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Topological properties of generalized contextual structures

TOPOLOGICKÉ VLASTNOSTI ZOBECNENÝCH KONTEXTOVÝCH STRUKTUR

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1. Introduction

Topological notions and methods could be applied in a wide variety of applications in different areas of Physics, Engineering and Computer Science. They could be used not only for formulating or solving scientific problems but also in the information processing where modern topological methods and algorithms play a significant part (the information processing is an area of mathematics concerned with the properties of the space preserved under the continuous deformations). These methods provide a different levels for solving problems starting from a problem formulating in a general language of mathematic, physics and other technical sciences and ending as a kernel of different digital applications and computer programs for the information processing. The general mathematical structures usually based on the continuous representation of real or complex numbers, classic spaces usually contain only "ideal" elements obtained as a result of calculating or approximating processes. Because of the digital nature of the most applications, mathematical structures in Computer Science are different from the mathematical structures traditionally used in Mathematics.

A new trend in Mathematics is studying of objects, that contain not the whole calculating process but only its parts (partial objects, finite objects could be observed in finite time). The inspiration comes from many areas of theoretical and practical disciplines; among them are Digital Topology, Theoretical and Mathematical Physics, Theoretical Computer Science, Domain Theory, Formal Concept Analysis, Object-oriented Programming. This thesis can not pay attention to all of these aspects. The main studied aspect will be the interaction of the three areas of Mathematics: Formal Concept Analysis, General Topology and Partial Metrics. The interaction of Formal Concept Analysis with a General Topology has been already studied a little bit in some articles. However, in basic definitions in the Formal Concept Analysis we can find more very interesting dependencies. Also in this thesis the research in the area of frameworks, De Groot duals, game theory was made. The areas common for all these dependencies are General Topology and Formal Concept Analysis.

Every topological space (X, τ) we could understand as a formal context $(X, \tau, \in)$ with a set of objects X, a set of attributes τ and an incidence relation $\in$. It is possible to generate topologies in a natural way on the attribute and object sets with help of the incidence relation by generating close or open subbases. And these topologies deserves an independent investigation. Many questions arise in this area, for example, how the general topological properties could be represented in the formal concept analysis language. And, on the other hand, what influence could the changes in a formal context bring to its topologies? A General Topology is a theoretical discipline, but Formal Concept Analysis is an area of Mathematics that has a lot of different applications. For example artificial intelligence, analysis and digital data processing, designing expert systems or work with databases. Geometric and metric properties of objects consistent with Euclidean – Hausdorff real world around us are not consistent with the new digital structures carrying information, and so the advanced methods used in general and digital topology could be very useful.

2. Topology as a Formal Context

There are many different mathematical disciplines. In this part of the research we intend to study the interaction of three of them: Formal Concept Analysis, Topology and Partial Metrics.

2.1. Generating the Left Topology

Definition 2.1 *(left and right topologies) Let $(X, A, \vdash)$ be a formal context. The topology τ on X, generated by its closed subbase $\{a'|a \in A\}$ is called the left topology on $(X, A, \vdash)$. Similarly, the right topology on $(X, A, \vdash)$ is the topology on A generated by the family $\{x'|x \in X\}$ used as its subbase for the closed sets.*

Results for the left and right topologies are symmetric. So we will pay attention here only to the left topology. The topological closure operator induced by this topology we will denote by *cl*. All closed sets in the left topology τ denote as $\mathcal{C}$.

Recall that a preorder of specialization on a topological space (X, τ) is the binary relation $\leq$ satisfying the condition $x \leq y \Leftrightarrow x \in \mathrm{cl}\{y\}$. We can rewrite this formula as

$$\mathrm{cl}\{y\} = \downarrow_{\leq} \{y\}.$$

Theorem 2.1 *If $(X, A, \vdash)$ is a formal context and τ is its left topology on X then the following sets are closed subbases for the topology τ:*

(1) $\mathcal{C}_1 = \{a'|a \in A\}$

(2) $\mathcal{C}_2 = \{F'|F \subseteq A\}$

(3) $\mathcal{C}_3 = \{P|P \text{ is an extent}\}$

The main result is the following theorem.

Theorem 2.2 *Let* $(X, A, \vdash)$ *be a formal context,* τ *be its left topology on* X*. Then for an arbitrary element* $p \in X$ *it holds* $\mathrm{cl}\{p\} = p''$*.*

From the previous theorem it follows that on the one-element sets a topological closure coincide with a second derivation operator.

Corollary 2.1 *Let* $(X, A, \vdash)$ *be a formal context,* τ *be its left topology on* X*,* $\leq$ *is a preorder of specialization on* X *equipped with the topology* τ*. The following statements for arbitrary elements* $x, y \in X$ *are equivalent:*

(1) $x \leq y$*,*

(2) $x \in \mathrm{cl}\{y\}$*,*

(3) $x \in y''$*,*

(4) $y' \subseteq x'$*,*

(5) $x'' \subseteq y''$*,*

Theorem 2.2 yields the possibility to construct the closure of one-element sets in an easy way. But what would it happen if we take an arbitrary set? The operators $''$ and *cl* need not necessarily be equivalent for all other sets. The second derivation operator has a lot of various properties, and it seems to be an additive operator. However, this is not true.

Example 2.1 *The second derivation operator is not additive even on finite sets.*

R	**a**	**b**	**c**
1	x		
2		x	
3			x

Table 1: Counterexample for non-additivity of second derivation operator

Theorem 2.3 *Let $(X, A, \vdash)$ be a formal context and τ is its left topology. Let us denote $Ext^F(X, A, \vdash)$ the set of all finite unions of extents. If set X is finite, then*

$$Ext^F(X, A, \vdash) = \mathcal{C}.$$

Example 2.2 *There exists a context $(X, A, \vdash)$ for which*

$$Ext^F(X, A, \vdash) \neq \mathcal{C}.$$

The following definition could be found for instance in [18].

Definition 2.2 *Let (X, τ) be a topological space. A set $B \subseteq X$ is a saturated set in the topology τ on X if it is an intersection of open sets.*

It is easy prove that if the binary relation $\leq$ is the preorder of specialization on the topological space (X, τ), then a set B is saturated if and only if

$$B = \uparrow_{\leq} \{B\} = \{x | x \in X, \text{ for some } a \in B \text{ it holds } a \leq x\}.$$

Theorem 2.4 *Let $(X, A, \vdash)$ be a formal context and (X, τ) be a left topology on it. Then for an arbitrary set $P \subseteq X$ the following statements are equivalent:*

(1) P *is a saturated set,*

(2) $P = \uparrow_{\leq} \{P\}$,

(3) $\{x | x \in X \setminus P, P \cap x'' \neq \varnothing\} = \varnothing$,

(4) $\forall x \in X \setminus P, P \cap x'' = \varnothing$.

We have already described in details a transformation from a formal context to the left topology. Let us have a look at the opposite task. We need to construct a context from a topology. The simplest context we can generate in an easy way by the definition of the left topology.

Theorem 2.5 *Let us take a topological space* (X, τ) *and* ζ *is its subbase. Then a formal context* $(X, \zeta, \in)$ *generates a left topology* (X, τ)*.*

Corollary 2.2 *Let* (X, τ) *be a topological space,* Ω *be a closed base for this topology. Then* $(X, \Omega, \in)$ *is a one of such contexts, where the left topology is* (X, τ)*.*

Definition 2.3 *Contexts are called topologically equivalent if they generate the same left topology.*

2.2. Partial Metrics on a Quotient Context

Let μ be a finite counting measure. The finite counting measure is an intuitive way to put a measure on any finite set. A measure of the set is taken to be a number of its elements:

$$\mu(A) = |A|.$$

Lemma 2.1 *Let us take an arbitrary finite set* A*. Let* μ *be the finite counting measure as on* A*. The function* $p : 2^A \times 2^A \longrightarrow \mathbb{R}^+$ *constructed as*

$$p(x, y) = \mu(x \cup y), \text{ where } x, y \subseteq A$$

is a partial metric on the set 2^A*.*

Consider a context $(X, A, \vdash)$ and a measure μ on the set A. A measure is a function defined on the appropriate σ-algebra $\Sigma \subseteq 2^A$. Let $\Sigma = \sigma(A \cup \{x' | x \in A\})$ be the smallest σ-algebra containing the set $A \cup \{x' | x \in A\}$.

Lemma 2.2 *Let us take a finite row-clarified formal context* $(X, A, \vdash)$*. Given a finite counting measure* $\mu : \Sigma \to \mathbb{R}^+$ *on the finite set* A *(where* Σ *is a* σ*-algebra on* A*), define a function* $p : X \times X \longrightarrow \mathbb{R}^+$

$$p(x, y) = \mu(x' \cup y'), \quad \text{where } x, y \in X.$$

Then p is a partial metric on X.

Lemma 2.3 *Let take a row-clarified context $(X, A, \vdash)$, where A is a finite set. Let us denote a counting finite measure μ on the set A. Let p be a partial metric on X generated by a counting finite measure μ on A. Then*

$$\ll_p = \preceq,$$

where $\preceq$ is a specialization preorder for a left topology generated on the context $(X, A, \vdash)$.

Corollary 2.3 *Let us take a row-clarified context $(X, A, \vdash)$, where A is a finite set. Then the specialization preorder $\preceq$ for a left topology is a partial order.*

Definition 2.4 *Let us take a context $(X, A, \vdash)$, where sets X, A are not necessarily finite sets. A relation $\vdash$ is a relation on $X \times A$. A function $\mu : \Sigma \to \mathbb{R}^+$ is a general (i.e., countably-additive) measure (where Σ is a σ-algebra). Relation $R_\vdash = \{(Q, W) | Q, W \subseteq A \text{ and } \mu(W \div Q) = 0\}$ is called an attribute relation.*

Lemma 2.4 *On the context $(X, A, \vdash)$ with given general, countably-additive measure μ on the set A, an attribute relation $R_\vdash$ is an equivalence relation on 2^A.*

Because $R_\vdash$ is an equivalence on the set A, than it has equivalence classes. We will denote equivalence classes as $[Q] = \{W | \mu(Q \div W) = 0\}$. And we can construct a quotient set $A|_{R_\vdash}$ with operations $\sqcap$ and $\sqcup$. Let us denote operations $\sqcap$ and $\sqcup$ as $[Q] \sqcap [W] = [Q \cap W]$ and $[Q] \sqcup [W] = [Q \cup W]$.

Lemma 2.5 *For the equivalence relation $R_\vdash$ on the formal context $(X, A, \vdash)$ operations $\sqcap$ and $\sqcup$ on a quotient set $A|_{R_\vdash}$ are well-defined. Operations are defined as follows: $[Q] \sqcap [W] = [Q \cap W]$ and $[Q] \sqcup [W] = [Q \cup W]$.*

The relation $R_\vdash$ is defined on the set A. But the goal was to define equivalence classes on the set X, which arise from the set A in a some way. The relation $R_\vdash$ in a natural way induces a relation $S_\vdash = \{(x, y)|x, y \in X, (x', y') \in R_\vdash\}$ on the set X. Generally speaking, we obtain relation $S_\vdash$ with help of renaming the elements of X and A. It is obvious, that $S_\vdash$ is the equivalence relation too. The equivalence classes on the set X are denoted as $[x]$. Now let us define a function $p : X|_{S_\vdash} \times X|_{S_\vdash} \rightarrow \mathbb{R}^+$ with help of the measure μ defined on the set A:

$$p([x], [y]) = \mu(x' \cup y')$$

Theorem 2.6 *For the formal context* $(X, A, \vdash)$ *a function* $p : X|_{S_\vdash} \times X|_{S_\vdash} \rightarrow \mathbb{R}^+$ *defined as* $p([x], [y]) = \mu(x' \cup y')$ *is a partial metric on the set* $X|_{S_\vdash}$.

Lemma 2.6 *Let* Σ *be a* σ*-algebra and* μ *is a measure. Then for the countable sequence* $A_i \in \Sigma$ *that* $\mu(A_i \div A_j) = 0, \quad i, j \in \mathbb{N}$ *sets* $\cup A_i$ *and* $\cap A_i$ *are measurable and*

$$\mu(\cup A_i) = \mu(\cap A_i) = \mu(A_1).$$

Theorem 2.7 *For a given formal context* $(X, A, \vdash)$, σ*-algebra* Σ *on* A *a measure* μ *on the set* A *generates a partial metric on the quotient context.*

3. A Context Structure Framework

3.1. Introduction

Modern topological methods are widely used in many recent scientific applications, including theoretical computer science, formal concept analysis, digital image analysis and processing, causal quantum structures and study of qualitative properties of certain differential equations. By means of these highly theoretical disciplines, topological results are also applied in the theory of parallel computation and concurrent processes, quantum algorithms, analysis of digital images in tomography, microscopy or echolocation, electron holography, quantum gravity and the theory of quantum topological insulators.

One of the most important aspects of the studied topological properties having some relationship to the above mentioned applications are the properties of compact sets, whose topological behavior is characterized (among others) by the well-known construction of the de Groot dual. J. Lawson and M. Mislove stated there a problem known as Problem 540, whether the sequence of iterated duals of τ is infinite or the process of taking duals terminates after finitely many steps with two topologies which are dual to each other. For a special case of T_1-topologies the problem had been solved in [12]. The problem in general was solved by Martin Kovár in 2001. He proved that for any topology it holds $\tau^{dd} = \tau^{dddd}$ [18]. In 2004 the result was improved by the same author to its (so far) final form $\tau^d = (\tau \vee \tau^{dd})^d$ [20]. Note that from this result it also follows that $\tau^d \subseteq \tau^{ddd}$ for any topology τ. It should be also noted that in [18] M. Kovár stated several natural questions regarding the dual topologies. Some of them were studied by Tomoo Yokoyama in his paper [37].

The questions of J.Lawson and M.Mislove related to the de Groot dual arise from study of various semantic models in the theoretical computer science, where the dual and the patch topologies are an important tools of investigation. Than it is natural to ask, whether the similar results could be obtained for more general structures. Another interesting direction of research was introduced by Bernhard Banaschewski [1], who replaced the usual frame structure by a more general, partially ordered structure called *preframe*, where the suprema exist for all non-empty up-directed subcollections. Taking some inspiration from the "classic" results of J. Lawson, M. Mislove, M. Kovár, T. Yokoyama, and from B. Banaschewski's preframe structure of opens of pretopological systems, we investigate the possibility of a construction analogous to the de Groot dual, but in a new, modified setting [2],[4],[6]. A possible range of applications could lie in improvements of the efficiency of some topological algorithms and investigation of the properties of certain causal structures, applicable in quantum gravity and the theory of quantum topological insulators.

3.2. De Groot Dual in Compactly Localic Structures

We will start with recalling some key notions and making several useful denotations.

Definition 3.1 *The Sierpiński frame* $\mathbf{2} = \{\bot, \top\}$ *is a set consisting of the two elements* $\top$ *– the top and* $\bot$ *– the bottom.*

Definition 3.2 *Let* (X, τ) *be a topological space. Then* τ^d *is called the de Groot dual (or co-compact) if it is a topology generated by the all compact saturated sets in* (X, τ) *used as its closed base.*

Definition 3.3 *Let $(P, \leq)$ be a partially ordered set (poset). The weak topology is a topology defined by taking the principal lower sets $\downarrow\{x\}$, for $x \in P$, as the closed subbase. Similarly, the weak^d topology is defined by taking the principal upper sets $\uparrow\{x\}$, for $x \in P$, as the closed subbase for a topology on P.*

It is well-known that in a locale $(X, A, \vdash)$, the set of points X may be represented as a family of all frame morphisms : $A \to \mathbf{2}$ and the relation $\vdash$ is defined by $x \vdash a \Leftrightarrow x(a) = \top$ for $x \in X$ and $a \in A$. The Hofmann-Mislove theorem says that there is 1-1 correspondence between the compact saturated sets in $(X, A, \vdash)$ and the functions from A to $\mathbf{2}$ that preserve directed joins and finite meets. Taking these functions as points and the elements of A as opens, we obtain a new structure $(X', A, \vdash')$ that redistributes the logic: The localic points are replaced by the compacts sets and the new relation $\vdash'$ preserves directed joins as well as finite joins on both sides. On the other hand, it should be noted that $(X', A, \vdash')$ need not be a topological system in the usual sense but a structure slightly different.

Definition 3.4 *A poset A is a preframe (or directly complete semilattice) if and only if A*

1. *is closed under (non-empty) directed joins,*
2. *is closed under finite meets (including the meet of the empty set),*
3. *binary meets distribute over directed joins.*

We write $\top = \bigwedge \varnothing$ (top) and $\bot = \bigvee \varnothing$ (bottom). A simple example of a preframe which is not a frame is given by the poset $P = \{\bot, 0\} \cup \mathbb{N}$ on the Figure 1.

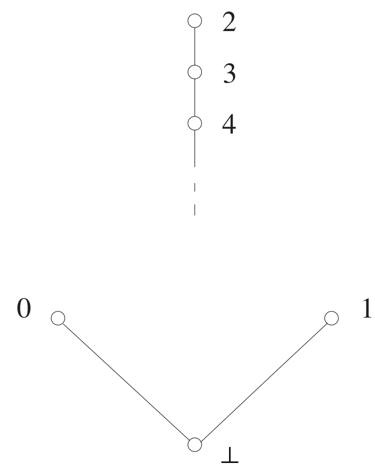

Figure 1: Preframe but not a frame

Definition 3.5 *Let X be a set, A be a preframe, and $\vdash$ be a subset of $X \times A$. We write $x \vdash a$ for $(x, a) \in \vdash$. Let the following conditions are satisfied:*

(i) If $B \subseteq A$ is non-empty and directed, then $(x \vdash \bigvee B) \Leftrightarrow (x \vdash b$ for some $b \in B)$.

(ii) If $C \subseteq A$ is non-empty and finite, then $(x \vdash \bigwedge C) \Leftrightarrow (x \vdash c$ for every $c \in C)$.

Then we say that the triple $(X, A, \vdash)$ is a pretopological system. The elements of A we call, similarly as in topological systems, opens. If $A \subseteq 2^X$ is ordered by the inclusion, $\varnothing, X \in A$ and the relation $\vdash$ is $\in$, then A is called a pretopology and $(X, A, \vdash)$ is a pretopological space.

Definition 3.6 *Let A be a poset. We denote by $\langle A \to \mathbf{2} \rangle \subseteq \mathbf{2}^A$ the set of all functions $A \to \mathbf{2}$ that preserve the non-empty directed joins and finite meets, whenever they exist. The elements of $\langle A \to \mathbf{2} \rangle$ we will call morphisms.*

Proposition 3.1 *The poset $\langle A \to \mathbf{2}\rangle$ forms a preframe of all morphisms of A to $\mathbf{2}$.*

Definition 3.7 *A pretopological system $(X, A, \vdash)$ is a compactly localic if $X = \langle A \to \mathbf{2}\rangle$ and $x \vdash a$ if and only if $x(a) = \top$ for $x \in \langle A \to \mathbf{2}\rangle$ and $a \in A$.*

Let us denote, similarly as in [19], $\mathrm{int}_X(a) = \{x |\, x \in X, x \vdash a\}$ for every $a \in A$.

Definition 3.8 *A set $K \subseteq X$ is compact in a pretopological system $(X, A, \vdash)$ if for every directed $B \subseteq A$ with $K \subseteq \mathrm{int}_X(\bigvee B) = \bigcup_{b\in B} \mathrm{int}_X(b)$, there is some $a \in B$ such that $K \subseteq \mathrm{int}_X(a)$.*

Definition 3.9 *A set $S \subseteq X$ is saturated in a pretopological system $(X, A, \vdash)$ if S is an intersection of the sets $\mathrm{int}_X(b)$, $b \in B$ for some $B \subseteq A$.*

One can easily check that the notions of compactness and saturation in pretopological systems slightly differ from their counterparts in topological systems if A is not a frame (although the sets $\mathrm{int}_X(a)$, $a \in A$ generate some underlying topology on X). If A is not a frame, the previously introduced notions need not coincide with compactness and saturation related to this topology. Although it is obviously possible to define a dualization of a general pretopological system, it is not so simple to choose the best construction from several possibilities since they are not easily comparable with the classical topological case.

Definition 3.10 *Let $(X, A, \vdash)$ be a compactly localic pretopological system. For any $x \in X$ and any $y \in \langle A \to \mathbf{2}\rangle$ we say that x is independent on y and write $x \models y$ if there is some $a \in A$ such that $y(a) = \top$ and $x \nvdash a$.*

Definition 3.11 *The dualization of a compactly localic pretopological system* $(X, A, \vdash)$ *we mean the triple* $(X, \langle A \to \mathbf{2} \rangle, \models)$.

We will see that under the condition that A is a frame, $(X, \langle A \to \mathbf{2} \rangle, \models)$ will correspond to its topological counterpart.

Theorem 3.1 *Let* A *be a frame and* $(X, A, \vdash)$ *be a compactly localic pretopological system. Then* $(X, \langle A \to \mathbf{2} \rangle, \models)$ *is also a pretopological system.*

Theorem 3.2 *Let* A *be a frame,* $(X, A, \vdash)$ *be a compactly localic pretopological system. Then the topology on* X *induced by* $\langle A \to \mathbf{2} \rangle$ *is dual to the topology on* X *induced by* A *in the usual sense and it equals to its weak*^{d} *topology.*

On the other hand, if A is a more general preframe than a frame, it is easy to see that the triple $(X, \langle A \to \mathbf{2} \rangle, \models)$ representing the dual of $(X, A, \vdash)$ even need not be a pretopological system and if so, it need not be compactly localic. Hence, the sequences of iterated dualizations are not possible in general in this setting. One possible idea how to fix this problem could be modifying the underlying set of points of the dual pretopological system. This idea leads to the following natural definition.

Definition 3.12 *A compactly localic dualization of a compactly localic pretopological system* $(X, A, \vdash)$ *is the pretopological system* $(X', \langle A \to \mathbf{2} \rangle, \Vdash)$, *where* $X' = \langle \langle A \to \mathbf{2} \rangle \to \mathbf{2} \rangle$ *and* $(u \Vdash y) \Leftrightarrow (u(y) = \top)$ *for* $u \in X'$ *and* $y \in \langle A \to \mathbf{2} \rangle$.

Now the iterated compactly localic dualizations exist for any compactly localic pretopological system, and they are fully represented by sequence of the posets of their opens: $\langle A \to \mathbf{2} \rangle$, $\langle \langle A \to \mathbf{2} \rangle \to \mathbf{2} \rangle$, $\langle \langle \langle A \to \mathbf{2} \rangle \to \mathbf{2} \rangle \to \mathbf{2} \rangle$, $\ldots$, etc.

3.3. Dualizations for the Posets of Opens

Now we will concentrate ourselves on the preframe structure of the opens of the pretopological counterpart of the de Groot dual. As we have shown in the previous section, the opens of the dual may be represented as certain maps from A to the Sierpiński frame $\mathbf{2}$, where A is the poset representing the opens of the original pretopological system.

Let A be a poset. We denote by $h_A : A \to \langle\langle A \to \mathbf{2}\rangle \to \mathbf{2}\rangle$ a mapping for which $h_A(a)(x) = x(a)$ for every $x \in \langle A \to \mathbf{2}\rangle$. The following theorem holds:

Theorem 3.3 *Let A be a poset. Then $h_A : A \to \langle\langle A \to \mathbf{2}\rangle \to \mathbf{2}\rangle$ is a morphism.*

On the other hand, we have the following two counterexamples; the corresponding posets are given by their Hasse diagrams on the Figure 3 and Figure 2.

Example 3.1 *There exists a preframe A such that h_A is not an epimorphism. Let $A = \omega + 1 = \{1, 2, \ldots, \omega\}$, where ω is the first infinite ordinal, with its natural linear order. Let $n' : A \to \mathbf{2}$ be a mapping with the $\top$-kernel $\{n, n+1, \ldots, \omega\}$ for every $n \in A$ and $(\omega+1)'$ be a mapping identically equal to $\bot$. The construction is illustrated by the Figure 2.*

Example 3.2 *There exists a preframe A such that h_A is not a monomorphism. The construction is illustrated by the Figure 3. That means, 0 is the bottom, ω is the top, $2k$ has two successors $2k+1$, $2k+2$ and $2k+1$ has a unique successor $2k+3$ for every $k \in \{0, 1, \ldots\}$.*

The positive results can be reached especially for the finite case:

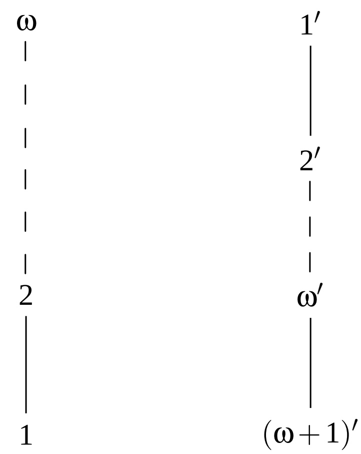

Figure 2: h_A is not an epimorphism

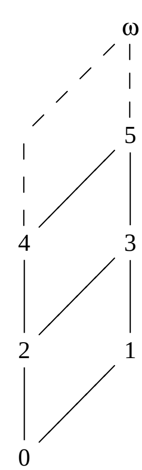

Figure 3: h_A is not a monomorphism

Theorem 3.4 *Let A be a finite preframe. Then $h_A : A \to \langle\langle A \to \mathbf{2}\rangle \to \mathbf{2}\rangle$ is an isomorphism.*

Corollary 3.1 *Let A be a finite poset. Then its iterated duals,*

$$\langle A \to \mathbf{2}\rangle\,,\ \textit{and}\ \langle\langle\langle A \to \mathbf{2}\rangle \to \mathbf{2}\rangle \to \mathbf{2}\rangle\,,$$

are isomorphic.

Now we will consider a more general case, when A is not necessarily finite. For every $v \in \langle\langle\langle A \to \mathbf{2}\rangle \to \mathbf{2}\rangle \to \mathbf{2}\rangle$ and $a \in A$ we put $h_A^*(v) = v \circ h_A$.

Theorem 3.5 *Let A be a poset. Then $h_A^* : \langle\langle\langle A \to \mathbf{2}\rangle \to \mathbf{2}\rangle \to \mathbf{2}\rangle \to \langle A \to \mathbf{2}\rangle$ is a morphism.*

We close the section by the main theorem, which is the an analogue of Martin Kovár's result $\tau^d \subseteq \tau^{ddd}$, proven in 2001 for the topological spaces [18]. Among others, from the next theorem it follows that $h_{\langle A \to \mathbf{2}\rangle} : \langle A \to \mathbf{2}\rangle \to \langle\langle\langle A \to \mathbf{2}\rangle \to \mathbf{2}\rangle \to \mathbf{2}\rangle$ is a monomorphism, which makes the analogy with the classical topological result from 2001 more obvious.

Theorem 3.6 *Let A be a poset. Then $h_A^* : \langle\langle\langle A \to \mathbf{2}\rangle \to \mathbf{2}\rangle \to \mathbf{2}\rangle \to \langle A \to \mathbf{2}\rangle$ is a retraction.*

4. Spatio-temporal Concepts of Framology

4.1. Introduction

In most applications, topology is usually not the first, primary structure, but the information which finally leads to the construction of the certain, for some purpose required topology, is filtered out by more or less thick layer of the other mathematical structures. This fact has some natural consequences:

1. For most important applied constructions the primary structure is sufficient and topology may be bypassed (in the cost of loss of some elegance).

2. Some topologically important information from the reality may be filtered out by the other, front-end mathematical structures and finally lost.

Obviously, our traditional topological conceptions of the world around us may be far from reality. Natural examples of such situations we can meet often in our everyday life, but usually they are ignored. For instance, as noted in [13], in nature or physical universe there are probably no existing, real points like in the classical Euclidean geometry. Points, as a useful mathematical abstraction, are infinitesimally small and thus cannot be measured or detected by any physical way. However, what we can be sure that really exists, there are various locations, containing concrete physical objects. We will call these locations *places*. Various places can overlap, they can be merged, embedded or glued together, so the theoretically understood virtual "observer" can visit multiple places simultaneously. For instance, the Galaxy, the Solar system, the Earth, (the territory of) Europe, the room in which the reader is present just now, are simple and natural examples of places conceived in our way. Certainly, in this sense, one can be present at many of these places at the same time, and, also certainly, there exist pairs of places, where the simultaneous presence of any physical objects is not possible. Thus, the presence of various physical objects connects these primarily free objects – our places – to the certain structure, which we call a *framework*, [22].

Let us recall the exact definition of this notion and formulate some its basic properties, illustrating how this structure is naturally connected with topology. On a simple example from game theory we will also demonstrate the difference between the really existing objects modeled by a framework and its virtual extension, having no direct counterpart in reality, represented by a topological space.

Definition 4.1 *Let P be a set, $\pi \subseteq 2^P$. We say that (P, π) is a framework. The elements of P we call places, the set π we call framology.*

Although every topological space is a framework by the definition, the primary interpretation of a framework is different from the usual interpretation of a topological space. The elements of the framology are not primarily considered as neighborhoods of places, although this seems to be also very natural. The framework structure is rather a special case of a formal context with the places as the objects, π as the set of attributes and $\in$ as the incidence relation.

There exists also a natural physical-like motivation of the structure: P represents the set of some locations, where an element of π is a "list" of locations containing certain physical object, say a particle, simultaneously.

The places primarily have no geometrical properties or meaning and they are not connected with any outer geometrical structure as the spacetime or so. The structure arises in an intrinsic way, just from the relation between elements of P given by the family π. The places may naturally overlap, contain each other or they may be glued together by presence of some physical object (for instance, a particle).

Definition 4.2 *Let (P, π) and (S, σ) be frameworks. A mapping $f : P \to S$ satisfying $f(\pi) \subseteq \sigma$ we call a framework morphism.*

Definition 4.3 *A framework (P, π) is T_0 if for every $x, y \in P$, $x \neq y$, there exists $U \in \pi$ such that $x \in U$, $y \notin U$ or $x \notin U$, $y \in U$.*

Definition 4.4 *Let (P, π) be a framework. Denote $P^d = \pi$ and $\pi^d = \{\pi(x) | \, x \in P\}$, where $\pi(x) = \{U | \, U \in \pi, x \in U\}$. Then (P^d, π^d) is the dual framework of (P, π). The places of the dual framework (P^d, π^d) we call abstract points or simply points of the original framework (P, π).*

The framework duality is a simple but handy tool for switching between the classical point-set representation (like in topological spaces) and point-less representation, introduced by the framework theory. More information regarding the framework structure the reader can find, for example, in [21] or [22].

4.2. Framework Topological Models

The framework structure could be also suitable for addressing the compatibility problem of various scales in physics and their different models. Since the points of the Universe probably do not exist in reality (although they are a useful mathematical abstraction), the abstract points of a framework only express certain relationships between places, which – in a contrast to points – can be really observed and which exclusively exist in the physical reality. Then various framologies and various topological models may peacefully coexist with help of the framework duality on a given set P of places. Let as formulate more precisely, what we mean by the topological model of a framework.

Definition 4.5 *Let (P, π) be a framework, (X, τ) be a topological space with the family $\mathcal{C}$ of closed sets. We say that (X, τ) is an open (closed, respectively) topological model for (P, π), if there exists a framework (S, σ) isomorphic to (P, π) and set $X' \subseteq X$ such that $S \subseteq \tau$ ($S \subseteq \mathcal{C}$, respectively) and $\sigma = \{\{U | \, U \in S, x \in U\} | \, x \in X'\}$.*

An inspiration for a construction of various frameworks one can find in game theory, theoretical or mathematical physics. In case of reader's interest the full description is stated in the full version of the thesis.

4.3. Approximations by finite frameworks

Now we will study the possibility of an approximation of any framework, say (P, π), by a directed system of finite frameworks. We will need to introduce the following notion.

Definition 4.6 *Let (X, α) be a framework, $Y \subseteq X$. Denote $\beta = \{U \cap Y | \, U \in \alpha\}$. Then (Y, β) is called the induced subframework of (X, α).*

We put $\pi_K = \{U \cap K | \, U \in \pi\}$ for every finite $K \subseteq P$. Ovbiously, if K, L are finite subsets of P and $K \subseteq L$, (K, π_K) is an induced subframework of (L, π_L) and both are induced subframeworks of the original framework

(P, π). The collection of finite frameworks (P_K, π_K) is directed by the set inclusion. Let

$$\sigma = \{W \,|\, W \subseteq P, W \cap K \in \pi_K \text{ for every finite } K \subseteq P\}.$$

Obviously, $\pi \subseteq \sigma$. Moreover, after a restriction to a finite family $K \subseteq P$ of places in the framework (P, π) there is no way how to distinguish between (P, π) and (P, σ), since

$$\{U \cap K \,|\, U \in \pi\} = \pi_K = \{W \cap K \,|\, W \in \sigma\}.$$

It could seem that it would be a good idea to approximate (P, π) by (P, σ). However, as we will show later, (P, σ) may contain too many abstract poins (that is, elements of σ) in comparison to (P, π).

Let $\mu \subseteq \sigma$ be the set of all maximal elements of σ. The framework (P, μ) could be another candidate for an approximation of (P, π).

Example 4.1 *Let $P = \mathbb{N}$ and let π be the set of all finite subsets of P. Then, respecting the previous denotations, $\sigma = 2^P$, and $\mu = \{P\}$.*

The following theorem now describes the approximation properties of our construction under very general topological conditions.

Theorem 4.1 *Let (X, τ) be a topological T_1 space, $\mathcal{C}$ the family of all closed sets. Let (P, π) be the dual framework of $(X, \mathcal{C})$. Then the dual of (P, μ) generates the Wallman compactification of (X, τ). More precisely, μ^d is a closed subbase of ωX.*

Among others the previous theorem means that for a compact T_1 topological space, its approximation by a suitable family of finite frameworks may achieve an arbitrary precision.

5. Topology as a Tool in Game Theory

That game theory could be a natural source of various situations in which framework and contextual structures may appear as very helpful in analysis of the underlying structure. The last chapter of the dissertation will be devoted to completing the last piece of stone to our mosaic. Now we will use topology as a tool for investigation in game theory. The most results included in this chapter have been recently published in a joint paper [23].

5.1. Introduction

Undominated strategies play an important role in game theory as well as in many related engineering and economical applications. The theorem ensuring the existence of undominated strategies in a normal form game under the assumption that the set of all strategies of a player is compact and the utility function is continuous, belongs to the well-known and fundamental results. It was stated in 1981 in Herve Moulin's comprehensive textbook on game theory [27], and essentially it was also contained and used in many other papers. The proof presented in the first edition of [27] was dependent on a combination of relatively non-trivial results from measure theory, metric topology and mathematical analysis. In the second, revised edition [28] of the same book, now there is stated a simplified proof using some topological argumentation together with Zorn's Lemma. However, the proof in [28] is unfortunately incorrect, since it implicitly uses a non-valid argument that every chain (that is, a linearly ordered set) contains a cofinal subsequence. The first uncountable ordinal ω_1 is a proper counterexample witnessing that in general it is not true. The mistake itself is not very critical for game theory, since in metric spaces, for which the classical results are usually formulated, the topology is first countable and hence the sequences are still sufficient to fully describe the topology by means of the convergence. Nevertheless, the mentioned fact itself, was a source of inspiration for a revision of of the original Moulin's Theorem leading to its our generalization and improvement. A natural question how substantial our improvement really is we will demonstrate on a simple example.

5.2. Main Results

A topological space X is *compact*, if every net or every filter base in X has a cluster point. For more detail and other equivalent and well-known characterizations of compactness, especially in terms of open covers, we refer the reader the monographs [7], [8], [29] and [35]. We also remark that in a modern approach to compactness, motivated by the growing interest of the theoretical computer scientists in topology, the Hausdorff separation axiom is no longer assumed as a part of the definition of compactness (see, for example, [35]). Recall that a topological space is *almost compact* [7] if every open filter base in X has a cluster point. It is clear from the definition that every compact space is almost compact but not vice versa, as the reader may check from a counterexample in [7]. Another counterexample we will present also in the next section.

The well-known fundamental theorem stated by Herve Moulin in [27] and [28] follows.

Theorem 5.1 *Let $G = (X_1, X_2, \ldots, X_n, u_1, u_2, \ldots, u_n)$ be a normal form game of n persons. Suppose that for some $i \in \{1, 2, \ldots, n\}$ there exist a compact topology on X_i in which the utility function u_i is a continuous,* *real valued function of the argument $x_i \in X_i$. Then the i-th player has an undominated strategy.*

Let us continue with the simple example. As we will show later, the existence of undominated strategies of both players is not a consequence of the classical Moulin's Theorem, but it follows from our generalization.

Example 5.1 *Consider a normal form game of two players with the same sets of strategies $X_1 = X_2 = [0,1) \times \{0\} \cup \{1\} \times \{0,1,\dots\}$. Let the corresponding utility functions of the players be*

$$u_1 = \frac{x_1}{x_1 + x_2} \cdot f(y_2), \text{ and } u_2 = \frac{x_2}{x_1 + x_2} \cdot g(y_1),$$

where f, g are arbitrary real-valued functions defined on $\{0\} \cup \mathbb{N}$. It is easy to see that the pairs $(1, n) \in X_i$, where $n \in \{0, 1, \dots\}$ and $i = 1, 2$, are equivalent, maximal and undominated strategies of the i-th player. However, although the utility functions u_i are continuous, the topology of X_i, induced from the real plane is not compact. For instance, the sequence $\{(1, n) | \, n = 0, 1, 2, \dots\}$ has no cluster point in X_i. Hence, the existence of undominated strategies of the i-th player is not a consequence of Moulin's theorem.

The reader may also notice that there is some additional space for improving the result stated in Theorem 5.1 yielded by a modification of its topological assumptions. For instance, the theorem will remain true, if one replaces the continuity of the utility function by its upper semi-continuity. This is a result due to H. Salonen [32]. He essentially used a characterization of compactness by the centered collections of sets (in other words, having the finite intersection property, [29]), or filters and filter bases, which are topologically equivalent to nets. A similar technique was also used in [31] for iteratively undominated strategies with the continuous utility function.

Another, and perhaps new natural improvement of Theorem 5.1 we receive by relaxing the condition of compactness. At least, we did not see such a modification of the original result in the literature.

Theorem 5.2 *Let $G = (X_1, X_2, \dots, X_n, u_1, u_2, \dots, u_n)$ be a normal form game of n players. Suppose that for some $i \in \{1, 2, \dots, n\}$, X_i is almost compact and the utility function u_i is a continuous, real valued function of the argument $x_i \in X_i$. Then the i-th player has an undominated strategy.*

Now, let us check the advantage of Theorem 5.2 over its original version. Notice that the game described in Example 5.1 cannot be easily covered by Theorem 5.1. Although the utility functions u_i are continuous, the topology of X_i, induced from the real plane is not compact (and certainly also not almost compact). For instance, the sequence $\{(1, n) | \, n = 0, 1, 2, \dots\}$ has no cluster point. Let us define another topology on X_i, where $i = 1, 2$, by the local base of a general point $(x, y) \in X_i$:

1. The point $(0, 0)$ has neighborhoods of the form $[0, \varepsilon) \times \{0\}$, $0 < \varepsilon < 1$.

2. For every $x \in (0, 1)$, the point $(x, 0)$ has neighborhoods of the form $(x - \varepsilon, x + \varepsilon) \times \{0\}$, $0 < \varepsilon < \min\{x, 1 - x\}$.

3. For every $n = 0, 1, \dots$, the point $(1, n)$ has neighborhoods having the form $(1 - \varepsilon, 1) \times \{0\} \cup \{(1, n)\}$, where $0 < \varepsilon < 1$.

The new topology on X_i is now similar to the Euclidean topology on the unit segment $[0, 1]$ but with one important difference – the right end point of the “segment” is present infinitely many times. The space X_i is T_1, but certainly non-Hausdorff and non-compact. Indeed, denoting $Y_n = [0,1) \times \{0\} \cup \{(1,n)\}$, the family $\{Y_n |\, n = 0,1,\dots\}$ is an open cover of X_i, having no finite subcover. However, we can show that the new topology is almost compact. Let Ω be an open cover of X_i. The subspace $Y_0 = [0,1] \times \{0\} \subseteq X_i$ is compact since it is homeomorphous with the unit segment $[0,1]$, so there exists a finite subfamily $\{U_1, U_2, \dots, U_k\} \subseteq \Omega$ with $Y_0 \subseteq \bigcup_{j=1}^{k} U_j$. Then there is $r \in \{1,2,\dots,k\}$ such that $(1,0) \in U_r$. But for every $n = 1,2,\dots$ it follows $(1,n) \in \operatorname{cl} U_r$, so the closures of $\{U_1, U_2, \dots, U_k\}$ cover X_i.Thus X_i is almost compact. The utility functions u_i are continuous functions of the argument (x_i, y_i) since they are continuous on the open subspaces $Y_n = [0,1) \times \{0\} \cup \{(1,n)\}$ of X_i, $n = 0,1,\dots$, homeomorphous to $[0,1]$. Hence, the existence of the undominated strategies now follow also from Theorem 5.2. Note that similar spaces as X_i are also known as examples of non-Hausdorff manifolds, [16], and they may appear also in sheaf theory and in certain problems of mathematical physics, [13].

6. Summary

Let us summarize the main results of this research. A formal context is a general structure that could represent other mathematical structures and the main properties would be enclosed in the set of attributes and the incidence relation. The topology could be represented as a formal context with help of the membership relation. It was shown that a second derivation operator and a closure operator the left topology coincide on the one-element subsets and on the extents. Suppose we have some piece of information and a formal context is an easy way of representing it in a cross-table view. At every particular moment of time, we do not know if that information was complete, because in some moment a new part of information could appear and it could completely change the structure. We put a partial metric on the formal context to measure that partial information. As it was shown with help of measure defined at the attribute set it is possible to construct a partial metric on the object set (and because of the duality we could construct a partial metric on the attribute set from the measure defined on the object set). A measure shows us not only the ordering information, but it shows us in a some way a quantity of the information. In every particular moment of time a part of information is finite and by the Lemma 2.2 we could construct a partial metric. But in general, some objects we do not distinguish and that is why we could not follow the same way. So, at first we construct a quotient context where objects would be induced by the equivalence classes and on that context we could apply the same method as it was showed in the Theorem 2.7. Every new formal context is formed by a new piece of information (objects added / deleted , attributes added/deleted , measure (weight) of attributes changed, the incidence relation changed). Then we compute a partial metric for every instance of the formal context. Because the instances of a formal context are different, we could not directly compare a partial metric $p(x, x)$ for an object x – before that we need to "renormalize" it. And after that we would be able to do the appropriate analysis of the objects.

In some applications a significant role play the properties of compact sets, whose topological behavior could be characterized by the de Groot dual. But nowadays it becomes more popular to study more general structures, because they appear in computer science. Thus in the following chapter we talk about pretopological systems and its duals. The opens of the dual may be represented as certain maps from A to the Sierpiński frame $\mathbf{2}$, where A is the poset representing the opens of the original pretopological system. Then we successfully defined an analogue of the De Groot dual for compactly localic pretopological systems and in a more general approach, also for any poset. We also proved an counterpart of M. Kovár's result $\tau^d \subseteq \tau^{ddd}$ proved for the general topological spaces (Theorem 3.6 for the general posets and The 3.4 for the special, finite case). But the result stated in Theorem 3.6 is best possible as it was showed in Example 3.1 and Example 3.2.

In the following chapter we have a look at the structure named framework. The most of the mathematical structures are classical point-set representations, but framework is a point-less. It is a pair of sets where first one is a set of places and the second one, called framology, is a family, whose members are collections of places. Each element of the framology can be interpreted as a collection of places where some object could be present at the same time. A natural range of applications of this structure are the spatio-temporal relationships in modern theoretical physics, for example as in quantum gravity. The main result of this chapter is Theorem 4.1, describing the approximation of a general framework by a directed family of finite frameworks from the point of view of the generated topologies and the framework duality.

In the last chapter we pay attention to game theory. Although topology and game theory seem to be far from each other, topological approach can be used with advantage in many proofs of game theory. For instance, the original proof of the well-known theorem of H. Moulin contained a wrong, set-theoretical assumption, that every chain has a cofinal subsequence. Using topological theory of convergence, expressed in terms of nets, and a certain, less usual modification of Zorn's Lemma, we were able to offer another, simpler proof, which yields even a slightly more general result – Theorem 5.2.

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Abstract

The main idea of this work was to find dependencies, relationships and analogies between several branches of mathematics. First part of the work is concerned to the relationship between Formal Concept Analysis, General Topology and Partial Metrics. A formal context is a very general mathematical structure that can represent other mathematical structures in a unified form. In a natural way, we could represent an information in a cross-table-like view of a formal context and generate a topology on an attribute and object sets.

In the second part the we study especially the pretopological systems as a special case of formal contexts. Among others, we construct and study an analogue of the de Groot dual for pretopological systems and posets, including its iteration properties.

The third part is devoted to a mathematical structure of framework, having a contextual nature. It consists of two sets, first one is a set of places, and the second one is a family of some its subsets, without the necessity of any external axioms. The structure is equipped with a simple duality construction, allowing to switch between the classical point-set representation (like in topological spaces) and the point-less representation of topological relationships. At the end of the chapter, we suggest and study how a framework could be approximated by a directed family of finite frameworks from the point of view of the generated topology.

In the last part the methods of general topology were used for correction and improvement of one of fundamental theorems in game theory. It was shown that in a normal form game if i-th player has a continuous utility function and if the set of his strategies is almost-compact then he has an undominated strategy.

Abstrakt

Hlavní myšlenkou práce bylo najít závislosti, vztahy a analogie mezi několika odvětvími matematiky. První část práce se týká vztahu mezi formální pojmovou analýzou, topologií a parciálními metrikami. Formální kontext je velice obecná matematická struktura, která může reprezentovat ostatní matematické struktury v jednotné a sjednocené formě. Přirozeným způsobem bychom mohli reprezentovat informaci podobně jako v tabulce, reprezentující formální kontext a generovat určité topologie na množinách atributů a objektů.

V druhé části studujeme především pretopologické systémy jako speciální případ formálních kontextů. Mimo jiné se zabýváme konstrukcí analogie de Grootova duálu pro pretopologické systémy a uspořádané množiny, včetně iterovaných vlastností.

Třetí část práce je zasvěcena struktuře framework, která má přirozenou strukturu formálního kontextu. Framework se skládá ze dvojice množin, z nichž první je množina míst a druhá obsahuje jistý systém podmnožin první množiny, aniž by bylo vyžadováno splnění nějakých axiómů. Struktura je opatřena jednoduchou konstrukcí duality, umožňující přepínání mezi klasickým, bodově-množinovým přístupem, podobně jako v topologii a bezbodovou reprezentací topologických vztahů. V závěru navrhujeme a studujeme, jak aproximovat libovolný framework pomocí usměrněného souboru konečných frameworků z hlediska generované topologie.

V poslední části práce používáme metody obecné topologie ke korekci a zlepšení jednoho ze základních teorémů teorie her. Dokázali jsme mimo jiné, že pro hru v normální formě, v níž má i-tý hráč spojitou výherní funkci a množina jeho strategií je skoro-kompaktní, má tento hráč nedominovanou strategii.