WEAKLY DELAYED SYSTEMS IN $\mathbb{R}^3$

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Abstract: The paper is concerned with a linear discrete system with delay

$$x(k+1) = Ax(k) + Bx(k-m),\ k = 0,1,\dots,$$

in $\mathbb{R}^3$. It is assumed that the system is weakly delayed. For one of the possible Jordan forms solution of an arbitrary initial problem is given.

Keywords: Discrete system, weakly delayed system, linear system, initial problem.

1 INTRODUCTION

In the paper, a linear system of difference equations

$$x(k+1) = Ax(k) + Bx(k-m),\ k = 0,1,\dots \tag{1}$$

is considered. In (1), $A = (a_{ij})_{i,j=1}^3$, $B = (b_{ij})_{i,j=1}^3$ are 3×3 constant matrices and $m \geq 1$ is a natural number.

The system (1) is assumed to be weakly delayed. The following definitions explain this notion.

Definition 1 *System* (1) *is called weakly delayed if the characteristic equations for* (1) *and for the system without delay* $x(k+1) = Ax(k)$ *have identical roots, that is, if, for every* $\lambda \in \mathbb{C} \setminus \{0\}$,

$$\det\left(A + \lambda^{-m}B - \lambda E\right) = \det\left(A - \lambda E\right),$$

where E *is a* 3×3 *unit matrix.*

Let the matrix A have eigenvalues $\lambda = \lambda_1$ with algebraic multiplicity 1 and $\lambda = \lambda_2$ with algebraic multiplicity 2 and with geometrical multiplicity 2. I.e. the Jordan form of A is

$$A:\ \Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}. \tag{2}$$

By Definition 1, the following are the conditions for the system to be weakly delayed.

Theorem 1 ([4]) *System* (1) *is weakly delayed iff*

$$b_{11} = 0, \tag{3}$$

$$b_{22} + b_{33} = 0, \tag{4}$$

$$b_{12}b_{21} + b_{13}b_{31} = 0, \tag{5}$$

$$b_{22}b_{33} - b_{23}b_{32} = 0, \tag{6}$$

$$b_{12}b_{23}b_{31} + b_{13}b_{21}b_{32} - b_{13}b_{22}b_{31} - b_{12}b_{21}b_{33} = 0. \tag{7}$$

In the paper, we give an explicit formula for the solution of the initial problem

$$x(0) = x_0 = \begin{pmatrix} x_{0,1} \\ x_{0,2} \\ x_{0,3} \end{pmatrix}, \dots, \; x(m) = x_m = \begin{pmatrix} x_{m,1} \\ x_{m,2} \\ x_{m,3} \end{pmatrix} \tag{8}$$

to system (1) where $x_{i,j}$, $i = 0, \dots, m$, $j = 1,2,3$, are real constants.

2 RESULTS

First, we transform (1) into a system without delay. Instead of the system (1), we investigated an equivalent system

$$x(k+1) = \Lambda x(k) + Bx(k-m) \tag{9}$$

with initial data (8). Define new dependent function $z^1, \dots, z^m$ by

$$\begin{aligned}
z^1(k) &= x(k-1), & &\Rightarrow & z^1(k+1) &= x(k),\\
z^2(k) &= x(k-2), & &\Rightarrow & z^2(k+1) &= x(k-1),\\
&\vdots \\
z^m(k) &= x(k-m), & &\Rightarrow & z^m(k+1) &= x(k-(m-1))
\end{aligned}$$

and, instead of (9), consider a system without delay

$$\begin{array}{rcllllll}
x(k+1) & = & \Lambda x(k) & & & & & +Bz^m(k),\\
z^1(k+1) & = & x(k), & & & & &\\
z^2(k+1) & = & & z^1(k), & & & &\\
z^3(k+1) & = & & & z^2(k), & & &\\
\vdots & & & & & \ddots & &\\
z^m(k+1) & = & & & & & z^{m-1}(k). &
\end{array}$$

Set

$$\begin{aligned}
y_i(k) &:= x_i(k), \; i = 1,2,3,\\
y_{j+3}(k) &:= z_j^1(k), \; j = 1,2,3,\\
y_{j+6}(k) &:= z_j^2(k), \; j = 1,2,3,\\
&\vdots\\
y_{j+3m}(k) &:= z_j^m(k), \; j = 1,2,3.
\end{aligned}$$

Then,

$$y(k+1) = \mathcal{A}y(k), \; k \geq 0 \tag{10}$$

where

$$\mathcal{A} = \begin{pmatrix} \Lambda & \Theta & \dots & \Theta & B \\ E & \Theta & \dots & \Theta & \Theta \\ \Theta & E & \dots & \Theta & \Theta \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Theta & \Theta & \dots & E & \Theta \end{pmatrix},$$

Θ is a 3×3 zero matrix and $y(k) = (y_1(k), \dots, y_{3m+3}(k))^T$.

2.1 Solution of the system (10)

The transformed initial conditions for the system (10) are $y(0) = y_0 = (x_0, \dots, x_m)^T$, see formula (8).

Transform (10) by formulas $y(k) = Sw(k)$ assuming that S is a regular transient matrix and $w(k)$ is a new dependent $3(m+1)$-dimensional vector into a system with a matrix of the Jordan form. We get

$$Sw(k+1) = \mathcal{A}Sw(k)$$

or

$$w(k+1) = \gamma w(k) \tag{11}$$

where

$$\gamma = S^{-1}\mathcal{A}S.$$

The initial data for (11) are

$$w(0) = S^{-1}y(0).$$

Then, the solution of the system (11) is

$$w(k) = \gamma^k w(0), \;\; k = 1, 2, 3, \dots \, . \tag{12}$$

We use the following auxiliary result.

Theorem 2 *Let a matrix A be of the type* (2) *and let the entries of a matrix B satisfy* (3)–(7)*. Then, the eigenvalues $\mu_i, i = 1, \dots, 3m+3$ of the matrix $\mathcal{A}$ are $\mu_1 = \lambda_1$, $\mu_2 = \mu_3 = \lambda_2$, $\mu_4 = \mu_5 = \dots = \mu_{3m+3} = 0$.*

The proof of the theorem is based on well-known properties of determinants.

When powers γ^k, $k = 1, 2, 3, \dots$ are computed, it is necessary to take into account the geometrical multiplicity of the zero eigenvalue of matrix B.

2.1.1 Case I - the geometrical multiplicity of B equals 1

Due to Theorem 2, we can assume that the transition matrix S is such that we get

$$\gamma = \gamma_1 = \begin{pmatrix} \Lambda_B^I & \Theta & \dots & \Theta & \Theta \\ \Theta & E & \dots & \Theta & \Theta \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Theta & \Theta & \dots & E & \Theta \\ \Theta & \Theta & \dots & \Theta & \Lambda \end{pmatrix}, \text{ where } \Lambda_B^I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

is the Jordan form of the matrix B. Obviously, the following are the powers of γ:

$$\gamma_1^2 = \begin{pmatrix} (\Lambda_B^I)^2 & \Theta & \dots & \Theta & \Theta \\ \Theta & E & \dots & \Theta & \Theta \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Theta & \Theta & \dots & E & \Theta \\ \Theta & \Theta & \dots & \Theta & \Lambda^2 \end{pmatrix}, \text{ where } (\Lambda_B^I)^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \Lambda^2 = \begin{pmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 2\lambda_2^1 \\ 0 & 0 & \lambda_3^2 \end{pmatrix},$$

and for $k \geq 3$

$$\gamma_1^k = \begin{pmatrix} \Theta & \Theta & \dots & \Theta & \Theta \\ \Theta & E & \dots & \Theta & \Theta \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Theta & \Theta & \dots & E & \Theta \\ \Theta & \Theta & \dots & \Theta & \Lambda^k \end{pmatrix}, \text{ where } \Lambda^k = \begin{pmatrix} \lambda_1^k & 0 & 0 \\ 0 & \lambda_2^k & k\lambda_2^{k-1} \\ 0 & 0 & \lambda_3^k \end{pmatrix}.$$

2.1.2 Case II - the geometrical multiplicity of B equals 2

Due to Theorem 2, we can assume that the transition matrix S is such that

$$\gamma = \gamma_2 = \begin{pmatrix} \Lambda_B^{II} & \Theta & \dots & \Theta & \Theta \\ \Theta & E & \dots & \Theta & \Theta \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Theta & \Theta & \dots & E & \Theta \\ \Theta & \Theta & \dots & \Theta & \Lambda \end{pmatrix}, \text{ where } \Lambda_B^{II} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since

$$(\Lambda_B^{II})^2 = \Theta,$$

the powers γ_2^k, for $k \geq 2$, are

$$\gamma_2^k = \begin{pmatrix} \Theta & \Theta & \dots & \Theta & \Theta \\ \Theta & E & \dots & \Theta & \Theta \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Theta & \Theta & \dots & E & \Theta \\ \Theta & \Theta & \dots & \Theta & \Lambda^k \end{pmatrix}.$$

2.2 Solution of the problem (1), (8)

The solution of (10) is

$$y(k) = Sw(k) = S\gamma_i^k w(0), \ k = 1, 2, 3, \dots$$

where $i = 1$ if the geometrical multiplicity of the zero eigenvalue of B equals 1 and $i = 2$ if the geometrical multiplicity of the zero eigenvalue of B equals 2. Using an auxiliary matrix

$$Q = (E, \underbrace{\Theta, \dots, \Theta}_{m}),$$

we can write the solution of the initial problem (1), (8) in the form

$$x(k) = QS\gamma_i^k w(0), \ i = 1, 2, \ k = 1, 2, 3, \dots, \tag{13}$$

where

$$w(0) = S^{-1} x(0). \tag{14}$$

Therefore we can formulate a theorem summarizing above computations.

Theorem 3 *Let the matrix A have the form* (2) *with one double real root* λ_1, $\lambda_2 = \lambda_3$, *let elements of the matrix B satisfy* (3)– (7). *Then, the solution of the initial problem* (1), (8) *is given by formula* (13) *where* $i = 1$ *if the geometrical multiplicity of the zero eigenvalue of B equals* 1 *and* $i = 2$ *if the geometrical multiplicity of the zero eigenvalue of B equals* 2 *and* $w(0)$ *is given by* (14).

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