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Mgr. Irena Hinterleitner

Selected Special Vector Fields and Mappings in Riemannian Geometry

Brno University of Technology
Faculty of Mechanical Engineering
Institute of Mathematics

Mgr. Irena Hinterleitner

Selected Special Vector Fields and Mappings in Riemannian Geometry

Vybraná speciální vektorová pole a zobrazení
v Reimanové geometrii

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Study field: Applied Mathematics
Supervisor: doc. RNDr. Miroslav Doupovec, CSc.
Opponents: prof. Andrey Popov, DrSc.
doc. RNDr. Jiří Tomáš, CSc.
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Contents

1. Introduction

The geometry of curved spaces, introduced in the 19^{th} century by B. Riemann, generalises Euclidian geometry in a natural way. Today its application in physics is very wide. Above all it provides the mathematical foundation of General Relativity, more recently it is applied for example in gauge field theory and in connection with σ models, popular in string theory.

Nowadays Riemannian geometry is usually formulated in terms of manifolds introduced by E. Cartan. Differential geometry on manifolds facilitates the formulation of mappings and can be written in coordinate free form, widely used in contemporary mathematics. In physics, nevertheless, calculations are mostly carried out in local coordinates.

Interesting actual parts of differential geometry are the study of diffeomorphisms and automorphisms of different types of geometric structures on smooth manifolds. In geometry the term morphism denotes a mapping between manifolds which preserves some characteristic properties. An important structure in differential geometry are affine and special Riemannian connections, the latter ones expressed by Christoffel symbols. These connections are very important and useful in physics. A generalisation of Riemannian geometry is Finsler geometry with Berwald connection.

In Riemannian space the natural generalisation of straight lines are geodesics. This is illustrated by their role in General Relativity: geodesics are trajectories of freely falling particles in curved space-time, replacing the rectilinear motion of free particles in flat Euclidian space. Today the theory of geodesics has reached the stadium of technical application in GPS, but their physical and mathematical significance is well known since the time of Bernoulli, Euler, and Lagrange. Already in Beltrami's lifetime geodesic-preserving morphisms were studied - *geodesic mappings*. T. Levi-Civita, who laid the foundations of this theory in tensor form, studied it from the point of view of modelling dynamical processes in mechanics. Presently, for example, Ferapontov [6] and Hall and Lonie [8, 9] continue working on this subject.

Much work is spent further on isometric, homothetic and conformal mappings, also on various generalisations of geodesic mappings, among them for example holomorphic-projective, quasi-geodesic, semi-geodesic, F-planar mappings, transformations and deformations. The theory of modelling physical-mechanical processes by quasi-geodesic mappings was generalised by A. Z. Petrov [19].

Results on related questions can be found in many monographs, reports and theses: [4, 5, 13, 14, 18, 19, 21, 23, 24].

The mathematical apparatus employed here is tensor calculus, which is used for global and local relations on n-dimensonal manifolds with affine connection, denoted in the following by A_n, and Riemannian manifolds, denoted by V_n. The signature of the metric of V_n can be indefinite, so under the notion of Riemannian manifolds we understand also pseudo-Riemannian manifolds, irrespective of the signature of their metric, as for example in the books [13, 14, 18, 19, 24].

The subject of this thesis are selected examples of Riemannian spaces with special symmetry properties, namely equidistant spaces and generalisations thereof and several kinds of diffeomorphisms which preserve certain geometric structures.

A major part is devoted to mappings between equidistant spaces. Equidistant spaces are characterised by the existence of certain vector fields, called concircular (see 2.8). In physics these spaces are represented by the spatially homogenous and isotropic cosmological models (Friedmann-Robertson-Walker-Lemaître models).

2. Basic Objects of Differential Geometry

In this section we present a brief overview of basic notations. A profound explanation can be found in the books [5, 15, 19, 21, 23, 24].

We present basic objects of differential geometry as manifolds with affine connections and (pseudo-) Riemannian manifolds, and also geodesics and F-planar curves. Another important field in this work is the theory of equidistant spaces. These spaces admit special concircular vector fields. The fundamental equation of a special ***concircular*** vector field $\boldsymbol{\xi}$ can be written in the form: $\nabla_X \boldsymbol{\xi} = \rho \cdot X, \quad \forall X \in TV_n$.
If $\rho = \text{const}$, then $\boldsymbol{\xi}$ is called ***convergent***.

The priority in the theory of concircular vector fields is usually ascribed to K. Yano [28]. But many results were found earlier.

Convergent vector fields were first studied by P.A. Shirokov [23].

Concircular vector fields were first studied by H.W. Brinkmann [2] (see [17, 19, 22], [H6]), in connection with conformal mappings of Einstein spaces. His results contain general properties of the metrics of Riemannian and pseudo-Riemannian spaces with concircular vector fields.

He showed that in a space V_n with non-isotropic concircular vector field ξ $(g(\xi,\xi)\neq 0)$, there exists a coordinate system x, in which the metric has the following form

$$\mathrm{d}s^2 = \frac{1}{f(x^1)}(\mathrm{d}x^1)^2 + f(x^1)\,\mathrm{d}\tilde{s}^2, \tag{2.1}$$

where $f \in C^1$; $(f \neq 0)$ is a function and $\mathrm{d}\tilde{s}^2 = \tilde{g}_{ab}(x^2,\cdots,x^n)\mathrm{d}x^a\mathrm{d}x^b$; $(a,b=2,\cdots,n)$ is the metric form of a certain Riemannian space $\widetilde{V}_{n-1}$.

After a transformation of the coordinate x^1 the metric (2.1) acquires the form

$$\mathrm{d}s^2 = a(x^1)\,(\mathrm{d}x^1)^2 + b(x^1)\,\mathrm{d}\tilde{s}^2, \tag{2.2}$$

where $a,b \in C^1$; $(a,b\neq 0)$ are functions. The transformation can be chosen such that $a(x^1)=\pm 1$.

H.W. Brinkmann showed that the space V_n with metric (2.1) is an Einstein space $\mathcal{E}_n$ (resp. a space $\mathcal{S}_n$ with constant curvature K) if and only if the following holds:

Condition 1. $\mathrm{d}\tilde{s}^2$ is a metric of an Einstein space $\tilde{\mathcal{E}}_{n-1}$ (resp. a space $\tilde{\mathcal{S}}_{n-1}$ with constant curvature $\tilde{K}$), and $f(x^1) = K\,(x^1)^2 + 2a\,x^1 + b$, where K, a and b are constants and moreover

$$\tilde{K} = \frac{\tilde{R}}{(n-1)(n-2)} = b\,K - a^2 \quad \text{and} \quad K = \frac{R}{n(n-1)}.$$

Here R and $\tilde{R}$ are the scalar curvatures of $\mathcal{E}_n$ and $\tilde{\mathcal{E}}_{n-1}$ (resp. $\mathcal{S}_n$ and $\tilde{\mathcal{S}}_{n-1}$).

We mention the application of equidistant spaces in cosmology. Homogenous and isotropic models are the simplest cosmological models characterised by constant spatial curvature. They were introduced and studied by Friedmann, Lemaître, Robertson and Walker [10].

3. $\varphi(\boldsymbol{Ric})$ - Vector Fields in Riemannian Spaces

In this section we study a class of special curvature-determined vector fields ($\varphi(\boldsymbol{Ric})$ ***- vector fields***) in Riemannian spaces, which are in some sense modifications of concircular vector fields and defined by

$$\nabla\varphi = \mu\,\boldsymbol{Ric}, \tag{3.3}$$

where μ is some constant and $\boldsymbol{Ric}$ is the Ricci tensor. We suppose that $\mu \neq 0$ and V_n is not an Einstein space.

We prove the following theorems

Theorem 1 *Riemannian or pseudo-Riemannian spaces* (M^n, g) *with a* $\varphi(\mathbf{Ric})$*- vector field of constant length have constant scalar curvature.*

Theorem 2 *If a* $\varphi(\mathbf{Ric})$ *- vector field exists together with a non-collinear concircular vector field on a Riemannian manifold, then the latter one is necessarily covariantly constant.*

Theorem 3 *In a non-Einsteinian symmetric pseudo-Riemannian space* V_n *with* $Ric(X, R(Y,Z)V) = 0$ *for all tangent fields* X, Y, Z, V *there exists locally a* $\varphi(\mathbf{Ric})$ *- vector field.*

An example of a pseudo-Riemannian space V_n satisfying the condition of Theorem 3 is a space with a metric of the following form

$$\mathrm{d}s^2 = \exp(2x^1)\{2\mathrm{d}x^1\mathrm{d}x^2 + e_3(\mathrm{d}x^3)^2 + \ldots + e_n(\mathrm{d}x^n)^2\}, \quad e_i = \pm 1, i = 3\ldots n.$$

Lemma 1 *Classical Riemannian spaces* V_n *with positive definite metric and the above properties (of Theorem 3) do not exist.*

Example of a $\varphi(\boldsymbol{Ric})$ - vector field: Our example is a non-isotropic generalization of an equidistant space, motivated by the Kasner vacuum metric in General Relativity [12]. For simplicity we have restricted ourselves to a $2+1$ dimensional Riemannian space with diagonal metric in the coordinates x^1, x^2, x^3,

$$\mathrm{d}s^2 = -(\mathrm{d}x^1)^2 + (x^1)^{2\cos\theta}(\mathrm{d}x^2)^2 + (x^1)^{2\sin\theta}(\mathrm{d}x^3)^2, \tag{3.4}$$

with the parameter θ conveniently restricted by $\theta \in \langle 0, 2\pi\rangle$.

In this space the $\varphi(\boldsymbol{Ric})$ vector field is given by the component

$$\varphi = ((1 - \cos\theta - \sin\theta)(x^1)^{-1}, 0, 0).$$

As special cases the model contains flat space for $\theta = 0, \pi/2,$; for $\theta = \pi/4, \pi, 5\pi/4, 3\pi/2$ the space is equidistant. These results were published in [H7].

4. Special Mappings Between Riemannian Spaces

4.1. Formalisms Concerning Diffeomorphisms of Manifolds

For the study of affine, conformal, geodesic and other mappings several formalisms were established , which are generally used for diffeomorphisms between spaces with affine connection. Now we describe the two most frequently used formalisms that we need for our work.

The Formalism of "Common Coordinate Systems". Assume a diffeomorphism $f\colon\ A_n \to \overline{A}_n$, where A_n and $\overline{A}_n$ are spaces with affine connections ∇ and $\overline{\nabla}$. We restricted ourselves to the study of the respective coordinate neighbourhoods U and $f(U)$.

The neighbourhoods U and $f(U)$ are said to be related by a common coordinate system x with respect to the map f, if two corresponding points $m \in U$ and $f(m) \in f(U)$ have the same coordinates $x = (x^1, \dots, x^n)$. We study the neighbourhood of each point $m \in A_n$.

Corresponding geometric objects on A_n and on $\overline{A}_n$ will be distinguished by a bar. For example $\overline{\Gamma}^h_{ij}$, $\overline{R}^h_{ijk}$ and $\overline{R}_{ij}$ are components of the affine connection, the Riemann and the Ricci tensor of the space $\overline{A}_n$.

The Formalism of the "Common Manifold". Assume a diffeomorphism $f\colon\ A_n \to \overline{A}_n$ as above. We can specify that A_n is a manifold M with the affine connection ∇ and $\overline{A}_n$ is a manifold $\overline{M}$ with the affine connection $\overline{\nabla}$.

Because f is a diffeomorphism between M and $\overline{M}$, from the point of view of topology we can take $\overline{M} \equiv M$. Then the spaces A_n and $\overline{A}_n$ are defined on the same manifold M with ∇ and $\overline{\nabla}$ determining two different spaces with affine connections.

Similarly we can define other geometric objects on this manifold M, as for example, a metric, a Riemann and a Ricci tensor, and others, which will be distinguished by a bar, as before.

Of particular interest in the following will be the differences between the connection components, which form a tensor, the ***deformation tensor***

$$P(X,Y) = (\overline{\nabla} - \nabla)(X,Y), \quad X,Y \in TM. \tag{4.1}$$

Of course, if the connections are symmetric, the deformation tensor is symmetric, too.

This formalism has a global meaning and is not connected with coordinate neighbourhoods as common coordinates. On the other hand, in view of the formalism described in the monographs by A.Z. Petrov [19], and R.W. Penrose and W. Rindler [18], with the aid of common coordinates it is possible to express not only local, but also global properties. These methods were used in fact in the work by K. Yano and S. Bochner [29], and many other authors, see [13, 14, 24].

4.2. Special Mappings of Equidistant Spaces

Of particular interest in the following are mappings between equidistant spaces. In this field some of the results of this work were obtained and published in [H3].

Consider a *special mapping* f between the equidistant space V_n with metric

$$\mathrm{d}s^2 = a(x^1)\,(\mathrm{d}x^1)^2 + b(x^1)\,\mathrm{d}\tilde{s}^2, \tag{4.2}$$

and $\overline{V}_n$ with metric

$$\mathrm{d}\overline{s}^2 = A(x^1)\,(\mathrm{d}x^1)^2 + B(x^1)\,\mathrm{d}\hat{s}^2, \tag{4.3}$$

in the common coordinate system x, where $a(x^1), b(x^1), A(x^1), B(x^1) \in C^1$ are non-zero functions,

$$\mathrm{d}\tilde{s}^2 = \tilde{g}_{ab}(x^2,\,\dots,x^n)\,\mathrm{d}x^a\mathrm{d}x^b \quad \text{and} \quad \mathrm{d}\hat{s}^2 = \hat{g}_{ab}(x^2,\,\dots,x^n)\,\mathrm{d}x^a\mathrm{d}x^b \quad (a,b=\overline{2,n})$$

are the metric forms of certain Riemannian spaces $\tilde{V}_{n-1}$ and $\hat{V}_{n-1}$.

The mapping $f\colon V_n \to \overline{V}_n$ is ***conformal***, if $\overline{g} = \mathrm{e}^{2\sigma(x)}g$, in common coordinates $\overline{g}_{ij}(x) = \mathrm{e}^{2\sigma(x)}g_{ij}(x)$, where $\sigma(x)$ is a function on V_n.

We have the following results.

Lemma 2 *The special mapping f between equidistant spaces V_n and $\overline{V}_n$ is conformal if and only if there exists a function $\varrho(x^1) \neq 0$ such that the metric form of $\overline{V}_n$ has the following form*

$$\mathrm{d}\overline{s}^2 = \varrho(x^1)\,\mathrm{d}s^2 = \varrho(x^1)\left(a(x^1)\,(\mathrm{d}x^1)^2 + b(x^1)\,\mathrm{d}\tilde{s}^2\right).$$

A ***harmonic map*** [27] is a smooth (C^∞) map $\varphi : V_m \to \overline{V}_n$ between Riemannian spaces which extremises the Dirichlet or energy integral

$$E(\varphi; D) = \frac{1}{2}\int_D |\mathrm{d}\varphi|^2\sqrt{\det g}\,\mathrm{d}^m x \tag{4.4}$$

with respect to variations of φ on compact domains D.

Analogously for a two dimensional V_m (4.4) has the form of the action integral in string theory, whose variation gives the worldsheet of a string, the analog of the worldline of a point particle. More generally, harmonic maps are extremal in relation to the energy functionals of sigma models, as described by J. Wood [27].

In the following we specialize to diffeomorphic mappings between spaces of equal dimension, ***harmonic diffeomorphisms***. From (4.4) it follows that a diffeomorphism from V_n onto $\overline{V}_n$ is harmonic if and only if the following conditions hold [25]: $P^h_{\alpha\beta}\, g^{\alpha\beta} = 0$.

For this special case we have the following theorem.

Theorem 4 *The special mapping f between an equidistant space V_n with a metric of the form (4.2), and another equidistant space $\overline{V}_n$ with the metric*

$$\mathrm{d}\overline{s}^2 = c \cdot a(x^1)\, b^{1-n}(x^1)\, (\mathrm{d}x^1)^2 + B\, \mathrm{d}\hat{s}^2, \ \textit{where} \quad c, B = \text{const} \neq 0$$

is harmonic if and only if the mapping between the subspaces $\tilde{V}_{n-1}$ and $\hat{V}_{n-1}$ is harmonic.

Moreover, if $\hat{g}_{ab}\, \tilde{g}^{ab} = \text{const}$, then for an arbitrary function $B(x^1) \in C^1$ there is a function $A(x^1)$ satisfying the ordinary differential equation

$$\frac{1}{a}\left(\frac{A'(x^1)}{A(x^1)} - \frac{a'(x^1)}{a(x^1)}\right) - \frac{1}{b}\left(\frac{B'(x^1)}{A(x^1)}\, \hat{g}_{ab}\, \tilde{g}^{ab} - (n-1)\, \frac{b'(x^1)}{a(x^1)}\right) = 0.$$

We consider ***equivolume mappings***, which were defined and studied by T.V. Zudina and S.E. Stepanov [30]. Such a mapping $f\colon\ V_n \to \overline{V}_n$ is characterized by the following condition $P^{\alpha}_{i\alpha}(x) = 0$.

Theorem 5 *The special non-affine mapping f between equidistant spaces V_n and $\overline{V}_n$ is equivolume if and only if $\tilde{V}_{n-1}$ admits an equivolume mapping on $\hat{V}_{n-1}$, and the metric of $\overline{V}_n$ has the following form*

$$\mathrm{d}\overline{s}^2 = \alpha \cdot a(x^1) \left(\frac{b(x^1)}{B(x^1)}\right)^{n-1} (\mathrm{d}x^1)^2 + B(x^1)\, \mathrm{d}\hat{s}^2,$$

where α is a non-zero constant, and $B(x^1)$ is a non-zero differentiable function.

4.3. Affine and Geodesic Mappings

First we introduce affine and geodesic mappings of ***spaces with affine connection***.

Definition 1 Let A_n and $\overline{A}_n$ be two spaces with affine connections ∇ and $\overline{\nabla}$. The diffeomorphism f from A_n to $\overline{A}_n$ is called a ***geodesic mapping*** if it maps all geodesic curves of A_n onto geodesic curves in $\overline{A}_n$.

If this mapping preserves also the natural parameters of geodesic curves, then f is called ***affine***.

We remark that the foundations of geodesic mappings were laid by J.L. Lagrange and E. Beltrami. Of particular importance in the theory of geodesic mappings are three fundamental directions:

1. the study of general dependences of geodesic mappings,
2. the integration of fundamental equations,
3. the study of geodesic mappings of special manifolds.

The main contributions to the study of general dependences of the theory of geodesic mappings come from the work of T. Levi-Civita, H. Weyl, T. Thomas, A.Z. Petrov, A.S. Solodovnikov, N.S. Sinyukov, J. Mikeš, V.S. Matveev, see [5, 13, 15, 19, 24, 29].

N.S. Sinyukov [24] found invariant (with respect to the choice of the coordinate system), necessary and sufficient conditions of intrinsic character for the existence of geodesic mappings between ***Riemannian spaces***. J. Mikeš and V.E. Berezovskij (see [13, 15]) solved these problems in the case of geodesic mappings from spaces with affine connections to Riemannian spaces. These conditions, nevertheless, have an implicit form. With respect to geodesic mappings the notion of the *degree of freedom* of Riemannian manifolds was defined. Practically this is the number of numeric parameters, on which the number of Riemannian manifolds $\overline{V}_n$, depends which map geodesically onto the given Riemannian manifold V_n, respectively A_n. A relation between the degree of freedom with respect to geodesic mappings and some isometric, homothetic and projective transformations was found [13].

The *second* direction in the theory of geodesic mappings, as said before, is the integration of fundamental equations. These questions are connected with the names U. Dini, T. Levi-Civita, P.A. Širokov, A.Z. Petrov, V.I. Golikov, A.V. Aminova, see [15, 19, 23, 24].

The *third* important direction is the study of geodesic mappings of special spaces, for example spaces with constant curvature, Einstein, equidistant, symmetric, recurrent, semisymmetric spaces and their generalizations, see the monographs and referential books [5, 13, 15, 19, 24].

Fundamental equations of geodesic mappings. It was proved that a necessary and sufficient condition for the existence of a geodesic mapping from A_n to $\overline{A}_n$, is $(\overline{\nabla} - \nabla)_X X = 2\,\psi(X)\,X,\ \forall X \in TA_n)$, where ψ is a linear form on A_n. A geodesic mapping is affine iff $\psi \equiv 0$.

In local coordinates the "coordinate-free" formula introduced above is equivalent to the following equation, named after ***Levi-Civita***, who effectively proved it, see [5, 13, 15, 19, 24]:

$$\overline{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \psi_i \delta^h_j + \psi_j \delta^h_i\,, \tag{4.5}$$

here $\psi_i(x)$ are the components of the linear form ψ.

We point out that in the case of geodesic mappings of the space A_n to the Riemannian space $\overline{V}_n$ with the metric tensor $\overline{g}_{ij}$ the equations (4.5) are equivalent to the following equations, which also bear the name of Levi-Civita:

$$\overline{g}_{ij,k} = 2\psi_k \overline{g}_{ij} + \psi_i \overline{g}_{jk} + \psi_j \overline{g}_{ik}, \tag{4.6}$$

here " , " denotes the covariant derivative in the space A_n.

Remark about possible applications of geodesic mappings: Recall that geodesics are the trajectories of test particles in a gravitational field. For this reason one can argue that the geometric quantities of two spaces V_n and $\overline{V}_n$, which are related by a geodesic mapping, describe the same gravitational field. In other words, geodesic mappings could be mere gauge transformations between different geometric realizations of one and the same physical situation. General Relativity, on the other hand, is not invariant under geodesic mappings, but there are recent attempts for geodesically invariant modifications [26].

Affine and geodesic mappings between equidistant spaces. After a detailed analysis we obtain the following results.

Lemma 3 *A mapping f between the equidistant spaces V_n with metric (4.2) and $\overline{V}_n$ is affine if and only if*

1) *the metric of* $\overline{V}_n$ *has the form* $\mathrm{d}\overline{s}^2 = \mathrm{const}\cdot \mathrm{d}s^2$*, i.e.* f *is homothetic, and* ***2)*** *in the case* $b(x^1) = \mathrm{const.}$*, the metric of* $\overline{V}_n$ *has the following form* $\mathrm{d}\overline{s}^2 = \alpha\, a(x^1)\,(\mathrm{d}x^1)^2 + \mathrm{d}\hat{s}^2$*, and the space* $\hat{V}_{n-1}$ *with metric* $\mathrm{d}\hat{s}^2$ *is affine to* $\tilde{V}_{n-1}$.

Theorem 6 *The special mapping* f *between equidistant spaces* V_n *and* $\overline{V}_n$ *is non-trivially geodesic if and only if* $\hat{V}_{n-1}$ *is homothetic to* $\tilde{V}_{n-1}$*, and the metric of* $\overline{V}_n$ *has the form*

$$\mathrm{d}\overline{s}^2 = \frac{p\, a(x^1)}{(1+q\, b(x^1))^2}\,(\mathrm{d}x^1)^2 + \frac{p\, b(x^1)}{1+q\, b(x^1)}\,\mathrm{d}\tilde{s}^2, \tag{4.7}$$

where p, q *are some constants such that* $p \neq 0$*,* $1+q\, b(x^1) \neq 0$*, and* $q\, b'(x^1) \not\equiv 0$*. From this follows that the function, which defines the geodesic mapping, has the form* $\psi = -\frac{1}{2}\ln|1+q\, b(x^1)|$.

Petrov's Hypothesis on Geodesic Mappings of Einstein Spaces. A.Z. Petrov [19] formulated the following **hypothesis**:

When an Einstein space V_n *with Minkowski signature of the metric maps geodesically onto another Einstein space* $\overline{V}_n$ *with Minkowski signature, then either the spaces* V_n *and* $\overline{V}_n$ *have constant curvature or the mapping is trivial.*

This hypothesis is trivial for $n = 2$ and $n = 3$, for $n = 4$ it was proved by several authors, see [15, 19]. For $n > 4$ a counterexample was constructed in [11]. We introduce this counterexample in a simple form in [H6], where the metrics of geodesically related Einstein spaces with Minkowski signature are given explicitly.

Results concerning geodesic mappings of Einstein spaces in the paper [H6] were quoted by G.S. Hall and D.P. Lonie [9].

5. Conformally-Projective Harmonic Diffeomorphisms

This chapter is devoted to a further kind of diffeomorphisms, which is of special interest. In the study of their properties we obtained the results published in the papers [H1,H2,H3,H5]. Our interest was motivated by the article by S.E. Stepanov and I.G. Shandra [25] about harmonic diffeomorphisms. In their paper they also considered compositions of conformal and geodesic mappings, which are at the same time harmonic.

Here in the first part we investigate the conditions of their existence in form of differential equations and find their solutions.

In the second part we apply this type of mapping to equidistant manifolds and show the Friedmann cosmological models as an explicit example.

5.1. Definition and Fundamental Equations

Definition 2 We call a composition of a conformal and a geodesic (projective) diffeomorphism between Riemannian spaces, which is harmonic, *conformally-projective harmonic.*

Necessary and sufficient conditions for the existence of these mappings are given in [25]. A diffeomorphism from an n-dimensional Riemannian space V_n onto a Riemannian space $\overline{V}_n$ is conformally-projective harmonic if and only if the conditions

$$\overline{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \varphi_i\delta^h_j + \varphi_j\delta^h_i - \tfrac{2}{n}\,\varphi^h g_{ij} \tag{5.8}$$

hold, where $\varphi_i = \partial_i\varphi(x)$ is a gradient-like covector and $\varphi^h = g^{h\alpha}\varphi_\alpha$.

By analysis of formula (5.8) the following theorem can be shown, giving the conditions for the existence of conformally-projective harmonic mappings in a different form.

Theorem 7 *A necessary and sufficient condition for* $f\colon\ V_n \to \overline{V}_n$ *to be conformally-projective harmonic is*

$$\overline{g}_{ij,k} = 2\varphi_k\overline{g}_{ij} + \varphi_i\overline{g}_{jk} + \varphi_j\overline{g}_{ik} - \tfrac{2}{n}\,(\overline{\varphi}_i g_{jk} + \overline{\varphi}_j g_{ik}), \tag{5.9}$$

where $\overline{g}_{ij}$ *are components of the metric tensor of* $\overline{V}_n$, $\overline{\varphi}_i = \varphi^\alpha\overline{g}_{\alpha i}$.

We obtained the fundamental equations of conformally-projective harmonic mappings in form of a system of first order differential equations in covariant derivatives of Cauchy type. We make use of the existence and uniqueness of the solutions of Cauchy type problems in covariant derivatives with the corresponding theory, which is analogous to the standard theory of Cauchy problems, presented in an appendix. In the studied case the solutions depend on at most $\frac{1}{2}(n+1)(n+2)$ independent parameters.

These equations are derived under the condition $\mathrm{Rank}\,\|\overline{g}_{ij} - \alpha g_{ij}\| > 1$, where α is a function. In the opposite case the following theorem holds

Theorem 8 *Let $f : V_n \to \overline{V}_n$ be a conformally-projective harmonic mapping. If* $\operatorname{Rank} \|\overline{g}_{ij} - \alpha g_{ij}\| \leq 1$ *holds, then V_n and $\overline{V}_n$ are equidistant and*

$$\overline{g}_{ij} = C\, \mathrm{e}^{2\varphi}\, g_{ij} + \beta(\varphi)\, \varphi_i \varphi_j, \tag{5.10}$$

where C is constant, β is a function of φ, which generates a gradient-like concircular vector φ_i. This solution depends maximally on $n+2$ independent real parameters.

5.2. Conformally-Projective Mappings of Equidistant Spaces

The following theorems were proved:

Theorem 9 *An equidistant manifold V_n with the metric*

$$ds^2 = (1 + q\, f(x^1))^{\frac{2}{n-2}} \left(e\,(dx^1)^2 + f(x^1)\, d\tilde{s}^2\right), \tag{5.11}$$

where $e \pm 1$, $f \in C^1$ ($f \neq 0$) is a function, $d\tilde{s}^2 = \tilde{g}_{ab}(x^2, \ldots, x^n)\, dx^a dx^b$ ($a, b = 2, \ldots, n$) is the metric of some $(n-1)$-dimensional Riemannian space $\tilde{V}_{n-1}$, admits a conformally-projective harmonic mapping on the Riemannian space $\overline{V}_n$ with the metric (5.12)

$$d\overline{s}^2 = \frac{ep}{(1+qf)^2}\,(dx^1)^2 + \frac{p\, f}{1+qf}\, d\tilde{s}^2, \tag{5.12}$$

where p, q are some constants such that $1 + qf \neq 0$, $p \neq 0$. If $qf' \not\equiv 0$, the mapping is nontrivial; otherwise it is affine.

Theorem 10 *The special non-affine mapping f between equidistant spaces V_n and $\overline{V}_n$ is conformally-projective harmonic if and only if $\tilde{V}_{n-1}$ admits a homothetic mapping on $\hat{V}_{n-1}$, and the metric of $\overline{V}_n$ has the following form*

$$\mathrm{d}\overline{s}^2 = \alpha \cdot a(x^1)\, \mathrm{e}^{4\,\frac{n-1}{n}\varphi(x^1)}\, (\mathrm{d}x^1)^2 + \beta \cdot b(x^1)\ \mathrm{e}^{2\varphi(x^1)} \mathrm{d}\tilde{s}^2,$$

where α, β are non-zero constants, and the function $\varphi(x^1)$ satisfies the ordinary differential equation

$$\beta\, n\, (b'(x^1) + 2b(x^1)\varphi') \cdot \mathrm{e}^{2\,\frac{2-n}{n}\,\varphi} - \alpha n\, b'(x^1) - 4\alpha\, b(x^1)\, \varphi' = 0.$$

6. On F-planar Mappings

Further results, published in [H4], were obtained with a type of mapping, which preserves F-planar curves. They generalize geodesics and geodesic mappings and are considered in Subsection 4.3, see [14, 16].

Theorem 11 *An F-planar mapping f from A_n onto $\overline{A}_n$ $(n > 2)$ preserves F-structures and is characterized by the following condition*

$$P(X, Y) = \psi(X)\,Y + \psi(Y)\,X + \varphi(X)\,FY + \varphi(Y)\,FX \qquad (6.13)$$

for any vector fields X, Y, where $P \stackrel{\text{def}}{=} \overline{\nabla} - \nabla$ is the deformation tensor field of f, ψ, φ are some linear forms and F is the F-structure, a tensor field of type $(1, 1)$.

Theorem 11 was proved by J. Mikeš and N.S. Sinyukov [14, 16] for finite dimension $n > 3$.

We can show a more rational proof of this Theorem for $n > 3$ and also a proof for $n = 3$. We show a counterexample for $n = 2$.

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9. Curriculum Vitae

Mgr. Irena Hinterleitner

Born: Odessa, Ukraine, 25 February 1980
Address: Oblá 65/1, 634 00 Brno
Mobile: 00420/774239488
E-mail: Hinterleitner.Irena@seznam.cz

Education:
1999-2004 – Applied Mathematics (master study), Faculty of Science, Palacky University Olomouc
2004 – Mgr degree
2005-today – Applied Mathematics (PhD. study), Institute of Mathematics, Faculty of Mechanical Engineering, Brno University of Technology

Pedagogical experience:
2005-today – Undergraduate courses in mathematics

10. Abstract

This PhD Thesis is devoted to the study of special vector fields and special mappings of (pseudo-) Riemannian spaces.

We defined and analyzed $\varphi(\boldsymbol{Ric})$ - vector fields on (pseudo-) Riemannian spaces, which are generalized concircular vector fields and characterized by the following condition $\nabla\varphi = \mu \cdot \mathrm{Ric}$, $\mu = \mathrm{const}$.

The metrics of equidistant spaces, which admit special mappings, in particular affine, conformal, geodesic, harmonic, equivolume, and conformally-projective harmonic mappings, are presented.

A counterexample to Petrov's hypothesis on geodesic mappings of Einstein spaces is constructed.

The fundamental equations of conformally-projective harmonic diffeomorphisms were formulated in the form of PDEs of Cauchy type and a new derivation of the fundamental equations of F-planar mappings between manifolds with affine connections was obtained.