$

Proof Let a_0, b_0, and ϱ be from the assertion of Propositions 2.2 and 2.3 with $h(t) = -[p(t)]_-$ for $t \in]a,b[$. Let, moreover, $\alpha \in [a,a_0[$ (resp., $\beta \in]b_0,b]$) and u be a solution of problem (1), (35) (resp., (1), (37)). By virtue of Propositions 2.2 and 2.3, the problem

$$v'' = -\big[p(t)\big]_- v - \big|q(t)\big|, \tag{39}$$
$$v(a) = 0, \qquad v(a_0) = \|u\|_{[\alpha,a_0]} \qquad \big(\text{resp., } v(b_0) = \|u\|_{[b_0,\beta]}, \qquad v(b) = 0\big)$$

has a unique solution v and, moreover, for any $t \in]a,a_0]$ (resp., $t \in [b_0,b[$), the estimate

$$\begin{gathered} 0 \le v(t) \le \varrho\bigg((t-a)\|u\|_{[\alpha,a_0]} + \int_a^t (s-a)\big|q(s)\big|\,ds + (t-a)\int_t^{a_0}\big|q(s)\big|\,ds\bigg) \\ \bigg(\text{resp., } 0 \le v(t) \le \varrho\bigg((b-t)\|u\|_{[b_0,\beta]} + \int_t^b (b-s)\big|q(s)\big|\,ds + (b-t)\int_{b_0}^t\big|q(s)\big|\,ds\bigg)\bigg) \end{gathered} \tag{40}$$

holds. Let us show that

$$\big|u(t)\big| \le v(t) \quad \text{for } t \in [\alpha,a_0] \ \big(\text{resp., for } t \in [b_0,\beta]\big). \tag{41}$$

Assume the contrary, let (41) be violated. Define

$$w(t) = \big|u(t)\big| - v(t) \quad \text{for } t \in [\alpha,a_0] \ \big(\text{resp., for } t \in [b_0,\beta]\big).$$

Then there exist $t_1 \in [\alpha,a_0[$ and $t_2 \in]t_1,a_0]$ (resp., $t_1 \in [b_0,\beta[$ and $t_2 \in]t_1,\beta]$) such that

$$w(t) > 0 \quad \text{for } t \in]t_1,t_2[, \tag{42}$$
$$w(t_1) = 0, \qquad w(t_2) = 0. \tag{43}$$

In view of (1), (39), and (42), it is clear that $w \in AC'_{\mathrm{loc}}(]t_1,t_2[)$ and

$$w''(t) = p(t)\big|u(t)\big| + q(t)\operatorname{sgn} u(t) + \big[p(t)\big]_- v(t) + \big|q(t)\big| \ge -\big[p(t)\big]_- w(t) \quad \text{for } t \in]t_1,t_2[.$$

Hence, by virtue of (43) and Proposition 2.2, we get $w(t) \le 0$ for $t \in]t_1,t_2[$, which contradicts (42). Therefore, (41) is fulfilled. The estimate (36) (resp., (38)) now follows from (40) and (41). □

Lemma 3.3 *Let* (6) *hold and problem* (1_0), (2) *have no nontrivial solution. Then there exist* $\bar{a}_0 \in]a,b[$, $\bar{b}_0 \in]\bar{a}_0,b[$, *and* $r_0 > 0$ *such that for any* $\alpha \in [a,\bar{a}_0]$ *and* $\beta \in [\bar{b}_0,b]$ *and any* q *satisfying* (4), *every solution* u *of equation* (1) *satisfying*

$$u(\alpha) = 0, \qquad u(\beta) = 0$$

admits the estimate

$$\big|u(t)\big| \le r_0 \int_a^b (s-a)(b-s)\big|q(s)\big|\,ds \quad \textit{for } t \in [\alpha,\beta].$$

Proof Suppose to the contrary that the lemma is not true. Then there exist sequences $\{a_n\}_{n=1}^{+\infty} \subset [a, \frac{a+b}{2}[$, $\{b_n\}_{n=1}^{+\infty} \subset]\frac{a+b}{2}, b]$, $\{q_n\}_{n=1}^{+\infty} \subset L_{\mathrm{loc}}(]a,b[)$, and $\{u_n\}_{n=1}^{+\infty} \subset AC'_{\mathrm{loc}}(]a,b[)$ such that (11) holds,

$$\lim_{n\to+\infty} a_n = a, \qquad \lim_{n\to+\infty} b_n = b, \tag{44}$$

$$u_n''(t) = p(t)u_n(t) + q_n(t) \quad \text{for } t \in]a,b[, \qquad u_n(a_n) = 0 \qquad u_n(b_n) = 0$$

and

$$\|u_n\|_{[a_n,b_n]} > n \int_a^b (s-a)(b-s)\big|q_n(s)\big|\,ds \quad \text{for } n = 1,2,\ldots. \tag{45}$$

Introduce the notation

$$\tilde{u}_n(t) = \frac{1}{\|u_n\|_{[a_n,b_n]}} u_n(t), \qquad \tilde{q}_n(t) = \frac{1}{\|u_n\|_{[a_n,b_n]}} q_n(t).$$

Then it is clear that

$$\|\tilde{u}_n\|_{[a_n,b_n]} = 1 \tag{46}$$

and

$$\tilde{u}_n''(t) = p(t)\tilde{u}_n(t) + \tilde{q}_n(t) \quad \text{for } t \in]a_n,b_n[, \qquad \tilde{u}_n(a_n) = 0, \qquad \tilde{u}_n(b_n) = 0. \tag{47}$$

Moreover, it follows from (45) that

$$\lim_{n\to+\infty} \int_a^b (s-a)(b-s)\big|\tilde{q}_n(s)\big|\,ds = 0 \tag{48}$$

and, consequently,

$$\lim_{n\to+\infty} \int_{\frac{a+b}{2}}^t \left(\int_{\frac{a+b}{2}}^s \tilde{q}_n(\xi)\,d\xi \right) ds = 0 \quad \text{for } t \in]a,b[. \tag{49}$$

By virtue of Lemma 3.1, (46), and (47),

$$\begin{aligned}(t-a)(b-t)\big|\tilde{u}_n'(t)\big| \le b-a &+ \int_a^b (s-a)(b-s)\big[p(s)\big]_-\,ds \\ &+ \int_a^b (s-a)(b-s)\big|\tilde{q}_n(s)\big|\,ds \quad \text{for } t \in]a_n,b_n[.\end{aligned}$$

Hence, in view of (44) and (48), the sequence $\{\tilde{u}_n'\}_{n=1}^{+\infty}$ is uniformly bounded in $]a,b[$ and, therefore, the sequence $\{\tilde{u}_n\}_{n=1}^{+\infty}$ is equicontinuous in $]a,b[$. Taking, moreover, into account (46), by virtue of the Arzelá-Ascoli lemma, we can assume, without loss of generality, that

$$\lim_{n\to+\infty} \tilde{u}_n(t) = u_0(t) \quad \text{uniformly in }]a,b[, \tag{50}$$

where $u_0 \in C(]a,b[)$ and, moreover,

$$\lim_{n\to+\infty} \tilde{u}_n'\left(\frac{a+b}{2}\right) = c_0. \tag{51}$$

By a direct calculation, one can easily verify that

$$\begin{aligned} \tilde{u}_n(t) = \tilde{u}_n\left(\frac{a+b}{2}\right) + \left(t - \frac{a+b}{2}\right)\tilde{u}_n'\left(\frac{a+b}{2}\right) \\ + \int_{\frac{a+b}{2}}^{t}\left(\int_{\frac{a+b}{2}}^{s}\left[p(\xi)\tilde{u}_n(\xi) + \tilde{q}_n(\xi)\right] d\xi\right) ds \quad \text{for } t \in]a,b[, \end{aligned}$$

whence, in view of (49)-(51), we get

$$u_0(t) = u_0\left(\frac{a+b}{2}\right) + \left(t - \frac{a+b}{2}\right)c_0 + \int_{\frac{a+b}{2}}^{t}\left(\int_{\frac{a+b}{2}}^{s} p(\xi)u_0(\xi)\, d\xi\right) ds \quad \text{for } t \in]a,b[.$$

Thus $u_0 \in AC'_{\mathrm{loc}}(]a,b[)$ and u_0 is a solution of equation (1_0).

Now let $a_0 \in]a,b[$, $b_0 \in]a_0,b[$, and $\varrho > 0$ be from the assertion of Lemma 3.2. Assume, without loss of generality, that $a_n < a_0$ and $b_n > b_0$ for any natural n. Then, by virtue of Lemma 3.2, (46), and (47), the estimates

$$\begin{aligned} \left|\tilde{u}_n(t)\right| &\le \varrho\left(t - a + \int_a^t (s-a)\left|\tilde{q}_n(s)\right| ds + (t-a)\int_t^{a_0}\left|\tilde{q}_n(s)\right| ds\right) \quad \text{for } t \in]a_n,a_0], \\ \left|\tilde{u}_n(t)\right| &\le \varrho\left(b - t + \int_t^b (b-s)\left|\tilde{q}_n(s)\right| ds + (b-t)\int_{b_0}^{t}\left|\tilde{q}_n(s)\right| ds\right) \quad \text{for } t \in [b_0,b_n[\end{aligned} \tag{52}$$

are fulfilled. Moreover, in view of (48), we have

$$\lim_{n\to+\infty}\left(\int_a^t (s-a)\left|\tilde{q}_n(s)\right| ds + (t-a)\int_t^{a_0}\left|\tilde{q}_n(s)\right| ds\right) = 0 \quad \text{for } t \in]a,a_0]$$

and

$$\lim_{n\to+\infty}\left(\int_t^b (b-s)\left|\tilde{q}_n(s)\right| ds + (b-t)\int_{b_0}^{t}\left|\tilde{q}_n(s)\right| ds\right) = 0 \quad \text{for } t \in [b_0,b[.$$

Taking, moreover, into account (50), we get from (52) that

$$\left|u_0(t)\right| \le \varrho(t-a) \quad \text{for } t \in]a,a_0] \quad \text{and} \quad \left|u_0(t)\right| \le \varrho(b-t) \quad \text{for } t \in [b_0,b[,$$

and thus u_0 satisfies the conditions

$$u_0(a) = 0, \qquad u_0(b) = 0.$$

On account of (44) and (48), there exist $\alpha_0 \in]a,a_0[$, $\beta_0 \in]b_0,b[$, and n_0 such that

$$a_n < \alpha_0, \quad \varrho\left(\alpha_0 - a + \int_a^{a_0}(s-a)\left|\tilde{q}_n(s)\right| ds\right) < 1 \quad \text{for } n > n_0$$

and

$$b_n > \beta_0, \quad \varrho\left(b - \beta_0 + \int_{b_0}^{b}(b-s)\big|\tilde{q}_n(s)\big|\,ds\right) < 1 \quad \text{for } n > n_0.$$

Then it follows from (52) that

$$\big|\tilde{u}_n(t)\big| < 1 \quad \text{for } t \in [a_n, \alpha_0] \cup [\beta_0, b_n], n > n_0.$$

Hence, in view of (46), $\|\tilde{u}_n\|_{[\alpha_0,\beta_0]} = 1$ for $n > n_0$. Taking now into account (50), we get $\|u_0\|_{[\alpha_0,\beta_0]} = 1$, and thus u_0 is a nontrivial solution of problem (1_0), (2). However, this contradicts an assumption of the lemma. □

4 Proofs of the main results

Proof of Theorem 1.1 To prove the theorem, it is sufficient to show that if problem (1_0), (2) has no nontrivial solution, then problem (1), (2) has at least one solution.

Let a_0, b_0, $\bar{a}_0$, $\bar{b}_0$, ϱ, and r_0 be from the assertions of Lemmas 3.2 and 3.3. Let, moreover, the sequences $\{a_n\}_{n=1}^{+\infty} \subset\,]a, \min\{a_0, \bar{a}_0\}[$ and $\{b_n\}_{n=1}^{+\infty} \subset\,]\max\{b_0, \bar{b}_0\}, b[$ be such that

$$\lim_{n\to+\infty} a_n = a, \qquad \lim_{n\to+\infty} b_n = b. \tag{53}$$

By virtue of Lemma 3.3, the problem

$$u'' = p(t)u; \qquad u(a_n) = 0, \qquad u(b_n) = 0$$

has no nontrivial solution. Hence, by virtue of Proposition 2.1, the problem

$$\begin{aligned} &u_n'' = p(t)u_n + q(t), \\ &u_n(a_n) = 0, \qquad u_n(b_n) = 0 \end{aligned} \tag{54}$$

has a unique solution u_n. Moreover, by virtue of Lemma 3.3, the estimate

$$\big|u_n(t)\big| \le r_1 \quad \text{for } t \in [a_n, b_n] \tag{55}$$

holds, where

$$r_1 = r_0 \int_a^b (s-a)(b-s)\big|q(s)\big|\,ds.$$

On the other hand, on account of Lemma 3.1 and (55), we have

$$(t-a)(b-t)\big|u_n'(t)\big| \le r_2 \quad \text{for } t \in [a_n, b_n], \tag{56}$$

where

$$r_2 = r_1\left(b - a + \int_a^b (s-a)(b-s)\big[p(s)\big]_-\,ds\right) + \int_a^b (s-a)(b-s)\big|q(s)\big|\,ds.$$

In view of (53), (55), and (56), the sequence $\{u_n\}_{n=1}^{+\infty}$ is uniformly bounded and equicontinuous in $]a,b[$. Hence, by virtue of the Arzelá-Ascoli lemma, we can suppose, without loss of generality, that

$$\lim_{n\to+\infty} u_n(t) = u_0(t) \quad \text{uniformly in }]a,b[, \tag{57}$$

where $u_0 \in C(]a,b[)$ and, moreover,

$$\lim_{n\to+\infty} u_n'\left(\frac{a+b}{2}\right) = c_0. \tag{58}$$

Taking into account (54), one can easily verify, by a direct calculation, that

$$\begin{aligned} u_n(t) &= u_n\left(\frac{a+b}{2}\right) + \left(t - \frac{a+b}{2}\right)u_n'\left(\frac{a+b}{2}\right) \\ &\quad + \int_{\frac{a+b}{2}}^{t}\left(\int_{\frac{a+b}{2}}^{s}\big[p(\xi)u_n(\xi) + q(\xi)\big]\,d\xi\right)ds \quad \text{for } t \in [a_n,b_n]. \end{aligned}$$

Hence, in view of (57) and (58), we get

$$\begin{aligned} u_0(t) &= u_0\left(\frac{a+b}{2}\right) + \left(t - \frac{a+b}{2}\right)c_0 \\ &\quad + \int_{\frac{a+b}{2}}^{t}\left(\int_{\frac{a+b}{2}}^{s}\big[p(\xi)u_0(\xi) + q(\xi)\big]\,d\xi\right)ds \quad \text{for } t \in]a,b[. \end{aligned}$$

Thus $u_0 \in AC'_{\mathrm{loc}}(]a,b[)$ and u_0 is a solution of equation (1).

Further, by virtue of Lemma 3.2 and (55), the inequalities

$$\big|u_n(t)\big| \le \varrho\left(r_1(t-a) + \int_a^t (s-a)\big|q(s)\big|\,ds + (t-a)\int_t^{a_0}\big|q(s)\big|\,ds\right) \quad \text{for } t \in]a_n,a_0]$$

and

$$\big|u_n(t)\big| \le \varrho\left(r_1(b-t) + \int_t^b (b-s)\big|q(s)\big|\,ds + (b-t)\int_{b_0}^{t}\big|q(s)\big|\,ds\right) \quad \text{for } t \in [b_0,b_n[$$

are fulfilled. Hence, on account of (57), we get

$$\begin{aligned} \big|u_0(t)\big| &\le \varrho\left(r_1(t-a) + \int_a^t (s-a)\big|q(s)\big|\,ds + (t-a)\int_t^{a_0}\big|q(s)\big|\,ds\right) \quad \text{for } t \in]a,a_0], \\ \big|u_0(t)\big| &\le \varrho\left(r_1(b-t) + \int_t^b (b-s)\big|q(s)\big|\,ds + (b-t)\int_{b_0}^{t}\big|q(s)\big|\,ds\right) \quad \text{for } t \in [b_0,b[, \end{aligned}$$

and thus $u_0(a) = 0$ and $u_0(b) = 0$. Consequently, u_0 is a solution of problem (1), (2). □

Proof of Theorem 1.2 According to Theorem 1.1, problem (1), (2) has a unique solution u. By virtue of Lemma 3.3, the estimate

$$\big|u(t)\big| \le r_0\int_a^b (s-a)(b-s)\big|q(s)\big|\,ds \quad \text{for } t \in [a,b]$$

holds. On the other hand, it follows from Lemma 3.1 that

$$(t-a)(b-t)\left|u'(t)\right| \leq \|u\|_{[a,b]}\left(b-a+\int_a^b (s-a)(b-s)\big[p(s)\big]_- \,ds\right) + \int_a^b (s-a)(b-s)\left|q(s)\right| ds \quad \text{for } t \in\,]a,b[.$$

The latter two inequalities imply (9) with

$$r = 1 + r_0\left(b-a+\int_a^b (s-a)(b-s)\big[p(s)\big]_- \,ds\right). \qquad \square$$

Proof of Corollary 1.1 By virtue of Theorem 1.1, problems (1), (2) and (10_n), (2) have unique solutions u and u_n, respectively. Let

$$v_n(t) = u_n(t) - u(t) \quad \text{for } t \in [a,b]. \tag{59}$$

Then it is clear that

$$v_n''(t) = p(t)v_n(t) + \tilde{q}_n(t) \quad \text{for } t \in\,]a,b[, \qquad v_n(a) = 0, \qquad v_n(b) = 0,$$

where

$$\tilde{q}_n(t) = q_n(t) - q(t) \quad \text{for } t \in\,]a,b[. \tag{60}$$

Hence, by virtue of Theorem 1.2,

$$\left|v_n(t)\right| + (t-a)(b-t)\left|v_n'(t)\right| \leq r\int_a^b (s-a)(b-s)\left|\tilde{q}_n(s)\right| ds \quad \text{for } t \in\,]a,b[.$$

Taking now into account (12), (59), and (60), we get (13) and (14). $\square$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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