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TOPOLOGICKÉ VLASTNOSTI ZOBECNĚNÝCH KONTEXTOVÝCH STRUKTUR

TOPOLOGICAL PROPERTIES OF GENERALIZED CONTEXT STRUCTURES

DIZERTAČNÍ PRÁCE

DOCTORAL THESIS

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Abstrakt

Práce je zaměřena na vzájemnou interakci několika odvětví matematiky. Hlavní myšlenkou práce bylo najít závislosti, vztahy a analogie mezi nimi. První část práce se týká vztahu mezi formální pojmovou analýzou, topologií a parciálními metrikami. Formální kontext je velice obecná matematická struktura, která může reprezentovat ostatní matematické struktury v jednotné a sjednocené formě. Přirozeným způsobem bychom mohli reprezentovat informaci podobně jako v tabulce, reprezentující formální kontext (s respektem ke všem množinově-teoretickým omezením) a generovat určité topologie na množinách atributů a objektů.

V druhé části studujeme především pretopologické systémy jako speciální případ formálních kontextů. Od topologických systémů se pretopologické systémy liší především obecnější uspořádanou strukturou na množině atributů, reprezentujících zobecněné otevřené množiny. Vlastnosti tohoto uspořádání podstatně ovlivňují chování celé struktury a proto mu věnujeme zvláštní pozornost v závěru kapitoly, kde se mj. zabýváme konstrukcí analogie de Grootova duálu, včetně jeho iterovaných vlastností.

Třetí část práce je zasvěcena struktuře framework, která má přirozenou strukturu formálního kontextu. Framework se skládá ze dvojice množin, z nichž první je množina míst a druhá obsahuje jistý systém podmnožin první množiny, aniž by bylo vyžadováno splnění nějakých axiómů. Struktura je opatřena jednoduchou konstrukcí duality, umožňující přepínání mezi klasickým, bodově-množinovým přístupem, podobně jako v topologii a bezbodovou reprezentací topologických vztahů. V závěru navrhujeme a studujeme, jak aproximovat libovolný framework pomocí usměrněného souboru konečných frameworků z hlediska generované topologie.

V poslední části práce používáme metody obecné topologie ke korekci a zlepšení jednoho ze základních teorémů teorie her. Dokázali jsme mimo jiné, že pro hru v normální formě, v níž má i-tý hráč spojitou výherní funkci a množina jeho strategií je skorokompaktní, má tento hráč nedominovanou strategii. Kromě tohoto výsledku v poslední a předposlední kapitole ukazujeme, že teorie her přirozeným způsobem generuje velmi obecné, například nehausdorffovské topologické a kontextové struktury, čímž posouvá tradiční chápání reality neobvyklým směrem.

Summary

This work is focused on the interaction of several branches of mathematics. The main idea was to find dependencies, relationships and analogies between them. First part of the work is concerned to the relationship between Formal Concept Analysis, General Topology and Partial Metrics. A formal context is a very general mathematical structure that can represent other mathematical structures in a unified form. In a natural way, we could represent an information in a cross-table-like view of a formal context (fully respecting all set-theoretical limitations) and generate a topology on an attribute and object sets.

In the second part the we study especially the pretopological systems as a special case of the formal contexts. They differ from topological systems especially by a more general poset structure of the set of attributes, representing the generalized open sets. Since the properties of this order structure are essential for the behavior of the whole structure, we pay them a special attention at the end of the chapter. Among others, we construct and study an analogue of the de Groot dual for posets, including its iteration properties.

The third part is devoted to a mathematical structure called framework that has a contextual nature. A framework consists of two sets, first one is a set of places, and the second one is a family of some its subsets, without the necessity of any external axioms to be fulfilled. The structure is equipped with a simple duality construction, allowing to switch between the classical point-set representation (like in topological spaces) and the point-less representation of topological relationships. At the end of the chapter, we suggest and study how a framework could be approximated by a directed family of finite frameworks from the point of view of the generated topology.

In the last part the general topology methods were used to correct and improve one of the fundamental theorems in the game theory. It was showed that in a normal form game if i-th player has a continuous utility function and if the set of his strategies is almost-compact then he has an undominated strategy. In addition to this result, in the last two chapters we show that game theory naturally generates very general, for instance non-Hausdorff topological and context structures, which shifts the traditional perception of reality in unexpected direction.

Klíčová slova

Asymetrická topologie, parciální metrika, teorie her, framology, formální pojmová analyza, framework, pretopologický systém, nedominovaná strategie, de Groot dual

Keywords

Asymmetric Topology, Partial Metric, Game Theory, Framology, Formal Concept Analysis, Framework, Pretopological System, Undominated Strategies, de Groot Dual

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I declare that I have elaborate my doctoral thesis on Topological Properties of Generalized Context Structures independently under supervision of the doctoral thesis doc. RNDr. Kovár Martin, Ph.D. and using literature and other information sources, which are all cited in the work and listed in the bibliography at the end of work. I furthermore declare that, concerning the creation of this doctoral thesis, I did not infringe the copyrights of third parties, in particular, I have not infringed any copyright. In particular, I have not unlawfully encroached on anyone's personal copyright and I am fully aware of the consequences in the case of breaking Regulation 11 and the following of the Copyright Act No 121/2000 Vol., including the possible consequences of criminal law resulted from Regulation 152 of Criminal Act No 140/1961 Vol.

In Brno ....

(author signature)

Mgr. Alena Chernikava

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Contents

1. Introduction

Topological notions and methods could be applied in a wide variety of applications in different areas of Physics, Engineering and Computer Science. They could be used not only for formulating or solving scientific problems but also in the information processing where modern topological methods and algorithms play a significant part (the information processing is an area of mathematics concerned with the properties of the space preserved under the continuous deformations). These methods provide a different levels for solving problems starting from a problem formulating in a general language of mathematic, physics and other technical sciences and ending as a kernel of different digital applications and computer programs for the information processing. The general mathematical structures usually based on the continuous representation of real or complex numbers, classic spaces usually contain only "ideal" elements obtained as a result of calculating or approximating processes. Because of the digital nature of the most applications, mathematical structures in Computer Science are different from the mathematical structures traditionally used in Mathematics.

A new trend in Mathematics is studying of objects, that do not contain the whole calculating process but only its parts (partial objects, finite objects could be observed in finite time). The inspiration comes from many areas of theoretical and practical disciplines among them are Digital Topology, Theoretical and Mathematical Physics, Theoretical Computer Science, Domain Theory, Formal Concept Analysis, Object-oriented Programming. This thesis can not pay attention to all of these aspects. The main studied aspect will be the interaction of the three areas of Mathematics: Formal Concept Analysis, General Topology and Partial Metrics. The interaction of Formal Concept Analysis with a General Topology has been already studied a little bit in some articles. We would like to mention Chu spaces that are the result of synthesis of formal concept and topological space. The main idea of that structure was introduced by M.Barr and then was developed by his student P.H. Chu in 80s. A Chu space on the set K is a triple (A, r, X) where A is a point set, X is a set of states and $r : A \times X \to K$ is a mapping that could be understated as K-valued binary relation between sets A and X. Then a topological space $(X\tau)$ is a Chu space $(X, \in, \tau)$ on an arbitrary two element set e.g. $\{True, False\}$. In the same way it is possible to represent an arbitrary formal context as a Chu space [7],[8],[9]. In [7] X. Chen and Q. Li introduce a notion of information base and study the relationships between elements in information base and approximable concepts of Chu space. The notion of approximable concept was introduced in [28] by G.-Q. Zhang and P. Hitzler and then developed in [59],[60],[61],[62]. However, in basic definitions in the Formal Concept Analysis we can find more very interesting dependencies. Also in this thesis the research in the area of frameworks, De Groot duals, game theory was made. The areas common for all these dependencies are General Topology and Formal Concept Analysis.

Every topological space (X, τ) we could understand as a formal context $(X, \tau, \in)$ with a set of objects X, a set of attributes τ and an incidence relation $\in$. It is possible to generate topologies in a natural way on the attribute and object sets with help of the incidence relation by generating close or open subbases. And these topologies deserve an independent investigation. Many questions arise in this area, for example, how the general topological properties could be represented in the formal concept analysis language. And, on the other hand, what influence could the changes in a formal context bring to its topologies? A General Topology is a highly theoretical discipline, but Formal Concept Analysis is an area of Mathematics that has a lot of different applications. For example, artificial intelligence, analysis and digital data processing, designing expert systems or work with databases. Geometric and metric properties of objects consistent with the Euclidean – Hausdorff real world around us are not consistent with the new digital structures, carrying information, and so the advanced methods used in general and digital topology could be very useful.

The first chapter mostly contains elementary background for the studied problems. In the Chapter 3 we describe the interaction of Topology, Formal Contexts and Partial Metrics. The Section 3.1 looks at the generating topologies on the formal context and the Section 3.2 looks at the constructing a partial metric on the formal context with help of measure. In the Chapter 4 we pay attention to the certain analogues of the de Groot duals in pretopological systems and in a more general approach, even for general posets. The following Chapter 5 describes the structure called framology, its duality and approximation properties in connection with topology and very general causality relationships, motivated by quantum gravity. The last Chapter 6 is concerned to game theory – especially to the one of the most fundamental results related to the existence of undominated strategies.

2. Basic Notions and Definitions

This work consists of several parts and this chapter provide theoretical background for them.

2.1. General Topology

For the reader's comfort and convenience, in this section we provide a short review of some basic notions in general topology. For more detail, see, for example, the monographs [20], [30], [56] and the paper [24] as a complementary resource. A reader, who is familiar with the most common topological notions or who is not interested in some details, may safely skip this short section.

Definition 2.1 *A topology is a family of sets τ satisfying the following two conditions*

1. *the intersection of any two members of τ is a member of τ,*
2. *union of the members of each subfamily of τ is a member of τ,*
3. *$X, \varnothing \in \tau$.*

The pair (X, τ) is called a topological space.

Definition 2.2 *The members of the topology τ on the set X are called open relative to τ. A subset $B \subseteq X$ is called closed if and only if its complement $X \setminus B$ is open.*

Definition 2.3 *The closure of a subset A of a topological space (X, τ) is the intersection of the members of the family of all closed sets containing A. Usually is denoted as $\bar{A}$ or $\mathrm{cl}\, A$.*

The function assigning to each subset A of a topological space (X, τ) the value $\bar{A} \subseteq X$ is called the closure operator. It is well-known that a closure operator can be equivalently described by so called *Kuratowski closure axioms*, see e.g. [30].

Definition 2.4 *(Closure axioms) A topological closure operator φ on a set X is a map assigning a closure $\varphi A \subseteq X$ to each subset $A \subseteq X$ if*

(c1) $\varphi\varnothing = \varnothing$,

(c2) $A \subseteq \varphi A$ *for each subset* $A \subseteq X$*,*

(c3) $\varphi\varphi A = \varphi A$ *for each subset* $A \subseteq X$*.*

(c4) $\varphi(A \cup B) = \varphi A \cup \varphi B$ *for every pair of subset* $A, B \subseteq X$*.*

It is not difficult to show that the purpose of the individual Kuratowski axioms is the following. Consider $\mathcal{C}$ the family of the subsets $A \subseteq X$, for which $\varphi A = A$. The axiom (c2) of the monotony of the closure operator provides the fact that $\mathcal{C}$ is closed under arbitrary intersections, which corresponds with the similar property of closed sets in a topological space. From this axiom it also follows that $X \in \mathcal{C}$. The closedness of $\mathcal{C}$ with respect to finite intersections is a consequence of (c4). The axiom (c3) ensures that for every $A \subseteq X$ it holds $\varphi A \in \mathcal{C}$. Finally, the purpose of (c1) is to claim that $\varnothing \in \mathcal{C}$. Hence, it is easy to see that the family of all closed sets in a topological space satisfies all the axioms (c1) – (c4) and vice versa, if all axioms (c1) – (c4) are satisfied for a family $\mathcal{C} \subseteq 2^X$, then the complements of the elements of $\mathcal{C}$ form a topology on X.

Definition 2.5 *A family* η *of sets is a base for the topology* τ*, or briefly an open base for* τ*, if the following holds:*

1. η *is a subfamily of* τ*,*

2. *for each point* x *of the space and each neighborhood* U *of* x *there is a member* V *of* η *such that* $x \in V \subset U$*.*

Instead of the previous definition it is often used its equivalent, but more comfortable formulation:

A subfamily η of a topology τ is an open base for τ if and only if each member of τ is the union of members of τ (open sets).

By the following theorem stated in [30] one can generate a topology on the set from the given arbitrary set.

Theorem 2.1 *If* ζ *is any non-void family of sets, then the family of all finite intersections of members of* ζ *is an open base for a topology for the set* $X = \cup\{S|S \in \zeta\}$*.*

According to this fact, a generalization of the open base has its natural place here.

Definition 2.6 *A family ζ of sets is called an open subbase for a topology τ if the family of finite intersections of members of ζ is an open base for τ.*

In other words a family ζ of sets is an open subbase for a topology τ if and only if each member of τ is the union of finite intersections of members of ζ. In a similar way it is possible to define the terms of closed base and closed subbase for the topology. It suffices just to replace open sets with closed sets, unions with intersections and vice versa. The mutual relationships of these notions are described by the following theorem.

Theorem 2.2 *Let X be a set, and $\zeta \subseteq 2^X$ its subsets. Let ζ^F be the family of all finite unions of elements of ζ (including the empty union, whose result is $\varnothing$). Then ζ^F is a base for the closed sets of some topology τ on X and ζ is closed subbase; or, in other words, the family $\sigma = \{X \setminus P | \, P \in \zeta^F\}$ is an open base for the topology τ.*

A natural and interesting generalization of topological spaces are minusspaces, introduced by J. de Groot in his comprehensive article [24]. Let us recall the definition.

Definition 2.7 *Let X be a set and $\rho \subseteq 2^X$ is a family of its subsets which are closed under finite unions and arbitrary intersections. Then the pair (X, ρ) is called a minusspace and the elements of the family ρ are its closed sets.*

In fact, minusspaces differ from the topological spaces in dropping the axiom stating that the sets $\varnothing, X$ are closed. One can, of course, extend every minusspace to a topological space by adding X and $\varnothing$ as closed sets. The Kuratowski closure axioms (c2)–(c4) naturally define a minusspace, however, this characterization of minusspaces is not equivalent, since there evidently exist minusspaces in which the underlying set X is not closed. Note that the notion of the closed base or subbase can be naturally extended also to minusspaces. Although the difference between minusspaces and topological spaces seems to be more or less cosmetic, it becomes more interesting if observed through a prism of the behavior of compact sets and their families.

Definition 2.8 *Let (X, τ) be a topological space. We say, that a subset $A \subseteq X$ is compact, if every open cover of A has a finite subcover.*

In particular, we do not assume any separation axiom as a part of the definition of compactness, in a consensus with the modern trend in general topology inspired and motivated by the problems of computer science, for instance as in the monograph [56]. There are many equivalent conditions ensuring compactness in topological spaces (see, e.g., [20], [30]), however, not all of them are suitable for transfer of the notion to minusspaces. Let ψ be a family of sets. We say that ψ has the *finite intersection property*, or briefly, that ψ has *f.i.p.*, if for every $P_1, P_2, \ldots, P_k \in \psi$ it follows $P_1 \cap P_2 \cap \cdots \cap P_k \neq \varnothing$. In some literature (for example, in [17]), a collection ψ with this property is called *centered*.

Definition 2.9 *Let (X, ρ) be a minusspace. We say that $A \subseteq X$ is compact, if for every family $\zeta \subseteq \rho$ such that the family $\{A\} \cup \zeta$ has f.i.p. it follows $A \cap (\bigcap \zeta) \neq \varnothing$.*

By modifying the well-known Alexander's Subbase Theorem [20] it is easy to show that the family ρ can be replaced by arbitrary closed subbase for ρ in the previous definition.

Definition 2.10 *Two minusspaces $(X, \mathcal{C})$, $(X, \mathcal{K})$ are called antispaces if $\mathcal{C}$ is the family of all compact sets for $(X, \mathcal{K})$ and vice versa, that is, $\mathcal{K}$ contains exactly all the compact sets for $(X, \mathcal{C})$.*

Now it is more clear why it could be interesting and helpful if one drops the underlying set X or $\varnothing$ from the family of closed sets for forming the pairs of antispaces – the requirement of compactness of X in the topology of the second minusspace would be too limiting. It can be easily observed from the following example.

Example 2.1 *The discrete space and the space with the cofinite topology, both on the same infinite set X, form an example of a pair of antispaces.*

In a discrete topology, the only compact sets are the finite sets, which are exactly those, which are closed in the cofinite minusspace. On the other hand, in the cofinite topology, all subsets of the underlying set X are compact. However, taking X as a closed set of the cofinite minusspace would not work, since it is not compact with respect to the discrete topology. Minusspaces are especially interesting from our point of view, since they naturally arise from formal contexts.

2.2. Formal Contexts

Formal concept analysis (FCA) was proposed by Rudolf Wille in 1982 as an attempt of restructuring lattice theory [57]. FCA works with data and data is described with a binary relationship between an object set and an attribute set. Such data appear in many areas of human activities and could be easily represented as a table. That is why practical applications were found in different fields including data mining, text mining, machine learning, hierarchical organization of web search results, software development and etc. The main goal of this theory is to restructure the data in some other form for better understanding, searching, analyzing. It is possible to say, that an object set, an attribute set and an incidence relation is some kind of an input structure and with help of formal concept analysis we could produce a concept lattice as an output structure. The concept lattice is a collection of concepts which are hierarchically ordered. But the main problem is, that this structure grows very fast and at some moment it is very difficult to analyze it. For more details the reader is referred to [4] and [57].

2.2.1. Formal Context, Formal Concept, Basic Properties

Definition 2.11 *A formal context is a triple* $(X, A, \vdash)$ *where* X, A *are sets and* $\vdash \subseteq X \times A$ *is a binary relation between them.*

In a formal concept analysis, the elements of X are called *objects* and the elements of A are called *attributes* of the context $(X, A, \vdash)$. The binary relation $\vdash$ is called the *incidence* relation. We say x has (the attribute) a or x satisfies a.

Definition 2.12 *Let* $(X, A, \vdash)$ *be a formal context,* $P \subseteq X$, $F \subseteq A$. *We put*

$$P' = \{a |\, a \in A, x \vdash a \textit{ for every } x \in P\}$$

and

$$F' = \{x |\, x \in X, x \vdash a \textit{ for every } a \in F\}.$$

Note: If $P = \{p\}$ *is a singleton, we simply write* $p' = P'$. *Similarly we write* $f' = F'$ *for* $F = \{f\}$. *The pair* (P, F) *is called a formal concept of the context* $(X, A, \vdash)$ *if* $P' = F$ *and* $F' = P$. *The mappings* $' : 2^X \to 2^A$ *and* $' : 2^A \to 2^X$ *we call the derivation operators.* P *is called the extent and* F *the intent of the concept* (P, F).

If (P, F) is a formal concept, than

1. F is the set of all common attributes for all objects in P,
2. P is the set of all objects that share all attributes in F.

The formal concept is a fundamental structure in FCA. As it was mentioned above formal concept could be represented as a cross-table, where formal concepts are maximal rectangles in it. Set of all formal concepts of a formal context is ordered with the subconcept-superconcept relation.

Definition 2.13 *For formal concepts* $(A_1, B_1), (A_2, B_2)$ *of formal context* $(X, A, \vdash)$ *we put* $(A_1, B_1) \leq (A_2, B_2)$ *if and only if* $A_1 \subseteq A_2$ *(or* $B_2 \subseteq B_1$*).*

So $(A_1, B_1) \leq (A_2, B_2)$ means that (A_1, B_1) is more specific and (A_2, B_2) is more abstract. $B_2 \subseteq B_1$ means that B_2 has less attributes than B_1 thus more objects could satisfy this condition((A_2, B_2) forms a concept). $A_1 \subseteq A_2$ means that A_1 has less objects than A_2 thus more attributes could satisfy this condition((A_1, B_1) forms a concept).

Definition 2.14 *Denote by* $\mathfrak{B}(X, A, \vdash)$ *the collection of all formal concepts of* $(X, A, \vdash)$*:*

$$\mathfrak{B}(X, A, \vdash) = \{(P, F) \in 2^X \times 2^A | P' = F, F' = P\}.$$

The set $\mathfrak{B}(X, A, \vdash)$ *equipped with the subconcept-superconcept order is called a concept lattice of context* $(X, A, \vdash)$*.*

The concept lattice $\mathfrak{B}(X, A, \vdash)$ represents all potentially interesting clusters of data information which are hidden in data $(X, A, \vdash)$.

Theorem 2.3 *(Main Theorem of concept lattices, Wille(1982))*

(1) $\mathfrak{B}(X, A, \vdash)$ *is a complete lattice with infima and suprema given by*

$$\bigwedge_{j \in J} (A_j, B_j) = (\bigcap_{j \in J} A_j, (\bigcup_{j \in J} B_j)'')$$

and

$$\bigvee_{j \in J} (A_j, B_j) = ((\bigcup_{j \in J} A_j)'', \bigcap_{j \in J} B_j)$$

(2) moreover, an arbitrary complete lattice $\mathbb{V} = (V, \leq)$ *is isomorphic to* $\mathfrak{B}(X, A, \vdash)$ *if and only if there are mappings* $\gamma : X \to V$*,* $\nu : A \to V$ *such that*

(i) $\gamma(X)$ *is* $\bigvee$*-dense in* V*,* $\nu(A)$ *is* $\bigwedge$*-dense in* V*.*

(ii) $\gamma(x) < \nu(x)$ *if and only if* $(x, y) \in \vdash$*.*

Now we will define the second derivation operator for a context $(X, A, \vdash)$ (by a composition of the first derivation operators):

(1) Map $'' : 2^X \to 2^X$ that for $P \subseteq X$, $P \mapsto P''$,

(2) Map $'' : 2^A \to 2^A$ that for $F \subseteq A$, $F \mapsto F''$.

The following lemma immediately follows from the basic theorem of the concept analysis.

Lemma 2.1 *The intersection of any family of extents is an extent.*

The derivation operators have interesting properties. They are summed up in the following proposition.

Proposition 2.1 *(Basic properties) Let $(X, A, \vdash)$ be a context and M, M_1, $M_2 \subseteq X$, $N, N_1, N_2 \subseteq A$ then*

(1) $M_1 \subseteq M_2 \Rightarrow M_2' \subseteq M_1'$, *(1')* $N_1 \subseteq N_2 \Rightarrow N_2' \subseteq N_1'$,

(2) $M \subseteq M''$, *(2')* $N \subseteq N''$,

(3) $M' = M'''$, *(3')* $N' = N'''$.

By the definition of the derivation operators it is easy to check all properties.

Definition 2.15 *A context $(X, A, \vdash)$ is called*

(1') row-clarified if for each $g, h \in X$ $g' = h'$ implies $g = h$,

(2') column-clarified if for each $m, n \in A$ $m' = n'$ implies $n = m$,

(3') clarified if it is column- and row-clarified.

If a context is row-clarified, it means, that there is no objects in object set represented with the same subset of attributes. It means that there is no objects, that we could not distinguish with the set of attributes. That means, that every object has a unique attribute representation. The same holds for the attributes in the column-clarified contexts.

2.2.2. Closure Properties

The closure properties of the second derivation operator deserve a special attention. The closure operator on the set is defined by the following three axioms as it is stated in [22].

Definition 2.16 *A closure operator φ on a set G is a map assigning a closure $\varphi X \subseteq G$ to each subset $X \subseteq G$ if* *(s1)* $X \subseteq Y \Rightarrow \varphi X \subseteq \varphi Y$ *for each subset* $X, Y \subseteq G$,

(s2) $X \subseteq \varphi X$ *for each subset* $X \subseteq G$,

(s3) $\varphi\varphi X = \varphi X$ *for each subset* $X \subseteq G$.

The closure operator on a set, defined by the previous definition we also refer more briefly as a *set closure operator*.

To understand the nature of the second derivation operator and to prove that it is a closure operator on a set we need the notion of a Galois connection.

Definition 2.17 *Let* $\varphi : P \to Q$ *and* $\psi : Q \to P$ *be maps between two ordered sets* $(P, \leq)$, $(Q, \leq)$. *Such pair of maps is called a Galois Connection between ordered sets, if satisfies next three conditions:*

(1) $p_1 \leq p_2 \Rightarrow \varphi p_1 \geq \varphi p_2$,

(2) $q_1 \leq q_2 \Rightarrow \psi q_1 \geq \psi q_2$,

(3) $p \leq \psi\varphi p$ *and* $q \leq \varphi\psi q$.

With help of the maps (that form a Galois connection) it is possible to define a closure operator by the following proposition.

Proposition 2.2 *Let* $\varphi : P \to Q$ *and* $\psi : Q \to P$ *form a Galois Connection between two ordered sets* $(P, \leq)$, $(Q, \leq)$. *Then the map* $A \mapsto \psi\varphi A$ *is a closure operator on* P *and the map* $B \mapsto \varphi\psi B$ *is a closure operator on* Q.

Theorem 2.4 *For a formal context the pair of first derivation operators forms a Galois connection.*

The proof is obvious. We need to compare the definition of the Galois connection and Proposition 2.1. From Proposition 2.2 it immediately follows that the second derivation operators are the closure operators on the object and the attribute sets respectively. This question is discussed in more detail in [22].

It is a natural question regarding the relationship between the set closure operator and a topology closure operator (Definition 2.4). It is obvious that $(c2) = (s2)$ and $(c3) = (s3)$ and $(c4) \Rightarrow (s1)$. Thus a topological closure is a set closure operator. It is clear that not every set closure is a topological closure. However, if we add the axiom $(c4)$ to axioms

$(s1) - (s3)$, could we obtain a topological closure? In other words, how to guarantee that axiom $(c1)$ holds? Unfortunately, in general, axioms $(s1) - (s3)$, $(c4)$ do not imply the axiom $(c1)$, as it is illustrated by the following example.

Example 2.2 *Let take an arbitrary set X and choose an arbitrary element $p \in X$. Consider the following operator $u(A) := A \cup \{p\}$. The axioms $(s1) - (s3)$, $(c4)$ hold, but the axiom $(c1)$ do not.*

What does $\varnothing''$ mean in the FCA language for the formal context $(X, A, \vdash)$? Let just compute it: $\varnothing'' = (\varnothing')' = X'$. But X' denotes all attributes that are common for all objects. Such attributes in the context table are represented by a fulfilled column. And there are two possible options:

1. $X' = \varnothing$, which holds if and only if there is no attribute common for all objects.

2. $X' \neq \varnothing$, which holds if and only if there is at least one attribute common for all objects. Such attributes carry a redundant information.

The same holds also for object set. That is why in this research we are interested especially in such contexts having no attribute (object) common for all objects (attributes). It means that axiom $c1$ holds for all such contexts. So from this point of view the closure operator on the set and topological closure operator are differ only in the axiom $c4$.

Remark 2.1 *Let us take a general context that possibly could have common attribute or common object. That means, that the Kuratowski axiom $c1$ does not hold. In this situation we can use minusspaces instead of topological spaces. In case of need, it is always possible to extend a minusspace to a topological space.*

2.3. Measures

A measure can be understood as a generalization of intuitive notions of length, area, volume, weight. In this research we use measure as a building stone for a certain general mathematical structure – a partial metric.

2.3.1. Algebras and σ-Algebras

In the measure theory σ-algebras play a significant role. This section introduces basic notions and definitions from that area.

Definition 2.18 *Let X be an arbitrary set. A collection Σ of subsets of X is an algebra on X if the following holds:*

(a) $X \in \Sigma$,

(b) for each set A that belongs to Σ, the set $X \backslash A$ also belongs to Σ,

(c) for each finite sequence $A_1, \ldots, A_n$ of sets that belong to Σ, the set $\cup_{i=1}^{n} A_i$ belongs to Σ.

The conditions (b), (c) required that Σ is closed under complementation and under the formation of finite unions. It is easy to check that a closure under complementation and a closure under the formation of finite unions together imply a closure under the formation of finite intersections. Thus in the definition of algebra instead of the condition (c) is possible to use the following condition (d) for each finite sequence $A_1, \ldots, A_n$ of sets that belong to Σ, the set $\cap_{i=1}^{n} A_i$ belongs to Σ. Again let X be an arbitrary set.

Definition 2.19 *A collection Σ of subsets of X is a σ-algebra on X if*

(a) $X \in \Sigma$,

(b) for each set A that belongs to Σ, the set $X \backslash A$ belongs to Σ,

(c) for each infinite countable sequence $\{A_i\}$ of sets that belong to Σ, the sets $\cup A_i$, $\cap A_i$ belong to Σ.

Thus a σ-algebra on X is a family of subsets of X that contains X and is closed under complementation, under the formation of countable unions and under the formation of countable intersections. Note that, the conditions $(a), (b)$ implies $\varnothing \in \Sigma$. Each σ-algebra on X is an algebra on X since, for example, the union of the finite sequence $A_1, A_2, \ldots, A_n$ is the same as the union of the infinite sequence $A_1, A_2, \ldots, A_n, A_n, A_n, \ldots$.

If Σ is a σ-algebra on the set X then a subset of X is called Σ-measurable if it belongs to Σ. The next example provide common used σ-algebras.

Example 2.3 *(a) Let X be a set, and let Σ be the collection of all subsets of X. Then Σ is a σ-algebra on X.*

(b) Let X be a set, and let $\Sigma = \{\varnothing, X\}$. Then Σ is a σ-algebra on X.

(c) Let X be an infinite set, and let Σ be the collection of all finite subsets of X. Then Σ does not contain X and is not closed under complementation; hence it is not an algebra (or a σ-algebra) on X.

(d) Let X be an uncountable set, and let Σ be the collection of all countable (i.e., finite or countably infinite) subsets of X. Then A does not contain X and is not closed under complementation; hence it is not even an algebra.

(e) Let X be a set, and let Σ be the collection of all subsets A of X such that either A or $X \backslash A$ is countable. Then Σ is a σ-algebra.

Proposition 2.3 *Let X be a set. Then the intersection of an arbitrary nonempty collection of σ-algebras on X is a σ-algebra on X.*

The reader should note that the union of a family of σ-algebras can fail to be a σ-algebra. This situation is described in details in [15].

Corollary 2.1 *Let X be a set, and let $\mathcal{F}$ be a family of subsets of X. Then there is a smallest σ-algebra on X that includes $\mathcal{F}$.*

The set Σ is the smallest σ-algebra on X that includes $\mathcal{F}$ means that Σ is a σ-algebra on X that includes $\mathcal{F}$ and that every σ-algebra on X that includes $\mathcal{F}$ also includes Σ. It is easy to check that the smallest σ-algebra on X that includes $\mathcal{F}$ is unique.

Definition 2.20 *The smallest σ-algebra on X that includes $\mathcal{F}$ is called the σ-algebra generated by $\mathcal{F}$ and is often denoted by $\sigma(\mathcal{F})$.*

Proposition 2.4 *Let X be a set, and let Σ be an algebra on X. Then Σ is a σ-algebra if either*

(a) Σ is closed under the formation of unions of increasing sequences of sets, or

(a) Σ is closed under the formation of intersections of decreasing sequences of sets.

2.3.2. Measures

Definition 2.21 *Let X be a set, and let Σ be a σ-algebra on X. A function μ whose domain is the σ-algebra Σ and whose values belong to the extended half-line $[0, +\infty]$ is said to be countably additive if it satisfies*

$$\mu(\cup_{i=1}^{\infty} A_i) = \Sigma_{i=1}^{\infty} \mu(A_i)$$

for each infinite sequence $\{A_i\}$ *of disjoint sets that belong to* Σ.

Definition 2.22 *A measure (or a countably additive measure) on* Σ *is a function* $\mu : \Sigma \to [0, +\infty]$ *that satisfies* $\mu(\varnothing) = 0$ *and is countably additive.*

Definition 2.23 *Let* Σ *be an algebra (not necessarily a* σ*-algebra) on the set* X. *A function* μ *whose domain is* Σ *and whose values belong to* $[0, +\infty]$ *is finitely additive if it satisfies*

$$\mu(\cup_{i=1}^{n} A_i) = \Sigma_{i=1}^{n} \mu(A_i)$$

for each finite sequence $A_1, \ldots, A_n$ *of disjoint sets that belong to* Σ.

Definition 2.24 *A finitely additive measure on the algebra* Σ *is a function* $\mu : A \to [0, +\infty]$ *that satisfies* $\mu(\varnothing) = 0$ *and is finitely additive.*

It is easy to check that every countably additive measure is finitely additive. There are, however, finitely additive measures that are not countably additive. Countably additive measures seem to be sufficient for almost all applications. We should emphasize that in this work the word "measure" (without modifiers) will always denote a countably additive measure. The expression "finitely additive measure" will always be written out in full.

Definition 2.25 *If* X *is a set, if* Σ *is a* σ*-algebra on* X, *and if* μ *is a measure on* Σ , *then the triplet* (X, Σ, μ) *is often called a measure space.*

Definition 2.26 *If* X *is a set and if* Σ *is a* σ*-algebra on* X, *then the pair* (X, Σ) *is often called a measurable space.*

If (X, Σ, μ) is a measure space, then function μ is called a measure on (X, Σ), or, if the σ-algebra Σ is clear from context, a measure on X.

Example 2.4

(a) Let X *be an arbitrary set, and let* Σ *be a* σ*-algebra on* X. *Define a function* $\mu : \Sigma \to [0, +\infty]$ *by letting* $\mu(A) = n$ *if* A *is a finite set with* n *elements and letting* $\mu(A) = +\infty$ *if* A *is an infinite set. Then* μ *is a measure; it is often called a counting measure on* (X, Σ).

(b) Let X be a nonempty set, and let Σ be a σ-algebra on X. Let x be a member of X. Define a function $\delta_x : \Sigma \to [0, +\infty]$ by letting $\delta_x(A) = 1$ if $x \in A$ and letting $\delta_x(A) = 0$ if $x \notin A$. Then δ_x is a measure; it is called a point mass concentrated at x.

Proposition 2.5 *Let (X, Σ, μ) be a measure space, and let A and B be subsets of X that belong to Σ and satisfy $A \subseteq B$. Then $\mu(A) \leq \mu(B)$. If in addition A satisfies $\mu(A) < +\infty$, then*

$$\mu(B \backslash A) = \mu(B) - \mu(A).$$

Let μ be a measure on a measurable space (X, Σ). Then μ is a finite measure if $\mu(X) < +\infty$ and is a σ-finite measure if X is the union of a sequence $A_1, A_2, \ldots$ of sets that belong to Σ and satisfy $\mu(A_i) < +\infty$ for each i. More generally, a set in Σ is σ-finite under μ if it is the union of a sequence of sets that belong to Σ and have finite measure under μ. The measure space (X, Σ, μ) is also called finite or σ-finite if μ is finite or σ-finite. If the measure space (X, Σ, μ) is σ-finite, then X is the union of a sequence $\{B_i\}$ of disjoint sets that belong to Σ and have finite measure under μ; such a sequence $\{B_i\}$ can be formed by choosing a sequence $\{A_i\}$ as in the definition of σ-finiteness, and then letting $B_1 = A_1$ and $B_i = A_i \backslash (\cup_{j=1}^{i-1} A_j)$ if $i > 1$.

Example 2.5 *(Dealing with σ-Finiteness) Note that the measure defined in Example 2.4(a) is finite if and only if the set X is finite and is σ-finite if and only if the set X is the union of a sequence of finite sets that belong to Σ. The measure defined in Example 2.4(b) is finite.*

The following propositions give some elementary but useful properties of measures.

Proposition 2.6 *Let (X, Σ, μ) be a measure space. If $\{A_k\}$ is an arbitrary sequence of sets that belong to Σ, then*

$$\mu(\cup_{k=1}^{\infty} A_k) \leq \Sigma_{k=1}^{\infty} \mu(A_k).$$

In other words, the countable additivity of μ implies the countable subadditivity of μ.

Proposition 2.7 *Let (X, Σ, μ) be a measure space.*

(a) If $\{A_k\}$ is an increasing sequence of sets that belong to Σ, then $\mu(\cup_k A_k) = \lim_k \mu(A_k)$.

(b) If $\{A_k\}$ is a decreasing sequence of sets that belong to Σ, and if $\mu(A_n) < +\infty$ holds for some n, then $\mu(\cap_k A_k) = \lim_k \mu(A_k)$.

Proposition 2.8 *Let (X, Σ) be a measurable space, and let μ be a finitely additive measure on (X, Σ). Then μ is a measure if either*

(a) $\lim_k \mu(A_k) = \mu(\cup_k A_k)$ holds for each increasing sequence $\{A_k\}$ of sets that belong to Σ, or

(b) $\lim_k \mu(A_k) = 0$ holds for each decreasing sequence $\{A_k\}$ of sets that belong to Σ and satisfy $\cap_k A_k = \varnothing$.

2.4. Partial Metrics and the Foundations of Asymmetric Topology

Partial metrics were introduced by S. G. Matthews in early 1990s. The main idea was to divide computational process on parts. In order to compute any given point $x \in X$, parts of this point have to be computed. It means, that the original space X we will extended with the parts of each point. A generalized metric space called a partial metric space is a structure that could carry such structure. It captures the structure of the original metric space and the additional partial points P by dropping the fundamental zero self-distance axiom. It is possible to distinguish the points of original space X and partial points P in the following way. The point x is a partial point, if $d(x, x) > 0$, and is the original point if $d(x, x) = 0$.

Definition 2.27 *Let X be a set, $\preceq$ a binary relation on X, which is reflexive and transitive, that is,*

(i) $x \preceq x$ for every $x \in X$,

(ii) if $x \preceq y$ and $y \preceq z$, then $x \preceq z$.

Then $\preceq$ is called a preorder on X and the pair $(X, \preceq)$ is said to be a preordered set.

The Zorn's Lemma is usually formulated for partially ordered set (briefly *posets*). However, we will rather need the following formulation of the Zorn's Lemma, which holds also for the preordered sets [43], [19].

Theorem 2.5 (Zorn's Lemma.) *A preordered set in which each chain (i.e., a linearly ordered subset) has an upper bound contains at least one maximal element.*

The classical proof of the well-known fact that Zorn's Lemma is (including in the formulation above) equivalent to the Axiom of Choice the reader can find in [19], or in a significantly simplified version in [41].

Definition 2.28 *Let (X, τ) be a topological space. For every $x, y \in X$ we put $x \ll y$ if and only if $x \in \mathrm{cl}\{y\}$. The binary relation $\ll$ is called the order (or preorder) of specialization.*

The term "asymmetric topology" is related to the *preorder of specialization*. It is the usual general topology but with accent to spaces in which the specialization preorder is non-trivial, that is, for spaces which are not T_1 in general. Of course, it is clear that for T_1 spaces the preorder of specialization is an antichain. As a founder of the discipline of asymmetric topology is usually considered Ralph Kopperman (see, e.g., [32]). Note that the binary relation $\ll$ from the previous definition is always reflexive and transitive but it is antisymmetric if and only if the space (X, τ) is T_0. In Computer Science the binary relation $\ll$ is used as an information order in which $x \ll y$ is interpreted as all the information contained in x is also contained in y. There is also studied a backward relationship between topology and the information order. The topology usually placed upon the underlying set will at least be T_0, and it will be consistent with the relation $\ll$ [42].

Definition 2.29 *Let X be a set equipped with a preorder $\ll$. The sets closed with respect to $\ll$ form a topology, which is called the topology of upwardly closed sets:*

$$T[\ll] = \{S | S \subseteq U, \textit{for all } x \in S, x \ll y \Rightarrow y \in S\}.$$

Thus, for example, for the usual partial order $\leq$ on $\mathbb{N} \cup \{\infty\}$ (the positive integers with infinity),

$$T[\leq] = \{\{n, n+1, ..., \infty\} | n \in \mathbb{N} \cup \{\infty\}\}.$$

Another term for the topology of upwardly closed sets is the *topology of upper-closed sets* or briefly *upper-closed topology*.

Definition 2.30 *A weakly order consistent topology is a topology τ on the underlying set X which is weaker than the topology of upwardly closed sets with respect to the preorder of specialization of the topological space (X, τ) .*

Our definition is slightly different (but equivalent) from the one used in [42], but more convenient since it uses a more general notion of the preorder of specialization. For more detail, the reader is referred to [42] and [23].

Definition 2.31 *Let X be a set. A quasi-metric on X is a function $q : X \times X \to \mathbb{R}$ such that for every $x, y \in X$ it holds*

(Q1) $x = y \Leftrightarrow q(x, y) = q(y, x) = 0,$

(Q2) $q(x, z) \leq q(x, y) + q(y, z).$

It is easy to see that the notion of quasi-metric arises from usual metric by dropping the axiom of symmetry.

Lemma 2.2 *For each quasi-metric $q : X \times X \to \mathbb{R}$ the binary relation $\ll_q$ on X defined by*

$$x \ll_q y \Leftrightarrow q(x, y) = 0$$

for all $x, y \in X$, is a partial order.

Lemma 2.3 *For each quasi-metric $q : X \times X \to \mathbb{R}$ the set of all open balls of the form*

$$B_q^\varepsilon(x) = \{y | y \in X, q(x, y) < \varepsilon\}$$

for each $x \in X$ and $\varepsilon > 0$ is the base for a weakly consistent topology $T[q]$ over $\ll_q$.

Lemma 2.4 *For each quasi-metric $q : X \times X \to \mathbb{R}$, the symmetrization function $q^S : X \times X \to \mathbb{R}$ for q, defined by*

$$q^S(x, y) = q(x, y) + q(y, x)$$

for every $x, y \in X$, is a metric on X, such that $T[q] \subseteq T[q^S]$.

Recall that a *contraction* on a metric space (X, d) is a mapping $f : X \to X$ such that there exist $c \in \mathbb{R}$, $0 \leq c < 1$, having the property $d(f(x), f(y)) \leq c \cdot d(x, y)$. The term of contraction can be naturally extended to a quasi-metric space if one replace a metric by a quasi-metric. Hence, there arises a natural question whether an analogue of the well-known Banach Fix-point Theorem holds also for quasi-metric spaces. The following result of S. Matthews [42] answers the question in the positive.

Lemma 2.5 *For each quasi-metric $q : X \times X \to \mathbb{R}$ such that q^S is complete, every contraction $f : X \to X$ has a unique fixed point.*

As it is noted in [42], each constant function is a contraction, and so the fixed point obtained by the quasi-metric contraction mapping theorem is not in general maximal with respect to $\ll_q$. Thus, as objects with totally defined information content will always be maximal this theorem cannot be used within Computer Science to prove that recursive definitions specify totally defined objects. The root of the problem here is that the quasi-metric gives us no way of measuring the definedness of an object, and so it yields no way of discussing total definedness. The next definition, due to S. Matthews, attempts to overcome this problem by introducing another, alternative generalization of a metric.

Definition 2.32 *A partial metric or pmetric is a function $p : X \times X \to \mathbb{R}$ such that, for every $x, y \in X$,*

(P1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,

(P2) $p(x, x) \leq p(x, y)$,

(P3) $p(x, y) = p(y, x)$,

(P4) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

The pmetric axioms $(P1)$ through $(P4)$ are intended to be a minimal generalization of the usual metric axioms such that each object does not necessarily have to have zero distance from itself. In this generalization the symmetry axiom is preserved as $(P3)$, but the usual triangle inequality was slightly generalized and took the form $(P4)$. Consequently, a metric is precisely a pmetric $p : X \times X \to \mathbb{R}$ such that, for all $x \in X$, $p(x, x) = 0$. Half of the usual first metric axiom is preserved as for all $x, y \in X$, $p(x, y) = 0 \Rightarrow x = y$. However, the converse implication does not generally hold. The value $p(x, x)$, referred to as the size or weight of x, is a feature used to describe the amount of information contained in x. The smaller $p(x, x)$ the more defined x is, being totally defined, if $p(x, x) = 0$.

Definition 2.33 *An open ball for a pmetric $p : X \times X \to \mathbb{R}$ is a set of the form*

$$B^p_\varepsilon(x) = \{y | y \in X, p(x, y) < \varepsilon\}$$

for each $\varepsilon > 0$ and each $x \in U$.

Note that, unlike their metric counterparts, some pmetric open balls may be empty. For example, if $p(x, x) > 0$ then $B^p_{p(x,x)}(x) = \varnothing$. In Matthews [42], there are stated the following theorems.

Theorem 2.6 *The set of all open balls of a pmetric $p : X \times X \to \mathbb{R}$ is an open base of some topology on X, denoted by $T[p]$.*

Theorem 2.7 *For each pmetric $p : X \times X \to \mathbb{R}$, open ball $B^p_\varepsilon(a)$, and $x \in B^p_\varepsilon(a)$, there exists $\delta > 0$ such that $x \in B^p_\delta(x) \subseteq B^p_\varepsilon(a)$.*

Using the last result it can be shown that each sequence $\{x_n | \, n = 1, 2, \ldots\} \subseteq X$ converges to an object $a \in X$ if and only if

$$\lim_{n\to\infty} p(x_n, a) = p(a, a).$$

Theorem 2.8 *Each topology $T[p]$ on X, induced by a pmetric $p : X \times X \to \mathbb{R}$, is T_0.*

It has been shown that a partial metric p can quantify the amount of information in an object x using the numerical measure $p(x, x)$, and also that p has an open ball topology.

Definition 2.34 *Let X be a set, $p : X \times X \to \mathbb{R}$ a pmetric on X. For every $x, y \in X$, we put*

$$x \ll_p y \Leftrightarrow p(x, x) = p(x, y).$$

Theorem 2.9 *For each pmetric $p : X \times X \to \mathbb{R}$, the binary relation $\ll_p$ is a partial order.*

If $x \ll y$ for an information order $\ll$ then y must have at least as much information as x. To see that $\ll_p$ does indeed have this property the following result can be obtained from axioms $(P1)$ and $(P2)$. For every $x, y \in X$ it holds

$$x \ll_p y \Rightarrow p(x, x) \geq p(y, y).$$

Thus totally defined objects are indeed maximal in the pmetric framework, and also derive an interesting result for chains. If $L = \{x_n | \, n = 1, 2, \ldots\} \subseteq X$ is a chain converging to $a \in X$, and if in addition,

$$\lim_{n\to\infty} p(x_n, x_n) = p(a, a),$$

then the least upper bound of L must exist, and it will be equal to a.

Theorem 2.10 *For each pmetric $p : X \times X \to \mathbb{R}$, it holds $T[p] \subseteq T[\ll_p]$, that is, $T[p]$ is a weakly order consistent topology over $\ll_p$.*

Theorem 2.11 *For each pmetric $p : X \times X \to \mathbb{R}$, it holds $T[p] = T[\ll_p]$ if and only if for each $x \in X$ there exists such $\varepsilon > 0$ that*

$$B^p_\varepsilon(x) = \{y|\, y \in X, x \ll_p y\} = \uparrow_{\ll_p}\{x\}.$$

From the previous theorem it follows that both topologies $T[p]$, $T[\ll_p]$ coincide if and only if each principal upper set (that is, an upper set generated by a singleton) in an open ball.

The following two theorems describe some mutual relations between pmetrics and quasi-metrics.

Theorem 2.12 *Let $p : X \times X \to \mathbb{R}$ be a pmetric. We put $q(x, y) = p(x, y) - p(x, x)$ for every $x, y \in X$. Then $q : X \times X \to \mathbb{R}$ is a quasi-metric such that $T[p] = T[q]$ and $\ll_p = \ll_q$.*

Theorem 2.13 *For each quasi-metric $q : X \times X \to \mathbb{R}$ defined on finite set X, there exists a pmetric $p : X \times X \to \mathbb{R}$ such that $T[p] = T[q]$ and $\ll_p = \ll_q$.*

An important consequence of the last theorem it is the fact that any finite partial order can be defined by a partial metric.

2.5. Game Theory

Game Theory aims to help us understand situations in which two or more decision-makers interact. Under the game everybody understands a competitive activity in which players interact with each other according to a set of rules. The major development of the game theory began in the 1920s with the work of the Emile Borel and John von Neumann. The publication "Theory of games and economic behavior" by von Neumann and Oskar Morgenstern in 1944 was the groundbreaking text that created the interdisciplinary research field of game theory. Then game-theoretic models began to be used in economic theory, political science, psychology, evolutionary theory, microeconomic theory and a wide range of other areas.

The theory of rational choice is a common component of models in game theory. The main idea is that a decision-maker chooses the best action according to his preferences among all available actions at every moment of time. In the theory of rational choice there is two components: a set A consisting of all actions available under some circumstances to a decision-maker and a payoff (utility) function, that represents a tool for comparing that actions with respect to a decision-maker's rationality choice. In any given situation a decision-maker has a subset of actions, from which a single element must be chosen. The decision-maker knows at every decision moment which action of the pair of any two actions he must take. In the worst case both actions are equally desirable. And we assume, that the transitive law holds here (if a decision-maker prefers action a to action b and action b to action c, then he prefers action a to action c). The payoff function could depend on some other person's outcome but it is not necessary. It is important to mention, that the payoff function do not tell us "how much" a decision maker prefers one action to another, because it converts only ordinal information.

All these rules and notions is possible to describe with help of the mathematical notations. It helps us to formulate any problems in the formal mathematical language. Usually in the game theory decision-makers are called players and actions are called strategies.

First, let us to recapitulate some basic notions from the theory of non-cooperative bimatrix games.

Definition 2.35 *A bimatrix game is a finite two-person game with a general sum of their payoff functions, which is represented as a pair of $m \times n$ matrices, $A = (a_{ij})$, $B = (b_{ij})$, or equivalently, as a $m \times n$ matrix (A, B), each of whose entries is an ordered pair (a_{ij}, b_{ij}). The entries a_{ij}, b_{ij} are the utility (or payoff) functions of the first and the second player, respectively, assuming they choose, respectively their i-th and j-th pure strategies.*

Definition 2.36 *A mixed strategy for a player is a probability distribution on the set of his pure strategies. In case that the player has only a finite number, say m, of pure strategies, a mixed strategy reduces to a vector $x = (x_1, x_2, \ldots, x_m)$, satisfying*

$$x_i \geq 0,$$
$$\sum_{i=1}^{m} x_i = 1.$$

Definition 2.37 *A pair of mixed strategies (x^*, y^*) for the bimatrix game (A, B) is said to be in equilibrium if, for any other mixed strategies, x and y,*

$$xAy^{*t} \leq x^*Ay^{*t},$$

$$x^* B y^t \leq x^* B y^{*t}.$$

The following theorem holds for bimatrix games and mixed strategies.

Theorem 2.14 *Every bimatrix game has at least one equilibrium point in mixed strategies.*

For the proof, the reader is referred, for instance, to [49], where there can be found some other details and related topics. It should be also noted that it is relatively easy to construct a counterexample of a bimatrix game which has no equilibrium point in pure strategies. Below there is simple, such an example.

(u, v)	A	B	C
A	$(2, 2)$	$(2, 4)$	$(2, 3)$
B	$(4, 2)$	$(3, 3)$	$(1, 5)$
C	$(3, 2)$	$(5, 1)$	$(0, 0)$

Table 2.1: A bimatrix game without equilibrium

In our future considerations, we will also work with general, normal form games of n persons and with the problem of dominance of strategies in these games.

Definition 2.38 *An n-person game G in a normal or strategic form is denoted by the $2n$-tuple $G = (X_1, X_2, \ldots, X_n, u_1, u_2, \ldots, u_n)$, where for each $i \in \{1, 2, \ldots, n\}$, X_i is a non-empty set of strategies of the i-th player and $u_i : \Pi_{j=1}^{n} X_j \to \mathbb{R}$ is his real valued utility (or payoff) function.*

Definition 2.39 *Let $i \in \{1, 2, \ldots, n\}$ and let $x_i, y_i \in X_i$ be some strategies of the i-th player. We say that the strategy y_i dominates the strategy x_i, if the following conditions hold:*

(1) For any selection of strategies $s_k \in X_k$, where $k \in \{1, 2, \ldots, n\}$, $k \neq i$,

$$u_i(s_1, s_2, \ldots, s_{i-1}, x_i, s_{i+1}, \ldots, s_n) \leq u_i(s_1, s_2, \ldots, s_{i-1}, y_i, s_{i+1}, \ldots, s_n).$$

(2) For each $k \in \{1, 2, \ldots, n\}$, $k \neq i$, there exists some strategy $t_k \in X_k$ such that

$$u_i(t_1, t_2, \ldots, t_{i-1}, x_i, t_{i+1}, \ldots, t_n) < u_i(t_1, t_2, \ldots, t_{i-1}, y_i, t_{i+1}, \ldots, t_n).$$

The strategy $x_i \in X_i$ *of the i-th player is said to be undominated if there is no strategy* $y_i \in X_i$ *which dominates* x_i.

It should be noted that this kind of dominance is sometimes referred as a *weak dominance*, in opposite to the *strict dominance*, which differs from the above defined notion at the condition (1) by the strict form $<$ of the inequality. Generally speaking a player's strategy strictly dominates another strategy if it is superior, no matter what the other players do.

Definition 2.40 *Two strategies* $x_i, y_i \in X_i$ *are called equivalent, if for any selection of strategies* $s_k \in X_k$, *where* $k \in \{1, 2, \ldots, n\}$, $k \neq i$, *it holds*

$$u_i(s_1, s_2, \ldots, s_{i-1}, x_i, s_{i+1}, \ldots, s_n) = u_i(s_1, s_2, \ldots, s_{i-1}, y_i, s_{i+1}, \ldots, s_n).$$

For more detail, see, for example, [21], [50],[51],[52]. Several natural questions may arise from these basic definitions. Does player has any undominated strategies? Under which circumstances can the undominated strategies exist? How can we find them? Some of these questions were described and solved by Herve Moulin in [45].

3. Topology as a Formal Context

There are many different mathematical disciplines. In this part of the research we intend to study an interaction of three of them: Formal Concept Analysis, Topology and Partial Metrics. It is possible to say that a formal context is a mathematical structure that is an source of some information (obvious and hidden). A formal context someone could understand as a mathematic-formal notion for the “table”. According to the different ways of extracting information from the formal context different topologies on different sets could arise. A General Topology studies the properties of topological spaces, constructing of spaces and mapping of spaces. At the beginning we need to define at least one way of obtaining a context from the topology and a topology from the context.

An arbitrary topological space for an arbitrary formal context we can construct in the next way. An arbitrary topological space (X, τ) we can interpret as a formal context $(X, \tau, \in)$ with the object set X, the attribute set τ and the incidence relation $\in$. Many other interpretations for a topological space can be found in the literature. On the contrary, from an arbitrary formal context we can construct in a natural way several different topologies with a specific properties. How could we compare objects? How could we compare information about object in different moments of time? How is it possible to measure stored information in objects at the same or different moments of time? Because information stored in formal context mostly is not a total information about objects we could describe a formal context with a partial metric.

The partial metric space contain not only the “ideal” total defined objects, but parts of objects. The main idea was to generalize metric by “dropping” the first axiom of the metric (zero self-distance). The zero self-distance axiom was replaced by a non-negative self distance axiom. What does it mean? A zero-self distance means that all information about this object/structure is already known. That is an “ideal” element. But positive self-distance means that we do not know all the details about the object. That is a partial element. The smaller self-distance is the more defined object is. As a simple example we could take a map. If we know were object is situated we could find that point. But if we do not have enough information we could only find an area where the point is situated. So, from this point of view that point is the totally defined object and the area is a partial defined object. This approach could be used in Computer Science, because at every particular moment of time we know(could compute) only a part of information about the object.

3.1. Generating the Left Topology

Definition 3.1 *(left and right topologies) Let* $(X, A, \vdash)$ *be a formal context. The topology* τ *on* X*, generated by its closed subbase* $\{a'|a \in A\}$ *is called the left topology on* $(X, A, \vdash)$*. Similarly, the right topology on* $(X, A, \vdash)$ *is the topology on* A *generated by the family* $\{x'|x \in X\}$ *used as its subbase for the closed sets.*

Results for the left and right topologies are symmetric. So we will pay attention here only to the left topology. The topological closure operator induced by this topology we will denote by cl. All closed sets in the left topology τ denote as $\mathcal{C}$.

Recall that a preorder of specialization on a topological space (X, τ) is the binary relation $\leq$ satisfying the condition $x \leq y \Leftrightarrow x \in \mathrm{cl}\{y\}$. We can rewrite this formula as

$$\mathrm{cl}\{y\} = \downarrow_{\leq} \{y\}.$$

As it was mentioned above the second derivation operator satisfies three axioms of closure $(s1) - (s3)$. If additionally if satisfies an axiom $(c4)$ then the second derivation operator is a topological closure operator. It is possible to say, that a second derivation operator plays the same significant role in the formal concept analysis, as a closure operator in the topology. The following question arises in a natural way. What is the difference between operators $''$ and cl? Under what conditions can these operators provide the same result? When is it possible to substitute one operator for the other? First of all we need to find fundamental connections between the formal context and a left topology build on it.

Theorem 3.1 *If* $(X, A, \vdash)$ *is a formal context and* τ *is its left topology on* X *then the following sets are closed subbases for the topology* τ*:*

(1) $\mathcal{C}_1 = \{a'|a \in A\}$

(2) $\mathcal{C}_2 = \{F'|F \subseteq A\}$

(3) $\mathcal{C}_3 = \{P|P \textit{ is an extent}\}$

Proof. Let us denote a topology generated by closed subbases $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ correspondingly τ_1, τ_2, τ_3. It is obvious that $\tau = \tau_1$. Let us prove the inclusion $\tau_1 \subseteq \tau_3 \subseteq \tau_2 \subseteq \tau_1$ in succession. If (P, F) is a formal concept of the context $(X, A, \vdash)$ then $P = F' \subseteq X$, $F = P' \subseteq A$. It immediately follows that $\mathcal{C}_3 \subseteq \mathcal{C}_2$ and $\tau_3 \subseteq \tau_2$. Let us take an arbitrary element $a \in A$ and denote $a' = P$, $F = P'$. Than $F' = P'' = a''' = a' = P$. Note that

(a', a'') is a formal concept and it implies $\mathcal{C}_1 \subseteq \mathcal{C}_3$ then it follows $\tau_1 \subseteq \tau_3$. It remained to prove $\tau_2 \subseteq \tau_1$. Let us take such $F \subseteq A$ that $F' = \bigcap_{a \in F} a'$. It means that F' is an intersection of sets from $\mathcal{C}_1$ closed in the topology τ_1 and then it is closed in the topology τ. Thus $\tau_2 \subseteq \tau_1$.

That means, that there are already three ways how to generate a topology from a formal context. The second one is more complicated. Because of the size. The number of elements in the subbase is the largest. The first one is good for understanding how does it work and for contexts with the small number of attributes. The third one is the most interesting, because it provides the relationship between the principal structure in the formal concept analysis - formal concept - and the basic structure in a general topology a subbase. This theorem provide a lot of possibilities. Also it should be noted that an arbitrary intersection of extents is a closed set and a finite union of extents is a closed set. It immediately follows from the fact that extents form a subbase for the topology.

Lemma 3.1 *Let (X, τ) be a topological space, $\mathcal{C}$ be a set of all closed sets, $\mathcal{C}_2$ be a subbase for $\mathcal{C}$ and $\mathcal{C}_1$ be a set of all finite unions of elements $\mathcal{C}_2$ (so $\mathcal{C}_1$ is a base for $\mathcal{C}$ generated from subbase $\mathcal{C}_2$). Then for an arbitrary element $p \in X$ holds*

$$\mathrm{cl}\{p\} = \bigcap\{C|\, C \in \mathcal{C}, p \in C\} = \bigcap\{C|\, C \in \mathcal{C}_1, p \in C\} = \bigcap\{C|\, C \in \mathcal{C}_2, p \in C\}.$$

Lemma 3.2 *Let $(X, A, \vdash)$ be a formal context and $x, y \in X$. Then $x \in y''$ if and only if $y' \subseteq x'$.*

Proof. Let us suppose that $x \in y''$. Then for every $a \in y'$ it holds $x \vdash a$ and it means that $a \in x'$. Then $y' \subseteq x'$. On the other side, let us suppose $y' \subseteq x'$. Let us take an arbitrary element $a \in y'$ then $a \in x'$. It means $x \vdash a$ and then $x \in y''$.

The main result is the following theorem.

Theorem 3.2 *Let $(X, A, \vdash)$ be a formal context, τ be its left topology on X. Then for an arbitrary element $p \in X$ it holds $\mathrm{cl}\{p\} = p''$.*

Proof. According to Lemma 3.1 it holds $\mathrm{cl}\{p\} = \bigcap\{a'|\, p \in a'\}$. Now we need to check relation between p'' and $\bigcap\{a'|\, p \in a'\}$. Suppose $x \in p''$. Let us take $a \in A$ that $p \in a'$. Now we need to prove that $x \in a'$. According to Lemma 3.2 we have $p' \subseteq x'$. Besides from formula $p \in a'$ it follows that $a \in p'$, and we can conclude $a \in x'$. It is same as

$x \in a'$, then $x \in \bigcap\{a' | p \in a'\}$. It follows that $p'' \subseteq \mathrm{cl}\{p\}$. On the other side, suppose $x \in \bigcap\{a' | p \in a'\}$. According to Lemma 3.2 we need to prove that $p' \subseteq x'$. Let us take an element $a \in p'$. It means that $p \in a'$. We know that the set a' is a closed set as an element of a closed subbase of the topological space (X, τ), it follows $\mathrm{cl}_\tau\{a\} \subseteq a'$. Then $x \in a'$ and it is equivalent to $a \in x'$. Now we can conclude that $p' \subseteq x'$. We checked that $\mathrm{cl}\{p\} \subseteq p''$.

From the previous theorem it follows that on the one-element sets a topological closure coincide with a second derivation operator.

Corollary 3.1 *Let $(X, A, \vdash)$ be a formal context, τ be its left topology on X, $\leq$ is a preorder of specialization on X equipped with the topology τ. The following statements for arbitrary elements $x, y \in X$ are equivalent:*

(1) $x \leq y$,

(2) $x \in \mathrm{cl}\{y\}$,

(3) $x \in y''$,

(4) $y' \subseteq x'$,

(5) $x'' \subseteq y''$,

Theorem 3.2 yields the possibility to construct the closure of one-element sets in an easy way. But what would it happen if we take an arbitrary set? The operators $''$ and *cl* need not necessarily be equivalent for all other sets. The second derivation operator has a lot of various properties, and it seems to be an additive operator. However, this is not true.

Example 3.1 *The second derivation operator is not additive even on finite sets.*

Let us take a context (X, A, R), where X is a set of objects, A is a set of attributes, R is a relation:

R	**a**	**b**	**c**
1	x		
2		x	
3			x

Table 3.1: Counterexample for non-additivity of second derivation operator

One-element subsets:

$1'' = \{a\}' = \{1\}$, $2'' = \{b\}' = \{2\}$, $3'' = \{c\}' = \{3\}$.

Two-element subsets:

$$\{1,2\}' = \{y|y \in A, xRy, \forall x \in \{1,2\}\} = \varnothing,$$

$$\{1,2\}'' = \varnothing' = \{1,2,3\}.$$

But here we already see, that the second derivation operator is not additive, because

$$\{1,2\}'' : \{1,2,3\} \neq 2'' \cup 3''.$$

Now, it could be easily seen, that the second derivation operator can differ from the topological closure operator generated by the left topology, not only on infinite sets, but also on finite sets (because a topological closure operator generated by the left topology is additive).

From this lemma arises a question: when are the second derivation operator and the closure operator the same?

Lemma 3.3 *Let $(X, A, \vdash)$ be a formal context. Then every extent is a closed set in the left topology:*

$$Ext(X, A, \vdash) \subseteq \mathcal{C}.$$

A closed set need not necessarily be an extent.

Proof. This $Ext(X, A, \vdash) \subseteq \mathcal{C}$ directly follows from the Theorem 3.1. For the second part of the theorem we would provide a counterexample. Let us take a finite context $(\{1,2,3\}, \{a,b,c,d\}, \vdash)$, where the relation $\vdash$ is represented by the table:

$\vdash$	**a**	**b**	**c**	**d**
1	x			
2		x	x	x
3			x	

Table 3.2: A counterexample – closed, but not an extent.

The set of all closed sets is $\mathcal{C} = \{\varnothing, \{1\}, \{2\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}$. The set of all extents is $Ext(X, A, \vdash) = \{\{1\}, \{2\}, \{2,3\}, \{1,2,3\}\}$. We see, that set $\{1,2\}$ is closed, but it is not an extent.

Theorem 3.3 *Let $(X, A, \vdash)$ be a formal context and τ is its left topology. Let us denote $Ext^F(X, A, \vdash)$ the set of all finite unions of extents. If set X is finite, then*

$$Ext^F(X, A, \vdash) = \mathcal{C}.$$

Proof. $Ext^F(X, A, \vdash) \subseteq \mathcal{C}$ is obviously true. Now it remains to prove $\mathcal{C} \subseteq Ext^F(X, A, \vdash)$. Let Y be an arbitrary closed set in the left topology. Then by the Theorem 3.1 from the left topology we can easily obtain

$$Y = \cap_{J_{arb}} \cup_{I_{fin}} M_{i,j}$$

where $M_{i,j} \in Ext(X, A, \vdash)$. Because X is a finite set, then J_{arb} is a finite set too. So we have

$$Y = \cap_{J_{fin}} \cup_{I_{fin}} M_{i,j} = \cup_{I_{fin}} \cap_{J_{fin}} M_{i,j}.$$

It means, that every closed set we can represent as a finite union of some extents.

Example 3.2 *There exists a context $(X, A, \vdash)$ for which*

$$Ext^F(X, A, \vdash) \neq \mathcal{C}.$$

Let Φ be a family containing $\varnothing$, all singletons $\{r\}$, where $r \in \mathbb{R}$ and closed intervals $[a, b]$, where $a, b \in \mathbb{R}$ and $a < b$. It is easy to see that Φ is closed under the finite intersections. Consider the context $(X, A, \vdash)$, where $X = \mathbb{R}, A = \Phi$ and $\vdash$ stands for the relation $\in$ (membership relation). Let us admit the following $a' = \{s|x \in X, x \in a\} = a$. If (P, F) is a formal concept of $(X, A, \vdash)$ then $P = F' = \{x|x \in \mathbb{R}, x \in a$ for all $a \in F\} = \cap_{a \in F} a' = \cap F \in \Phi$. The $\cap F \in \Phi$ holds, because $F \subseteq \Phi$ and obvious the intersection of F again lies in Φ. Hence, $Ext(X, A, \vdash) \subseteq \Phi$. Conversely, let $a \in A = \Phi$. Since (a', a'') is a formal concept, $a' = a$ is an extent. So $Ext(X, A, \vdash) = \Phi$. We will show that Φ^F is not closed under infinite intersections. Consider the sequence $D_0, D_1, D_2, \ldots$ where $D_0 = [0, 1], D_1 = [0, 1/3] \cup [2/3, 1], D_2 = [0, 1/9] \cup [2/9, 3/9] \cup [6/9, 7/9] \cup [8/9, 1] \ldots$ Set $\mathbb{D} = \cap_{i=0}^{\infty} D_i$. It is easy to check that $D_0, D_1, D_2, \dots \in \Phi^F$ but $\mathbb{D}$, which forms so called Cantor Discontinuum, is not a union of finitely many closed intervals. Thus $Ext^F(X, A, \vdash \ ) = \Phi^F$ is not closed under arbitrary intersection, so it cannot be equal to $\mathcal{C}$.

The following definition could be found for instance in [33].

Definition 3.2 *Let (X, τ) be a topological space. A set $B \subseteq X$ is a saturated set in the topology τ on X if it is an intersection of open sets.*

It is easy prove that if the binary relation $\leq$ is the preorder of specialization on the topological space (X, τ), then a set B is saturated if and only if

$$B = \uparrow_{\leq} \{B\} = \{x | x \in X, \text{ for some } a \in B \text{ it holds } a \leq x\}.$$

Theorem 3.4 *Let $(X, A, \vdash)$ be a formal context and (X, τ) be a left topology on it. Then for an arbitrary set $P \subseteq X$ the following statements are equivalent:*

(1) P is a saturated set,

(2) $P = \uparrow_{\leq} \{P\}$,

(3) $\{x | x \in X \smallsetminus P, P \cap x'' \neq \varnothing\} = \varnothing$,

(4) $\forall x \in X \smallsetminus P, P \cap x'' = \varnothing$.

Proof. The equivalence of the (1) and (2) expressions had been already mentioned above.

At first let us have a look at the definition of the upper-set $\uparrow_{\leq} \{P\}$. $\uparrow_{\leq} \{P\} = \{x| \text{ for some } a \in P \text{ holds } a \leq x, x \in X\} = \{x| \text{ for some } a \in P \text{ holds } a \in x'', x \in X\}$ $= \{x | x \in X, P \cap x'' \neq \varnothing\} = \{x | x \in P, P \cap x'' \neq \varnothing\} \cup \{x | x \in X \smallsetminus P, P \cap x'' \neq \varnothing\} = P \cup \{x | x \in X \smallsetminus P, P \cap x'' \neq \varnothing\}$. If $P = \uparrow_{\leq} \{P\}$ we can conclude that $\{x | x \in X \smallsetminus P, P \cap x'' \neq \varnothing\} \subset P$ must hold, because $P \cup \{x | x \in X \smallsetminus P, P \cap x'' \neq \varnothing\} = P$. But there is only one possibility $\{x | x \in X \smallsetminus P, P \cap x'' \neq \varnothing\} = \varnothing$ (because obviously it is a subset of $X \smallsetminus P$). Conversely, if $\{x | x \in X \smallsetminus P, P \cap x'' \neq \varnothing\} = \varnothing$, it is clear that $P = \uparrow_{\leq} \{P\}$.

The equivalence of the (3) and (4) is obvious.

We have already described in details a transformation from a formal context to the left topology. Let us have a look at the opposite task. We need to construct a context from a topology. The simplest context we can generate in an easy way by the definition of the left topology.

Theorem 3.5 *Let us take a topological space (X, τ) and ζ is its subbase. Then a formal context $(X, \zeta, \in)$ generates a left topology (X, τ).*

Corollary 3.2 *Let (X, τ) be a topological space, Ω be a closed base for this topology. Then $(X, \Omega, \in)$ is a one of such contexts, where the left topology is (X, τ).*

Definition 3.3 *Contexts are called topologically equivalent if they generate the same left topology.*

So we can describe a topology as a context with a membership relation. Can we obtain all possible contexts, which generate the same left topology? Or in the other words, can we find all topologically equivalent contexts? Let us split this question into several parts. Suppose that an incidence relation is a membership relation. Can we find all topological equivalent contexts with membership relation?

Lemma 3.4 *For every topology τ we denote a set of all closed bases and all closed subbases as χ. Then the set of all contexts (where relation is a membership relation) with the the same left topologies has the form $\{(X, A, \in) | A \in \chi\}$.*

Proof. Let us prove the Lemma by the contradiction. Suppose that there is a context $(X, B, \in) \notin \{(X, A, \in)|A \in \chi\}$ which has the left topology τ. Then $\{b'|b \in B\}$ (where $'$ means the derivation operator) is a closed subbase of τ. And obviously $b = b'$. So the set B is a subbase for the left topology τ.

Now, consider the context $(X, A, \vdash)$. We will replace every attribute $a \in A$ with $a' \subset X$. That will not change the structure of the context, it is just a renaming of the attribute set. And we will obtain context $(X, A_1, \vdash_1)$. In this context set A_1 consists of some subsets of X. And for this context for every attribute $a \in A_1$ holds $a' = a$. Now we can easily replace relation $\vdash_1$ by relation $\in$. Again, the structure of the context stays the same. So by means of the "renaming" we can describe every context in terms of the membership relation. The only problem is the duplicated attributes. But we can fix it if we assume, that we will work with a column-clarified context (row-clarified contexts) for the generating left(right) topologies.

3.2. Partial Metrics on a Quotient Context

Let μ be a finite counting measure. The finite counting measure is an intuitive way to put a measure on any finite set. A measure of the set is taken to be a number of its elements:

$$\mu(A) = |A|.$$

Lemma 3.5 *Let us take an arbitrary finite set A. Let μ be the finite counting measure as on A. The function $p : 2^A \times 2^A \longrightarrow \mathbb{R}^+$ constructed as*

$$p(x, y) = \mu(x \cup y), \text{ where } x, y \subseteq A$$

is a partial metric on the set 2^A.

Proof. We need to check all axioms of a partial metric.

(1) Notice that the measure is a monotonic function, so $x \cap y \subseteq y$ implies $\mu(y) \geq \mu(x \cap y)$. Using this fact we can deduce $p(x, y) = \mu(x \cup y) = \mu(x) + \mu(y) - \mu(x \cap y) \geq \mu(x) = p(x, x)$.

(2) Axiom $(p2)$ is clear.

(3) Let us calculate $p(x, z) - [p(x, y) + p(y, z) - p(y, y)] = \mu(x \cup z) - [\mu(x \cup y) + \mu(y \cup z) - \mu(y)] = \mu(x) + \mu(z) - \mu(x \cap z) - [\mu(x) + \mu(y) - \mu(x \cap y) + \mu(y) + \mu(z) - \mu(y \cap z) - \mu(y)] = -\mu(x \cap z) - \mu(y) + \mu(x \cap y) + \mu(y \cap z) = -\mu(x \cap z \cap y) - \mu((x \cap z)\backslash y) - [\mu((y \cap x) \cap z) + \mu((y \cap x)\backslash z) + \mu((y \cap z)\backslash x) + \mu((y\backslash x)\backslash z)] + \mu((x \cap y) \cap z) + \mu((x \cap y)\backslash z) + \mu((y \cap z) \cap x) + \mu((y \cap z)\backslash x) = -\mu((x \cap z)\backslash y) - \mu((y\backslash x)\backslash z)]$. So we have $p(x, z) - [p(x, y) + p(y, z) - p(y, y)] = -\mu((x \cap z) - y) - \mu((y - x) - z)] < 0$ and it means that $p(x, z) < p(x, y) + p(y, z) - p(y, y)$

(4) for a finite case it is obvious (but for infinite sets it does not hold).

It is necessary to mention that, if we take the same subsets ($x = y$), than a partial metric could be calculated as $p(x, x) = \mu(x \cup x) = \mu(x)$. And now we have a good tool for combining a partial metric and a left topology generated by a formal context. Let us return to it. In a natural way we can introduce a partial metric on the set X. We can easily see that information that is carried in a partial-metric function is enough. Let us take a finite context $(X, A, \vdash)$. We have already build a left topology. Now we are interested in a relationship between a partial metric and a left topology. The left topology is defined on set X. Let us define a partial metric on it. Because of the structure of a formal context in general, we can easily rename objects (or attributes) with set of attributes (respectively objects). So let us rephrase previous lemma for our purposes. Consider a context $(X, A, \vdash)$ and a measure μ on the set A. A measure is a function defined on the appropriate σ-algebra $\Sigma \subseteq 2^A$. According to the notation of Definition 2.20, let $\Sigma = \sigma(A \cup \{x'|x \in A\})$ be the smallest σ-algebra containing the set $A \cup \{x'|x \in A\}$.

Lemma 3.6 *Let us take a finite row-clarified formal context* $(X, A, \vdash)$. *Given a finite counting measure* $\mu : \Sigma \to \mathbb{R}^+$ *on the finite set* A *(where* Σ *is a* σ*-algebra on* A*), define a function* $p : X \times X \longrightarrow \mathbb{R}^+$

$$p(x, y) = \mu(x' \cup y'), \qquad \text{where } x, y \in X.$$

Then p *is a partial metric on* X.

Proof. Again we need to check all axioms of a partial metric. Similary as in previous Lemma 3.5 we can easily check axioms $(p1), (p2), (p3)$.
For $(p4)$ first we will prove that for all $x, y \in X, \quad x = y$ implies $p(x, y) = p(x, x) = p(y, y)$. It is obvious(from the definition of the derivation operators on a formal context) that $x = y$ implies $x' = y'$ and then $x' = y' = x' \cup y'$. Then $\mu(x') = \mu(y') = \mu(x' \cup y')$. Thus $p(x, y) = p(x, x) = p(y, y)$.

The most interesting part is to prove that

$$\text{for all } x, y, \in X \quad p(x, y) = p(x, x) = p(y, y)$$

implies $x = y$. From $p(x, y) = p(x, x) = p(y, y)$ we can obtain $\mu(x') = \mu(y') = \mu(x' \cup y')$. Thus $\mu(x' \backslash y') = \mu(y' \backslash x') = 0$. Then it implies $x' \subseteq y'$ since μ is a finite counting measure. And similarly $y' \subseteq x'$. Thus $x' = y'$. And finally $x = y$ simply because the formal context $(X, A, \vdash)$ is row-clarified. Thus for all $x, y, \in X \quad p(x, y) = p(x, x) = p(y, y) \Rightarrow x = y$.

Lemma 3.7 *Let take a row-clarified context $(X, A, \vdash)$, where A is a finite set. Let us denote a counting finite measure μ on the set A. Let p be a partial metric on X generated by a counting finite measure μ on A. Then*

$$\ll_p = \preceq,$$

where $\preceq$ is a specialization preorder for a left topology generated on the context $(X, A, \vdash)$.

Proof. Suppose that $x \ll_p y$, then by the definition 2.34 of the binary relation $\ll_p$ we have $p(x, x) = p(x, y)$. The partial metric p is generated by the counting finite measure μ, hence $\mu(x') = \mu(x' \cup y')$. Then $\mu(x') = \mu(x') + \mu(y') - \mu(x' \cap y')$ and then $\mu(x' \cap y') = \mu(y')$. Now we divide the set y' into two disjoint sets $y' = (y' \cap x') \cup (y' \backslash x')$. Then $\mu(y') = \mu(y' \backslash x') + \mu(x' \cap y')$ and it immediately follows $\mu(y' \backslash x') = 0$. Because μ is a counting finite measure then $y' \subseteq x'$. And by the corollary 3.1 we have $x \preceq y$.
Now let us suppose that $x \preceq y$. By the corollary 3.1 this is equivalent to $y' \subseteq x'$. It is obvious that $y' \backslash x' = \varnothing$. Then by the property of measure holds $\mu(y' \backslash x') = 0$. From the $y' = (y' \backslash x') \cup (x' \cap y')$ by the definition of the measure we obtain $\mu(y') = \mu(y' \backslash x') + \mu(x' \cap y')$. Hence $\mu(y') = \mu(x' \cap y')$. By adding a $\mu(x')$ to the every side of equation we have $\mu(x') + \mu(y') = \mu(x') + \mu(x' \cap y')$. Thus $\mu(x') + \mu(y') - \mu(x' \cap y') = \mu(x')$. Then $\mu(x') = \mu(x' \cup y')$. Then we obtain $p(x, x) = p(x, y)$. And $x \ll_p y$ follows by the definition 2.34 of $\ll_p$ binary relation.

Corollary 3.3 *Let us take a row-clarified context* $(X, A, \vdash)$*, where* A *is a finite set. Then the specialization preorder* $\preceq$ *for a left topology is a partial order.*

Proof. Immediately follows from the previous Lemma and Theorem 2.9.

Now we have a possibility for constructing a partial order on any set, with help of some additional finite set with defined counting finite measure. Now let us consider a context $(X, A, \vdash)$, where the set A is not necessarily be a finite set. What is the difference from the previous case? The main problem is in the $(p4)$ axiom of a partial metric. The problem is that in general $\mu(a) = 0$ does not induce $a = \varnothing$, and if a context is not row-clarified, then $x' = y'$ does not imply $x = y$. It means that such objects we can hardly distinguish. So from this point of view they are equivalent. In a natural way we can define equivalence classes and construct a quotient context.

Definition 3.4 *Let us take a context* $(X, A, \vdash)$*, where sets* X, A *are not necessarily finite sets. A relation* $\vdash$ *is a relation on* $X \times A$*. A function* $\mu : \Sigma \to \mathbb{R}^+$ *is a general (i.e., countably-additive) measure (where* Σ *is a* σ*-algebra). Relation* $R_\vdash = \{(Q, W) | Q, W \subseteq A \text{ and } \mu(W \div Q) = 0\}$ *is called an attribute relation.*

Lemma 3.8 *On the context* $(X, A, \vdash)$ *with given general, countably-additive measure* μ *on the set* A*, an attribute relation* $R_\vdash$ *is an equivalence relation on* 2^A*.*

Proof. We will check the axioms of the relation of equivalence.

(1) Reflexivity. Let us take a pair (Q, Q). It is obvious that $Q \div Q = \varnothing$ then $\mu(Q \div Q) = 0$.

(2) Symmetry. Let us take a pair $(Q, W) \in R_\vdash$, where $Q \neq W$. It means that $\mu(W \div Q) = 0$, then $\mu(Q \div W) = 0$, then $(W, Q) \in R_\vdash$

(3) Transitivity. Let us take pairs $(Q, W), (W, T) \in R_\vdash$. We need to prove that $(Q, T) \in R_\vdash$, this is equivalent to $\mu(Q \div T) = 0$. At first let us prove that $\mu(Q \backslash T) = 0$. The difference between sets Q and T we can represented as $Q \backslash T = ((Q \backslash W) \cup (Q \cap W)) \backslash T = ((Q \backslash W) \backslash T) \cup ((Q \cap W) \backslash T)$. Let us have a look at the first part $(Q \backslash W) \backslash T$. $(Q \backslash W) \backslash T \subseteq Q \backslash W$ implies $\mu((Q \backslash W) \backslash T) = 0$ because measure is a monotonic function and $\mu(Q \backslash W) = 0$. Now let us have a look at the second part $(Q \cap W) \backslash T$. $(Q \cap W) \backslash T \subseteq W \backslash T$ implies $\mu((Q \cap W) \backslash T) = 0$ because measure is a monotonic function and $\mu(W \backslash T) = 0$. Hence $\mu(Q \backslash T) = \mu((Q \backslash W) \backslash T) + \mu((Q \cap W) \backslash T) = 0$. In the same way it easy to obtain $\mu(T \backslash Q) = 0$. Thus $(Q, T) \in R_\vdash$.

Because $R_\vdash$ is an equivalence on the set A, than it has equivalence classes. We will denote equivalence classes as $[Q] = \{W|\mu(Q \div W) = 0\}$. And we can construct a quotient set $A|_{R_\vdash}$ with operations $\sqcap$ and $\sqcup$. Let us denote operations $\sqcap$ and $\sqcup$ as

$$[Q] \sqcap [W] = [Q \cap W]$$

and

$$[Q] \sqcup [W] = [Q \cup W].$$

Lemma 3.9 *For the equivalence relation $R_\vdash$ on the formal context $(X, A, \vdash)$ operations $\sqcap$ and $\sqcup$ on a quotient set $A|_{R_\vdash}$ are well-defined. Operations are defined as follows:*

$$[Q] \sqcap [W] = [Q \cap W]$$

and

$$[Q] \sqcup [W] = [Q \cup W].$$

Proof. Operation $\sqcup$ is well-defined if $\eta_1 R_\vdash \eta_2$ and $\xi_1 R_\vdash \xi_2$ implies $((\eta_1 \cup \xi_1), (\eta_2 \cup \xi_2)) \in R_\vdash$. It means that we need to prove $\mu((\eta_1 \cup \xi_1) \div (\eta_2 \cup \xi_2)) = 0$. Let us have a look at $(\eta_1 \cup \xi_1) \backslash (\eta_2 \cup \xi_2)$. By simple transformations we can obtain $(\eta_1 \cup \xi_1) \backslash (\eta_2 \cup \xi_2) = (\eta_1 \cup \xi_1) \backslash \eta_2 \backslash \xi_2 = (\eta_1 \backslash \eta_2 \backslash \xi_2) \cup (\xi_1 \backslash \eta_2 \backslash \xi_2) = (\nabla_1 \backslash \xi_2) \cup (\xi_1 \backslash \xi_2 \backslash \eta_2) = (\nabla_1 \backslash \xi_2) \cup (\Delta_1 \backslash \eta_2)$, where $\nabla_1 = \eta_1 \backslash \eta_2$ and $\Delta_1 = \xi_1 \backslash \xi_2$. It is obvious that $\eta_1 R_\vdash \eta_2$ and $\xi_1 R_\vdash \xi_2$ implies respectively $\mu(\nabla_1) = 0$ and $\mu(\Delta_1) = 0$. Thus $\mu(\nabla_1 \backslash \xi_2) = 0$ and $\mu(\Delta_1 \backslash \eta_2) = 0$. And hence $\mu((\eta_1 \cup \xi_1) \backslash (\eta_2 \cup \xi_2)) = 0$. In the same way $\mu((\eta_2 \cup \xi_2) \backslash (\eta_1 \cup \xi_1)) = 0$ could be obtained. And it means that $((\eta_1 \cup \eta_2), (\xi_1 \cup \xi_2)) \in R_\vdash$. Now we can conclude, that the operation $\sqcup$ is well-defined. By the same token we could prove that the operation $\sqcap$ is well-defined too.

The relation $R_\vdash$ is defined on the set A. But the goal was to define equivalence classes on the set X, which arise from the set A in a some way. The relation $R_\vdash$ in a natural way induces a relation $S_\vdash = \{(x, y)|x, y \in X, (x', y') \in R_\vdash\}$ on the set X. Generally speaking, we obtain relation $S_\vdash$ with help of renaming the elements of X and A. It is obvious, that $S_\vdash$ is the equivalence relation too. The equivalence classes on the set X are denoted as $[x]$. Now let us define a function $p : X|_{S_\vdash} \times X|_{S_\vdash} \to \mathbb{R}^+$ with help of the measure μ defined on the set A:

$$p([x], [y]) = \mu(x' \cup y')$$

Theorem 3.6 *For the formal context $(X, A, \vdash)$ a function $p : X|_{S_\vdash} \times X|_{S_\vdash} \to \mathbb{R}^+$ defined as*

$$p([x], [y]) = \mu(x' \cup y')$$

is a partial metric on the set $X|_{S_\vdash}$.

Proof. We need to check all axioms of a partial metric

1. Notice that measure is a monotonic function, then $x' \cap y' \subseteq y'$ implies $\mu(y') \geq \mu(x' \cap y')$. Using this fact we can deduce $p([x], [y]) = \mu(x' \cup y') = \mu(x') + \mu(y') - \mu(x' \cap y') > \mu(x') = p([x], [x])$.

2. This axiom is obvious

3. Let us compute $p([x], [z]) - [p([x], [y]) + p([y], [z]) - p([y], [y])] = \mu(x' \cup z') - [\mu(x' \cup y') + \mu(y' \cup z') - \mu(y')] = \mu(x') + \mu(z') - \mu(x' \cap z') - [\mu(x') + \mu(y') - \mu(x' \cap y') + \mu(y') + \mu(z') - \mu(y' \cap z') - \mu(y')] = -\mu(x' \cap z') - \mu(y') + \mu(x' \cap y') + \mu(y' \cap z') = -\mu(x' \cap z' \cap y') - \mu((x' \cap z') \backslash y') - [\mu((y' \cap x') \cap z') + \mu((y' \cap x') \backslash z') + \mu((y' \cap z') \backslash x') + \mu((y' \backslash x') \backslash z')] + \mu((x' \cap y') \cap z') + \mu((x' \cap y') \backslash z') + \mu((y' \cap z') \cap x') + \mu((y' \cap z)' \backslash x') = -\mu((x' \cap z') \backslash y') - \mu((y' \backslash x') \backslash z')]$. So we have $p([x], [z]) - [p([x], [y]) + p([y], [z]) - p([y], [y])] = -\mu((x' \cap z') \backslash y') - \mu((y' \backslash x') \backslash z') < 0$ and it means that $p([x], [z]) < p([x], [y]) + p([y], [z]) - p([y], [y])$.

4. As it was mentioned above, without equivalence classes this does not hold in a general case. But with equivalence classes on the set X function p satisfies this axiom. Suppose $[x] = [y]$, then obviously $p([x], [y]) = p([x], [x]) = p([y], [y])$. Now suppose that $p([x], [y]) = p([x], [x]) = p([y], [y])$. Thus $\mu(x' \cup y') = \mu(x') = \mu(y')$. Doing as in lemma[cislo] we can obtain $\mu(x' \backslash y') = \mu(y' \backslash x') = 0$. This means $[x'] = [y']$. And this equality by the definition of the equivalence classes on the set X immediately implies $[x] = [y]$.

Function p satisfies (p1)-(p4) axioms of a partial metric.

Now we are close to the construction of a quotient context for the formal context $(X, A, \vdash\ )$. The main goal of using a quotient context is to "eliminate" objects that behave in the same way. The main idea is to replace the set X with a quotient set $X|_{S_\vdash}$. But what has to happen to the attribute set A? There are two options. The first option is to change the attribute set to the quotient attribute set. Then some information would be lost, and we will not have any possibility to restore neither the original context neither full information. The other option is to keep remaining the attribute set unchanged. Hereupon we could save as much information as it is possible. So, the best option is the second one. And the last element of the formal context is the relation $\vdash$. It is obvious that it could not remain the same. We need to construct an induced binary relation $\models$.

A measure μ is a function defined on a σ-algebra $\Sigma \subseteq 2^A$ as $\mu : \Sigma \to \mathbb{R}^+$. Sets of the form $x' \subseteq A$ must be measurable, that means $x' \in \Sigma \quad \forall x \in X$. The relation $S_\vdash$ on X is described as $S_\vdash = \{(x, y)|\mu(x' \div y') = 0\}$. For a context $(X, A, \vdash)$ and measure μ a quotient context is a context of the form $(X|_{S_\vdash}, A, \models)$. A binary relation $\models$ is possible to define in two different ways and each of them has its advantages and disadvantages. A maximal envelope means that we create a super object $p = [x]$, that has an attribute a if at least one object has has attribute. If we represent that context in cross-table wiew we would have a maximal relation $\vdash$. But a minimal envelope means that object $p = [x]$ has an attribute a onle if all object from the equivalence class has that attribute. It means, that relation $\vdash$ would have a minimal number of elements.

1. "Union" - Maximal envelope
   $[x] \models a \iff \exists y \in [x]$, that $y \vdash a(a \in y')$. So a relation has the following form $\models = \{([x], a)|\exists y \in [x]$, that $a \in y'\} = \{([x], a)|a \in \cup_{y\in[x]} y'\}$. If we compare this denotation with the definition of the first derivation operator on a formal context, then we can say, that $[x]'_\models = \cup_{y\in[x]} y'$.

2. "Intersection" - Minimal envelope
   $[x] \models a \iff \forall y \in [x]$, that $y \vdash a(a \in y')$. So a relation has the following form $\models = \{([x], a)|\forall y \in [x]$, that $a \in y'\} = \{([x], a)|a \in \cap_{y\in[x]} y'\}$. If we compare this denotation with the definition of the first derivation operator on a formal context, then we can say, that $[x]'_\models = \cap_{y\in[x]} y' = [x]'$.

"Union" means that we generalize equivalent objects with the super object, that coincide with every "internal" object. But "Intersection" just creates a new "minimal" object. At this step, let us sum up. We have a context $(X, A, \vdash)$, where sets X, A are not necessarily finite sets, a formal context need not necessarily be row-clarified, $\vdash$ is a relation on $X \times A$. A function $\mu : \Sigma \to \mathbb{R}$ is a general countably-additive measure. Measure is a function defined on the appropriate σ-algebra $\Sigma \subseteq 2^A$, so $A \in \Sigma$, and for every object $x \in X$, set x' lies in Σ, as we already stated above. The equivalence relation $S_\vdash$ was constructed on the set X. Then a new quotient context $(X_{S_\vdash}, A, \models)$ was defined. Because the new quotient context must behave well, it is necessary to prove that the set $[x]'$ is measurable $\forall x \in X$. Formal concepts may arise or change in some applications at some moment of physical time. While the set of objects is usually permanent or changes only a little, the set of attributes may significantly increase depending on time. In most cases, the changes are only finite in a time interval of a finite length. Hence, a countable set A of attributes can cover all these changes and it can be used as a universal attribute set in a model of such applications. Note that for the proof of the following lemma, the assumption of countability of the set A is essential.

Lemma 3.10 *Let Σ be a σ-algebra and μ is a measure. Then for the countable sequence $A_i \in \Sigma$ that $\mu(A_i \div A_j) = 0, \quad i, j \in \mathbb{N}$ sets $\cup A_i$ and $\cap A_i$ are measurable and*

$$\mu(\cup A_i) = \mu(\cap A_i) = \mu(A_1).$$

Proof. At first let us prove that $\mu(\cup A_i) = \mu(A_1)$. A proof consists of two parts.

1. Suppose that $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ where $\mu(A_{i+1} \backslash A_i) = 0$. Let us denote $B_i = A_{i+1} \backslash A_i$ and then $A_{i+1} = B_i \cup A_i$ (because of inclusion). We should notice, that sets B_i and A_i are disjoint. Also it is obvious that $\mu(B_i) = 0$. For every $j \in \mathbb{N}$ set A_j could be represented as $A_j = A_{j-1} \cup B_{j-1} = \cdots = A_1 \cup B_1 \cup \cdots \cup B_{j-1}$. Then $A_j \backslash A_1 = \cup_{n=1}^{j-1} B_n$ because sets $A_1, B_1, \ldots, B_{j-1}$ are disjoint. Thus $\mu(A_j \backslash A_1) = \mu(\cup_{n=1}^{j-1} B_n) = \cup_{n=1}^{j-1} \mu(B_n) = 0$. Let us calculate $\mu(\cup_{i=1}^{\infty} A_i) = \mu(A_1 \cup (\cup_{i=2}^{\infty} A_i)) = \mu(A_1 \cup ((\cup_{i=2}^{\infty} A_i) \backslash A_1)) = \mu(A_1 \cup (\cup_{i=2}^{\infty} (A_i \backslash A_1))$. Hence $\mu(\cup_{i=1}^{\infty} A_i) = \mu(A_1) + \Sigma_{i=2}^{\infty} \mu(A_i \backslash A_1) = \mu(A_1)$

2. Suppose we have an arbitrary countable sequence of sets $\{A_i\}$. Let us construct a system of sets $\{E_i\}$ by denoting $E_i = \cup_{k=1}^{i} A_k$. Let us calculate a set $E_{i+1} \backslash E_i = \cup_{k=1}^{i+1} A_k \backslash \cup_{j=1}^{i} A_j = \cup_{k=1}^{i+1} (A_k \backslash \cup_{j=1}^{i} A_j) \subseteq \cup_{k=1}^{i+1} (A_k \backslash A_i) \subseteq \cup_{k=1}^{i+1} (A_k \div A_i)$. Thus $\mu(E_{i+1} \backslash E_i) \leq \Sigma_{k=1}^{i+1} \mu(A_k \div A_i) = 0$. Hence $\mu(E_{i+1} \backslash E_i) = 0$. By the construction $E_1 \subseteq E_2 \subseteq E_3 \subseteq \ldots$ holds. It is obvious, that $\mu(\cup_{i=1}^{\infty} A_i) = \mu(\cup_{i=1}^{\infty} E_i)$. Hence $\mu(\cup_{i=1}^{\infty} A_i) = \mu(E_1) = \mu(A_1)$.

We have proved $\mu(\cup A_i) = \mu(A_1)$. Let us prove the other part $\mu(\cap A_i) = \mu(A_1)$. Again a proof consists of two parts.

1. Suppose that $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$ where $\mu(A_i \backslash A_{i+1}) = 0$. It is obvious that $A_1 \backslash A_j = (A_1 \backslash A_2) \cup (A_2 \backslash A_j) = (A_1 \backslash A_2) \cup (A_2 \backslash A_3) \cup \cdots \cup (A_{j-1} \backslash A_j)$. The sets of

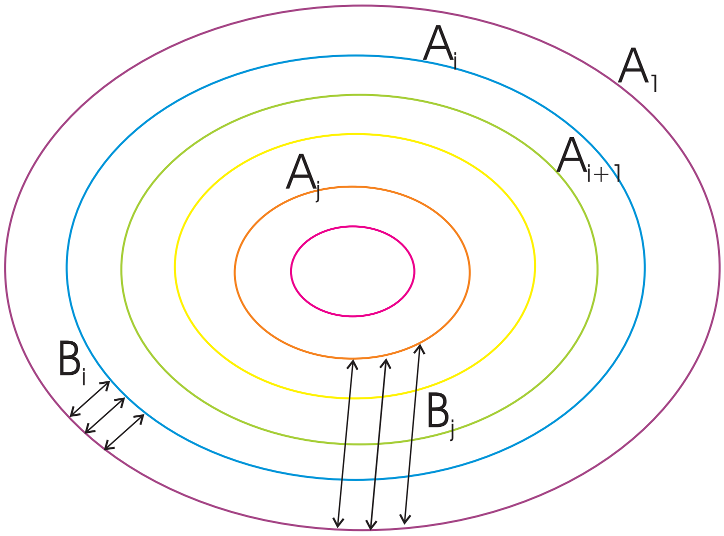


Figure 3.1: Decreasing sequence of A_i, $i = 1, 2, \ldots$.

the form $A_{j-1} \backslash A_j$ are disjoint because $A_{i+1} \subseteq A_i$. Thus $\mu(A_1 \backslash A_j) = \mu((A_1 \backslash A_2) \cup (A_2 \backslash A_3) \cup \cdots \cup (A_{j-1} \backslash A_j)) = \mu(A_1 \backslash A_2) + \mu(A_2 \backslash A_3) + \cdots + \mu(A_{j-1} \backslash A_j) = 0$. So $\mu(A_1 \backslash A_j) = 0$. Let us have a look at $A_1 \backslash (\cap_{i=1}^{\infty} A_i)$. $A_1 \backslash (\cap_{i=1}^{\infty} A_i) = \cup_{i=1}^{\infty} (A_1 \backslash A_i)$. Denote $A_1 \backslash A_i$ as B_i. Then $B_i \subseteq B_{i+1}$. Thus $\mu(A_1 \backslash (\cap_{i=1}^{\infty} A_i)) = \mu(\cup_{i=1}^{\infty} (A_1 \backslash A_i)) = \mu(\cup_{i=1}^{\infty} (B_i))$. If we want to use the previous case with the union, we need to show, that $\mu(B_i \div B_j) = 0$. Suppose that $j > i$. $\mu(B_i \div B_j) = \mu(B_j \backslash B_i) = \mu((A_1 \backslash A_j) \backslash (A_1 \backslash A_i)) = 0$ because $(A_1 \backslash A_j) \backslash (A_1 \backslash A_i) \subseteq (A_1 \backslash A_j)$ and $\mu(A_1 \backslash A_j) = 0$. Thus $\mu(\cup_{i=1}^{\infty} (B_i)) = \mu(B_1) = \mu(\varnothing) = 0$. Then $\mu(A_1 \backslash (\cap_{i=1}^{\infty} A_i)) = 0$. Thus $\mu(A_1) = \mu(\cap_{i=1}^{\infty} A_i)$.

2. Suppose we have an arbitrary countable sequence of sets $\{A_i\}$. Let us construct a system of sets $\{E_i\}$ by denoting $E_i = \cap_{k=1}^{i} A_k$. Notice that for arbitrary sets B, C holds $(B \cap C) \backslash C = \varnothing$. Let us calculate a set $E_{i+1} \backslash E_i = \cap_{k=1}^{i+1} A_k \backslash \cap_{j=1}^{i} A_j = (A_{i+1} \cap (\cap_{j=1}^{i} A_j)) \backslash \cap_{j=1}^{i} A_j = \varnothing$. Hence $\mu(E_{i+1} \backslash E_i) = 0$. By the construction $E_1 \supseteq E_2 \supseteq E_3 \supseteq \ldots$ holds. It is obvious, that $\mu(\cap_{i=1}^{\infty} A_i) = \mu(\cap_{i=1}^{\infty} E_i)$. Hence $\mu(\cap_{i=1}^{\infty} A_i) = \mu(E_1) = \mu(A_1)$

Theorem 3.7 *For a given formal context $(X, A, \vdash)$, σ-algebra Σ on A a measure μ on the set A generates a partial metric on the quotient context.*

3.3. Conclusion

During the data analysis someone could say, that some attributes of the object are more significant than others. How we could include such information on our mathematical model? The most appropriate mathematical structure is a countably-additive measure. No one knows how this significance could change in the time. For every object a set $\{a'|a \in A, x \in a'\}$ is a set of some neighborhoods. In this chapter we started a study of an informational structure named formal context in terms of topology. For the finite case, the topologically closed sets are generated as unions of extents. We also described the relationships between closure operator, second derivation operator and saturation. The opposite task of generating contexts from the topology was mentioned. Topology was described in terms of contexts with the membership relation – "be an element". In Computer Science information appears by parts. At every particular moment of time, we do not know if that information was complete, because in some moment a new part of information could appear and it could completely change the structure of information. A new formal context (with new information) we will call a new instance of our formal context. How does the partial metric work in this case? For a appropriate interpretation we need to build up a sequence of formal contexts. Every new formal context is a context with new information (objects added / deleted , attributes added/deleted , measure (weight) of attributes changed, incidence relation changed). Then we compute a partial metric for every instance of formal context. Because the instances of a formal context are different, we could not directly compare a partial metric $p(x, x)$, before that we need to "renormalize" it. And then we would be able to do some analysis of the objects.

4. A Context Structure Framework

4.1. Introduction

Modern topological methods are widely used in many recent scientific applications, including theoretical computer science, formal concept analysis, digital image analysis and processing, causal quantum structures and study of qualitative properties of certain differential equations. By means of these highly theoretical disciplines, topological results are also applied in the theory of parallel computation and concurrent processes, quantum algorithms, analysis of digital images in tomography, microscopy or echolocation, electron holography, quantum gravity and the theory of quantum topological insulators.

One of the most important aspects of the studied topological properties having some relationship to the above mentioned applications are the properties of compact sets, whose topological behavior is characterized (among others) by the well-known construction of the de Groot dual. Recall that for a given topological space (X, τ), a topology τ^d, generated by the family of all compact saturated sets used as its closed base, is called the de Groot dual of the original topology τ. Its importance for applications (especially in theoretical computer science) is witnessed by the paper of Jimmie Lawson and Michael Mislove included in the monograph [44]. J. Lawson and M. Mislove stated there a problem known as Problem 540, whether the sequence of iterated duals of τ is infinite or the process of taking duals terminates after finitely many steps with two topologies which are dual to each other. For a special case of T_1-topologies the problem had been solved in [26]. The problem in general was solved by Martin Kovár in 2001. He proved that for any topology it holds $\tau^{dd} = \tau^{dddd}$ [33]. In 2004 the result was improved by the same author to its (so far) final form $\tau^d = (\tau \vee \tau^{dd})^d$ [35]. Note that from this result it also follows that $\tau^d \subseteq \tau^{ddd}$ for any topology τ. It should be also noted that in [33] M. Kovár stated several natural questions regarding the dual topologies. Some of them were studied by Tomoo Yokoyama in his paper [58].

The questions of J.Lawson and M.Mislove related to the de Groot dual arise from study of various semantic models in the theoretical computer science, where the dual and the patch topologies are an important tools of investigation. Than it is natural to ask, whether the similar results could be obtained for more general structures. Another interesting direction of research was introduced by Bernhard Banaschewski [3], who replaced the usual frame structure by a more general, partially ordered structure called *preframe*, where the suprema exist for all non-empty up-directed subcollections. Taking some inspiration from the "classic" results of J. Lawson, M. Mislove, M. Kovár, T. Yokoyama, and from B. Banaschewski's preframe structure of opens of pretopological systems, we investigate the possibility of a construction analogous to the de Groot dual, but in a new, modified setting [10],[12],[14]. A possible range of applications could lie in improvements of the efficiency of some topological algorithms and investigation of the properties of certain causal structures, applicable in quantum gravity and the theory of quantum topological insulators.

4.2. De Groot Dual in Compactly Localic Structures

We will start with recalling some key notions and making several useful denotations.

Definition 4.1 *The Sierpiński frame* $\mathbf{2} = \{\bot, \top\}$ *is a set consisting of the two elements* $\top$ *– the top and* $\bot$ *– the bottom.*

Recall that a set $A \subseteq X$ is saturated (see Definition 3.2) in a topological space (X, τ), if it is an intersection of open sets. It is easy to see, that a set is saturated if and only if it is an upper set with respect to the specialization preorder (introduced in Definition 2.28). As we stated in Definition 2.8, a set $C \subseteq X$ is compact if any its open cover admits a finite subcover. We note that in our approach compactness is considered without any separation axiom (in general sense of paper [32], but originally in [24],[25],[26] a T_1-axiom was assumed).

Several times we will also mention the notions of a *topological system*, and a *locale* as its special case, in order to put our considerations into a wider context. However, we do not directly work with locales, so it is sufficient to refer the reader to [56] for the precise definition in case of her or his interest.

Definition 4.2 *Let* (X, τ) *be a topological space. Then* τ^d *is called the de Groot dual (or co-compact) if it is a topology generated by the all compact saturated sets in* (X, τ) *used as its closed base.*

Definition 4.3 *Let* $(P, \leq)$ *be a partially ordered set (poset). The weak topology is a topology defined by taking the principal lower sets* $\downarrow\{x\}$*, for* $x \in P$*, as the closed subbase.*

Similarly, the weak^d topology is defined by taking the principal upper sets $\uparrow\{x\}$, for $x \in P$, as the closed subbase for a topology on P.

It is well-known that in a locale $(X, A, \vdash)$, the set of points X may be represented as a family of all frame morphisms : $A \to \mathbf{2}$ and the relation $\vdash$ is defined by $x \vdash a \Leftrightarrow x(a) = \top$ for $x \in X$ and $a \in A$. The Hofmann-Mislove theorem says that there is 1-1 correspondence between the compact saturated sets in $(X, A, \vdash)$ and the functions from A to $\mathbf{2}$ that preserve directed joins and finite meets. Taking these functions as points and the elements of A as opens, we obtain a new structure $(X', A, \vdash')$ that redistributes the logic: The localic points are replaced by the compacts sets and the new relation $\vdash'$ preserves directed joins as well as finite joins on both sides. On the other hand, it should be noted that $(X', A, \vdash')$ need not be a topological system in the usual sense but a structure slightly different. We will specify it in a more detail. For the reader's comfort at first we provide the definition of the frame.

Definition 4.4 *A poset A is a frame if and only if*

1. *Every subset has a join,*
2. *Every finite subset has a meet,*
3. *Binary meets distribute over joins.*

Definition 4.5 *A poset A is a preframe (or directly complete semilattice) if and only if A*

1. *is closed under (non-empty) directed joins,*
2. *is closed under finite meets (including the meet of the empty set),*
3. *binary meets distribute over directed joins.*

We write $\top = \bigwedge \varnothing$ (top) and $\bot = \bigvee \varnothing$ (bottom). A simple example of a preframe which is not a frame is given by the poset $P = \{\bot, 0\} \cup \mathbb{N}$ on the Figure 4.1

In [56] one can find the following definition of a topological system.

Definition 4.6 *Let X be a set, A be a frame and $\vdash$ be a subset of $X \times A$. We write $x \vdash a$ for $(x, a) \in \vdash$ and say "x satisfies a". Let the following conditions are satisfied:*

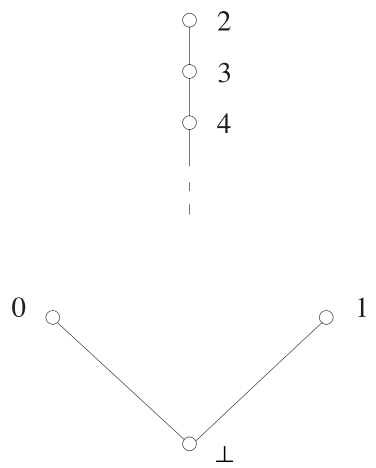


Figure 4.1: Preframe but not a frame

(i) If $B \subseteq A$*, then* $(x \vdash \bigvee B) \Leftrightarrow (x \vdash b$ *for some* $b \in B)$*.*

(ii) If $C \subseteq A$ *is finite, then* $(x \vdash \bigwedge C) \Leftrightarrow (x \vdash c$ *for every* $c \in C)$*.*

Then we the triple $(X, A, \vdash)$ *is said to be a topological system.*

Modifying slightly the previous definition, we receive the definition of a pretopological system.

Definition 4.7 *Let* X *be a set,* A *be a preframe, and* $\vdash$ *be a subset of* $X \times A$*. We write* $x \vdash a$ *for* $(x, a) \in \vdash$*. Let the following conditions are satisfied:*

(i) If $B \subseteq A$ *is non-empty and directed, then* $(x \vdash \bigvee B) \Leftrightarrow (x \vdash b$ *for some* $b \in B)$*.*

(ii) If $C \subseteq A$ *is non-empty and finite, then* $(x \vdash \bigwedge C) \Leftrightarrow (x \vdash c$ *for every* $c \in C)$*.*

Then we say that the triple $(X, A, \vdash)$ *is a pretopological system. The elements of* A *we call, similarly as in topological systems, opens. If* $A \subseteq 2^X$ *is ordered by the inclusion,* $\varnothing, X \in A$ *and the relation* $\vdash$ *is* $\in$*, then* A *is called a pretopology and* $(X, A, \vdash)$ *is a pretopological space.*

Definition 4.8 *Let* A *be a poset. We denote by* $\langle A \to \mathbf{2} \rangle \subseteq \mathbf{2}^A$ *the set of all functions* $A \to \mathbf{2}$ *that preserve the non-empty directed joins and finite meets, whenever they exist. The elements of* $\langle A \to \mathbf{2} \rangle$ *we will call morphisms.*

Proposition 4.1 *The poset* $\langle A \to \mathbf{2} \rangle$ *forms a preframe of all morphisms of* A *to* $\mathbf{2}$*.*

Proof. We will show that $\langle A \to \mathbf{2}\rangle$ has the non-empty directed joins, all finite meets (including the meet of the empty set) and that the meets distribute over the directed joins.

Let $Y \subseteq \langle A \to \mathbf{2}\rangle$ be a non-empty and directed set. Let $f(a) = \bigvee_{y\in Y} y(a)$ for every $a \in A$. We will show that $f = \bigvee Y$ in $\langle A \to \mathbf{2}\rangle$.

First, we must show that $f \in \langle A \to \mathbf{2}\rangle$. Let $B \subseteq A$ be a non-empty and directed set. Then

$$f(\bigvee B) = \bigvee_{y\in Y} y(\bigvee B) = \bigvee_{y\in Y}\bigvee_{b\in B} y(b) = \bigvee_{b\in B}\bigvee_{y\in Y} y(b) = \bigvee_{b\in B} f(b).$$

So f preserves non-empty directed joins. Let $C \subseteq A$ be a non-empty and finite set. At the first step, let us prove $f(\bigwedge C) \le \bigwedge_{c\in C} f(c)$. If $f(\bigwedge C) = \bot$ holds, than it is obvious. Suppose that we have

$$f(\bigwedge C) = \bigvee_{y\in Y} y(\bigwedge C) = \bigvee_{y\in Y}\bigwedge_{c\in C} y(c) = \top$$

which implies that there exists some $y_1 \in Y$ such that for every $c \in C$ it follows $y_1(c) = \top$. Then $\top = \bigwedge_{c\in C}\bigvee_{y\in Y} y(c) = \bigwedge_{c\in C} f(c)$ which gives $f(\bigwedge C) \le \bigwedge_{c\in C} f(c)$. At the second step, let us prove $f(\bigwedge C) \ge \bigwedge_{c\in C} f(c)$. If $\bigwedge_{c\in C} f(c) = \bot$ holds, than it is obvious. Suppose we have

$$\bigwedge_{c\in C} f(c) = \bigwedge_{c\in C}\bigvee_{y\in Y} y(c) = \top.$$

Then for every $c \in C$ there is some $y_c \in Y$ with $y_c(c) = \top$. Since Y is directed and C is finite, there exists some $y_1 \in Y$ such that $y_1 \ge y_c$ for every $c \in C$. Hence, for every $c \in C$ it follows $y_1(c) = \top$. Then

$$\top = \bigvee_{y\in Y}\bigwedge_{c\in C} y(c) = \bigvee_{y\in Y} y(\bigwedge C) = f(\bigwedge C)$$

which implies that $f(\bigwedge C) \ge \bigwedge_{c\in C} f(c)$. Now we have $f(\bigwedge C) = \bigwedge_{c\in C} f(c)$, thus f preserves also non-empty finite meets and so it is a member of $\langle A \to \mathbf{2}\rangle$. Now, let $u \in \langle A \to \mathbf{2}\rangle$ be an upper bound of Y. Then, for every $a \in A$ and every $y \in Y$ it follows that $u(a) \ge y(a)$, which gives $u(a) \ge \bigvee_{y\in Y} y(a) = f(a)$ and, consequently, $u \ge f$. So f is a correctly defined supremum of Y in $\langle A \to \mathbf{2}\rangle$.

Suppose that $Z \subseteq \langle A \to \mathbf{2}\rangle$ is a non-empty and finite set. Let $g(a) = \bigwedge_{z\in Z} z(a)$ for every $a \in A$. We will show that $g = \bigwedge Z$ in $\langle A \to \mathbf{2}\rangle$.

First, we must show that $g \in \langle A \to \mathbf{2} \rangle$. Let $B \subseteq A$ be a non-empty and directed set. At the first step, let us prove $g(\bigvee B) \leq \bigvee_{b \in B} g(b)$. If $g(\bigvee B) = \bot$ holds, than it is obvious. Suppose that we have

$$g(\bigvee B) = \bigwedge_{z \in Z} z(\bigvee B) = \bigwedge_{z \in Z} \bigvee_{b \in B} z(b) = \top$$

which implies that for every $z \in Z$ there is $b_z \in B$ with $z(b_z) = \top$. Since Z is finite and B is directed, there is some $b_1 \in B$ such that $b_1 \geq b_z$ for every $z \in Z$. Then $z(b_1) = \top$ for every $z \in Z$, which implies that

$$\top = \bigvee_{b \in B} \bigwedge_{z \in Z} z(b) = \bigvee_{b \in B} g(b).$$

Hence, $g(\bigvee B) \leq \bigvee_{b \in B} g(b)$. At the second step let us prove $g(\bigvee B) \geq \bigvee_{b \in B} g(b)$. If $\bigvee_{b \in B} g(b) = \bot$ holds than that is obvious. Suppose we have

$$\bigvee_{b \in B} g(b) = \bigvee_{b \in B} \bigwedge_{z \in Z} z(b) = \top.$$

Then, there exists $b_1 \in B$ such that $z(b_1) = \top$ for every $z \in Z$. Then

$$\top = \bigwedge_{z \in Z} \bigvee_{b \in B} z(b) = \bigwedge_{z \in Z} z(\bigvee B) = g(\bigvee B).$$

It follows that $g(\bigvee B) \geq \bigvee_{b \in B} g(b)$. Hence, together with the previously proven converse inequality, we have $g(\bigvee B) = \bigvee_{b \in B} g(b)$. Now, let $C \subseteq A$ be a non-empty and finite set. Then

$$g(\bigwedge C) = \bigwedge_{z \in Z} z(\bigwedge C) = \bigwedge_{z \in Z} \bigwedge_{c \in C} z(c) = \bigwedge_{c \in C} \bigwedge_{z \in Z} z(c) = \bigwedge_{c \in C} g(c).$$

Since g preserves both non-empty directed joins and non-empty finite meets, it follows that g is a weak morphism and so $g \in \langle A \to \mathbf{2} \rangle$. Let $l \in \langle A \to \mathbf{2} \rangle$ be a lower bound of Z. Then, for every $a \in A$ and every $z \in Z$ it follows that $l(a) \leq z(a)$, which gives $l(a) \leq \bigwedge_{z \in Z} z(a) = g(a)$ and, consequently, $l \leq g$. Therefore, g is a correctly defined infimum of Z in $\langle A \to \mathbf{2} \rangle$.

Finally, we will show that the binary meets distribute over the directed joins in $\langle A \to \mathbf{2} \rangle$. Let $x \in \langle A \to \mathbf{2} \rangle$ and $Y \subseteq \langle A \to \mathbf{2} \rangle$ be directed. Then $(x \wedge (\bigvee Y))(a) = x(a) \wedge (\bigvee Y)(a) = x(a) \wedge (\bigvee_{y \in Y} y(a)) = \bigvee_{y \in Y}(x(a) \wedge y(a)) = \bigvee_{y \in Y}((x \wedge y)(a)) = (\bigvee_{y \in Y}(x \wedge y))(a)$ for every $a \in A$, which implies $x \wedge (\bigvee Y) = \bigvee_{y \in Y}(x \wedge y)$. By the definition, $\langle A \to \mathbf{2} \rangle$ is a preframe.

Example 4.1 *Let A be a preframe. We put $x \vdash a$ if and only if $x(a) = \top$ for $x \in \langle A \to \mathbf{2} \rangle$ and $a \in A$. Then $(\langle A \to \mathbf{2} \rangle, A, \vdash)$ is a pretopological system.*

It can be shown that similarly as in locales, the pretopological system constructed in the previous example is fully determined by A.

Definition 4.9 *A pretopological system $(X, A, \vdash)$ is a compactly localic if $X = \langle A \to \mathbf{2} \rangle$ and $x \vdash a$ if and only if $x(a) = \top$ for $x \in \langle A \to \mathbf{2} \rangle$ and $a \in A$.*

Let us denote, similarly as in [34], $\mathrm{int}_X(a) = \{x | \, x \in X, x \vdash a\}$ for every $a \in A$.

Definition 4.10 *A set $K \subseteq X$ is compact in a pretopological system $(X, A, \vdash)$ if for every directed $B \subseteq A$ with $K \subseteq \mathrm{int}_X(\bigvee B) = \bigcup_{b \in B} \mathrm{int}_X(b)$, there is some $a \in B$ such that $K \subseteq \mathrm{int}_X(a)$.*

Definition 4.11 *A set $S \subseteq X$ is saturated in a pretopological system $(X, A, \vdash)$ if S is an intersection of the sets $\mathrm{int}_X(b)$, $b \in B$ for some $B \subseteq A$.*

One can easily check that the notions of compactness and saturation in pretopological systems slightly differ from their counterparts in topological systems if A is not a frame (although the sets $\mathrm{int}_X(a)$, $a \in A$ generate some underlying topology on X). If A is not a frame, the previously introduced notions need not coincide with compactness and saturation related to this topology. Although it is obviously possible to define a dualization of a general pretopological system, it is not so simple to choose the best construction from several possibilities since they are not easily comparable with the classical topological case.

Definition 4.12 *Let $(X, A, \vdash)$ be a compactly localic pretopological system. For any $x \in X$ and any $y \in \langle A \to \mathbf{2} \rangle$ we say that x is independent on y and write $x \models y$ if there is some $a \in A$ such that $y(a) = \top$ and $x \nvdash a$.*

Under the inspiration of the construction of the de Groot dual of a topological space (see e.g. [44], [5], [24] or [32]) and constructions studied in [34] we could define the dual (pretopological) system.

Definition 4.13 *The dualization of a compactly localic pretopological system $(X, A, \vdash)$ we mean the triple $(X, \langle A \to \mathbf{2} \rangle, \models)$.*

We will see that under the condition that A is a frame, $(X, \langle A \to \mathbf{2}\rangle, \models)$ will correspond to its topological counterpart.

Theorem 4.1 *Let A be a frame and $(X, A, \vdash)$ be a compactly localic pretopological system. Then $(X, \langle A \to \mathbf{2}\rangle, \models)$ is also a pretopological system.*

Proof. Let $Y \subseteq \langle A \to \mathbf{2}\rangle$ be a non-empty and directed set. It follows that $x \models \bigvee Y$ if and only if there exists $a \in A$ such that $x \nvdash a$ and $(\bigvee Y)(a) = \bigvee_{y\in Y} y(a) = \top$. But this is true if and only if there is some $a \in A$ and $y \in Y$, such that $x \nvdash a$ and $y(a) = \top$. This is equivalent to existence of some $y \in Y$ such that $x \models y$.

Let $Z \subseteq \langle A \to \mathbf{2}\rangle$ be a non-empty and finite set. It follows that $x \models \bigwedge Z$ if and only if there exists $a \in A$ such that $x \nvdash a$ and $(\bigwedge Z)(a) = \bigwedge_{z\in Z} z(a) = \top$. But this holds if and only if there is some $a \in A$ such that $x \nvdash a$ and $z(a) = \top$ for every $z \in Z$. This implies that $x \models z$ for every $z \in Z$. Conversely, let $x \models z$ for every $z \in Z$. Then, for every $z \in Z$, there exists $a_z \in A$ such that $z(a_z) = \top$ and $x \nvdash a_z$. We put $a = \bigvee_{z\in Z} a_z$. Then $z(a) = \top$ and $x \nvdash a$ for every $z \in Z$, which is equivalent to $x \models \bigwedge Z$. Therefore, $(X, \langle A \to \mathbf{2}\rangle, \models)$ is a pretopological system.

Before proving the main theorem, we need the following lemma.

Lemma 4.1 *Let $(X, A, \vdash)$ be a compactly localic pretopological system. Then the non-empty compact saturated sets in $(X, A, \vdash)$ are the sets of the form*

$$\uparrow\{x\} = \{y |\, y \in \langle A \to \mathbf{2}\rangle, y \geq x\},$$

where $x \in X$.

Proof. The fact that $(X, A, \vdash)$ is a compactly localic pretopological system means that $X = \langle A \to \mathbf{2}\rangle$. Let $x \in X = \langle A \to \mathbf{2}\rangle$ and let $B \subseteq A$ be an arbitrary directed set such that $\uparrow\{x\} \subseteq \bigcup_{b\in B} \mathrm{int}_X(b)$. Then there is some $b \in B$ such that $x \in \mathrm{int}_X(b)$. From the definition of the set $\mathrm{int}_X(b)$ it follows that $x \vdash b$, then by the definition of the compactly localic pretopological system we obtain $x(b) = \top$. Then $y(b) = \top$ holds for all $y \in\uparrow\{x\}$, because y is an element of an upper set. Then $y \in \mathrm{int}_X(b)$ for every $y \in\uparrow\{x\}$ and hence, $\uparrow\{x\} \subseteq \mathrm{int}_X(b)$. Then $\uparrow\{x\}$ is compact.

Suppose that $z \in X$ and $z \notin\uparrow\{x\}$. Then $z \not\geq x$, so there is some $a \in A$ such that $x(a) = \top$ and $z(a) = \bot$. Then $\uparrow\{x\} \subseteq \mathrm{int}_X(a)$ but $z \notin \mathrm{int}_X(a)$. Hence, $\uparrow\{x\} = \mathrm{int}_X(a)$ and $\uparrow\{x\}$ is saturated.

Let $K \subseteq X$ be non-empty, compact and saturated. We put for every $a \in A$

$$w_K(a) = \begin{cases} \top, & \text{if } x \vdash a \text{ for every } x \in K, \\ \bot, & \text{otherwise.} \end{cases}$$

We will show that $w_K \in \langle A \to \mathbf{2} \rangle$. Obviously, it follows $w_K(\bot) = \bot$ and $w_K(\top) = \top$, which means that the empty meets and joins are preserved whenever they exist. Let $B \subseteq A$ be non-empty and directed. We will show that both expressions $w_K(\bigvee B)$, $\bigvee_{b \in B} w_K(b)$ are equal by proving that they reach the same value $\top$ equivalently. Suppose that $w_K(\bigvee B) = \top$. Then $x \vdash \bigvee B$ for every $x \in K$. Then $K \subseteq \mathrm{int}_X(\bigvee B)$. Since K is compact, there exists $b_K \in B$ such that $K \subseteq \mathrm{int}_X(b_K)$. It follows that $w_K(b_K) = \top$ and hence $\bigvee_{b \in B} w_K(b) = \top$. Conversely, suppose that $\bigvee_{b \in B} w_K(b) = \top$. Then, there is some $b_K \in B$ such that $w_K(b_K) = \top$. If follows that $K \subseteq \mathrm{int}_X(b_K)$. Since $b_K \leq \bigvee B$ it follows that $K \subseteq \mathrm{int}_X(\bigvee B)$, which gives $w_K(\bigvee B) = \top$.

Let $C \subseteq A$ be non-empty and finite. Similarly as in the previous paragraph, we will check the equality of the expressions $w_K(\bigwedge C)$, $\bigwedge_{c \in C} w_K(c)$ by showing, that they have the same value $\top$ equivalently. Then $w_K(\bigwedge C) = \top$ if and only if $x \models \bigwedge C$ for every $x \in K$. This is true if and only if $x \models c$ for every $x \in K$ and $c \in C$. But this is equivalent to $w_K(c) = \top$ for every $c \in C$. This holds if and only if $\bigwedge_{c \in C} w_K(c) = \top$.

It follows that w_K preserves the directed joins and finite meets, so $w_K \in X = \langle A \to \mathbf{2} \rangle$. Let $y \in K$. If $w_K(a) = \top$, then $y(a) = \top$, so $y \geq w_K$. Hence $K \subseteq \uparrow \{w_K\}$. Conversely, let $y \in \uparrow \{w_K\}$. Then $y \geq w_K$. Suppose that $y \notin K$. Since K is saturated, there is $a \in A$ such that $K \subseteq \mathrm{int}_X(a)$ and $y \notin \mathrm{int}_X(a)$. Then $w_K(a) = \top$, but $y(a) = \bot$ which contradicts to $y \geq w_K$. Hence $y \in K$. It follows that $K = \uparrow \{w_K\}$.

Theorem 4.2 *Let A be a frame, $(X, A, \vdash)$ be a compactly localic pretopological system. Then the topology on X induced by $\langle A \to \mathbf{2} \rangle$ is dual to the topology on X induced by A in the usual sense and it equals to its weak^d topology.*

Proof. Let $\tau_X(A)$ be the topology on X induced by the sets $\mathrm{int}_X(a)$ for $a \in A$. Then the notions of compactness and saturation in $(X, A, \vdash)$ coincide with the usual topological notions related to $\tau_X(A)$ and the preorder of specialization on $X = \langle A \to \mathbf{2} \rangle$ is its own order $\leq$. It follows from Lemma 1 that the non-empty compact saturated sets in $\tau_X(A)$ are precisely the upper sets $\uparrow \{x\}$. Let us describe the extents of the opens of $(X, \langle A \to \mathbf{2} \rangle, \models)$. Let $y \in X$, $x \in \langle A \to \mathbf{2} \rangle$. Then $y \models x$ if and only if there is some $a \in A$ such that $x(a) = \top$

and $y \nVdash a$. But $X = \langle A \to \mathbf{2} \rangle$, so $y \nVdash a$ means $y(a) = \bot$. Hence, $y \models x$ if and only if $y \ngeq x$, so $\mathrm{int}_X(x) = X \setminus \uparrow\{x\}$.

On the other hand, if A is a more general preframe than a frame, it is easy to see that the triple $(X, \langle A \to \mathbf{2} \rangle, \models)$ representing the dual of $(X, A, \vdash)$ even need not be a pretopological system and if so, it need not be compactly localic. Hence, the sequences of iterated dualizations are not possible in general in this setting. One possible idea how to fix this problem could be modifying the underlying set of points of the dual pretopological system. This idea leads to the following natural definition.

Definition 4.14 *By a compactly localic dualization of a compactly localic pretopological system, say $(X, A, \vdash)$, we mean the pretopological system $(X', \langle A \to \mathbf{2} \rangle, \Vdash)$, where $X' = \langle \langle A \to \mathbf{2} \rangle \to \mathbf{2} \rangle$ and $(u \Vdash y) \Leftrightarrow (u(y) = \top)$ for $u \in X'$ and $y \in \langle A \to \mathbf{2} \rangle$.*

Now the iterated compactly localic dualizations exist for any compactly localic pretopological system, and they are fully represented by sequence of the posets of their opens: $\langle A \to \mathbf{2} \rangle$, $\langle \langle A \to \mathbf{2} \rangle \to \mathbf{2} \rangle$, $\langle \langle \langle A \to \mathbf{2} \rangle \to \mathbf{2} \rangle \to \mathbf{2} \rangle$, $\ldots$, etc.

4.3. Dualizations for the Posets of Opens

Now we will concentrate ourselves on the preframe structure of the opens of the pretopological counterpart of the de Groot dual. As we have shown in the previous section, the opens of the dual may be represented as certain maps from A to the Sierpiński frame $\mathbf{2}$, where A is the poset representing the opens of the original pretopological system.

Let A be a poset. We denote by $h_A : A \to \langle \langle A \to \mathbf{2} \rangle \to \mathbf{2} \rangle$ a mapping for which $h_A(a)(x) = x(a)$ for every $x \in \langle A \to \mathbf{2} \rangle$. The following theorem holds:

Theorem 4.3 *Let A be a poset. Then $h_A : A \to \langle \langle A \to \mathbf{2} \rangle \to \mathbf{2} \rangle$ is a morphism.*

Proof. Suppose that there exists the greatest element $\bigwedge \varnothing \in A$. It follows that $h_A(\bigwedge \varnothing)(x) = x(\bigwedge \varnothing) = \top$ for every morphism $x \in \langle A \to \mathbf{2} \rangle$, so $h_A(\bigwedge \varnothing) = \top$.

Let $B \subseteq A$ be non-empty and directed and suppose that there exists $\bigvee B \in A$. Let $x \in \langle A \to \mathbf{2} \rangle$. Then $h_A(\bigvee B)(x) = x(\bigvee B) = \bigvee_{b \in B} x(b) = \bigvee_{b \in B} h_A(b)(x) = (\bigvee_{b \in B} h_A(b))(x)$, which implies that $h_A(\bigvee B) = \bigvee_{b \in B} h_A(b)$.

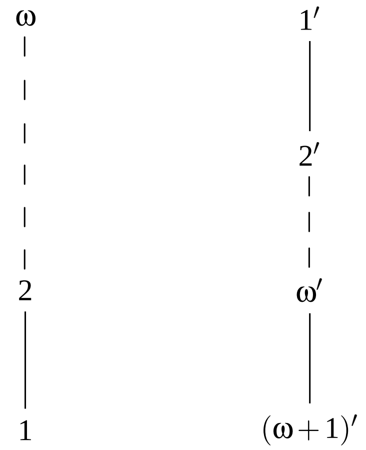


Figure 4.2: h_A is not an epimorphism

Let $C \subseteq A$ be non-empty, finite and assume that there exists $\bigwedge C \in A$. Let $x \in \langle A \to \mathbf{2} \rangle$. It follows $h_A(\bigwedge C)(x) = x(\bigwedge C) = \bigwedge_{c\in C} x(c) = \bigwedge_{c\in C} h_A(c)(x) = (\bigwedge_{c\in C} h_A(c))(x)$, which implies that $h_A(\bigwedge C) = \bigwedge_{c\in C} h_A(c)$.

Since h_A preserves all non-empty directed joins and all finite meets, it follows that h_A is a morphism.

On the other hand, we have the following two counterexamples; the corresponding posets are given by their Hasse diagrams on the Figure 4.3 and Figure 4.2:

Example 4.2 *There exists a preframe A such that h_A is not an epimorphism. Let $A = \omega + 1 = \{1, 2, \dots, \omega\}$, where ω is the first infinite ordinal, with its natural linear order. Let $n' : A \to \mathbf{2}$ be a mapping with the $\top$-kernel $\{n, n+1, \dots, \omega\}$ for every $n \in A$ and $(\omega+1)'$ be a mapping identically equal to $\bot$. The construction is illustrated by the Figure 4.2.*

Since every morphism is an isotone mapping and the constant mapping with the empty $\top$-kernel, $(\omega+1)' =$ **False**, *is not a morphism, it is not difficult to check that $\langle A \to \mathbf{2} \rangle = \{\omega', \dots, 2', 1'\}$. Notice that $\langle A \to \mathbf{2} \rangle$ is linearly ordered by the set inclusion of the corresponding $\top$-kernels of its elements. For every $x \in \langle A \to \mathbf{2} \rangle$ we put*

$$p(x) = \begin{cases} \top, & \text{for } x > \omega' \\ \bot, & \text{for } x = \omega'. \end{cases}$$

Obviously p is a morphism, so $p \in \langle\langle A \to \mathbf{2} \rangle \to \mathbf{2}\rangle$. But for every $a \in A$ and every $x \in \langle A \to \mathbf{2} \rangle$ it follows

$$h_A(a)(x) = x(a) = \begin{cases} \top \text{ for } x \geq a' \\ \bot \text{ for } x < a'. \end{cases}$$

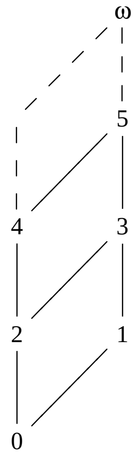


Figure 4.3: h_A is not a monomorphism

Therefore, there is no $a \in A$ such that $p = h_A(a)$, which implies that h_A is not a surjection.

Example 4.3 *There exists a preframe A such that h_A is not a monomorphism.*

That means, 0 is the bottom, ω is the top, $2k$ has two successors $2k+1$, $2k+2$ and $2k+1$ has a unique successor $2k+3$ for every $k \in \{0, 1, \dots\}$. Since for every $Y \subseteq A$ infinite it follows $\bigvee Y = \omega$, the binary meets distribute over all joins. It follows that A is preframe (moreover, a frame).

Let $x \in \langle A \to 2 \rangle$. We will show that $x(0) = x(1)$. If $x(0) = \top$, then also $x(1) = x(0 \vee 1) = x(0) \vee x(1) = \top \vee x(1) = \top$. Suppose that $x(2k) = \bot$ for every $k \in \{0, 1, \dots\}$. The set $S = \{2, 4, \dots\}$ infinite and directed. It follows that $x(\omega) = x(\bigvee S) = \bigvee_{s \in S} x(s) = \bigvee_{s \in S} \bot = \bot$. Then $x(1) = x(1 \wedge \omega) = x(1) \wedge x(\omega) = x(1) \wedge \bot = \bot$. Finally, suppose that k is the greatest number from $\{0, 1, \dots\}$ such that $x(2k) = \bot$. But $\bot = x(2k) = x((2k+2) \wedge (2k+1)) = x(2k+2) \wedge x(2k+1) = \top \wedge x(2k+1)$, which implies $x(2k+1) = \bot$. Then $x(1) = x(1 \wedge (2k+1)) = x(1) \wedge x(2k+1) = x(1) \wedge \bot = \bot$. Therefore, $h_A(0) = h_A(1)$ which implies that h_A is not injective.

The positive results can be reached especially for the finite case:

Theorem 4.4 *Let A be a finite preframe. Then $h_A : A \to \langle \langle A \to \mathbf{2} \rangle \to \mathbf{2} \rangle$ is an isomorphism.*

Proof. Let $p \in \langle \langle A \to \mathbf{2} \rangle \to \mathbf{2} \rangle$. We put $x_1 = \bigwedge \mathrm{Ker}_t(p)$ and $a_1 = \bigwedge \mathrm{Ker}_t(x_1)$, where Ker_t is a denotation for the $\top$-kernel of a mapping with its co-domain equal to a subset of **2**. Since A is finite, also $\mathrm{Ker}_t(p)$ is a finite set; say $\mathrm{Ker}_t(p) = \{y_1, y_2, \dots y_k\}$. Then $p(x_1) = p(y_1 \wedge p_2 \wedge \cdots \wedge p_k) = p(y_1) \wedge p(y_2) \wedge \cdots \wedge p(y_k) = \top$, which means that x_1 is the least element of $\mathrm{Ker}_t(p)$. Similarly, a_1 is the least element of $\mathrm{Ker}_t(x_1)$.

We claim that $h_A(a_1) = p$. Indeed, for every $y \in \langle A \to \mathbf{2} \rangle$ it follows $p(y) = \top \Leftrightarrow y \in \mathrm{Ker}_t(p) \Leftrightarrow x_1 \le y \Leftrightarrow \mathrm{Ker}_t(x_1) \subseteq \mathrm{Ker}_t(y) \Leftrightarrow a_1 \in \mathrm{Ker}_t(y) \Leftrightarrow h_A(a_1)(y) = y(a_1) = \top$. Hence, $h_A(a_1) = p$ which means that h_A is surjective.

Suppose that there exists $a_2 \in A$ such that also $h_A(a_2) = p$. Then $x_1(a_2) = \top$, which implies that $a_1 \le a_2$. Suppose that $a_1 \ne a_2$ and let

$$z(a) = \begin{cases} \top, & \text{for } a \ge a_2 \\ \bot, & \text{otherwise.} \end{cases}$$

Since A is finite, z obviously preserves all directed joins. Then $z(\bigwedge \varnothing) = \top$ since $a_1 < a_2 \le \bigwedge \varnothing$. Let $a, b \in A$. Then $z(a \wedge b) = \top \Leftrightarrow a \wedge b \ge a_2 \Leftrightarrow a \ge a_2$ and $b \ge a_2 \Leftrightarrow z(a) = \top$ and $z(b) = \top \Leftrightarrow z(a) \wedge z(b) = \top$. Hence, $z(a \wedge b) = z(a) \wedge z(b)$. It follows that z preserves also all finite meets. Then $z \in \langle A \to \mathbf{2} \rangle$, but $h_A(a_1)(z) = z(a_1) = \bot \ne \top = z(a_2) = h_A(a_2)(z)$, which is a contradiction. Therefore, $a_1 = a_2$, which implies that h_A is injective.

Corollary 4.1 *Let A be a finite poset. Then its iterated duals*

$$\langle A \to \mathbf{2} \rangle$$

and

$$\langle\langle\langle A \to \mathbf{2} \rangle \to \mathbf{2} \rangle \to \mathbf{2} \rangle$$

are isomorphic.

Now we will consider a more general case, when A is not necessarily finite. For every $v \in \langle\langle\langle A \to \mathbf{2} \rangle \to \mathbf{2} \rangle \to \mathbf{2} \rangle$ and $a \in A$ we put $h_A^*(v) = v \circ h_A$. Although the information formulated by the following lemma is essentially contained already in Lemma 4.1, for the purpose of our main result, we need to formulate it in a slightly different way, with more detail.

Lemma 4.2 *Let A be a poset. Then the following conditions hold:*

(i) $\langle A \to \mathbf{2} \rangle$ forms a preframe of all morphisms of A to **2**.

(ii) For a directed set $Y \subseteq \langle A \to \mathbf{2}\rangle$, and $a \in A$, it holds $(\bigvee Y)(a) = \bigvee_{y \in Y} y(a)$.

(iii) For a non-empty finite set $Z \subseteq \langle A \to \mathbf{2}\rangle$, and $a \in A$, $(\bigwedge Z)(a) = \bigwedge_{z \in Z} z(a)$.

(iv) $\bigwedge \varnothing$, the top element of $\langle A \to \mathbf{2}\rangle$, is represented by the constant mapping **True** $:$ *$A \to \mathbf{2}$, identically equal to $\top$.*

Proof. First, let us prove (ii). Let $Y \subseteq \langle A \to \mathbf{2}\rangle$ be non-empty and directed. Let $f(a) = \bigvee_{y \in Y} y(a)$ for every $a \in A$. We will show that $f = \bigvee Y$ in $\langle A \to \mathbf{2}\rangle$. First, we must prove that $f \in \langle A \to \mathbf{2}\rangle$. Let $B \subseteq A$ be non-empty and directed, such that $\bigvee B$ exists in A. Then $f(\bigvee B) = \bigvee_{y \in Y} y(\bigvee B) = \bigvee_{y \in Y} \bigvee_{b \in B} y(b) = \bigvee_{b \in B} \bigvee_{y \in Y} y(b) = \bigvee_{b \in B} f(b)$, so f preserves non-empty directed joins. Let $C \subseteq A$ be non-empty and finite. Suppose that $\bigwedge C$ exists in A. Then $f(\bigwedge C) = \bigvee_{y \in Y} y(\bigwedge C) = \bigvee_{y \in Y} \bigwedge_{c \in C} y(c) = \top$ implies that there exist some $y_1 \in Y$, such that for every $c \in C$ it follows $y_1(c) = \top$. Then $\top = \bigwedge_{c \in C} \bigvee_{y \in Y} y(c) = \bigwedge_{c \in C} f(c)$ which implies $f(\bigwedge C) \leq \bigwedge_{c \in C} f(c)$. Conversely, suppose that $\bigwedge_{c \in C} f(c) = \bigwedge_{c \in C} \bigvee_{y \in Y} y(c) = \top$. Then for every $c \in C$ there is some $y_c \in Y$ with $y_c(c) = \top$. Since Y is directed and C is finite, there exist some $y_1 \in Y$ such that $y_1 \geq y_c$ for every $c \in C$. Hence, for every $c \in C$ it follows $y_1(c) = \top$. Then $\top = \bigvee_{y \in Y} \bigwedge_{c \in C} y(c) = \bigvee_{y \in Y} y(\bigwedge C) = f(\bigwedge C)$ which implies that $f(\bigwedge C) \geq \bigwedge_{c \in C} f(c)$. Now we have $f(\bigwedge C) = \bigwedge_{c \in C} f(c)$, so f preserves also non-empty finite meets. It remains to check the preservation of the empty meet. Suppose that A has the greatest element $\bigwedge \varnothing \in A$. Then $f(\bigwedge \varnothing) = \bigvee_{y \in Y} y(\bigwedge \varnothing) = \bigvee_{y \in Y} \top = \top$. Hence, f is an element of $\langle A \to \mathbf{2}\rangle$, and, clearly, an upper bound of Y in $\langle A \to \mathbf{2}\rangle$. Now, let $u \in \langle A \to \mathbf{2}\rangle$ be another upper bound of Y. Then, for every $a \in A$ and every $y \in Y$ it follows that $u(a) \geq y(a)$, which gives $u(a) \geq \bigvee_{y \in Y} y(a) = f(a)$ and, consequently, $u \geq f$. So f is a correctly defined supremum of Y in $\langle A \to \mathbf{2}\rangle$.

Now, let us show (iii). Suppose that $Z \subseteq \langle A \to \mathbf{2}\rangle$ is non-empty and finite. Let $g(a) = \bigwedge_{z \in Z} z(a)$ for every $a \in A$. We will show that $g = \bigwedge Z$ in $\langle A \to \mathbf{2}\rangle$. First, we must show that $g \in \langle A \to \mathbf{2}\rangle$. Let $B \subseteq A$ be non-empty and directed, such that $\bigvee B$ exists in A. Then $g(\bigvee B) = \bigwedge_{z \in Z} z(\bigvee B) = \bigwedge_{z \in Z} \bigvee_{b \in B} z(b) = \top$ implies that for every $z \in Z$ there is $b_z \in B$ with $z(b_z) = \top$. Since Z is finite and B is directed, there is some $b_1 \in B$ such that $b_1 \geq b_z$ for every $z \in Z$. Then $z(b_1) = \top$ for every $z \in Z$, which implies that $\top = \bigvee_{b \in B} \bigwedge_{z \in Z} z(b) = \bigvee_{b \in B} g(b)$. Hence, $g(\bigvee B) \leq \bigvee_{b \in B} g(b)$. Conversely, suppose that $\bigvee_{b \in B} g(b) = \bigvee_{b \in B} \bigwedge_{z \in Z} z(b) = \top$. Then, there exists $b_1 \in B$, such that $z(b_1) = \top$ for every $z \in Z$. Then $\top = \bigwedge_{z \in Z} \bigvee_{b \in B} z(b) = \bigwedge_{z \in Z} z(\bigvee B) = g(\bigvee B)$. It follows that

$g(\bigvee B) \geq \bigvee_{b\in B} g(b)$ and hence, together with the previously proved (converse) inequality, we have $g(\bigvee B) = \bigvee_{b\in B} g(b)$. Now, let $C \subseteq A$ be non-empty and finite, having $\bigwedge C \in A$. Then $g(\bigwedge C) = \bigwedge_{z\in Z} z(\bigwedge C) = \bigwedge_{z\in Z} \bigwedge_{c\in C} z(c) = \bigwedge_{c\in C} \bigwedge_{z\in Z} z(c) = \bigwedge_{c\in C} g(c)$. Finally, suppose that A has the greatest element $\bigwedge \varnothing \in A$. Then $g(\bigwedge \varnothing) = \bigwedge_{z\in Z} z(\bigwedge \varnothing) = \bigwedge_{z\in Z} \top = \top$. It follows that g is an element of $\langle A \to \mathbf{2}\rangle$, and, clearly, a lower bound of Z in $\langle A \to \mathbf{2}\rangle$. Let $l \in \langle A \to \mathbf{2}\rangle$ be a lower bound of Z. Then, for every $a \in A$ and every $z \in Z$ we have $l(a) \leq z(a)$, which gives $l(a) \leq \bigwedge_{z\in Z} z(a) = g(a)$ and, consequently, $l \leq g$. Therefore, g is a correctly defined infimum of Z in $\langle A \to \mathbf{2}\rangle$.

Regarding (iv), the mapping **True**, constantly equal to $\top$, obviously preserves all non-empty directed joins and all finite meets, so $\langle A \to \mathbf{2}\rangle$ also has the greatest element $\wedge\varnothing$. Note that **True** does not preserve the empty join, but it is not required.

Finally, to show (i), it remains to check that binary meets distribute over directed joins in $\langle A \to \mathbf{2}\rangle$. Let $x \in \langle A \to \mathbf{2}\rangle$ and $Y \subseteq \langle A \to \mathbf{2}\rangle$ be directed. Then $(x\wedge(\bigvee Y))(a) = x(a)\wedge(\bigvee Y)(a) = x(a)\wedge(\bigvee_{y\in Y} y(a)) = \bigvee_{y\in Y}(x(a)\wedge y(a)) = \bigvee_{y\in Y}((x\wedge y)(a)) = (\bigvee_{y\in Y}(x\wedge y))(a)$ for every $a \in A$, which implies $x \wedge (\bigvee Y) = \bigvee_{y\in Y}(x \wedge y)$. By the definition, $\langle A \to \mathbf{2}\rangle$ is a preframe.

Theorem 4.5 *Let A be a poset. Then $h^*_A : \langle\langle\langle A \to \mathbf{2}\rangle \to \mathbf{2}\rangle \to \mathbf{2}\rangle \to \langle A \to \mathbf{2}\rangle$ is a morphism.*

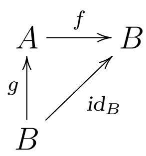
Proof. Let $v \in \langle\langle\langle A \to \mathbf{2}\rangle \to \mathbf{2}\rangle \to \mathbf{2}\rangle$. By the definition and Theorem 4.3, $h^*_A(v) = v \circ h_A$ is a morphism as a composition of two morphisms. So it is an element of $\langle A \to \mathbf{2}\rangle$, which means that h^*_A maps $\langle\langle\langle A \to \mathbf{2}\rangle \to \mathbf{2}\rangle \to \mathbf{2}\rangle$ into $\langle A \to \mathbf{2}\rangle$. It remains to show that it preserves the directed joins and the finite meets whenever they exist.

Let $V \subseteq \langle\langle\langle A \to \mathbf{2}\rangle \to \mathbf{2}\rangle \to \mathbf{2}\rangle$ be non-empty and directed. The poset, representing the triple dual $\langle\langle\langle A \to \mathbf{2}\rangle \to \mathbf{2}\rangle \to \mathbf{2}\rangle$ is a preframe by Lemma 4.2, so the supremum $\bigvee V$ exists. Denote $V' = \{v \circ h_A | \, v \in V\}$. The set V' is a non-empty directed subset of $\langle A \to \mathbf{2}\rangle$ and again, by Lemma 4.2, it follows that $\bigvee V'$ exists. Moreover, $\bigvee V' = \bigvee_{v\in V} v \circ h_A = \bigvee_{v\in V} h^*_A(v)$. We will show that $h^*_A(\bigvee V)$ and $\bigvee V'$ are the same maps. For any $a \in A$ it holds $h^*_A(\bigvee V)(a) = ((\bigvee V)\circ h_A)(a) = (\bigvee V)(h_A(a))$. Using the statement (ii) of Lemma 4.2, we may continue: $(\bigvee V)(h_A(a)) = \bigvee_{v\in V} v(h_A(a)) = \bigvee_{v\in V}((v \circ h_A)(a)) = \bigvee_{w\in V'} w(a)$. Again, by the same statement (ii) of Lemma 4.2, we get $\bigvee_{w\in V'} w(a) = (\bigvee V')(a)$. Hence, it follows $h^*_A(\bigvee V)(a) = (\bigvee V')(a)$ for every $a \in A$ and so $h^*_A(\bigvee V) = \bigvee V' = \bigvee_{v\in V} h^*_A(v)$.

Let $C \subseteq \langle\langle\langle A \to \mathbf{2}\rangle \to \mathbf{2}\rangle \to \mathbf{2}\rangle$ be non-empty and finite. Since $\langle\langle\langle A \to \mathbf{2}\rangle \to \mathbf{2}\rangle \to \mathbf{2}\rangle$ is a preframe, the infimum $\bigwedge C$ exists. Let $C' = \{c \circ h_A | \, c \in C\}$. The set C' is a finite subset of $\langle A \to \mathbf{2}\rangle$ and again, by Lemma 4.2, it follows that $\bigwedge C'$ exists. We have $\bigwedge C' = \bigwedge_{c \in C} c \circ h_A = \bigwedge_{c \in C} h_A^*(c)$. We will prove that $h_A^*(\bigwedge C)$ and $\bigwedge C'$ are the same maps. Let $a \in A$. Then $h_A^*(\bigwedge C)(a) = ((\bigwedge C) \circ h_A)(a) = (\bigwedge C)(h_A(a))$. Using the statement (iii) of Lemma 4.2, we may continue as follows: $(\bigwedge C)(h_A(a)) = \bigwedge_{c \in C} c(h_A(a)) = \bigwedge_{c \in C}((c \circ h_A)(a)) = \bigwedge_{d \in C'} d(a)$. Using the statement (iii) of Lemma 4.2 once more, we have $\bigwedge_{d \in C'} d(a) = (\bigwedge C')(a)$. Therefore, $h_A^*(\bigwedge C)(a) = (\bigwedge C')(a)$ for every $a \in A$ and so $h_A^*(\bigwedge C) = \bigwedge C' = \bigwedge_{c \in C} h_A^*(c)$.

To complete the proof, as the last step we need to prove the preservation of the empty meet. However, the empty meet $\bigwedge \varnothing$, considered as an element of $\langle\langle\langle A \to \mathbf{2}\rangle \to \mathbf{2}\rangle \to \mathbf{2}\rangle$, is a constant mapping, identically equal to $\top$ which takes arguments from $\langle\langle A \to \mathbf{2}\rangle \to \mathbf{2}\rangle$. Composed with h_A, we get a mapping defined on A, which is constantly equal to $\top$, that is, the top element **True** of $\langle A \to \mathbf{2}\rangle$.

Definition 4.15 *Suppose that there exist morphisms $f : A \to B$, $g : B \to B$ such that $f \circ g = id_B$. Then f is called a retraction and g is called a coretraction.*



It is easy to see that every retraction is an epimorphism and every coretraction is a monomorphism (see, e.g.,[1]).

We close the section by the main theorem, which is the an analogue of Martin Kovár's result $\tau^d \subseteq \tau^{ddd}$, proven in 2001 for the topological spaces [33]. Among others, from the next theorem it follows that $h_{\langle A \to \mathbf{2}\rangle} : \langle A \to \mathbf{2}\rangle \to \langle\langle\langle A \to \mathbf{2}\rangle \to \mathbf{2}\rangle \to \mathbf{2}\rangle$ is a monomorphism, which makes the analogy with the classical topological result from 2001 more obvious.

Theorem 4.6 *Let A be a poset. Then $h_A^* : \langle\langle\langle A \to \mathbf{2}\rangle \to \mathbf{2}\rangle \to \mathbf{2}\rangle \to \langle A \to \mathbf{2}\rangle$ is a retraction.*

Proof. We will show that $h_A^* \circ h_{\langle A \to \mathbf{2}\rangle} = id_{\langle A \to \mathbf{2}\rangle}$. Take $x \in \langle A \to \mathbf{2}\rangle$. Then $(h_A^* \circ h_{\langle A \to \mathbf{2}\rangle})(x) = h_A^*(h_{\langle A \to \mathbf{2}\rangle}(x)) = h_{\langle A \to \mathbf{2}\rangle}(x) \circ h_A$. Let $a \in A$. Then $((h_A^* \circ h_{\langle A \to \mathbf{2}\rangle})(x))(a) =$

$(h_{\langle A \to \mathbf{2} \rangle}(x) \circ h_A)(a) = (h_{\langle A \to \mathbf{2} \rangle}(x))(h_A(a)) = (h_A(a))(x) = x(a)$. Now we have $(h_A^* \circ h_{\langle A \to \mathbf{2} \rangle})(x) = x$, which completes the proof.

$$\begin{array}{ccc} \langle\langle\langle A \to \mathbf{2}\rangle \to \mathbf{2}\rangle \to \mathbf{2}\rangle & \xrightarrow{h_A^*} & \langle A \to \mathbf{2}\rangle \\ {\scriptstyle h_{\langle A \to \mathbf{2}\rangle}} \uparrow & \nearrow {\scriptstyle \mathrm{id}_{\langle A \to \mathbf{2}\rangle}} & \\ \langle A \to \mathbf{2}\rangle & & \end{array}$$

Remark 4.1 *For the topological space* (X, τ)*, let* $f(S) = \mathrm{cl}^{ddd}(S)$*,* $g(S) = \mathrm{cl}^d(S)$ *for every* $S \in 2^X$*, where* cl^d*,* cl^{ddd} *are the corresponding closure operators in the iterated de Groot duals* (X, τ^d)*,* (X, τ^{ddd}) *of* (X, τ)*, respectively. Denote by* $\mathcal{C}^d$*,* $\mathcal{C}^{ddd}$ *the families on all closed sets of the topological spaces* (X, τ^d)*,* (X, τ^{ddd}) *respectively. Then* $f : \mathcal{C}^d \to \mathcal{C}^{ddd}$*,* $g : \mathcal{C}^{ddd} \to \mathcal{C}^d$ *and* $g \circ f = \mathrm{id}_{\mathcal{C}^d}$*. In topological spaces, this an equivalent formulation of the result* $\tau^d \subseteq \tau^{ddd}$ *mentioned above.*

The previous remark illustrates the correspondence between the original topological version of the Problem 540 in Open Problems in Topology due to J. Lawson and M. Mislove, and solved by Martin Kovár, and our recent result, stated in Theorem 4.6. Since both results are reached by completely different techniques, the following open question has its natural place here.

Question 4.1 *Does there exist a generalized formulation, unifying both problems, that is, the problem of the iterated de Groot duals in topological spaces, and that one, addressed by Theorem 4.6, allowing a common, elucidating proof for them?*

4.4. Conclusion

We have successfully defined an analogue of the De Groot dual for compactly localic pretopological systems and in a more general approach, also for any poset. We also proved an adequate counterpart of M. Kovár's result $\tau^d \subseteq \tau^{ddd}$ proved for the general topological spaces (Theorem 4.6 for the general posets and The 4.4 for the special, finite case).

However, the counterexamples in Example 4.2 and Example 4.3 show that the result stated in Theorem 4.6 is best possible in some sense, since it cannot be strengthen with replacing $\langle A \to \mathbf{2}\rangle$ by A and taking the iterated dualizations by one step down. Just like for general topologies, it is not true that $\tau \subseteq \tau^{dd}$ [33].

5. Spatio-temporal Concepts of Framology

Throughout this section we will use the usual terminology of general topology which has been already used in the previous chapters and recalled especially in Section 2.1. For more detail the reader is referred to [17] or [20]. A special attention we pay to the notion of compactness, which we use, in a consensus with a modern approach to general topology, without the Hausdorff separation axiom as a part of its definition.

A source of inspiration for this part of dissertation there were some ideas of formal concept analysis developed by B. Ganter and R. Wille and formulated in [22], which we also mention in the introductory section, Section 2.2, which, however, were adjusted to description of spatial and spatio-temporal relationship in a similar way as in general topology. Foundation for this approach was given by Martin Kovár in his paper [36], which were further developed and studied in our joint work [37].

5.1. Introduction

In most applications, topology is usually not the first, primary structure, but the information which finally leads to the construction of the certain, for some purpose required topology, is filtered by more or less thick layer of the other mathematical structures. This fact has some natural consequences:

1. For most important applied constructions the primary structure is sufficient and topology may be bypassed (in the cost of loss of some elegance).

2. Some topologically important information from the reality may be filtered out by the other, front-end mathematical structures and finally lost.

Obviously, our traditional topological conceptions of the world around us may be far from reality. Natural examples of such situations we can meet often in our everyday life, but usually they are ignored. For instance, as noted in [27], in nature or physical universe there are probably no existing, real points like in the classical Euclidean geometry. Points, as a useful mathematical abstraction, are infinitesimally small and thus cannot be measured or detected by any physical way. However, what we can be sure that really exists, there are various locations, containing concrete physical objects. We will call these locations *places*. Various places can overlap, they can be merged, embedded or glued together, so the theoretically understood virtual "observer" can visit multiple places simultaneously. For instance, the Galaxy, the Solar system, the Earth, (the territory of) Europe, the room in which the reader is present just now, are simple and natural examples of places conceived in our way. Certainly, in this sense, one can be present at many of these places at the same time, and, also certainly, there exist pairs of places, where the simultaneous presence of any physical objects is not possible. Thus, the presence of various physical objects connects these primarily free objects – our places – to the certain structure, which we call a *framework*, [37].

Let us recall the exact definition of this notion and formulate some its basic properties with examples, illustrating how this structure is naturally connected with topology. On a simple example from game theory we will also demonstrate the difference between the really existing objects modeled by a framework and its virtual extension, having no direct counterpart in reality, represented by a topological space.

Definition 5.1 *Let P be a set, $\pi \subseteq 2^P$. We say that (P, π) is a framework. The elements of P we call places, the set π we call framology.*

Although every topological space is a framework by the definition, the primary interpretation of a framework is different from the usual interpretation of a topological space. The elements of the framology are not primarily considered as neighborhoods of places, although this seems to be also very natural. The framework structure is rather a special case of a formal context with the places as the objects, π as the set of attributes and $\in$ as the incidence relation.

There exists also a natural physical-like motivation of the structure: P represents the set of some locations, where an element of π is a "list" of locations containing certain physical object, say a particle, simultaneously. The places primarily have no geometrical properties or meaning and they are not connected with any outer geometrical structure as the spacetime or so. The structure arises in an intrinsic way, just from the relation between elements of P given by the family π. The places may naturally overlap, contain each other or they may be glued together by presence of some physical object (for instance, a particle).

Definition 5.2 *Let (P,π) and (S,σ) be frameworks. A mapping $f : P \to S$ satisfying $f(\pi) \subseteq \sigma$ we call a framework morphism.*

Definition 5.3 *A framework (P,π) is T_0 if for every $x, y \in P$, $x \neq y$, there exists $U \in \pi$ such that $x \in U$, $y \notin U$ or $x \notin U$, $y \in U$.*

Definition 5.4 *Let (P,π) be a framework. Denote $P^d = \pi$ and $\pi^d = \{\pi(x)|\, x \in P\}$, where $\pi(x) = \{U|\, U \in \pi, x \in U\}$. Then (P^d,π^d) is the dual framework of (P,π). The places of the dual framework (P^d,π^d) we call abstract points or simply points of the original framework (P,π).*

The following theorem is explicitly stated in [37] and we repeat it for the reader's comfort.

Theorem 5.1 *Let (P,π) be a framework. Then (P^{dd},π^{dd}) is isomorphic to the quotient of (P,π). Moreover, if (P,π) is T_0, then (P^{dd},π^{dd}) and (P,π) are isomorphic.*

Proof. We denote $R = P^d = \pi$, $\rho = \pi^d = \{\pi(x)|\, x \in P\}$, $S = R^d = \rho$, $\sigma = \rho^d = \{\rho(x)|\, x \in R\}$. Then (S,σ) is the double dual of (P,π). It remains to show, that (S,σ) is isomorphic to some quotient of (P,π).

For every $x \in P$, we put $f(x) = \pi(x)$. Then $f : P \to S$ is a surjective mapping. It is easy to show, that f is a morphism. Indeed, if $U \in \pi$, then $f(U) = \{\pi(x)|\, x \in U\} = \{\pi(x)|\, x \in P, U \in \pi(x)\} = \{V|\, V \in \rho, U \in V\} = \rho(U) \in \sigma$. Therefore, $f(\pi) \subseteq \sigma$, which means that f is an epimorphism of the framework (P,π) onto (S,σ).

Now, we define $x \sim y$ for every $x, y \in P$ if and only if $f(x) = f(y)$. Then $\sim$ is an equivalence relation on P. For every equivalence class $[x] \in P_\sim$ we put $h([x]) = f(x)$. The mapping $h : P_\sim \to S$ is correctly defined, moreover, it is a bijection. The verification that h is a framework isomorphism is standard, but, because of completeness, it has its natural place here. The quotient framology on $P_\sim$ is $g(\pi)$, where $g : P \to P_\sim$ is the quotient map. The quotient map g satisfies the condition $h \circ g = f$. Let $W \in g(\pi)$. There exists $U \in \pi$ such that $W = g(U)$. Then $h(W) = h(g(U)) = f(U) \in \sigma$. Hence $h(g(\pi)) \subseteq \sigma$, which means that $h : P_\sim \to S$ is a framework morphism. Conversely, let $W \in \sigma = \{\rho(U)|\, U \in \pi\}$. We will show that $h^{-1}(W) \in g(\pi)$. By the previous paragraph, $\rho(U) = f(U)$ for every $U \in \pi$, so there exists $U \in \pi$, such that $W = f(U) = h(g(U))$. Since h is a bijection, it follows that $h^{-1}(W) = g(U) \in g(\pi)$. Hence, also $h^{-1} : S \to P_\sim$ is a framework morphism, so the frameworks $(P_\sim, g(\pi))$ and (S,σ) are isomorphic.

Now let us consider the special case when (P, π) is T_0. Suppose that $f(x) = f(y)$ for some $x, y \in P$. Then $\pi(x) = \pi(y)$, which is possible only when $x = y$. Then the relation $\sim$ is the diagonal relation, and the quotient mapping g is an isomorphism.

The framework duality is a simple but handy tool for switching between the classical point-set representation (like in topological spaces) and point-less representation, introduced by the framework theory. More information regarding the framework structure the reader can find, for example, in [36] or [37].

Fore the reader's convenience we recall here some topological notions already introduced in Section 2.1. A family Φ of sets has the finite intersection property or shortly the f.i.p., if every its finite subfamily has a non-empty intersection. A topological space is said to be compact, if every its open cover admits of a finite subcover, or equivalently, if every family of closed sets with f.i.p has a non-empty intersection. Well-known Alexander's subbase lemma [20] ensures that the family of all open or closed sets, respectively, can be replaced by its corresponding open or closed subbase, respectively.

5.2. Framework Topological Models

As we already mentioned before, the framework duality is a simple but handy tool for switching between the classical point-set representation (like in topological spaces) and the point-less representation of topological relationships. The framework structure could be also suitable for addressing the compatibility problem of various scales in physics and their different models. Since the points of the Universe probably do not exist in reality (although they are a useful mathematical abstraction), the abstract points of a framework only express certain relationships between places, which – in a contrast to points – can be really observed and which exclusively exist in the physical reality. Then various framologies and various topological models may peacefully coexist with help of the framework duality on a given set P of places. Let as formulate more precisely, what we mean by the topological model of a framework.

Definition 5.5 *Let (P, π) be a framework, (X, τ) be a topological space with the family $\mathcal{C}$ of closed sets. We say that (X, τ) is an open (closed, respectively) topological model for (P, π), if there exists a framework (S, σ) isomorphic to (P, π) and set $X' \subseteq X$ such that $S \subseteq \tau$ ($S \subseteq \mathcal{C}$, respectively) and $\sigma = \{\{U | \, U \in S, x \in U\} | \, x \in X'\}$.*

Example 5.1 *Let p_i, $i \in \mathbb{Z}$ be pairwise distinct elements. We put $P = \{p_i | \, i \in \mathbb{Z}\}$, $\pi = \{\{p_i, p_{i+1}\} | \, i \in \mathbb{Z}\}$. Then the framework (P, π) has many open as well as closed topological models, including the real line $\mathbb{R}$ equipped with the Euclidean topology and the Khalimsky line, that is, the set $\mathbb{Z}$ with the topology generated by its open base $\tau_K = \{\dots, \{1\}, \{1, 2, 3\}, \{3\}, \{3, 4, 5\}, \{5\}, \dots\}$. For an easy proof, one can the places p_i identify with non-empty, open (or closed, respectively) overlapping sets in a topological space, such that p_i has the non-empty intersection only with p_{i-1}, p_i and p_{i+1}. In addition, in case of Khalimsky topology, one can simulate various scales by taking p_i with more or less elements, although the original framework (P, π) still remains the same.*

For the game-theoretic terminology and background used in the following example, we refer the reader to the standard publication [49]. See also Section 2.5 of Chapter 1, for the absolute, game-theoretical minimum. Some more recent and advanced survey of the topic the reader can find, for example, in [46]. The next example illustrates the contrast between the objects observable in reality and their virtual extension having no counterpart in existing objects.

Example 5.2 *Consider the following (bimatrix) game, a modification of the well-known Prisoner's dilemma. During the interrogation, each of the two players, the prisoners, can use one of the three possible strategies: The strategy A, which means to plead guilty; the strategy B, consisting of remaining silent; and the strategy C in which the prisoner will blame his accomplice. The questioning may repeat at most n times. The game terminates when any prisoner pleads guilty, if he blames his accomplice or if the number of examinations would exceed n in the next step. During the game, a player can use the following sequences of strategies:*

$$A,$$

$$B, \dots, B,$$

$$B, \dots, B, A,$$

$$B, \dots, B, C,$$

$$C,$$

where the number of B's in each sequence $B, \dots, B$ is at least 1, and the length of each sequence is at most n. So, the player can use either the pure strategies A, B, C or he *can combine B with A or C, but he cannot combine the strategies A and C together. We may consider the pure strategies as the places of a framework* $P = \{A, B, C\}$*. The possible combinations of various strategies define a framology structure on P, say* $\pi = \{\{A\}, \{B\}, \{C\}, \{A, B\}, \{B, C\}\}$.

Recall that a pair (x^*, y^*) *of strategies of two players is an equilibrium, if for any other pair* (x, y) *of strategies it holds*

$$u(x, y^*) \leq u(x^*, y^*),$$

and

$$v(x^*, y) \leq v(x^*, y^*),$$

where u, v are the utility functions of both players. The possible existence of an equilibrium depends on the concrete functions u, v, which, for pure strategies, have the form of 3×3 *matrices.*

(u, v)	A	B	C
A	$(2, 2)$	$(2, 4)$	$(2, 3)$
B	$(4, 2)$	$(3, 3)$	$(1, 5)$
C	$(3, 2)$	$(5, 1)$	$(0, 0)$

Table 5.1: A bimatrix game without equilibrium

Since for some bimatrix games (for instance, for the game with the utility functions given by Table 5.1) the equilibrium may not exist in pure strategies, in game theory it is studied the probability extension of the game. An extended, so called mixed strategy, is a probability distribution on the set of pure strategies of the player. It is well-known that in mixed strategies, every bimatrix game has at least one equilibrium point [49]. We leave to the reader to show as an exercise, that in our example with the matrices given by Table 1 the equilibrium in mixed strategies is $x^* = y^* = (\frac{1}{9}, \frac{1}{3}, \frac{5}{9})$ *for both players with the corresponding utility value* $u(x^*, y^*) = v(x^*, y^*) = 2$.

In this concrete case, given by our example, a general mixed strategy is given by a vector $(x_1, x_2, x_3) \in \mathbb{R}^3$ *such that* $x_i \geq 0$ *for each i and* $x_1 + x_2 + x_3 = 1$*. In this notation,*

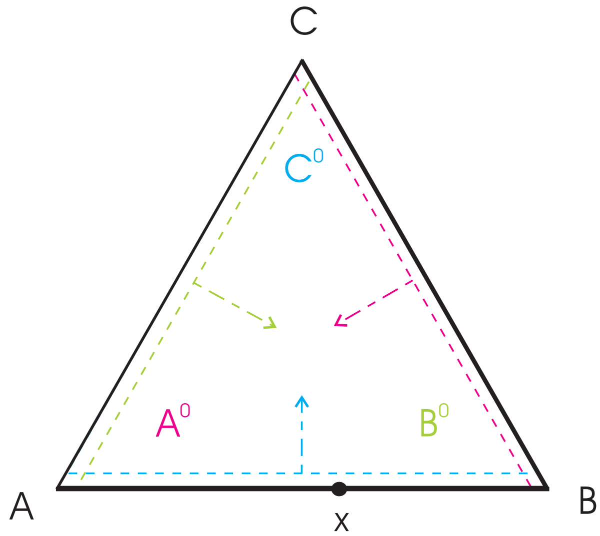


Figure 5.1: Topological model of a framework

one may represent the pure strategies as $A = (1, 0, 0)$, $B = (0, 1, 0)$ *and* $C = (0, 0, 1)$. *Then all possible mixed strategies of a player lie in the triangle* X *in* $\mathbb{R}^3$ *having the vertices* A, B, C. *The set* $X \subseteq \mathbb{R}^3$ *is naturally equipped with the Euclidean topology* τ *induced from* $\mathbb{R}^3$. *With respect to this topology, the line segments* AB, BC, AC *are closed in* X, *so their complements* $A^0 = X \setminus BC$, $B^0 = X \setminus AC$, $C^0 = X \setminus AB$ *are open. We put* $S = \{A^0, B^0, C^0\}$, $X' = AB \cup BC$. *Then* $\sigma = \{\{U | U \in S, x \in U\} | x \in X'\} = \{\{A^0\}, \{B^0\}, \{C^0\}, \{A^0, B^0\}, \{B^0, C^0\}\}$. *Thus* (S, σ) *is a framework isomorphic to* (P, π) *and* (X, τ) *is an open topological model for* (P, π).

The only strategies that may really happen and which can be only observed are the pure strategies A, B, C. *In a strict sense, therefore only the structure of* (P, π) *can be perceived in reality via the concrete progress of the game or of several its repetitions. The most of the points of the topological space* (X, τ) *are only 'virtual' since such strategies can never happen. If the game is repeated several times, some of the points lying in the interiors of the segments , represent the sequences of the strategies, which can be played. The framework represents all these inner points by the abstract points* $\{A, B\}$, $\{B, C\}$ *of the framology* π.

Note that although the most mixed strategies cannot happen, the probability model represented by (X, τ) still has its own sense, since it can yield some recommendation to the players how they should act, though it need not be a satisfactory solution.

Another inspiration for a construction of the various frameworks one can find in theoretical or mathematical physics. The following notion, introduced by J. D. Christensen and L. Crane in [16] is motivated by the research in quantum gravity.

Definition 5.6 *A causal site* $(S, \sqsubseteq, \prec)$ *is a set* S *of regions equipped with two binary relations* $\sqsubseteq$, $\prec$, *where* $(S, \sqsubseteq)$ *is a partial order having the binary suprema* $\sqcup$ *and the least element* $\bot \in S$, *and* $(S \setminus \{\bot\}, \prec)$ *is a strict partial order (i.e., anti-reflexive and transitive), linked together by the following axioms, which are satisfied for all regions* $a, b, c \in S$*:*

1. $b \sqsubseteq a$ *and* $a \prec c$ *implies* $b \prec c$*,*
2. $b \sqsubseteq a$ *and* $c \prec a$ *implies* $c \prec b$*,*
3. $a \prec c$ *and* $b \prec c$ *implies* $a \sqcup b \prec c$*.*
4. *There exits* $b_a \in S$*, called cutting of* a *by* b*, such that*
   (a) $b_a \prec a$ *and* $b_a \sqsubseteq b$*;*
   (b) *if* $c \in S$*,* $c \prec a$ *and* $c \sqsubseteq b$ *then* $c \sqsubseteq b_a$*.*

It is interesting that every causal site generates a compact T_1 topology, which can be effectively proved by using the framework theory. There is a natural way how a causal site defines a framework. Let $(P, \sqsubseteq, \prec)$ be a causal site and let us define appropriate framework structure on P. We say that a subset $F \subseteq P$ set is centered, if for every $x_1, x_2, \ldots, x_k \in F$ there exists $y \in P$, $y \neq \bot$ satisfying $y \sqsubseteq x_i$ for every $i = 1, 2, \ldots, k$. If $\mathcal{L} \subseteq 2^P$ is a chain of centered subsets of P linearly ordered by the set inclusion $\subseteq$, then $\bigcup \mathcal{L}$ is also a centered set. Then every centered $F \subseteq P$ is contained in some maximal centered $M \subseteq P$. Let π be the family of all maximal centered subsets of P. Now, consider the framework (P, π) and its dual (P^d, π^d). Let (X, τ) be the topological space with $X = P^d = \pi$ and the topology τ generated by its closed subbase (that is, a subbase for the closed sets) π^d.

The following theorem is due to Martin Kovár and it is contained in [36] and also in our joint paper [37]. Since the proof of the theorem, inspired by the construction of the Wallman-type or Shanin compactification (see, e.g., [17] or [20] for the Walman-type compactification and [48] for the Shanin's slightly different, but equivalent approach) became a source of inspiration of the main theorem of the next section, we repeat it, with the permition of its author, for the readers convenience.

Theorem 5.2 *The topological space (X, τ), corresponding to the framework (P^d, π^d) and the causal site $(P, \sqsubseteq, \prec)$, is compact T_1.*

Proof. By the well-known Alexander's subbase lemma, for proving the compactness of (X, τ) it is sufficient to show, that any subfamily of π^d having the f.i.p., has nonempty intersection. The subbase for the closed sets of (X, τ) has the form $\pi^d = \{\pi(x) | \, x \in P\}$, so any subfamily of π^d can be indexed by a subset of P. Let $F \subseteq P$ and suppose that for every $x_1, x_2, \dots, x_k \in F$ we have

$$\pi(x_1) \cap \pi(x_2) \cap \dots \cap \pi(x_k) \neq \varnothing.$$

Then there exists $U \in \pi$ such that $U \in \pi(x_1) \cap \pi(x_2) \cap \dots \cap \pi(x_k)$, so $x_i \in U$ for every $i = 1, 2, \dots, k$. Since U is a (maximal) centered family, there exists $\perp \neq y \in P$ such that $y \sqsubseteq x_i$ for every $i = 1, 2, \dots, k$. Thus, F is a centered family, contained in some maximal centered family $M \subseteq P$. Then we have $M \in \pi$, so

$$M \in \bigcap_{x \in M} \pi(x) \subseteq \bigcap_{x \in F} \pi(x) \neq \varnothing.$$

Hence, (X, τ) is compact.

Let $U, V \in X = \pi$, $U \neq V$. Since both are maximal centered subfamilies of P, none of them can contain the other one. So, there exist $x, y \in P$ such that $x \in U \setminus V$ and $y \in V \setminus U$. Then $U \in \pi(x)$, $V \notin \pi(x)$, $V \in \pi(y)$, $U \notin \pi(y)$. Thus, $X \setminus \pi(x)$, $X \setminus \pi(y)$ are open sets in (X, τ) containing just one of the points U, V. So the topological space (X, τ) satisfies the T_1 axiom.

In [37] we also show, that every globally hyperbolic Lorentzian manifold defines a causal site for which the topology generated by (P^d, π^d) coincides with the de Groot dual of the manifold topology. Since the manifold topology is locally compact, both topologies coincide on every compact set. The important consequence for physics it is that both topologies are exactly the same at finite distances, so there is probably no physical way, experiment or observation, how these topologies can be distinguished one from each other. On the other hand, we show in [37] that there exist non-Lorentzian causal sites, that is, causal sites which cannot be generated by any Lorentzian manifold. Hence, the corresponding framework (P, π) may capture the topological structure of very general, alternative models of space-time.

Since the proof of the existence of an appropriate causal site in a globally hyperbolic Lorentzian manifold requires a relatively deep results and knowledge of differential geometry, which is too far from the main theme of this dissertation, these results are not included and we rather refer the reader, in case of interest, to [37].

5.3. Approximations by finite frameworks

Now we will study the possibility of an approximation of any framework, say (P, π), by a directed system of finite frameworks. We will need to introduce the following notion.

Definition 5.7 *Let (X, α) be a framework, $Y \subseteq X$. Denote $\beta = \{U \cap Y | U \in \alpha\}$. Then (Y, β) is called the induced subframework of (X, α).*

We put $\pi_K = \{U \cap K | U \in \pi\}$ for every finite $K \subseteq P$. Ovbiously, if K, L are finite subsets of P and $K \subseteq L$, (K, π_K) is an induced subframework of (L, π_L) and both are induced subframeworks of the original framework (P, π). The collection of finite frameworks (P_K, π_K) is directed by the set inclusion. Let

$$\sigma = \{W | W \subseteq P, W \cap K \in \pi_K \text{ for every finite } K \subseteq P\}.$$

Obviously, $\pi \subseteq \sigma$. Moreover, after a restriction to a finite family $K \subseteq P$ of places in the framework (P, π) there is no way how to distinguish between (P, π) and (P, σ), since

$$\{U \cap K | U \in \pi\} = \pi_K = \{W \cap K | W \in \sigma\}.$$

It could seem that it would be a good idea to approximate (P, π) by (P, σ). However, as we will show later, (P, σ) may contain too many abstract poins (that is, elements of σ) in comparison to (P, π).

Let $\lambda \subseteq \sigma$ be a chain linearly ordered by the set inclusion. We put $L = \bigcup \lambda$. Clearly, $L \subseteq P$. If $K \subseteq P$ is finite, then also the set $L \cap K = \bigcup_{W \in \lambda}(W \cap K)$ is finite. Denote $L \cap K = \{x_1, x_2, \ldots, x_k\}$. Then for every $i \in \{1, 2, \ldots, k\}$, there is some $W_i \in \lambda$ with $x_i \in W_i$. But λ is a chain, so there is the greatest element, say $W_m \in \sigma$, among all $W_1, W_2, \ldots, W_k$ with respect to $\subseteq$. Then $L \cap K = W_m \cap K \in \pi_K$. Thus $L \in \sigma$, so L is the upperbound of the chain λ. By Zorn's Lemma, every element $W \in \sigma$ is contained in some maximal element $M \in \sigma$. Let $\mu \subseteq \sigma$ be the set of all maximal elements of σ. The framework (P, μ) could be another candidate for an approximation of (P, π).

Example 5.3 *Let $P = \mathbb{N}$ and let π be the set of all finite subsets of P. Then, respecting the previous denotations, $\sigma = 2^P$, and $\mu = \{P\}$.*

Proof. It is obvious, that $\sigma \subseteq 2^P$. Let $W \in 2^P$. For every finite $K \subseteq P$, $W \cap K \in \pi_K = 2^K$, so $W \in \sigma$. However, the set $\sigma = 2^P$ has only one maximal element with respect to the set inclusion, P.

The following theorem now describes the approximation properties of our construction under very general topological conditions.

Theorem 5.3 *Let (X, τ) be a topological T_1 space, $\mathcal{C}$ the family of all closed sets. Let (P, π) be the dual framework of $(X, \mathcal{C})$. Then the dual of (P, μ) generates the Wallman compactification of (X, τ). More precisely, μ^d is a closed subbase of ωX.*

Proof. We have $P = \mathcal{C}$ and $\pi = \{\mathcal{C}(x) |\, x \in X\}$, where $\mathcal{C}(x) = \{C |\, C \in \mathcal{C}, x \in C\}$. We will show that every element of σ is a family of closed sets of the topological space (X, τ), having f.i.p. Let $W \in \sigma$ and let $K \subseteq W$ be finite. Then $K = W \cap K \in \pi_K$, so there exists $y \in X$ such that $K = \mathcal{C}(y) \cap K$. Then $K \subseteq \mathcal{C}(y)$, which gives $y \in \bigcap K \neq \varnothing$. Hence, W has f.i.p and its closedness follows from the fact that $W \subseteq P = \mathcal{C}$.

Let us show that $\pi \subseteq \mu$. Suppose that for some $W \in \sigma$ we have $\mathcal{C}(x) \subseteq W$. Since (X, τ) is a T_1 space, $\{x\} \in \mathcal{C}(x) \subseteq W$. But W has f.i.p, so every its element must contain x. Then $W \in \mathcal{C}(x)$, so $W = \mathcal{C}(x)$. Therefore, $\mathcal{C}(x)$ is a maximal element of σ, that is, $\pi \subseteq \mu$.

Let η be the family of all maximal collections of closed sets having f.i.p. Note that, in other words, η is the family of all ultra-closed filters on (X, τ). We will show that $\eta = \mu$. As the first step, we will prove that $\eta \subseteq \mu$. Let $U \in \eta$. Take any finite $K \subseteq P = \mathcal{C}$ and denote $L = U \cap K$. The set L contains only finitely many elements of U. The family U has f.i.p., so $\bigcap L \neq \varnothing$. Denote $D = \bigcup(K \smallsetminus L)$. The set D is closed (and could be possibly empty, if $K = L$). Suppose that $\bigcap L = \bigcap(L \cup \{D\})$. Consider the family $U \cup \{D\}$. If $M \subseteq U \cup \{D\}$ is finite, then $M \smallsetminus \{D\}$ and also $(M \smallsetminus \{D\}) \cup L$ are finite subsets of U, so $\varnothing \neq \bigcap((M \smallsetminus \{D\}) \cup L) = \bigcap(M \smallsetminus \{D\}) \cap (\bigcap L) = (\bigcap(M \smallsetminus \{D\})) \cap (\bigcap(L \cup \{D\})) = (\bigcap(M \smallsetminus \{D\})) \cap D \cap (\bigcap L) = (\bigcap M) \cap (\bigcap L) \subseteq \bigcap M$. Then $U \cup \{D\}$ has f.i.p. In particular, $D \neq \varnothing$, which implies that $K \neq L$. Then $K \smallsetminus L \neq \varnothing$, and $U \cap (K \smallsetminus L) = \varnothing$. It follows from the maximality of U that $D = \bigcup(K \smallsetminus L) \notin U$. Then $U \cup \{D\}$ is a strictly greater family than U. This contradicts to the maximality of U. Therefore, there is some $z \in X$ such that $z \in \bigcap L$, $z \notin D$. Then $L \subseteq \mathcal{C}(z)$, but $(K \smallsetminus L) \cap \mathcal{C}(z) = \varnothing$. That means

$U \cap K = L = L \cap \mathcal{C}(z) = K \cap \mathcal{C}(z) \in \pi_K$. Then $U \in \sigma$. By definition of the set μ, there exists $W \in \mu$ such that $U \subseteq W$. But as we have shown above, W is a family of closed sets having f.i.p. By maximality of U, we have $U = W$, so $U \in \mu$. Therefore, $\eta \subseteq \mu$.

Conversely, let $U \in \mu$. Because also $U \in \sigma$, the family σ consits of closed sets and has f.i.p. Then there exists some $W \in \eta$ with $U \subseteq W$. By the previous paragraph, we have $W \in \mu \subseteq \sigma$. But U is a maximal element of σ, so $U = W$ and $U \in \eta$. Together we finally have $\mu = \eta$.

Now, consider the framework (P^d, μ^d). It holds $P^d = \mu$, $\mu^d = \{\mu(C) | \, C \in \mathcal{C}\}$, where $\mu(C) = \{U | U \in \mu, C \in U\}$. Consider the topological space (Y, ϑ), where $Y = P^d$ and its topology is generated by its closed subbase μ^d. Consider $\Psi \subseteq \mathcal{C}$, such that for every $C_1, C_2, \ldots, C_k \in \Psi$ it holds

$$\mu(C_1) \cap \mu(C_2) \cap \cdots \cap \mu(C_k) \neq \varnothing.$$

There exist $U \in \mu$ (depending on the selection of C_1, C_2, $\ldots C_k$), such that $C_1, C_2, \ldots C_k \in U$, so $\varnothing \neq \bigcap_{i=1}^{k} C_i \in U$. Then Ψ has f.i.p., so there exists a maximal family $W \subseteq \mathcal{C} = P$, having f.i.p. and containing Ψ. By the previous paragraph, $W \in \mu$. Now, if $C \in \Psi$, then also $C \in W$, which gives $W \in \mu(C)$ and so

$$W \in \bigcap_{C \in \Psi} \mu(C) \neq \varnothing.$$

Therefore, (Y, ϑ) is compact.

Finally, consider the mapping $f : X \to Y$, where $f(x) = \mathcal{C}(x)$. Clearly, f is an injection. Indeed, for $x \neq y$ we have $\{x\} \in \mathcal{C}(x)$ but $\{x\} \notin \mathcal{C}(y)$, so $f(x) \neq f(y)$. Let $C \in \mathcal{C}$. Then $f^{-1}(\mu(C)) = \{x | \, x \in X, f(x) \in \mu(C)\} = \{x | \, x \in X, \mathcal{C}(x) \in \mu(C)\} = \{x | \, x \in X, C \in \mathcal{C}(x)\} = \{x | \, x \in X, x \in C\} = C$, so f is continuous. Further, for any $D \in \mathcal{C}$, $f(D) = \{\mathcal{C}(x) | \, x \in D\} = \{\mathcal{C}(x) | \, x \in X, \mathcal{C}(x) \in \mu(D)\} = f(X) \cap \mu(D)$, so f is also a closed mapping. Then f is a homeomorphous embedding of (X, τ) to the compact space (Y, ϑ). Moreover, $f(X) = \pi$, so the elements of X and the families $\mathcal{C}(x)$, which constitute the principal ultra-closed filters generated by the elements of X, may be identified. Consider the set $\mu \setminus \pi$. Then every its element $W \in \mu \setminus \pi$ is vanishing (that is, non-convergent) – otherwise, because of maximality, $W = \mathcal{C}(z)$, where z is the unique element from $\bigcap W$, which would imply $W \in \pi$. But then Y is the underlying set of the Wallman compactification of (X, τ) and $\mu^d = \{\mu(C) | \, C \in \mathcal{C}\}$ is its closed base.

Among others the previous theorem means that for a compact T_1 topological space, its approximation by a suitable family of finite frameworks may achieve an arbitrary precision.

5.4. Conclusion

Our results in this chapter have some relatively interesting consequences for characterization of the topology that is perceived in the physical Universe and which is present at the background of various physical phenomena. If we take the usual, "smooth", locally Euclidean and Hausdorff topology of Lorentzian manifolds as our reference point, it seem that in reality we perceive rather its de Groot dual. This topology is weaker than the manifold topology, and from some point of view it is more nice, since it is compact and superconnected, but at the cost of loss of Hausdorffness. On compact sets, however, both topologies coincide, so at finite distances they cannot be (probably) distinguished by any physical experiment or measurement. Hence, the difference is rather philosophical. The perceived topology, as the extrapolation of our finite experience with the Universe or Nature is always compact, even for non-compact, relativistic models of space-time, which corresponds to the notion of the potential infinity. Our experience with the Nature is unbounded but finite, so also the perceived topology is unbounded but "finite" in the sense of compactness. On the other hand, the usual manifold topology of globally hyperbolic Lorentzian manifold is rather connected with the actual infinity. However, even in our model of "extrapolated finite experience" the most "points" are only virtual in the following sense: for the whole history of our experience, no event can happen at most of them, similarly, as it holds for the mixed strategies in Example 5.2.

Because of the 'contextual nature' of the framework structure, it is very appropriate for representing in computer memory, especially in connection with Theorem 5.3. One can imagine a space probe exploring unknown regions of the Universe, while successively adjusting its topological model by a sequence of finite frameworks, a computer program exploring unknown data and continuously mapping their structure, and many other similar applications.

6. Topology as a Tool in Game Theory

In the previous chapter we illustrated by Example 5.2 that game theory is a natural source of various situations in which framework and contextual structures may appear as very helpful in analysis of the underlying structure. In Chapter 1, especially in its section 2.2, and also in Chapter 3, we demonstrated how topology and the context structures are mutually connected. The last chapter of the dissertation will be devoted to completing the last piece of stone to our mosaic. Now we will use topology as a tool for investigation in game theory. The most results included in this chapter have been recently published in a joint paper [38], but due to a page limit in a rather shortened version. The full version, nowhere published in this complete form, now follows.

6.1. Introduction

Undominated strategies play an important role in game theory as well as in many related engineering and economical applications. The theorem ensuring the existence of undominated strategies in a normal form game under the assumption that the set of all strategies of a player is compact and the utility function is continuous, belongs to the well-known and fundamental results. Perhaps it could be difficult to say when the result was published first – at least, it was stated in 1981 in Herve Moulin's comprehensive textbook on game theory [45], and essentially it was also contained and used in many other papers. The proof presented in the first edition of [45] was dependent on a combination of relatively non-trivial results from measure theory, metric topology and mathematical analysis. In the second, revised edition [46] of the same book, now there is stated a simplified proof using some topological argumentation together with Zorn's Lemma. However, the proof in [46] is unfortunately incorrect, since it implicitly uses a non-valid argument that every chain (that is, a linearly ordered set) contains a cofinal subsequence. The first uncountable ordinal ω_1 is a proper counterexample witnessing that in general it is not true. The mistake itself is not very critical for game theory, since in metric spaces, for which the classical results are usually formulated, the topology is first countable and hence the sequences are still sufficient to fully describe the topology by means of the convergence. Nevertheless, the mentioned fact itself, was a source of inspiration for a revision of of the original Moulin's Theorem leading to its our generalization and improvement. A natural question how substantial our improvement really is we will demonstrate on a simple example.

6.2. Topological and Order-theoretic Background

For the reader's convenience and comfort, let us to recapitulate or summarize some necessary facts for our next considerations. More elementary topological notions are introduced and explained in Section 2.1 in Chapter 1.

A binary relation on a set is called a *preorder*, if it is reflexive and transitive (and not necessarily antisymmetric). Let A be a non-empty set, $\preccurlyeq$ be a preorder on A such that for every $x, y \in A$ there exists $z \in A$ with $x \preccurlyeq z$ and $y \preccurlyeq z$. Then we say that $(A, \preccurlyeq)$ is a *directed* set. A *net* in a topological space X is an arbitrary mapping from a directed set to the space X. Recall that if $f : X \to Y$ is a continuous mapping between topological spaces X, Y and φ is a net in X, having a cluster point $x \in X$, then $f \circ \varphi$ is a net in Y, having the corresponding cluster point $y = f(x) \in Y$. A family Φ of non-empty sets is called a filter base if any intersection of two sets belonging to Φ contains a subset from Φ. Let X be a topological space. We say that $p \in X$ is a θ-cluster point of a filter base Φ in X, if for every closed neighborhood H of p and every $F \in \Phi$, the intersection $H \cap F$ is non-empty. Similarly, p is a θ-cluster point of a net $\varphi(A, \preccurlyeq)$, if for each closed neighborhood H of p and for each $a \in A$, there exists $b \in A$, $b \succcurlyeq a$, such that $\varphi(b) \in H$. Taking the φ-images of the principal upper sets $\uparrow a = \{b |\, b \in A, b \succcurlyeq a\}$ one can easily convert the net $\varphi(A, \preccurlyeq)$ into a filter base, while the corresponding convergence and θ-convergence notions will be preserved.

A topological space X is *compact*, if every net or every filter base in X has a cluster point. For more detail and other equivalent and well-known characterizations of compactness, especially in terms of open covers, we refer the reader to Section 2.1 and the monographs [17], [20], [48] and [56]. We also remark that in a modern approach to compactness, motivated by the growing interest of the theoretical computer scientists in topology, the Hausdorff separation axiom is no longer assumed as a part of the definition of compactness (see, for example, [56]). Recall that a topological space is *almost compact* [17] if every open filter base in X has a cluster point. It is clear from the definition that every compact space is almost compact but not vice versa, as the reader may check from a counterexample in [17]. Another counterexample we will present also in the next section.

The real line $\mathbb{R}$, if not otherwise specified, we consider as a topological space equipped with the natural, Euclidean topology, generated by all open intervals.

6.3. Main Results

We will start with the correction of the proof of the well-known theorem, stated by Herve Moulin in [45] and [46].

Theorem 6.1 *Let $G = (X_1, X_2, \ldots, X_n, u_1, u_2, \ldots, u_n)$ be a normal form game of n persons. Suppose that for some $i \in \{1, 2, \ldots, n\}$ there exist a compact topology on X_i in which the utility function u_i is a continuous, real valued function of the argument $x_i \in X_i$. Then the i-th player has an undominated strategy.*

Proof. For two strategies $x_i, y_i \in X_i$ we put $x_i \preccurlyeq y_i$ if they satisfy the condition (1) of the definition of dominance. It is easy to see that $\preccurlyeq$ is a preorder on X_i. Let $L \subseteq X_i$ be an arbitrary linearly preordered subset of X_i (that is, for every $a, b \in L$, it holds $a \preccurlyeq b$, or $b \preccurlyeq a$). Let l be the identity mapping on X_i, restricted to L. Then l is a net in a compact topological space X_i, so l has a cluster point $p \in X_i$.

Now, suppose that the strategies $s_k \in X_k$ of the other players, $k \neq i$, are arbitrarily chosen, but fixed in this paragraph. We denote $u'_i(x_i) = u_i(s_1, s_2, \ldots, s_{i-1}, x_i, s_{i+1}, \ldots, s_n)$. The composition $u'_i \circ l$ is a net in $\mathbb{R}$, and since u'_i is a continuous function of x_i, $r = u'_i(p)$ is its cluster point in $\mathbb{R}$. Suppose, for a moment, that there exist some $t \in L$ with $u'_i(p) < u'_i(t)$. Then also $r = u'_i(p) < u'_i(t) \leq u'_i(s)$ for all $s \succcurlyeq t$, which contradicts to the fact, that r is a cluster point of the net $u'_i \circ l$. So $u'_i(t) \leq u'_i(p)$, or, in other words, the condition (1) is satisfied for the strategy p and for all strategies $t \in L$.

By the definition of the preorder $\preccurlyeq$, p is an upper bound of L. By Zorn's Lemma, there is a maximal element m in the preordered set $(X_i, \preccurlyeq)$. This completes the proof, since the strategy, maximal with respect to $\preccurlyeq$, cannot be dominated.

Let us continue with the following simple example. As we will show later, the existence of undominated strategies of both players is not a consequence of the classical Moulin's Theorem, but it follows from our generalization.

Example 6.1 *Consider a normal form game of two players with the same sets of strategies* $X_1 = X_2 = [0,1) \times \{0\} \cup \{1\} \times \{0,1,\dots\}$. *Let the corresponding utility functions of the players be*

$$u_1 = \frac{x_1}{x_1 + x_2} \cdot f(y_2), \text{ and } u_2 = \frac{x_2}{x_1 + x_2} \cdot g(y_1),$$

where f, g *are arbitrary real-valued functions defined on* $\{0\} \cup \mathbb{N}$. *It is easy to see that the pairs* $(1, n) \in X_i$, *where* $n \in \{0, 1, \dots\}$ *and* $i = 1, 2$, *are equivalent, maximal and undominated strategies of the* i*-th player. However, although the utility functions* u_i *are continuous, the topology of* X_i, *induced from the real plane is not compact. For instance, the sequence* $\{(1,n) |\, n = 0, 1, 2, \dots\}$ *has no cluster point in* X_i. *Hence, the existence of undominated strategies of the* i*-th player is not a consequence of Moulin's theorem.*

There are many possible ways how the game could be realized. A concrete background scenario of the game may consist, for instance, in a hypothetical struggle of two enemy countries, extracting the same oil reservoir. The oil wells of each country produce x_1 or x_2 units of oil per a time unit and the total volume of oil which can be extracted from the reservoir is for simplicity equal to 1. In addition, the i-th country may decide to destroy partially the output production of its opponent during its transportation to the market by a military action, but the country can decide to do it only after reaching its own technological maximum of production (that is, when $x_i = 1$). Each attack will decrease the transported oil supplies by multiplication of some coefficient less than 1. The i-th player may choose the number of his attacks as the second component y_i of his strategy $(x_i, y_i) \in X_i$.

From a more general point of view, probably the most significant interpretation of the game it could be a duopolistic competition over market share with patent wars. The first component x_i of the strategy (x_i, y_i) of the i-th player may represent the market share, while the second component y_i can be interpreted as obstructions extracting the profit of the player's opponent, in particular litigation over patent rights.

The reader may also notice that there is some additional space for improving the result stated in Theorem 6.1 yielded by a modification of its topological assumptions. For instance, the theorem will remain true, if one replaces the continuity of the utility function by its upper semi-continuity. This is a result due to H. Salonen [51]. He essentially used a characterization of compactness by the centered collections of sets (in other words, having the finite intersection property, [48]), or filters and filter bases, which are topologically equivalent to nets. A similar technique was also used in [50] for iteratively undominated strategies with the continuous utility function.

Another, and perhaps new natural improvement of Theorem 6.1 we receive by relaxing the condition of compactness. At least, we did not see such a modification of the original result in the literature.

For our intention to replace compactness by almost compactness and also for a better understanding of some other aspects of the previous example, we will need the following lemma. The contents of the lemma is already known – it is essentially contained in (but rather split between) the book [17] and the paper [55]. Useful are also comments in [29]. We present the result here with a proof in order to repeat and concentrate some ideas of these resources at one place for the reader's convenience.

Lemma 6.1 *Let (X, τ) be a topological space. The following conditions are equivalent:*

(i) (X, τ) is almost compact.

(ii) Every filter base in X has a θ-cluster point.

(iii) Every net in X has a θ-cluster point.

(iv) Every open cover of X has a finite subfamily whose union is dense in X.

Proof. Suppose (i) and let Φ be a filter base in X. The family $\Psi = \{U | \, U \in \tau$, there exists $F \in \Phi$ with $F \subseteq U\}$ is an open filter base and so it has a cluster point, say $p \in X$. Let H be a closed neighborhood of p. We will show that $H \cap F \neq \varnothing$ for every $F \in \Phi$. Suppose conversely, that $F \subseteq X \setminus H$ for some $F \in \Phi$. Then $X \setminus H \in \Psi$ and so $p \in \operatorname{cl}(X \setminus H)$. But this is not possible since $p \in \operatorname{int} H$ and $(\operatorname{int} H) \cap (X \setminus H) = \varnothing$. Hence, it follows (ii).

Consider (ii) and take a net $\varphi(A, \preccurlyeq)$ in X. The family $\Phi = \{\varphi(\uparrow a) | \, a \in A\}$ is a filter base with a θ-cluster point, say $p \in X$. Let H be a closed neighborhood of p and let $a \in A$. Then $H \cap \varphi(\uparrow a) \neq \varnothing$, so there is some $b \in A$, $b \succcurlyeq a$, with $\varphi(b) \in H$. It means that p is a θ-cluster point of $\varphi(A, \preccurlyeq)$ and (iii) holds.

Assume (iii) and take an open cover Ω of X. Let Ω^F be the family of all finite unions of elements of Ω. The family Ω^F is directed by the set inclusion. Suppose that for every $U \in \Omega^F$ the set $X \setminus \operatorname{cl} U$ is non-empty, so it contains some element $\varphi(U)$. The net $\varphi(\Omega^F, \subseteq)$ has a θ-cluster point, say $p \in X$. Since Ω^F is also a cover, there is some $V \in \Omega^F$ containing p. By the definition of the θ-cluster point, there exists $W \in \Omega^F$, $W \supseteq V$, such that $\varphi(W) \in \operatorname{cl} V$. But it also holds $\varphi(W) \in X \setminus \operatorname{cl} W$, so $\varnothing \neq (X \setminus \operatorname{cl} W) \cap V \subseteq (X \setminus W) \cap W$, which is not possible. Then some element of Ω^F must be dense in X.

Finally, suppose (iv). Let Ψ be an open filter base in X with no cluster point. Then $\bigcap\{\operatorname{cl} U | \, U \in \Psi\} = \varnothing$, so $\Omega = \{X \setminus \operatorname{cl} U | \, U \in \Psi\}$ is an open cover of X and since Ψ is a filter base, it is directed by the inclusion. By (iv), there exists $U \in \Psi$, such that

$X = \mathrm{cl}(X \setminus \mathrm{cl}\, U)$. Since $X \setminus U$ is a closed set containing $(X \setminus \mathrm{cl}\, U)$, it contains also its closure and so $X \setminus U = X$. But this is not possible according to the fact, that a filter base contains only non-empty elements. Therefore, Ψ has a cluster point and (i) now follows.

From the previous lemma it also follows the well-known fact that for regular spaces the compactness and almost compactness coincide. On the other hand, there exist a Hausdorff almost compact space which is not compact, as the reader may check for instance in [17]. Hausdorff almost compact spaces are also known under another terminology as H-closed spaces (also in [17], or [55]).

Theorem 6.2 *Let $G = (X_1, X_2, \ldots, X_n, u_1, u_2, \ldots, u_n)$ be a normal form game of n players. Suppose that for some $i \in \{1, 2, \ldots, n\}$, X_i is almost compact and the utility function u_i is a continuous, real valued function of the argument $x_i \in X_i$. Then the i-th player has an undominated strategy.*

Proof. Repeating the notation and the introductory considerations of the proof of Theorem 6.1, from the assumption that X_i is almost compact we may conclude that the net l has a θ-cluster point $p \in X_i$. By the definition it means that for every closed neighborhood H of p and every $t \in L$ there exists some $s \in L$, $s \succcurlyeq t$, with $l(s) \in H$.

Under the same assumption regarding the strategies s_k, $k \neq i$ and the same meaning of $u_i'(x_i)$, suppose for a while, that there is some $t \in L$ with $u_i'(p) < u_i'(t)$. Take $c \in \mathbb{R}$ such that $u_i'(p) < c < u_i'(t)$. Because of continuity of u_i', $H = u_i'^{-1}((-\infty, c])$ is a closed set in X_i whose interior contains p. Since p is a θ-cluster point of l, there exists $s \in L$, $s \succcurlyeq t$, such that $s = l(s) \in H$. But then $u_i'(s) \in (-\infty, c]$, which is not possible, because the relation $s \succcurlyeq t$ means that $c < u_i'(t) \leq u_i'(s)$. Consequently, p is an upper bound of L. Now, Zorn's Lemma completes the proof as in the previous section.

Now, let us check the advantage of Theorem 6.2 over its original version. Notice that the game described in Example 6.1 cannot be easily covered by Theorem 6.1. Although the utility functions u_i are continuous, the topology of X_i, induced from the real plane is not compact (and certainly also not almost compact, as it follows from the note after Lemma 6.1). For instance, the sequence $\{(1, n) | \, n = 0, 1, 2, \ldots\}$ has no cluster point. Let us define another topology on X_i, where $i = 1, 2$, by the local base of a general point $(x, y) \in X_i$:

1. The point $(0, 0)$ has neighborhoods of the form $[0, \varepsilon) \times \{0\}$, $0 < \varepsilon < 1$.

2. For every $x \in (0,1)$, the point $(x,0)$ has neighborhoods of the form $(x-\varepsilon, x+\varepsilon) \times \{0\}$, $0 < \varepsilon < \min\{x, 1-x\}$.

3. For every $n = 0, 1, \dots$, the point $(1,n)$ has neighborhoods having the form $(1-\varepsilon, 1) \times \{0\} \cup \{(1,n)\}$, where $0 < \varepsilon < 1$.

The new topology on X_i is now similar to the Euclidean topology on the unit segment $[0,1]$ but with one important difference – the right end point of the "segment" is present infinitely many times. The space X_i is T_1, but certainly non-Hausdorff and non-compact. Indeed, denoting $Y_n = [0,1) \times \{0\} \cup \{(1,n)\}$, the family $\{Y_n | \, n = 0, 1, \dots\}$ is an open cover of X_i, having no finite subcover. However, we can show that the new topology is almost compact. Let Ω be an open cover of X_i. The subspace $Y_0 = [0,1] \times \{0\} \subseteq X_i$ is compact since it is homeomorphous with the unit segment $[0,1]$, so there exists a finite subfamily $\{U_1, U_2, \dots, U_k\} \subseteq \Omega$ with $Y_0 \subseteq \bigcup_{j=1}^{k} U_j$. Then there is $r \in \{1, 2, \dots, k\}$ such that $(1,0) \in U_r$. But for every $n = 1, 2, \dots$ it follows $(1,n) \in \operatorname{cl} U_r$, so the closures of $\{U_1, U_2, \dots, U_k\}$ cover X_i. By the condition (iv) of Lemma 6.1, X_i is almost compact. The utility functions u_i are continuous functions of the argument (x_i, y_i) since they are continuous on the open subspaces $Y_n = [0,1) \times \{0\} \cup \{(1,n)\}$ of X_i, $n = 0, 1, \dots$, homeomorphous to $[0,1]$. Hence, the existence of the undominated strategies now follow also from Theorem 6.2. Note that similar spaces as X_i are also known as examples of non-Hausdorff manifolds, [31], and they may appear also in sheaf theory and in certain problems of mathematical physics, [27].

6.4. Conclusion

Our previous considerations show that our generalization of Moulin's Theorem significantly extends the class of applicable tasks or problems. In addition they together with Example 6.1 demonstrate that non-Hausdorff and non-Euclidean topologies are really very natural, just from the real life. We may close the chapter by a remark, yielding some outlook for a possible further research in the topic.

The topological structure on the strategy set X_i possibly need not be given only from outside, but may be also an important intrinsic characterization of the game itself. The partial utility functions $u_i' : X_i \to \mathbb{R}$ (derived from u_i for the concrete selection of the strategies of the other players) are associated with the weakest topology on X_i

in which they are continuous, so called *initial* topology. Studying the conditions under which this topology is compact or almost compact may lead to another, perhaps new characterizations of the existence of the undominated strategies.

7. Summary

Let us summarize the main results of this research. A formal context is a general structure able to represent other mathematical structures in a way that the main properties would be enclosed in the set of attributes and the incidence relation. Every topological space, for instance, can be represented as a formal context with help of the membership relation. It was shown that a second derivation operator and a closure operator the left topology coincides on the one-element subsets and on the extents. Suppose we have some piece of information and a formal context is an easy way of representing it in a cross-table view. At every particular moment of time, we do not know if that information was complete, because in some moment a new part of information could appear and it could completely change the structure. We put a partial metric on the formal context to measure that partial information. As it was shown with help of measure defined on the attribute set, it is possible to construct a partial metric on the object set (and because of the duality we could construct a partial metric on the attribute set from the measure defined on the object set). A measure shows us not only the ordering information, but it shows us in some way the quantity of information. In every particular moment of time, a part of information is finite and by the Lemma 3.6 we could construct a partial metric. But in general, some objects we could not distinguish and that is why the we could not follow in the same way. So, at first we construct a quotient context were objects would be induced by the equivalence classes and on that context we could apply the same method as it was showed in the Theorem 3.7. Every new formal context is formed by new piece of information (objects added / deleted , attributes added/deleted , measure (weight) of attributes changed, the incidence relation changed). Then we can compute a partial metric for every instance of formal context. Because the instances of a formal context are different, we could not directly compare a partial metric $p(x, x)$ for an object x – before that we need to “renormalize” it. After that, we would be able to do the appropriate analysis of the objects.

In some applications a significant role play properties of compact sets, whose topological behavior could be characterized by the de Groot dual. But nowadays it becomes more popular to study more general structures, because they frequently appear in computer science. Thus in the following chapter we discuss the pretopological systems and their de Groot-like duals. The opens of the dual may be represented as certain maps from A to the Sierpiński frame $\mathbf{2}$, where A is the poset representing the opens of the original pretopological system. Then it was successfully defined an analogue of the De Groot dual for compactly localic pretopological systems and in a more general approach, also for any poset. We also proved an counterpart of M. Kovár's result $\tau^d \subseteq \tau^{ddd}$, originally proved for the general topological spaces (Theorem 4.6 for the general posets and Theorem 4.4 for the special, finite case). Note that the result stated in Theorem 4.6 is best possible as it is documented by Example 4.2 and Example 4.3.

As a special case of formal contexts, in the next chapter we study the structure of frameworks. Unlike the most of the traditional mathematical structures are used in their classical point-set representations, framework is a point-less structure by its nature. However, equipped with a simple but handy duality construction, it can be used for convenient switching between point-less and the traditional, point-set approach. A natural range of applications of this structure are the spatio-temporal relationships in modern theoretical physics, for example as in quantum gravity. The main result of this chapter is Theorem 5.3, describing the approximation of a general framework by a directed family of finite frameworks from the point of view of the generated topologies and the framework duality. This result is suitable, for instance, for representing various spatio-temporal models of reality in finite computer memory. The chapter also contains an example inspired by game theory, illustrating the difference between the really existing objects and their virtual extension, arising from extrapolation of our finite experience.

In the last chapter we pay attention to game theory itself. At the first sight, the topology and game theory seem to be far from each other. However, the topological notions and methods can be used with an advantage in many proof techniques, including game theory. For example, one of the fundamental theorems stated by H. Moulin was proved with help of topological methods. Nevertheless, the original proof contained a wrong, set-theoretical assumption, that every chain (that is, a linearly ordered set) has a cofinal subsequence. Using topological theory of convergence, expressed in terms of nets and a certain, less usual modification of Zorn's Lemma were able to offer another, simpler proof, which yields even a slightly more general result - Theorem 6.2. By this theorem, the i-th player in a normal form game with the continuous utility function and almost compact (originally compact) set of strategies has an undominated strategy. As it is shown in Example 6.1, our result extends the range of applicability of Moulin's Theorem to a wider class of applications and also illustrates, that non-Hausdorff topologies naturally arise in game theory from the real problems.

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