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Diferenční rovnice se zpožděním a jejich aplikace

Delay difference equations and their applications

Zkrácená verze Ph.D. Thesis

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Diferenční rovnice se zpožděním, diferenciální rovnice se zpožděním, asymptotické chování, stabilita, Θ-metoda.

KEYWORDS

Delay difference equation, delay differential equation, asymptotic behaviour, stability, the Θ-metohod.

MÍSTO ULOŽENÍ DISERTAČNÍ PRÁCE

Oddělení vědy a výzkumu FSI VUT v Brně, Technická 2, 616 69 Brno

OBSAH

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1 INTRODUCTION

Mathematical modeling of various problems via delay differential equations (DDEs) is a conventional and classical topic. It turns out to be useful especially in the situation, where the mathematical description of investigated systems depends not only on the position of a system in the current time, but also in a preceding time. In such a case the modeling via ordinary differential equations (ODEs) turns out to be insufficient. There are many interesting applications of these equations in various areas ranging from the control theory to industrial problems (see, e.g [27]).

It is known that DDEs can be solved analytically only in some exceptional cases. There are no special types of DDEs and no computational methods (analogous to basic methods utilized for ODEs such as the variation of constants method, the separation of variables method and others) which can produce the exact solution. Therefore, the qualitative and numerical methods of solving of DDEs are of a fundamental importance, even in the study of basic (linear) types of DDEs.

Roughly speaking, basic numerical methods for DDEs originates from the corresponding procedures for ODEs, where some additional operations (especially the interpolation of delayed terms) are necessary. The resulting formulae are then delay difference equations. Their previous qualitative investigation is rather rare because - contrary to DDEs - there do not exist many original significant applications for this type of difference equations. Therefore it is just a numerical discretization of DDEs which motivates the investigation of delay difference equations.

The presented Ph.D. thesis are divided into six chapters. Their content is formed by scientific papers [8] and [20-22] published or accepted for the publication in international journals. Other related author's publications are [18] and [19].

1.1 OBJECTIVES OF THE THESIS

The aim is to discuss some properties of the numerical solution of a special delay differential equation in the form

$$y'(t) = ay(t) + by(\lambda t), \quad 0 < \lambda < 1, \qquad t \geq 0, \tag{1.1}$$

where $a, b \in \mathbb{C}$, which appears as a mathematical model of several problems (see, e.g. [32]). Among these applications we mention a technical problem on railways (see [37]) which gave the name to (1.1) - namely the pantograph equation. This thesis discusses its Θ-method discretizations which lead to delay difference equations with some specific properties (in particular, all the discretizations discussed throughout this thesis are difference equations of a variable order).

The first aim of this thesis is to formulate the asymptotic estimate of the solutions of the discretized pantograph equation. Our goal is not only to improve so far known results, we also wish to formulate the results which are "nonimprovable". More precisely, we derive the asymptotic bounds for the discretized pantograph equation which coincide with the corresponding estimates for the exact pantograph equation.

Another goal of this thesis is to analyse a change of the qualitative behaviour of the discretized pantograph equation. In particular, we want to give a mathematical explanation for a sudden change of its stability properties and perform some calculations concerning this phenomenon.

1.2 THE DERIVATION OF THE Θ-METHOD FOR LINEAR DDES

We consider the differential equation with a delayed argument in the form

$$y'(t) = a(t)y(t) + b(t)y(\tau(t)), \quad t \geq t_0, \tag{1.2}$$

where $a(t), b(t), \tau(t)$ are continuous (possibly complex-valued) functions on $[t_0, \infty)$ and $\tau(t)$ is a differentiable function which is strictly monotonically increasing and satisfying $\tau(t_0) = t_0$, $\tau(t) < t$ for all $t > t_0$.

The popular discretization of the equation (1.2) is the well-known Θ-method involving both Euler methods and the trapezoidal rule as particular cases. The derivation of this method is fully described in [19, 21]. Now we only sketch it.

Let $h > 0$ be the stepsize. We denote $\tau_n := \tau(t_0 + nh)$, $\bar{\tau}_n := (\tau_n - t_0)/h$, $a_n := a(t_0 + nh)$ and $b_n := b(t_0 + nh)$. We emphasize that the value τ_n is not usually a grid point. Then the explicit Euler discretization of (1.2) yields

$$y_{n+1} = y_n + ha_n y_n + hb_n y^h(\tau_n), \tag{1.3}$$

where $y_n \approx y(t_0 + nh)$ and the value $y^h(\tau_n)$ is given by the linear interpolation utilizing the left and right neighbours of τ_n, namely

$$y^h(\tau_n) = (1 - r_n) y_{\lfloor \frac{\tau_n - t_0}{h} \rfloor} + r_n y_{\lfloor \frac{\tau_n - t_0}{h} \rfloor + 1}, \tag{1.4}$$

where $r_n := \frac{\tau_n - t_0}{h} - \lfloor \frac{\tau_n - t_0}{h} \rfloor$ and the symbol $\lfloor \ \rfloor$ means the integer part.

Similarly, the implicit Euler discretization of (1.2) yields

$$y_{n+1} = y_n + ha_{n+1}y_{n+1} + hb_{n+1}y^h(\tau_{n+1}). \tag{1.5}$$

The linear combination of (1.3) and (1.5) yields the Θ-method in the form

$$y_{n+1} = y_n + h((1-\Theta)a_n y_n + \Theta a_{n+1}y_{n+1} + (1-\Theta)b_n y^h(\tau_n) + \Theta b_{n+1}y^h(\tau_{n+1})), \tag{1.6}$$

where $\Theta \in [0, 1]$ and instead of $y^h(\tau_n), y^h(\tau_{n+1})$ we substitute the corresponding terms by use of (1.4). Note that the equation (1.6) was derived using the procedure stated in [33].

Let $1 - \Theta h a_{n+1} \neq 0$. Then the equation (1.6) can be also rewritten as

$$y_{n+1} = R_n y_n + S_n \left(\beta_n y_{\lfloor \bar{\tau}_n \rfloor} + \alpha_n y_{\lfloor \bar{\tau}_n \rfloor + 1} + \widehat{\beta}_n y_{\lfloor \bar{\tau}_{n+1} \rfloor} + \widehat{\alpha}_n y_{\lfloor \bar{\tau}_{n+1} \rfloor + 1} \right), \tag{1.7}$$

$n = 0, 1, \ldots$, where

$$R_n := \frac{1 + (1 - \Theta)ha_n}{1 - \Theta ha_{n+1}}, \qquad S_n := \frac{b_n h}{1 - \Theta ha_{n+1}} \qquad (1.8)$$

and

$$\alpha_n := (1 - \Theta)(\bar{\tau}_n - \lfloor \bar{\tau}_n \rfloor), \qquad \beta_n := 1 - \Theta - \alpha_n,$$

$$\widehat{\alpha}_n := \frac{b_{n+1}}{b_n}\Theta(\bar{\tau}_{n+1} - \lfloor \bar{\tau}_{n+1} \rfloor), \qquad \widehat{\beta}_n := \frac{b_{n+1}}{b_n}\Theta - \widehat{\alpha}_n. \qquad (1.9)$$

In the sequel we consider the formulae arising from (1.7).

2 MAIN RESULTS

In this section we present the main results of the thesis. Subsection 2.1 describes the asymptotic estimate of the solutions of the discretized nonautonomous pantograph equation. In the next subsections, we discuss some possible extensions of this result. Some of these extensions are quite straightforward (e.g. the involvement of several proportional delays into our considerations), while others require some additional operations.

2.1 THE ASYMPTOTIC BEHAVIOUR OF THE Θ-METHOD FOR THE NONAUTONOMOUS PANTOGRAPH EQUATION

We consider the nonautonomous pantograph equation as the particular case of (1.2) via the choice $\tau(t) = \lambda t, 0 < \lambda < 1$ in the form

$$y'(t) = a(t)y(t) + b(t)y(\lambda t), \quad t \geq 0. \qquad (2.1)$$

The discretization procedure sketched in Subsection 1.2 yields the recurrence relation

$$y_{n+1} = R_n y_n + S_n \left(\beta_n y_{\lfloor \lambda n \rfloor} + \alpha_n y_{\lfloor \lambda n \rfloor + 1} + \widehat{\beta}_n y_{\lfloor \lambda(n+1) \rfloor} + \widehat{\alpha}_n y_{\lfloor \lambda(n+1) \rfloor + 1} \right), \quad (2.2)$$

$n = 0, 1, \ldots$, where R_n, S_n are given by (1.8) and the relations (1.9) become

$$\alpha_n := (1 - \Theta)(\lambda n - \lfloor \lambda n \rfloor), \qquad \beta_n := 1 - \Theta - \alpha_n,$$

$$\widehat{\alpha}_n := \frac{b_{n+1}}{b_n}\Theta(\lambda(n + 1) - \lfloor \lambda(n + 1) \rfloor), \qquad \widehat{\beta}_n := \frac{b_{n+1}}{b_n}\Theta - \widehat{\alpha}_n.$$

This section presents the result formulating the upper bound of the solutions y_n of (2.2). To describe this asymptotic estimate we introduce the inequality

$$|S_n|(|\beta_n|\varrho_{\lfloor \lambda n \rfloor} + |\alpha_n|\varrho_{\lfloor \lambda n \rfloor + 1} + |\widehat{\beta}_n|\varrho_{\lfloor \lambda(n+1) \rfloor} + |\widehat{\alpha}_n|\varrho_{\lfloor \lambda(n+1) \rfloor + 1}) \leq (1 - |R_n|)\,\varrho_n,$$
$$(2.3)$$

$n = 0, 1, \dots$, which plays the key role in our investigations. To simplify the analysis we further assume that

$$\tilde{S} := \sup_{n \in \mathbb{Z}^+} (|S_n|) < \infty, \qquad \tilde{R} := \sup_{n \in \mathbb{Z}^+} (|R_n|) < 1$$

$$\tilde{\eta} := \sup_{n \in \mathbb{Z}^+} (|\beta_n| + |\alpha_n| + |\widehat{\beta}_n| + |\widehat{\alpha}_n|) < \infty. \tag{2.4}$$

If we put

$$\tilde{\gamma} := \frac{\tilde{S}\tilde{\eta}}{1 - \tilde{R}}, \tag{2.5}$$

then we can present the explicit form of a solution of (2.3).

Proposition 2.1 *Consider the inequality (2.3) and assume that (2.4) holds. Then the sequence*

$$\varrho_n = \begin{cases} \left(n - \frac{1+\lambda}{1-\lambda}\right)^{-\log_\lambda \tilde{\gamma}} & \text{for } \tilde{\gamma} \geq 1, \\ \left(n + \frac{1}{1-\lambda}\right)^{-\log_\lambda \tilde{\gamma}} & \text{for } 0 < \tilde{\gamma} < 1 \end{cases} \tag{2.6}$$

defines the positive solution of (2.3) for all $n \in \mathbb{Z}^+$, $n \geq (1 + \lambda)/(1 - \lambda)$.

Now we can state the main assertion of this subsection formulating the asymptotic estimate of all solutions y_n of (2.2).

Theorem 2.2 *Let y_n be a solution of the delay difference equation (2.2), where we assume the validity of the hypothesis (2.4) and let $\tilde{\gamma}$ be given by (2.5). Then*

$$y_n = O\left(n^{-\log_\lambda \tilde{\gamma}}\right) \qquad \text{as } n \to \infty. \tag{2.7}$$

The significance of the hypothesis (2.4) consists in the fact that it provides the explicit form of a solution ϱ_n of the inequality (2.3) and thus enables us to formulate the effective asymptotic criterion for the Θ-method (2.2). Let us emphasize that some assumptions involved in (2.4) can be omitted: however, the searching for a suitable solution ϱ_n of (2.3) without assuming (2.4) can be generally a difficult task. If we succeed in this process, then the Theorem 2.2 essentially says that

$$y_n = O(\varrho_n) \qquad \text{as } n \to \infty$$

for any solution y_n of (2.2).

The application of the Theorem 2.2 to the significant autonomous case $a(t) \equiv a$ and $b(t) \equiv b$ will be discussed in Section 3.

2.2 THE ASYMPTOTIC ANALYSIS OF THE Θ-METHOD FOR THE EQUATION (2.1) WITH A GENERAL DELAY

We focus on the asymptotic investigation of the Θ-method

$$y_{n+1} = R_n y_n + S_n \left(\beta_n y_{\lfloor \tau_n \rfloor} + \alpha_n y_{\lfloor \tau_n \rfloor + 1} + \widehat{\beta}_n y_{\lfloor \tau_{n+1} \rfloor} + \widehat{\alpha}_n y_{\lfloor \tau_{n+1} \rfloor + 1}\right), \tag{2.8}$$

which originates from the discretization of the differential equation

$$y'(t) = a(t)y(t) + b(t)y(\tau(t)), \quad t \geq t_0, \tag{2.9}$$

involving a general delayed argument (see the equation (1.2)).

The asymptotic investigation of equations (2.9) and (2.8) is less developed than the study of their particular cases (2.1) and (2.2). Among papers related to our discussions on (2.9) we refer to papers [7, 9, 14, 36], where some asymptotic estimations for the equation (2.9) with infinite time lag (i.e. such that $\limsup(t - \tau(t)) = \infty$ as $t \to \infty$) have been performed. The derivation of the corresponding Θ-method discretization (2.8) as well as discussions on the stability analysis of (2.8) belong to the topics of papers [6, 15].

To analyse the asymptotics of (2.8), we have to appropriately modify the key inequality (2.3). As it might be expected, the relation

$$|S_n| \left(|\beta_n| \varrho_{\lfloor \bar{\tau}_n \rfloor} + |\alpha_n| \varrho_{\lfloor \bar{\tau}_n \rfloor + 1} + |\widehat{\beta}_n| \varrho_{\lfloor \bar{\tau}_{n+1} \rfloor} + |\widehat{\alpha}_n| \varrho_{\lfloor \bar{\tau}_{n+1} \rfloor + 1} \right) \leq (1 - |R_n|) \varrho_n, \tag{2.10}$$

$n = 0, 1, \ldots$ seems to be the natural replacement of (2.3). To confirm this conjecture we start with the searching for a suitable solution of (2.10). On this account we consider the auxiliary functional equation

$$\varphi(\tau(t)) = \kappa \varphi(t), \qquad \kappa = \tau'(t_0), \quad t \geq t_0 \tag{2.11}$$

which is usually referred to as the Schröder equation. It is known (see, e.g. [28]) that if $\tau \in C^2([t_0, \infty))$, $\tau(t_0) = t_0$, $\tau(t) < t$ for all $t > t_0$, τ' is positive on $[t_0, \infty)$ and $\tau'(t_0) < 1$, then there exists a unique strictly increasing and continuously differentiable solution φ of (2.11) satisfying $\varphi'(t_0) = 1$. This solution is given by the formula

$$\varphi(t) = \lim_{n \to \infty} \kappa^{-n}(\tau^n(t) - t_0), \qquad t \geq t_0, \tag{2.12}$$

where τ^n means the n-th iterate of τ. In the sequel we mention a slightly modified version of this result, where further condition on τ (namely τ' nonincreasing) is imposed to ensure some additional properties of φ. We utilize these properties in the proof of the main result of this section.

Proposition 2.3 *Let $\tau \in C^2([t_0, \infty))$ be such that $\tau(t_0) = t_0$, $\tau(t) < t$ for all $t > t_0$, τ' is positive and nonincreasing on $[t_0, \infty)$ and $\tau'(t_0) < 1$. Then the function φ defined by (2.12) is the solution of (2.11) such that φ' is positive, continuous and nonincreasing on $[t_0, \infty)$ and, furthermore, $\varphi'(t)/\varphi(t) \leq 1/(t - t_0)$ for all $t > t_0$.*

Throughout this subsection we shall assume that all the assumptions imposed on τ in the Proposition 2.3 are satisfied and φ is the function defined by (2.12) with the properties guaranteed by the Proposition 2.3. Then we consider the differential equation

(2.9), its Θ-method discretization (2.8) and the inequality (2.10). To formulate the upper bound of the solutions of (2.8) it is necessary to present the exact form of the solutions of (2.10).

Proposition 2.4 *Consider the inequality (2.10) and assume that (2.4) holds. Further, let $t^* \geq t_0$ be a (unique) real root of the equation $t - \tau(t + h) = h$ and let $k^* = \lfloor (t^* - t_0)/h \rfloor + 1$. Then*

$$\varrho_n = \begin{cases} (\varphi(t_0 + (n - k^*)h))^{-\log_\kappa \tilde{\gamma}} & \text{for } \tilde{\gamma} \geq 1, \\ (\varphi(t_0 + (n + k^*)h))^{-\log_\kappa \tilde{\gamma}} & \text{for } 0 < \tilde{\gamma} < 1, \end{cases} \quad (2.13)$$

where $\tilde{\gamma}$, $\tilde{\eta}$ are given by (2.5) and (2.4) respectively, defines the solution of (2.10). Moreover, if $\tilde{\gamma} \geq 1$, then $\Delta \varrho_n$ is nonnegative and if $0 < \tilde{\gamma} < 1$, then $\Delta \varrho_n$ is negative and nondecreasing.

Remark 2.5 *The sequence (2.13) is defined for all $n \geq k^*$ provided $\tilde{\gamma} \geq 1$. If $0 < \tilde{\gamma} < 1$, then ϱ_n defines the solution of (2.10) for all $n \geq 0$.*

Now we can formulate the following generalization of the Theorem 2.2.

Theorem 2.6 *Let y_n be a solution of (2.8), where we assume the validity of the hypothesis (2.4), let $\tilde{\gamma}$ be given by (2.5) and let $\kappa = \tau'(t_0)$. Then*

$$y_n = O\left((\varphi(n))^{-\log_\kappa \tilde{\gamma}}\right) \qquad \text{as } n \to \infty, \quad (2.14)$$

where φ is given by (2.12).

We can verify that the Theorem 2.6 actually represents the direct generalization of the Theorem 2.2. Indeed, if $\tau(t) = \lambda t$, $0 < \lambda < 1$, $t \geq 0$, then all the assumptions of the Proposition 2.3 are satisfied and the corresponding Schröder equation

$$\varphi(\lambda t) = \lambda \varphi(t), \qquad t \geq 0$$

admits the identity function as the required solution. Now obviously the asymptotic property (2.14) becomes (2.7).

To illustrate the applicability of the Theorem 2.6 also to other types of delays we consider the differential equation (2.9) with the power delayed argument in the form

$$y'(t) = a(t)y(t) + b(t)y(t^\omega), \quad t \geq 1, \quad (2.15)$$

where $0 < \omega < 1$ is a real scalar and a, b are nonzero continuous functions on $[1, \infty)$. Considering the Θ-method discretization of (2.15) based on the formula (2.8) we can present the following consequence of the Theorem 2.6.

Corollary 2.7 *Let y_n be a solution of the Θ-method discretization of (2.15), where we assume the validity of the hypothesis (2.4) and let $\tilde{\gamma}$ be given by (2.5). Then*

$$y_n = O\left((\log n)^{-\log_\omega \tilde{\gamma}}\right) \qquad \text{as } n \to \infty.$$

For other results discussing this type of asymptotics of differential equations with a power deviating argument we refer to the papers [36] and [13].

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2.3 THE ASYMPTOTIC ANALYSIS OF THE Θ-METHOD FOR THE EQUATION (2.1) WITH SEVERAL DELAYS

In this subsection, we discuss the numerical properties of the equation (2.1) with several proportional delays. For the sake of simplicity we consider the corresponding equation with constant coefficients. The extension to the nonautonomous case can be easily done via the modified technique utilized in the proof of the corresponding result in Subsection 2.1.

We consider the DDE

$$y'(t) = ay(t) + \sum_{i=1}^{k} b_i y(\lambda_i t), \quad t \geq 0, \tag{2.16}$$

where a, $b_i \neq 0$ are complex scalars, $0 < \lambda_i < 1$ are real scalars, $i \in \{1, 2, \ldots, k\}$. We focus on delay difference equations arising from (2.16) by use of the Θ-method discretization.

Using the procedure analogical with the procedure of derivation of (1.7) we arrive at

$$y_{n+1} = Ry_n + \sum_{i=1}^{k} S_i \left(\beta_{n,i} y_{\lfloor \lambda_i n \rfloor} + \alpha_{n,i} y_{\lfloor \lambda_i n \rfloor + 1} + \widehat{\beta}_{n,i} y_{\lfloor \lambda_i(n+1) \rfloor} + \widehat{\alpha}_{n,i} y_{\lfloor \lambda_i(n+1) \rfloor + 1} \right), \tag{2.17}$$

$n = 0, 1, \ldots$, where $y_n \approx y(nh)$, $h > 0$ is the stepsize,

$$R := \frac{1 + (1-\Theta)ha}{1 - \Theta ha}, \qquad S_i := \frac{hb_i}{1 - \Theta ha}$$

and

$$\alpha_{n,i} := (1-\Theta)(\lambda_i n - \lfloor \lambda_i n \rfloor), \; \beta_{n,i} := 1 - \Theta - \alpha_{n,i},$$

$$\widehat{\alpha}_{n,i} := \Theta(\lambda_i(n+1) - \lfloor \lambda_i(n+1) \rfloor), \; \widehat{\beta}_{n,i} := \Theta - \widehat{\alpha}_{n,i}.$$

Now we present the inequality which is useful in our further calculations. It is analogous to (2.3) and has the form:

$$\sum_{i=1}^{k} |S_i| \left(|\beta_{n,i}| \varrho_{\lfloor \lambda_i n \rfloor} + |\alpha_{n,i}| \varrho_{\lfloor \lambda_i n \rfloor + 1} + |\widehat{\beta}_{n,i}| \varrho_{\lfloor \lambda_i(n+1) \rfloor} + |\widehat{\alpha}_{n,i}| \varrho_{\lfloor \lambda_i(n+1) \rfloor + 1} \right)$$

$$\leq (1 - |R|)\varrho_n, \tag{2.18}$$

$n = 0, 1, \ldots$. Assuming

$$|R| < 1 \tag{2.19}$$

we can formulate the following assertion.

Lemma 2.8 *Let (2.19) hold. Then the sequence*

$$\varrho_n := \begin{cases} \left(n - \frac{1+\lambda}{1-\lambda}\right)^{-\log_\lambda \tilde{\gamma}} & \text{for } \tilde{\gamma} \geq 1, \\ \left(n + \frac{1}{1-\lambda}\right)^{-\log_\lambda \tilde{\gamma}} & \text{for } 0 < \tilde{\gamma} < 1, \end{cases}$$

where

$$\lambda := \begin{cases} \max(\lambda_1, \lambda_2, \ldots \lambda_k) & \text{for } \tilde{\gamma} \geq 1, \\ \min(\lambda_1, \lambda_2, \ldots \lambda_k) & \text{for } 0 < \tilde{\gamma} < 1 \end{cases} \tag{2.20}$$

and

$$\tilde{\gamma} := \frac{\sum_{i=1}^k |S_i|}{1 - |R|},$$

is a solution of the inequality (2.18).

The main result of this subsection is the following

Theorem 2.9 *Let y_n be a solution of (2.17), where $|R| < 1$, $S_i \neq 0$ and $0 < \lambda_i < 1$ for all $i \in \{1, 2, \ldots, k\}$. Further let λ be given by (2.20). Then*

$$y_n = O\left(n^{-\log_\lambda \tilde{\gamma}}\right) \qquad \text{as } n \to \infty, \qquad \tilde{\gamma} := \frac{\sum_{i=1}^k |S_i|}{1 - |R|}.$$

2.4 THE ASYMPTOTIC ANALYSIS OF THE Θ-METHOD FOR THE EQUATION (2.1) WITH A FORCING TERM

We extend the problem of the study of numerical discretization of (2.1) to the non-homogenous case. We consider the equation

$$y'(t) = ay(t) + by(\lambda t) + f(t), \quad t \geq 0, \tag{2.21}$$

where $a, b \neq 0$ are complex numbers, $0 < \lambda < 1$ is a real number and f is a complex-valued function. The generalization to the case $a = a(t)$, $b = b(t)$ is analogous to procedures utilized in the proof of the Theorem 2.2.

The corresponding Θ-method discretization of (2.21) arises as a simple modification of (2.2) in the form

$$\begin{aligned} y_{n+1} = &\ Ry_n + S\left(\beta_n y_{\lfloor \lambda n \rfloor} + \alpha_n y_{\lfloor \lambda n \rfloor + 1} + \widehat{\beta}_n y_{\lfloor \lambda(n+1) \rfloor} + \widehat{\alpha}_n y_{\lfloor \lambda(n+1) \rfloor + 1}\right) \\ &+ \frac{(1-\Theta)hf_n + \Theta h f_{n+1}}{1 - \Theta ah}, \quad n = 0, 1, \ldots, \end{aligned} \tag{2.22}$$

where

$$R := \frac{1 + (1-\Theta)ha}{1 - \Theta ha}, \qquad S := \frac{bh}{1 - \Theta ha}$$

and

$$\alpha_n := (1-\Theta)(\lambda n - \lfloor \lambda n \rfloor), \qquad \beta_n := 1 - \Theta - \alpha_n,$$

$$\widehat{\alpha}_n := \Theta(\lambda(n+1) - \lfloor \lambda(n+1) \rfloor), \qquad \widehat{\beta}_n := \Theta - \widehat{\alpha}_n.$$

The useful inequality (2.3) becomes

$$|S| \left(|\beta_n| \varrho_{\lfloor \lambda n \rfloor} + |\alpha_n| \varrho_{\lfloor \lambda n \rfloor + 1} + |\widehat{\beta}_n| \varrho_{\lfloor \lambda(n+1) \rfloor} + |\widehat{\alpha}_n| \varrho_{\lfloor \lambda(n+1) \rfloor + 1} \right) \le (1 - |R|) \varrho_n, \tag{2.23}$$

$n = 0, 1, \ldots$. To find its solution ϱ_n, we have to specify the meaning of the symbols $\tilde{S}, \tilde{\eta}$ and $\tilde{R}$ occurring in the hypothesis (2.4). Obviously $\tilde{S} = |S|$, $\tilde{\eta} = 1$ and $\tilde{R} = |R|$, where we assume $|R| < 1$. Thus the solution of (2.23) is given by (2.6), i.e.

$$\varrho_n = \begin{cases} \left(n - \frac{1+\lambda}{1-\lambda} \right)^{-\log_\lambda \tilde{\gamma}} & \text{for } \tilde{\gamma} \ge 1, \\ \left(n + \frac{1}{1-\lambda} \right)^{-\log_\lambda \tilde{\gamma}} & \text{for } 0 < \tilde{\gamma} < 1, \end{cases}$$

where $\tilde{\gamma}$ in (2.5) becomes

$$\tilde{\gamma} := \frac{|S|}{1 - |R|}.$$

Now we present the main result of this section. We introduce the following assumption

$$f_n = O(n^\nu) \qquad \text{as } n \to \infty \tag{2.24}$$

valid for a suitable real scalar $\nu < -\log_\lambda \tilde{\gamma}$. Then we can formulate the following

Theorem 2.10 *Let y_n be a solution of (2.22), where $|R| < 1$, $S \ne 0$ and $0 < \lambda < 1$. Further let (2.24) hold. Then*

$$y_n = O\left(n^{-\log_\lambda \tilde{\gamma}} \right) \qquad \text{as } n \to \infty, \qquad \tilde{\gamma} = \frac{|S|}{1 - |R|}.$$

2.5 STABILITY ANALYSIS OF THE EULER FORMULA FOR THE PANTOGRAPH EQUATION

In this subsection, we analyse a change of the qualitative behaviour of the numerical solution of the scalar pantograph equation

$$y'(t) = ay(t) + by(\lambda t), \quad 0 < \lambda < 1 \tag{2.25}$$

which is based on the implicit Euler discretization in the form

$$y_{n+1} = \mathcal{R} y_n + \mathcal{S} y_{\lfloor \lambda(n+1) \rfloor}, \qquad n = 0, 1, 2, \ldots, \tag{2.26}$$

where

$$\mathcal{R} := \frac{1}{1 - ah}, \quad \mathcal{S} := \frac{bh}{1 - ah}, \tag{2.27}$$

$h > 0$ is the stepsize. The derivation of this equation is sketched in the Subsection 1.2. Indeed, the formula (2.26) originates from (1.5) by use of $\tau_n = \lambda n h$ and $y^h(\tau_{n+1}) = y_{\lfloor \lambda(n+1) \rfloor}$. Note also that there is some analogy between (2.27) and (1.8). The relation

(2.27) corresponds to the case of the constant coefficients and the choice $\Theta = 1$ in (1.8).

Assume that a, b are real scalars, $|a| + b < 0$ and $0 < 1 - \lambda << 1$. Then the numerical solution of (2.25) has a tendency to tend to zero solution, but after reaching a certain critical index this tendency vanishes and the solution is "blowing up". Our next investigation is inspired by the paper [34], where this phenomenon (familiarly referred to as the numerical nightmare) has been investigated using the explicit Euler method. In the connection with the studied problem we can mention the other useful sources [1-3,10,17,30,31]

The difference equation (2.26) is of an increasing order, but for

$$n \in I_m := \left(\frac{m + \lambda - 1}{1 - \lambda}, \frac{m + \lambda}{1 - \lambda} \right], \quad m \in \mathbb{Z}^+$$

the order is fixed to the value m. Then we can rewrite the equation (2.26) as a three-term difference equation

$$y_{n+1} - \mathcal{R} y_n - \mathcal{S} y_{n-m} = 0, \qquad n \in I_m, \tag{2.28}$$

where $\mathcal{R}, \mathcal{S}$ are given by (2.27)

In the sequel we find the maximal order $m^\star$ of the difference equation (2.28), where the condition for the asymptotic stability of its solutions is still guaranteed, but starting from $m = m^\star + 1$ is no more valid.

It is well-known that the solution of linear difference equation (2.28) is asymptotically stable if and only if all the zeros of the corresponding characteristic polynomial

$$P(\mu) = \mu^{m+1} - \mathcal{R} \mu^m - \mathcal{S} \tag{2.29}$$

lie inside a unit disk. To analyse (2.29), we have used the Schur-Cohn criterion (see, e.g. [10]), which enables us to formulate the following result:

Theorem 2.11 *Let $|a| + b < 0$, $h < 1/(a + |b|)$ and let the values $\mathcal{R}, \mathcal{S}$ be given by (2.27). Then all the roots of the polynomial (2.29) lie inside the unit disk if and only if*

$$m \le m^\star := \begin{cases} \lfloor m_0 \rfloor, & m_0 \notin \mathbb{Z}^+, \\ m_0 - 1, & m_0 \in \mathbb{Z}^+, \end{cases}$$

where

$$m_0 = 2 \arctan \left(-\frac{\sqrt{4\mathcal{R}^2 - (1 + \mathcal{R}^2 - \mathcal{S}^2)^2}}{1 - \mathcal{S}^2 - \mathcal{R}^2 + 2\mathcal{R}\mathcal{S}} \right) \Bigg/ \arcsin \frac{\sqrt{4\mathcal{R}^2 - (1 + \mathcal{R}^2 - \mathcal{S}^2)^2}}{2\mathcal{R}}. \tag{2.30}$$

Furthermore,

$$\lim_{h \to 0} m^* h = \frac{2}{(b^2 - a^2)^{1/2}} \arctan \frac{(b^2 - a^2)^{1/2}}{a - b}. \tag{2.31}$$

Hence, under the assumptions introduced in the Theorem 2.11, the solution of (2.26) has a tendency to reach the zero solution for $n \leq n^\star = \left\lfloor \frac{m^\star + \lambda}{1 - \lambda} \right\rfloor$. For $n > n^\star$ this tendency vanishes.

The presented result can be applied also for the explicit Euler discretization of (2.25) in the form

$$y_{n+1} - \mathcal{R}y_n - \mathcal{S}y_{\lfloor \lambda n \rfloor} = 0.$$

In this case it is enough to consider $\mathcal{R} = 1 + ah$ and $\mathcal{S} = -bh$ in (2.28).

We emphasize that the above mentioned result improves the result derived in [34] for the explicit Euler discretization of the pantograph equation. While the relation (2.31) is reported in [34], the relation (2.30) is our contribution to this discussion. We emphasize that in our case the expression for $m^\star$ does not depend on the sign of a. Moreover contrary to the corresponding result in [34] we derive the exact expression for m_0. Consequently, we can compute the critical index $n^* = \left\lfloor \frac{m^* + \lambda}{1 - \lambda} \right\rfloor$ exactly.

Now we present the example illustrating the contribution of the Theorem 2.11.

Example 2.12 We consider the initial value problem

$$y'(t) = -0.1y(t) - y(0.99t), \qquad t \geq 0, \qquad y(0) = 1, \qquad (2.32)$$

and its implicit Euler discretization

$$y_{n+1} = \frac{100}{101}y_n - \frac{10}{101}y_{\lfloor 0.99(n+1) \rfloor}, \qquad n = 0, 1, 2, \ldots, \qquad y_0 = 1 \qquad (2.33)$$

with stepsize $h = 0.1$. Now we plot the solution of (2.33). For a better graphic illustration we denote $y^h(t)$ as the linear interpolation of $\{y_n\}_{n=0}^{n=\infty}$, i.e.

$$y^h(t) = \frac{(n+1)h - t}{h}y_n + \frac{t - nh}{h}y_{n+1}, \qquad t \in [nh, (n+1)h], \qquad (2.34)$$

$n = 0, 1, 2, \ldots$ and consider $t \in [0, 1270]$. The Fig. 2.1 illustrates the behaviour of the solution of (2.33). For a better representation of the character of this solution we present the Fig. 2.2. This figure plots the values $(t, \log_{10}(|y^h(t)| + \varepsilon))$ where $\varepsilon = 2.23 \times 10^{-308}$. It follows from the Fig. 2.2 that for $nh \in (100, 300)$ the values of y^h are already less then 10^{-40}. Considering such small values the solution of the problem (2.33) approximates the zero solution and it seems that the calculation could be finished. However if $nh > 300$, the solution increases quickly (in absolute values).

Using the Theorem 2.11 we are able to find the change point

$$n^*h = \left\lfloor \frac{m^* + \lambda}{1 - \lambda} \right\rfloor h = 1699h = 169.9,$$

where the character of this solution changes. Moreover we can find the point

$$t^* = \lim_{h \to 0} \left\lfloor \frac{m^* + \lambda}{1 - \lambda} \right\rfloor h = 167.9,$$

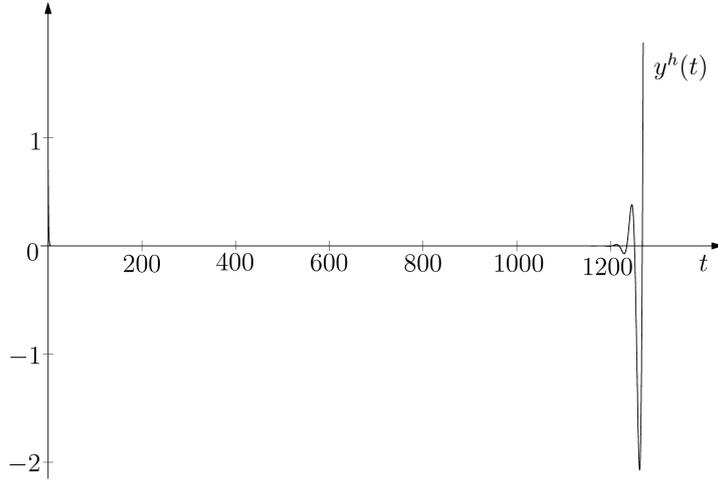where the character of the exact solution of the problem (2.32) changes.
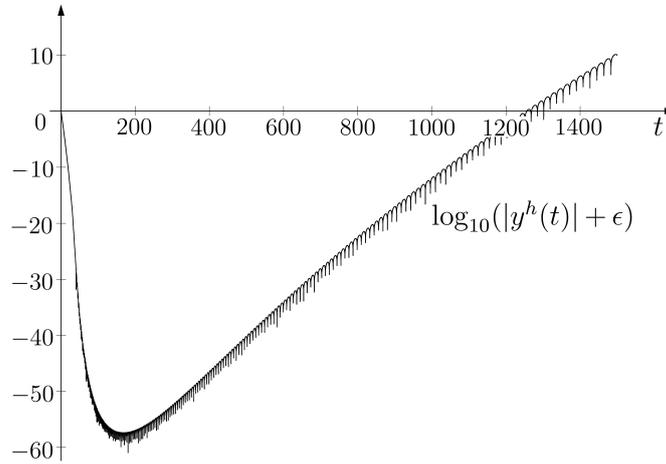
Fig. 2.1: The solution $y^h(t)$.

Fig. 2.2: The solution $y^h(t)$ on the logarithmic scale.

3 SOME CONSEQUENCES AND FINAL REMARKS

In this section, we mention several comparisons and numerical consequences concerning the asymptotic estimates of the exact pantograph equation and its Θ-method discretization. We consider the scalar pantograph equation

$$y'(t) = ay(t) + by(\lambda t), \quad t \geq 0, \tag{3.1}$$

where $0 < \lambda < 1$ and assume that Re $a < 0, b \neq 0$. We note that this equation is a particular case of each of the equations (2.1), (2.9), (2.16) and (2.21) considered in the previous section.

The application of the Θ-method to the equation (3.1) yields the recurrence

$$y_{n+1} = Ry_n + S\left(\beta_n y_{\lfloor \lambda n \rfloor} + \alpha_n y_{\lfloor \lambda n \rfloor + 1} + \widehat{\beta}_n y_{\lfloor \lambda(n+1) \rfloor} + \widehat{\alpha}_n y_{\lfloor \lambda(n+1) \rfloor + 1}\right), \tag{3.2}$$

$n = 0, 1, \ldots$, where R, S are given by

$$R := \frac{1 + (1 - \Theta)ah}{1 - \Theta ah}, \qquad S := \frac{bh}{1 - \Theta ah} \tag{3.3}$$

and

$$\alpha_n := (1 - \Theta)(\lambda n - \lfloor \lambda n \rfloor), \qquad \beta_n := 1 - \Theta - \alpha_n,$$
$$\widehat{\alpha}_n := \Theta(\lambda(n+1) - \lfloor \lambda(n+1) \rfloor), \qquad \widehat{\beta}_n := \Theta - \widehat{\alpha}_n.$$

The aim of the following subsections is to illustrate the contribution of the asymptotic results mentioned in previous sections to the numerical investigation of the equation (3.1) and present comparisons with the known results as well.

3.1 THE ASYMPTOTIC ESTIMATE FOR THE EXACT AND DISCRETIZED PANTOGRAPH EQUATION.

The important theoretical question about numerical approximations is the problem whether the numerical and exact solution admit a related asymptotic behaviour on the unbounded domain. Recall that the qualitative behaviour of the solutions of the exact equation (3.1) is well known (see, e.g. [16, 25, 26]) and can be described as follows:

Theorem 3.1 *Let y be a solution of the equation (3.1), where* $\mathrm{Re}\, a < 0$, $b \neq 0$ *and* $0 < \lambda < 1$. *Then*

$$y(t) = O\left(t^{-\log_\lambda |b/a|}\right) \qquad \text{as } t \to \infty. \tag{3.4}$$

Moreover, if $y(t) = o(t^{-\log_\lambda |b/a|})$ as $t \to \infty$, then y is the zero solution.

In other words, the estimate (3.4) is nonimprovable.

Now we are going to formulate the corresponding discrete estimate following from the Theorem 2.2 (as well as from the Theorem 2.6, the Theorem 2.9 or the Theorem 2.10).

Corollary 3.2 *Let y_n be a solution of the discretization (3.2) with R, S given by (3.3), where*

$$2\mathrm{Re}\, a < (2\Theta - 1)|a|^2 h, \tag{3.5}$$

$b \neq 0$ and $0 < \lambda < 1$. Then

$$y_n = O\left(n^{-\log_\lambda \tilde{\gamma}}\right) \qquad \text{as } n \to \infty, \qquad \tilde{\gamma} = \frac{|b|h}{|1 - \Theta ah| - |1 + (1 - \Theta)ah|}. \tag{3.6}$$

The condition (3.5) seems to be analogical with the condition $\mathrm{Re}\, a < 0$ in the Theorem 3.1. Let $\mathrm{Re}\, a < 0$. Then (3.5) is fulfilled for any $h > 0$ if and only if $1/2 \leq \Theta \leq 1$. Assuming $0 \leq \Theta < 1/2$, the condition (3.5) represents the restriction on the stepsize h and has the form

$$h < \frac{2\mathrm{Re}\, a}{(2\Theta - 1)|a|^2}.$$

The natural question arises, namely what is the relation between the upper bound (3.4) derived in [16, 25, 26] for the exact solution of (3.1) and our upper bound (3.6) derived for its numerical solution.

Answering our question we first consider the case where a is a real constant (b can be complex). Then we can observe that $\tilde{\gamma}$ occurring in (3.6) becomes

$$\tilde{\gamma} = \begin{cases} |b/a| & \text{for } (1 - \Theta)h|a| \leq 1, \\ h|b|/(2 + h|a|(2\Theta - 1)) & \text{for } (1 - \Theta)h|a| > 1. \end{cases}$$

Hence the value $|b/a|$ known from the asymptotic description of the exact scalar pantograph holds for discretization (3.2) with the modest restriction on the stepsize h. In particular, if $\Theta = 1$ (the case of implicit Euler method), both estimates (3.4) and (3.6) coincides for any $h > 0$. This result is a significant extension of the asymptotic stability property of the implicit Euler method. We have shown that this method preserves not only the convergence to zero, but also the decay rate of the exact solution. For related papers concerning these questions we refer to [5, 11, 12, 23, 24, 35].

Now we consider the case, where both parameters a, b are complex. If Im $a \neq 0$, then the previous relation for $\tilde{\gamma}$ is no longer valid and it turns out that $\tilde{\gamma}$ is always greater then $|b/a|$ and $h|b|/(2 + h|a|(2\Theta - 1))$ provided $(1 - \Theta)h|a| \leq 1$ and $(1 - \Theta)h|a| > 1$, respectively.

3.2 THE COMPARISON WITH OTHER ASYMPTOTIC ESTIMATES FOR THE Θ-METHOD DISCRETIZATION OF (3.1)

The asymptotic investigation of the discretized pantograph equation is rare. To our knowledge, the only paper dealing with the asymptotics of the Θ-method discretization is [33]. However, this paper discusses the Θ-method discretization on the quasi-geometric mesh (characterized by the property $\lim_{n \to \infty} h_n = \infty$). Considering the asymptotics of the Θ-method on the uniform mesh, we can mention papers [6] and [29] dealing with the trapezoidal rule and Euler discretization of (3.1), respectively.

To compare the estimate (3.6) with the relevant estimate presented in [6] we need to make some minor modifications. The reason is that the discretization of (3.1) utilized in [6] is slightly different from the formula (3.2). The mentioned discretization has the form

$$y_{n+1} = Ry_n + S\left(\tilde{\beta}_n y_{\lfloor \lambda n \rfloor} + \tilde{\alpha}_n y_{\lfloor \lambda n \rfloor + 1}\right), \tag{3.7}$$

where R, S are given by (3.3) and

$$\tilde{\beta}_n := 1 - \tilde{\alpha}_n, \qquad \tilde{\alpha}_n := \lambda n - \lfloor \lambda n \rfloor + \Theta \lambda.$$

Since the discretization studied in [6] originates from the formula (3.7), we first reformulate the Corollary 3.2 for such a discretization. To perform this, we denote

$$\eta = \eta(\Theta, \lambda) := \sup_{n \in \mathbb{Z}^+} \left(|\tilde{\beta}_n| + |\tilde{\alpha}_n|\right) < \infty.$$

The next lemma yields the explicit form of η and can be found in the particular case $\Theta = 1/2$ in [6, Theorem 6].

Lemma 3.3 *Let $0 < \lambda < 1, 0 \le \Theta \le 1$. Then the function $\eta(\Theta, \lambda)$ has the following values:*

$$\eta(\Theta, \lambda) = \begin{cases} 1, & \lambda = K/L, \ \Theta K \le 1, \ K, L \in \mathbb{Z}^+ \text{are relatively prime,} \\ 1 + 2\Theta\lambda - \frac{2}{L}, & \lambda = K/L, \ \Theta K \ge 1, \\ & K, L \in \mathbb{Z}^+, \ K, L \ne 1 \text{ are relatively prime,} \\ 1 + 2\Theta\lambda, & \lambda \text{ irrational.} \end{cases}$$

$$(3.8)$$

Using this we can reformulate the Corollary 3.2 for the discretization (3.7) as follows:

Corollary 3.4 *Let y_n be a solution of the discretization (3.7) with R, S given by (3.3), where (3.5) holds, let $b \ne 0$ and $0 < \lambda < 1$. Then*

$$y_n = O\left(n^{-\log_\lambda \gamma^*}\right) \qquad \text{as } n \to \infty, \qquad \gamma^* = \frac{|S|\eta}{1 - |R|}, \qquad (3.9)$$

where $\eta = \eta(\Theta, \lambda)$ is given by (3.8).

Note that the estimate (3.9) can be weaker than the estimate (3.6) because the value of η can be grater then one. It follows from the Lemma 3.3 that if $\Theta K \le 1$ and $\lambda = K/L$ where $K, L \in \mathbb{Z}^+$ are relatively prime, then $\eta(\Theta, \lambda) = 1$. In this case the asymptotic estimates (3.9) and (3.6) coincide.

Now we can discuss the main goal of this subsection, namely the comparison of our estimate (3.9) with the relevant result from [6] describing the asymptotics of (3.7) for $\Theta = 1/2$. On this account, we introduce the notation.

$$\gamma := |R| + \eta|S|,$$

where $\eta = \eta(1/2, \lambda)$ is given by (3.8). Now we can read [6, Theorem 5] as follows:

Corollary 3.5 *Let y_n be a solution of the discretization (3.7) with R, S given by (3.3), where $\operatorname{Re} a < 0$, $b \ne 0$ and $0 < \lambda < 1$. Further, let $\gamma \le 1$. Then*

$$y_n = O\left(n^{-\log_\lambda \gamma}\right) \qquad \text{as } n \to \infty, \qquad \gamma = |R| + \eta|S|. \qquad (3.10)$$

Let us emphasize that this result have been derived in a more general case when the equation (3.1) and its discretization (3.7) involve the neutral term. On the other hand, the Corollary 3.5 discusses only the case $\gamma \le 1$ and $\Theta = 1/2$.

Now we can easily compare our relation (3.9) with the asymptotic estimate (3.10) derived in [6, Theorem 5] under the assumption $\gamma \le 1$. Considering this assumption we get

$$\gamma^* = \frac{|S|\eta}{1 - |R|} \le |R| + \eta|S| = \gamma,$$

where the equality sign between γ^* and γ occurs if and only if $\gamma = 1$. In particular, substituting the values R and S from (3.3) into the inequality $\gamma^* \le 1$ we can easily check that the solution of (3.7) is bounded if

$$\operatorname{Re} a < 0, \qquad \eta|b| + \frac{4\operatorname{Re} a}{|2 + ha| + |2 - ha|} \le 0,$$

which is the same stability condition as the one derived in [6] and [15] by use of the inequality $\gamma \le 1$. However, considering the asymptotic stable case ($\gamma < 1$), the formula (3.9) provides a stronger asymptotic estimate than the formula (3.10) yields. More precisely, both formulae affirm the algebraic decay of y_n, but the asymptotic property (3.9) guarantees a stronger decay rate.

The next example illustrates the previous comparison. We specify the parameters a, b in (3.1) and discuss the upper bounds for (3.7) with the stepsize $h = 0.05$ and $\Theta = 1/2$.

Example 3.6 We choose $a = -1$ and $b = -0.5$ in (3.1) and consider the initial value problem

$$y'(t) = -y(t) - 0.5\, y(t/2), \qquad t \ge 0, \qquad y(0) = 1. \tag{3.11}$$

Then the corresponding discretization (3.7) becomes

$$y_{n+1} = \begin{cases} Ry_n + \frac{1}{4}S(y_{\frac{n-1}{2}} + 3y_{\frac{n+1}{2}}), & \text{n is positive and odd,} \\ Ry_n + \frac{1}{4}S(3y_{\frac{n}{2}} + y_{\frac{n}{2}+1}), & \text{n is nonnegative and even,} \end{cases} \tag{3.12}$$

$$y_0 = 1,$$

where the symbols R and S have been introduced in (3.3). Then the asymptotic estimates (3.10) and (3.9) become

$$y_n = O\left(n^{-0.0356}\right) \qquad \text{as } n \to \infty \tag{3.13}$$

and

$$y_n = O\left(n^{-1}\right) \qquad \text{as } n \to \infty, \tag{3.14}$$

respectively. Our next intention is the computational presentation of the estimate (3.14) and its graphic comparisons with the estimate (3.13) as well as with the real behaviour of the discretization (3.12). To make the estimate (3.14) more applicable from the computational viewpoint it is necessary to specify the O-term in (3.14), i.e. determine a constant $L_1 > 0$ such that

$$|y_n| \le L_1 n^{-1} \qquad \text{for all } n \text{ large enough.}$$

After some calculations we can precise the upper bound (3.14) for the solution y_n of (3.12) in the form

$$|y_n| \le 3.3834\, n^{-1} \qquad \text{for all } n \text{ large enough}$$

(more precisely, for $n = 150, 151, \dots$).

Now we consider the estimate (3.13). To obtain a sharp majorant constant we choose such L_2 that the values of y_n and its estimate $L_2 n^{-0.0356}$ coincide for $n = 150$. This implies

$$|y_n| \le 0.0144 \, n^{-0.0356} \qquad \text{for all } n \text{ large enough}$$

Now the gap between both asymptotic results can be simply illustrated by the following figure. We use here (2.34) and consider and consider $t \in [7.5, 400]$ (note that the left-end point $t = 7.5$ corresponds to the relation $t = 150 \, h$). The Fig. 3.1 plots the numerical solution y^h of (3.11) as well as its upper bounds $g(t) = 3.3834 \, h t^{-1} \approx 0.1692 \, t^{-1}$ and $f(t) = 0.0143 \, h^{0.0356} t^{-0.0356} \approx 0.0129 \, t^{-0.0356}$.
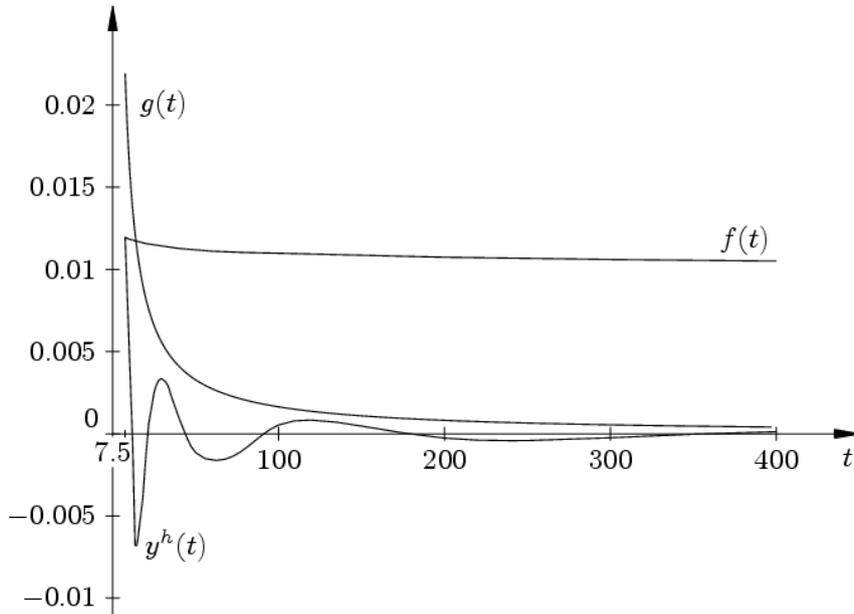

Fig. 3.1: The solution y^h and its upper bounds

3.3 CONCLUSION

The aim of this thesis was to present some qualitative properties of delay difference equations and their applications to the numerical analysis of given DDEs. A special attention was paid to the scalar pantograph equation

$$y'(t) = ay(t) + by(\lambda t), \quad 0 < \lambda < 1, \qquad t \ge 0$$

and its various modifications. We described the qualitative (mostly asymptotic and stability) properties of its Θ-method discretization (3.2). We compared these properties with the behaviour of the exact (differential) pantograph equation, which enabled us to formulate some numerical consequences of these qualitative results. Some comparisons with the known relevant results have been done and some illustrating examples have been involved as well. Furthermore, using the Schur-Cohn criterion on the

asymptotic stability of the solutions we analysed a family of three-term difference equations and discussed a specific stability phenomenon for the Euler discretization of the pantograph equation.

The common investigation of the properties of the studied differential equations and its difference analogies (obtained via a suitable numerical discretization) was the unified viewpoint of the considerations and results mentioned in this thesis. This is an important aspect of the modern theory of dynamic equations on time scales (see [4] and [5]). Therefore, this theory can motivate us to other extensions of our previous results.

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1. **Publikace v časopisech s impakt faktorem**

    [ČJ]   J. Čermák and J. Jánský, *On the asymptotics of the trapezoidal rule for the pantograph equation,* Math. Comp. **78** (2009), 2107–2126.

2. **Publikace v ostatních časopisech indexovaných v Mathematical Reviews a Zentralblatt MATH**

    [J1]   J. Jánský, *Some numerical investigations of differential equations with several proprotional delays*, Stud. Univ. Žilina Math. Ser. **23** (2009), 59–66.

    [JK]   J. Jánský and P. Kundrát, *The stability analysis of a discretized pantograph equation*, Math. Bohem. to appear.

3. **Ostatní publikace**

    [J2]   J. Jánský, *Difference equation with a proportional delay*, Sborník 5. konference o matematice a fyzice na VŠT, (2007), 141-146.

[J3]   J. Jánský, *Typy diskretizace rovnice pantografu*, Sborník z 16 semináře Moderní matematické metody v inženýrství, (2007), 105-109.

[J4]   J. Jánský, *The Θ-methods for the delay differential equations*, Sborník Aplimat 2008, (2008), 165-174.

ABSTRACT

This thesis discusses the qualitative properties of some delay difference equations. These equations originate from the Θ-method discretizations of differential equations with a delayed argument. Our purpose is to analyse the asymptotic properties of these numerical solutions and formulate their upper bounds. We also discuss stability properties of the studied discretizations. Several illustrating examples and comparisons with the known results are presented as well.