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REPRESENTATION OF SOLUTION OF LINEAR DISCRETE SYSTEMS WITH DELAY

REPREZENTACE ŘEŠENÍ LINEÁRNÍCH DISKRÉTNÍCH SYSTÉMŮ SE ZPOŽDĚNÍM

Short version of Ph.D. thesis

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1 Introduction

The dissertation is concerned with the representation of solutions of systems of discrete equations with delays. In this field, many valuable results have been achieved recently, which are also useful in applications of discrete systems, e.g., for solving problems in the control theory. The motivation for writing this dissertation was the results of the papers [6,7], which the dissertation extends.

The dissertation is devoted to problems of representation of solutions of linear discrete systems containing delays as partial cases of a general system

\[ x(k+1) = Ax(k) + Bx(k-m) + Cx(k-n) + f(k), \quad k \in \mathbb{N} \cup \{0\} \]  

(1)

where \( A, B \) and \( C \) are constant square matrices and \( f \) is a given nonhomogeneity. The main tool to derive appropriate formulas is a so-called discrete matrix delayed exponential (and its generalizations). Along with the system (1), the dissertation considers some of its special cases and discusses the influence of impulses at given points on the solution.

Summary of the current state of this problem is given in Chapter 2 (for details, please, see the relevant chapters of the dissertation). In addition to the papers [6, 7], other papers (devoted to discrete equations as well as differential equations) are close to the issues considered by the dissertation. We refer, e.g., to the papers [1–5, 8–19] and to the references therein. Among them, the papers most related to the dissertation’s topic include [14,18].

Chapter 3 considers an initial Cauchy problem

\[ x(k+1) = Ax(k) + Bx(k-m) + f(k), \quad k \in \mathbb{N} \cup \{0\}, \]  

(2)

\[ x(k) = \varphi(k), \quad k = -m, -m+1, \ldots, 0 \]  

(3)

where \( A, B \) are constant square matrices. The problem of representation of the solution of (2), (3) is solved under the assumption that impulses are acting on solution at prescribed points. The main result is given in Theorem 3.8.

In Chapter 4 two generalizations of discrete matrix delayed exponential for two delays are given (in Definition 4.1 and Definition 4.3). For both generalized discrete matrix delayed exponentials, their main properties are proved (in Theorem 4.2 and Theorem 4.5). Differences between two definitions of such exponentials naturally lead to different formulas for representation of initial Cauchy problem in Chapter 5 (Theorems 5.1, 5.2, 5.4, 5.5). The exponential given by Definition 4.1 corresponds the definition of the discrete matrix delayed exponential for a single delay, but its application needs the existence of an inverse \((B + C)^{-1}\). From this point of view, the second definition of discrete matrix delayed exponential is better, as no assumption on the existence of an inverse is necessary.

Throughout the dissertation, is used following notation: For integers \( s, t, s \leq t \), we define a set \( \mathbb{Z}^t_s := \{s, s+1, \ldots, t-1, t\} \). Similarly, we define sets \( \mathbb{Z}^t_{-\infty} := \{\ldots, t-1, t\} \) and \( \mathbb{Z}^\infty_s := \{s, s+1, \ldots\} \). The function \( \lfloor \cdot \rfloor \) is the floor integer function.

Define binomial coefficients as usual, i.e., for \( n \in \mathbb{Z} \) and \( k \in \mathbb{Z} \),

\[
\binom{n}{k} := \begin{cases} 
\frac{n!}{k!(n-k)!} & \text{if } n \geq k \geq 0, \\
0 & \text{otherwise.}
\end{cases}
\]
2 Current State

A “by-steps” method or “method of steps” is one of the basic methods of the theory of differential equations with delay to find a solution to the initial problem. It is effective especially for linear equations and their systems. In [12], method of steps is formalized for linear differential systems with a constant matrix and with a single delay. This formalization was achieved by utilizing a delayed matrix exponential.


A discrete version of delayed matrix exponential was defined in [6, 7]. In addition to the definition of a discrete matrix exponential its application is considered to solutions of initial-value problems for linear discrete systems with a single delay and representations of solutions are obtained. It also served as a useful tool for solving problems of control theory in [10]. A generalization of discrete delayed matrix exponential to several delays can be found in [14]. In [18], discrete delayed matrix exponential is used to investigate the stability of delay difference equations.

In the following sections, we give an overview of the known results, which are then used or generalized in Chapters 3, 4 and 5 of the dissertation.

2.1 Discrete Matrix Delayed Exponential

In the dissertation, we use a special matrix function called a discrete matrix delayed exponential. Such a discrete matrix function was first defined in [6, 7].

Definition 2.1. For an \( n \times n \) constant matrix \( B \), \( k \in \mathbb{Z} \) and fixed \( m \in \mathbb{N} \), we define a discrete matrix delayed exponential \( e^{Bk}_m \) as follows:

\[
e^{Bk}_m := \begin{cases} 
\Theta & \text{if } k \in \mathbb{Z}_{-m-1}^- \\
I + \sum_{i=1}^{\ell} B^i \left( k - m (i - 1) \right) & \text{if } \ell = 0, 1, 2, \ldots, k \in \mathbb{Z}_{(\ell-1)(m+1)+1}^{(m+1)} 
\end{cases}
\]

where \( \Theta \) is the \( n \times n \) null matrix and \( I \) is the \( n \times n \) unit matrix.

Theorem 2.2 (Theorem 2.1 in [7]). Let \( B \) be a constant \( n \times n \) matrix. Then, for \( k \in \mathbb{Z}_{-m}^\infty \),

\[
\Delta e^{Bk}_m = B e^{B(k-m)}_m.
\]
2.2 Solutions of Linear Discrete Systems with Single Delay

Consider an initial Cauchy problem
\[ \Delta x(k) = Bx(k - m) + f(k), \quad k \in \mathbb{Z}_0^\infty, \]  
\[ x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-m}^0, \]  
where \( m \geq 1 \) is a fixed integer, \( B = (b_{ij}) \) is a constant \( n \times n \) matrix, \( x: \mathbb{Z}_{-m}^\infty \to \mathbb{R}^n \), \( f: \mathbb{Z}_0^\infty \to \mathbb{R}^n \), \( \varphi: \mathbb{Z}_{-m}^0 \to \mathbb{R}^n \), and \( \Delta x(k) = x(k + 1) - x(k) \).

With the aid of discrete matrix delayed exponential, we will derive formulas for solutions of the homogeneous and nonhomogeneous initial problem (4), (5).

Consider first a homogeneous initial problem (4), (5), i.e., the problem
\[ \Delta x(k) = Bx(k - m), \quad k \in \mathbb{Z}_0^\infty, \]  
\[ x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-m}^0. \]

**Theorem 2.3** (Theorem 3.1 in [7]). Let \( B \) be a constant \( n \times n \) matrix. Then, a solution of the problem (6), (7) can be expressed as
\[ x(k) = e^{Bk}_m \varphi(-m) + \sum_{j=-m+1}^0 e^{B(k-m-j)}_m \Delta \varphi(j - 1), \]

where \( k \in \mathbb{Z}_{-m}^\infty \).

Now we consider a nonhomogeneous initial Cauchy problem (4), (5), i.e. the problem
\[ \Delta x(k) = Bx(k - m) + f(k), \quad k \in \mathbb{Z}_0^\infty, \]  
\[ x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-m}^0. \]

We get this solution, in accordance with the theory of linear equations, as the sum of a solution of the adjoint homogeneous problem (6), (7) (satisfying the same initial data) and a particular solution of (8) being zero on the initial interval. Therefore, we are going to find such a particular solution \( x_p(k), k \in \mathbb{Z}_{-m}^\infty \) of the initial Cauchy problem
\[ \Delta x(k) = Bx(k - m) + f(k), \quad k \in \mathbb{Z}_0^\infty, \]  
\[ x(k) = 0, \quad k \in \mathbb{Z}_{-m}^0. \]

**Theorem 2.4** (Theorem 3.5 in [7]). The solution \( x = x_p(k) \) of the initial Cauchy problem (10), (11) can be represented on \( \mathbb{Z}_{-m}^\infty \) in the form
\[ x_p(k) = \sum_{i=1}^k e^{B(k-m-i)}_m f(j - 1). \]

Combining the results of Theorem 2.3 and Theorem 2.4, we get immediately

**Theorem 2.5** (Theorem 3.6 in [7]). On \( \mathbb{Z}_{-m}^\infty \), the solution \( x = x(k) \) of the initial Cauchy problem (8), (9) can be represented in the form
\[ x(k) = e^{Bk}_m \varphi(-m) + \sum_{j=-m+1}^0 e^{B(k-m-j)}_m \Delta \varphi(j - 1) + \sum_{j=1}^k e^{B(k-m-j)}_m f(j - 1). \]
2.3 Solutions of Linear Discrete Systems with Single Delay – Generalization

Consider a linear discrete system

\[ x(k + 1) = Ax(k) + Bx(k - m) + f(k) \]  

where \( m \geq 1 \) is a fixed integer, \( k \in \mathbb{Z}_0^\infty \), \( A = (a_{ij}) \), \( \det A \neq 0 \) and \( B = (b_{ij}) \) are constant \( n \times n \) matrices with the commutative property

\[ AB = BA, \]

\( f : \mathbb{Z}_0^\infty \rightarrow \mathbb{R}^n \), \( x : \mathbb{Z}_{-m}^\infty \rightarrow \mathbb{R}^n \). Together with equation (12), we consider an initial Cauchy problem

\[ x(k) = \varphi(k) \]  

with a given \( \varphi : \mathbb{Z}_0^m \rightarrow \mathbb{R}^n \).

Consider first a homogeneous initial problem (12), (13), i.e., the problem

\[ x(k + 1) = Ax(k) + Bx(k - m), \quad k \in \mathbb{Z}_0^\infty, \]

\[ x(k) = \varphi(k), \quad k \in \mathbb{Z}_0^m. \]  

**Theorem 2.6** (Theorem 3.1 in [6]). Let \( A, B \) be constant \( n \times n \) matrices, \( AB = BA \) and \( \det A \neq 0 \). Then, the solution of (14), (15) can be expressed as

\[ x(k) = X_0(k)A^{-m}\varphi(-m) + A^m \sum_{j=-m+1}^{0} X_0(k - m - j) [\varphi(j) - A\varphi(j - 1)] \]  

where \( k \in \mathbb{Z}_0^\infty \), \( X_0(k) = A^k e^{B_1^k} \) and \( B_1 = A^{-1}BA^{-m} \).

Now we consider the nonhomogeneous initial Cauchy problem (12), (13), i.e.,

\[ x(k + 1) = Ax(k) + Bx(k - m) + f(k), \quad k \in \mathbb{Z}_0^\infty, \]

\[ x(k) = \varphi(k), \quad k \in \mathbb{Z}_0^m. \]  

We derive a solution of this problem as the sum of a solution of the adjoint homogeneous problem (14), (15) (satisfying the same initial data) and a particular solution of (17) being zero on the initial interval. Therefore, we will try to find such a particular solution.

We find a solution \( x = x_p(k), k \in \mathbb{Z}_0^\infty \) of the problem

\[ x(k + 1) = Ax(k) + Bx(k - m) + f(k), \quad k \in \mathbb{Z}_0^\infty, \]

\[ x(k) = 0, \quad k \in \mathbb{Z}_0^m. \]  

**Theorem 2.7** (Theorem 3.4 in [6]). Let \( A, B \) be constant \( n \times n \) matrices, \( AB = BA \) and \( \det A \neq 0 \). Then, a solution \( x = x_p(k) \) of the initial Cauchy problem (19), (20) can be represented on \( \mathbb{Z}_0^\infty \) in the form

\[ x_p(k) = A^m \sum_{i=1}^{k} X_0(k - m - j)f(j - 1) \]  

where \( X_0(k) = A^k e^{B_1^k} \) and \( B_1 = A^{-1}BA^{-m} \).
Combining the results of Theorem 2.6 and Theorem 2.7, we get immediately

**Theorem 2.8** (Theorem 3.5 in [6]). On $\mathbb{Z}_{-m}^\infty$, a solution $x = x(k)$ of the initial Cauchy problem (17), (18) can be represented in the form

\[
x(k) = X_0(k)A^{-m}\varphi(-m) + A^m \sum_{j=-m+1}^{0} X_0(k - m - j) [\varphi(j) - A\varphi(j - 1)] \\
+ A^m \sum_{i=1}^{k} X_0(k - m - j)f(j - 1)
\]

where $X_0(k) = A^k e^{B_1 k}$ and $B_1 = A^{-1}BA^{-m}$.

## 3 Solutions of Linear Discrete Systems with Impulses

In this chapter we present results on representation of solutions of linear discrete systems with impulses. The results of Sections 3.1 – 3.3 are published in [20, 22–24, 26] while the results in Section 3.4 are new.

Consider an initial Cauchy problem

\[
\begin{align*}
\Delta x(k) &= Bx(k - m) + f(k), \quad k \in \mathbb{Z}_0^\infty, \\
      x(k) &= \varphi(k), \quad k \in \mathbb{Z}_{-m}^0
\end{align*}
\]

(21) (22)

where $m \geq 1$ is a fixed integer, $B = (b_{ij})$ is a constant $n \times n$ matrix, $x: \mathbb{Z}_{-m}^\infty \to \mathbb{R}^n$, $f: \mathbb{Z}_0^\infty \to \mathbb{R}^n$, $\varphi: \mathbb{Z}_{-m}^0 \to \mathbb{R}^n$ and $\Delta x(k) = x(k+1) - x(k)$.

We assume that impulses are acting on $x$ at some prescribed points. Particularly, the problem (21), (22) is considered if impulses are focused on the first point of every interval $\mathbb{Z}_{(\ell-1)(m+1)+1}^{\ell(m+1)}$:

\[
x((\ell - 1)(m+1) + 1) = x((\ell - 1)(m+1) + 1 - 0) + J_\ell,
\]

$\ell \geq 1, \ell = \left\lfloor \frac{k+m}{m+1} \right\rfloor, k \in \mathbb{Z}_{0}^\infty, J_\ell \in \mathbb{R}^n$ (results are given in Theorems 3.1, 3.2), or on the $p$-th point of such intervals:

\[
x((\ell - 1)(m+1) + p) = x((\ell - 1)(m+1) + p - 0) + J_\ell,
\]

$p \in \{1, 2, 3, \ldots, m+1\}, \ell \geq 1, \ell = \left\lfloor \frac{k+m}{m+1} \right\rfloor, k \in \mathbb{Z}_{0}^\infty, J_\ell \in \mathbb{R}^n$ (results are given in Theorems 3.3, 3.4), or, in a general case, impulses are added to each point $k$ (Theorems 3.5, 3.6).

In Section 3.4 further generalization is given of the results from Section 3.3. The problem considered has a more general form

\[
\begin{align*}
x(k+1) &= Ax(k) + Bx(k - m) + f(k), \quad k \in \mathbb{Z}_0^\infty, \\
x(k) &= \varphi(k), \quad k \in \mathbb{Z}_{-m}^0, \\
x(k + 1) &= Cx(k + 1 - 0) + J_{k+1}, \quad k \in \mathbb{Z}_{0}^\infty
\end{align*}
\]

(23) (24) (25)

and the main assumption is that matrices $A$ and $B$ commute ($AB = BA$) and $ACB = BCA$ (Theorem 3.8).
3.1 Problem (21), (22) with Impulses at Points Having the Form 
\((\ell - 1)(m + 1) + 1\)

We will consider problem (21), (22) with impulses \(J_\ell \in \mathbb{R}^n\) added to \(x\) at points having the form \((\ell - 1)(m + 1) + 1\) where the index \(\ell \geq 1\) is defined as \(\ell = \left\lfloor \frac{k + m}{m+1} \right\rfloor\) for every \(k \in \mathbb{Z}_0^\infty\), i.e., we set
\[
x((\ell - 1)(m + 1) + 1) = x((\ell - 1)(m + 1) + 1 - 0) + J_\ell
\]
and investigate the solutions of both the homogeneous and nonhomogeneous problem (21), (22), (26).

**Theorem 3.1.** Let \(B\) be a constant \(n \times n\) matrix, \(m\) be a fixed integer, \(J_\ell \in \mathbb{R}^n\), \(\ell \geq 1\), \(\ell = \left\lfloor \frac{k + m}{m+1} \right\rfloor\). Then, the solution of the homogeneous initial Cauchy problem with impulses
\[
\Delta x(k) = Bx(k - m), \quad k \in \mathbb{Z}_0^\infty, \\
x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-m}^0, \\
x((\ell - 1)(m + 1) + 1) = x((\ell - 1)(m + 1) + 1 - 0) + J_\ell
\]
can be expressed in the form:
\[
x(k) = e^{Bk}_m \varphi(-m) + \sum_{j=-m+1}^{0} e^{B(k-m-j)}_m \varphi(j-1) + \sum_{q=1}^{\ell} J_q e^{B(k-q(m+1))}_m 
\]
where \(k \in \mathbb{Z}_{-m}^\infty\).

**Theorem 3.2.** Let \(B\) be a constant \(n \times n\) matrix, \(m\) be a fixed integer, \(J_\ell \in \mathbb{R}^n\), \(\ell \geq 1\), \(\ell = \left\lfloor \frac{k + m}{m+1} \right\rfloor\). Then, the solution of the nonhomogeneous initial Cauchy problem with impulses
\[
\Delta x(k) = Bx(k - m) + f(k), \quad k \in \mathbb{Z}_0^\infty, \\
x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-m}^0, \\
x((\ell - 1)(m + 1) + 1) = x((\ell - 1)(m + 1) + 1 - 0) + J_\ell
\]
can be expressed in the form:
\[
x(k) = e^{Bk}_m \varphi(-m) + \sum_{j=-m+1}^{0} e^{B(k-m-j)}_m \varphi(j-1) + \sum_{j=1}^{k} e^{B(k-m-j)}_m f(j-1) + \sum_{q=1}^{\ell} J_q e^{B(k-q(m+1))}_m 
\]
where \(k \in \mathbb{Z}_{-m}^\infty\).
3.2 Problem (21), (22) with Impulses at Points Having the Form \((\ell - 1)(m + 1) + p\)

We will consider the problem (21), (22) with impulses \(J_\ell \in \mathbb{R}^n\) added to \(x\) at points having the form \(((\ell - 1)(m + 1) + p\) where \(p\) is a fixed integer from the set \(\{1, 2, 3, \ldots, m + 1\}\) and the index \(\ell \geq 1\) is defined as \(\ell = \left\lfloor \frac{k+m}{m+1} \right\rfloor\) for every \(k \in \mathbb{Z}_0^\infty\), i.e., we set

\[
x((\ell - 1)(m + 1) + p) = x((\ell - 1)(m + 1) + p - 0) + J_\ell
\]

and investigate the solutions of both the homogeneous and nonhomogeneous problems (21), (22), (27).

The following theorems generalize the results from Section 3.1 where a particular case of this problem (if \(p = 1\)) was solved.

**Theorem 3.3.** Let \(B\) be a constant \(n \times n\) matrix, \(m\) be a fixed integer, \(p\) be a fixed integer from the set \(\{1, 2, 3, \ldots, m + 1\}\), \(J_\ell \in \mathbb{R}^n\), \(\ell \geq 1\), \(\ell = \left\lfloor \frac{k+m}{m+1} \right\rfloor\). Then, the solution of the homogeneous initial Cauchy problem with impulses

\[
\Delta x(k) = Bx(k - m), \quad k \in \mathbb{Z}_0^\infty, \\
x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-m}, \\
x((\ell - 1)(m + 1) + p) = x((\ell - 1)(m + 1) + p - 0) + J_\ell,
\]

can be expressed in the form:

\[
x(k) = e^{Bk}_m \varphi(-m) + \sum_{j=-m+1}^{0} e^{B(k-m-j)}_m \Delta \varphi(j - 1) + \sum_{q=1}^{\ell} J_q e^{B(k-(p-1)-q(m+1))}_m
\]

where \(k \in \mathbb{Z}_0^{-m}\).

**Theorem 3.4.** Let \(B\) be a constant \(n \times n\) matrix, \(m\) be a fixed integer, \(p\) be a fixed integer from the set \(\{1, 2, 3, \ldots, m + 1\}\), \(J_\ell \in \mathbb{R}^n\), \(\ell \geq 1\), \(\ell = \left\lfloor \frac{k+m}{m+1} \right\rfloor\). Then, the solution of the nonhomogeneous initial Cauchy problem with impulses

\[
\Delta x(k) = Bx(k - m) + f(k), \quad k \in \mathbb{Z}_0^\infty, \\
x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-m}, \\
x((\ell - 1)(m + 1) + p) = x((\ell - 1)(m + 1) + p - 0) + J_\ell,
\]

can be expressed in the form:

\[
x(k) = e^{Bk}_m \varphi(-m) + \sum_{j=-m+1}^{0} e^{B(k-m-j)}_m \Delta \varphi(j - 1)
\]

\[
+ \sum_{j=1}^{k} e^{B(k-m-j)}_m f(j - 1) + \sum_{q=1}^{\ell} J_q e^{B(k-(p-1)-q(m+1))}_m
\]

where \(k \in \mathbb{Z}_0^{-m}\).
3.3 Problem (21), (22) with Impulses at Each Point

We will consider the problem (21), (22) with impulses $J_{k+1} \in \mathbb{R}^n$ added to $x$ in every $k \in \mathbb{Z}_1$, i.e., we set

$$x(k+1) = x(k+1-0) + J_{k+1}$$

(28)

and investigate the solutions of both the homogeneous and nonhomogeneous problems (21), (22), (28).

**Theorem 3.5.** Let $B$ be a constant $n \times n$ matrix, $m$ be a fixed integer, $J_i \in \mathbb{R}^n$. Then, the solution of the homogeneous initial Cauchy problem with impulses

$$\Delta x(k) = Bx(k-m), \quad k \in \mathbb{Z}_0^\infty,$$

$$x(k) = \varphi(k), \quad k \in \mathbb{Z}_0^0,$$

$$x(k+1) = x(k+1-0) + J_{k+1}, \quad k \in \mathbb{Z}_0^\infty$$

can be expressed in the form:

$$x(k) = e^{Bk} \varphi(-m) + \sum_{j=-m+1}^0 e^{B(k-m-j)} \Delta \varphi(j-1) + \sum_{i=1}^k J_i e^{B(k-(i+m))}$$

where $k \in \mathbb{Z}_m^\infty$.

**Theorem 3.6.** Let $B$ be a constant $n \times n$ matrix, $m$ be a fixed integer, $J_i \in \mathbb{R}^n$. Then, the solution of the nonhomogeneous initial Cauchy problem with impulses

$$\Delta x(k) = Bx(k-m) + f(k), \quad k \in \mathbb{Z}_0^\infty,$$

$$x(k) = \varphi(k), \quad k \in \mathbb{Z}_0^0,$$

$$x(k+1) = x(k+1-0) + J_{k+1}, \quad k \in \mathbb{Z}_0^\infty$$

can be expressed in the form:

$$x(k) = e^{Bk} \varphi(-m) + \sum_{j=-m+1}^0 e^{B(k-m-j)} \Delta \varphi(j-1) + \sum_{j=1}^k e^{B(k-m-j)} f(j-1) + \sum_{i=1}^k J_i e^{B(k-(i+m))}$$

where $k \in \mathbb{Z}_m^\infty$.

3.4 Problem (23) – (25) with Impulses at Each Point

Consider the discrete system

$$x(k+1) = Ax(k) + Bx(k-m) + f(k), \quad k \in \mathbb{Z}_0^\infty,$$

(32)

where $m \geq 1$ is a fixed integer, $A = (a_{ij})$ and $B = (b_{ij})$ are regular constant $n \times n$ matrices with the commutative property

$$AB = BA,$$
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\( f : \mathbb{Z}_0^\infty \to \mathbb{R}^n \) and \( x : \mathbb{Z}_{-m}^\infty \to \mathbb{R}^n \).

Together with equation (32), we will consider the initial Cauchy problem

\[ x(k) = \varphi(k), \quad k \in \mathbb{Z}^0_{-m}, \quad (33) \]

with a given \( \varphi : \mathbb{Z}^0_{-m} \to \mathbb{R}^n \).

We will consider problem (32), (33) together with the condition

\[ x(k + 1) = Cx(k + 1 - 0) + J_{k+1} \quad (34) \]

where \( k \in \mathbb{Z}_0^\infty \), \( C = (c_{ij}) \) is a regular constant \( n \times n \) matrix and \( J_i \in \mathbb{R}^n \) are impulses.

We assume that, for matrices \( A \), \( B \) and \( C \), equality

\[ ACB = BCA \]

holds.

**Remark 3.7.** For \( A = C = I \), the problem (32) – (34) turns into (29) – (31) considered in Theorem 3.6.

**Theorem 3.8.** Let \( A \), \( B \), \( C \) be constant \( n \times n \) matrices with the property \( ACB = BCA \), \( m \) be a fixed integer and \( J_i \in \mathbb{R}^n \). Then, the solution of the initial Cauchy problem with impulses

\[ x(k + 1) = Ax(k) + Bx(k - m) + f(k), \quad k \in \mathbb{Z}_0^\infty, \quad (35) \]
\[ x(k) = \varphi(k), \quad k \in \mathbb{Z}^0_{-m}, \quad (36) \]
\[ x(k + 1) = Cx(k + 1 - 0) + J_{k+1}, \quad k \in \mathbb{Z}_0^\infty \quad (37) \]

can be expressed in the form:

\[ x(k) = X_0(k)(CA)^{-m}\varphi(-m) \]
\[ + (CA)^m \sum_{j=-m+1}^{0} X_0(k - m - j) [\varphi(j) - (CA)\varphi(j - 1)] \]
\[ + (CA)^m \sum_{i=1}^{k} X_0(k - m - i) [Cf(i - 1) + J_i] \quad (38) \]

where \( k \in \mathbb{Z}_1^\infty \), \( X_0(k) = (CA)^k e_{m}^{B_1 k} \), \( B_1 = (CA)^{-1}CB(CA)^{-m} \).

**Remark 3.9.** For \( k \in \mathbb{Z}^0_{-m} \), the problem (35) – (37) is equivalent to the problem (14), (15) and, by Theorem 2.6, formula (38) is equal to formula (16), i.e.,

\[ x(k) = X_0(k)A^{-m}\varphi(-m) + A^m \sum_{j=-m+1}^{0} X_0(k - m - j) [\varphi(j) - A\varphi(j - 1)] \]

where \( X_0(k) = A^k e_{m}^{B_1 k} \), \( B_1 = A^{-1}BA^{-m} \).
4 Discrete Matrix Delayed Exponentials for Two Delays

In this chapter we define two discrete matrix delayed exponentials for two delays and we present their main properties. Results of this chapter were already published in [21, 25].

4.1 Discrete Matrix Delayed Exponential $e_{BCk}^{mn}$

We define a discrete matrix function $e_{BCk}^{mn}$ called the discrete matrix delayed exponential for two delays $m, n \in \mathbb{N}$, $m \neq n$ and for two $r \times r$ commuting constant matrices $B, C$.

**Definition 4.1.** Let $B, C$ be constant $r \times r$ matrices with the property $BC = CB$ and let $m, n \in \mathbb{N}$, $m \neq n$ be fixed integers. We define a discrete $r \times r$ matrix function $e_{BCk}^{mn}$ called the discrete matrix delayed exponential for two delays $m, n$ and for two $r \times r$ constant matrices $B, C$ as follows:

$$e_{BCk}^{mn} := \begin{cases} 
\Theta & \text{if } k \in \mathbb{Z}_{-\infty}^{-\max\{m,n\}-1}, \\
I & \text{if } k \in \mathbb{Z}_{0}^{0}, \\
I + (B + C) \sum_{i=0, j=0}^{p(k)-1, q(k)-1} B^iC^j \left( \binom{i+j}{i} \left( k - mi - nj \right) \right) & \text{if } k \in \mathbb{Z}_{1}^{1}
\end{cases}$$

where

$$p(k) := \left\lfloor \frac{k + m}{m + 1} \right\rfloor, \quad q(k) := \left\lfloor \frac{k + n}{n + 1} \right\rfloor$$

and $\Theta$ is the $r \times r$ null matrix, $I$ is the $r \times r$ unit matrix.

The main property of $e_{BCk}^{mn}$ is given by the following theorem.

**Theorem 4.2.** Let $B, C$ be constant $r \times r$ matrices with the property $BC = CB$ and let $m, n \in \mathbb{N}$, $m \neq n$ be fixed integers. Then,

$$\Delta e_{BCk}^{mn} = Be_{BC(k-m)}^{mn} + C e_{BC(k-n)}^{mn}$$

holds for $k \in \mathbb{Z}_{0}^{0}$.

4.2 Discrete Matrix Delayed Exponential $\tilde{e}_{BCk}^{mn}$

Analyzing the applicability of $e_{BCk}^{mn}$ to a representation of the solution to initial problem (39), (40) we see that, unfortunately, this does not lead to satisfactory results because, as we will see below, an additional condition $\det(B + C) \neq 0$ is necessary. A small difference in the definition results in representations of solutions of initial problems without this assumption.

Now we give a second definition of a discrete matrix delayed exponential for two delays $\tilde{e}_{BCk}^{mn}$.
Definition 4.3. Let $B, C$ be constant $r \times r$ matrices with the property $BC = CB$ and let $m, n \in \mathbb{N}$, $m < n$ be fixed integers. We define a discrete $r \times r$ matrix function $\tilde{e}^{BC}_{mn}$ called the discrete matrix delayed exponential for two delays $m, n$ and for two $r \times r$ constant matrices $B, C$ as follows:

\[
\tilde{e}^{BC}_{mn}(k) := \begin{cases} 
\Theta & \text{if } k \in \mathbb{Z}_{-\infty}^{-1}, \\
I & \text{if } k \in \mathbb{Z}_0^m, \\
I + B \sum_{i=0,j=0}^{p(k)-1,q(k)-1} B^i C^j \left( \begin{array}{c} i+j \\ i \end{array} \right)_k \left( k - m - mi - nj \right)_{i+j+1} \\
+ C \sum_{i=0,j=0}^{p(k)-1,q(k)-1} B^i C^j \left( \begin{array}{c} i+j \\ i \end{array} \right)_k \left( k - n - mi - nj \right)_{i+j+1} & \text{if } k \in \mathbb{Z}_{m+1}^\infty
\end{cases}
\]

where $p(k) := \left\lfloor \frac{k + m}{m + 1} \right\rfloor$, $q(k) := \left\lfloor \frac{k + n}{n + 1} \right\rfloor$ and $\Theta$ is the $r \times r$ null matrix, $I$ is the $r \times r$ unit matrix.

Remark 4.4. For $k \in \mathbb{Z}_0^n$, it is easy to deduce that $\tilde{e}^{BC}_{mn}(k) = e^{B(k-m)}_m$.

The main property of $\tilde{e}^{BC}_{mn}$ is given by the following theorem.

Theorem 4.5. Let $B, C$ be constant $r \times r$ matrices with the property $BC = CB$ and let $m, n \in \mathbb{N}$, $m < n$ be fixed integers. Then,

\[
\Delta \tilde{e}^{BC}_{mn} = B\tilde{e}^{BC}_{mn}(k-m) + C\tilde{e}^{BC}_{mn}(k-n)
\]

holds for $k \in \mathbb{Z}_0^n$.

5 Solutions of Linear Discrete Systems with Two Delays

In this chapter, we deal with the discrete system

\[
\Delta x(k) = Bx(k-m) + Cx(k-n) + f(k)
\]

where $m, n \in \mathbb{N}$, $m < n$ are fixed, $k \in \mathbb{Z}_0^n$, $B = (b_{ij})$, $C = (c_{ij})$ are constant $r \times r$ matrices, $f: \mathbb{Z}_0^n \to \mathbb{R}^r$ is a given $r \times 1$ vector, and $x: \mathbb{Z}_0^n \to \mathbb{R}^r$.

Together with equation (39), we consider an initial Cauchy problem

\[
x(k) = \varphi(k)
\]

with a given $\varphi: \mathbb{Z}_0^n \to \mathbb{R}^r$.

With the aid of both discrete matrix delayed exponentials, we give formulas for the solutions of homogeneous and nonhomogeneous initial problems (39), (40). The results of this chapter are published in [25].
5.1 Homogeneous Initial Problem

Consider a homogeneous initial Cauchy problem

\[
\begin{align*}
\Delta x(k) &= Bx(k-m) + Cx(k-n), \quad k \in \mathbb{Z}_{\infty}^0, \\
x(k) &= \varphi(k), \quad k \in \mathbb{Z}_{-n}^0.
\end{align*}
\]

(41) (42)

First, we derive formulas for the solution of (41), (42) with the aid of the discrete matrix delayed exponential \( e^{BCk}_{mn} \) and then with the aid of the discrete matrix delayed exponential \( \tilde{e}^{BCk}_{mn} \).

**Theorem 5.1.** Let \( B, C \) be constant \( r \times r \) matrices such that \( BC = CB \), \( \det(B + C) \neq 0 \), and let \( m, n \in \mathbb{N} \), \( m < n \) be fixed integers. Then, the solution of the initial Cauchy problem (41), (42) can be expressed in the form

\[
x(k) = \sum_{j=0}^{n} e^{BC(k+j)}_{mn} v_j,
\]

where \( k \in \mathbb{Z}_{-n}^\infty \) and

\[
\begin{align*}
v_0 &= \varphi(-n) - \sum_{s=1}^{n} v_s, \\
v_\ell &= (B + C)^{-1} \left[ \Delta \varphi(-\ell) - \sum_{t=1}^{n-\ell} \Delta e^{BCt}_{mn} v_{t+\ell} \right], \quad \ell \in \mathbb{Z}_{n}^1.
\end{align*}
\]

Now we express the solution of the homogeneous Cauchy problem by \( \tilde{e}^{BC(k)}_{mn} \). In this case, the condition \( \det(B + C) \neq 0 \) is not necessary.

**Theorem 5.2.** Let \( B, C \) be constant \( r \times r \) matrices with \( BC = CB \) and let \( m, n \in \mathbb{N} \), \( m < n \) be fixed integers. Then, the solution of the initial Cauchy problem (41), (42) can be expressed in the form

\[
x(k) = \sum_{j=0}^{n} \tilde{e}^{BC(k+j)}_{mn} w_j,
\]

where \( k \in \mathbb{Z}_{-n}^\infty \) and

\[
\begin{align*}
w_0 &= \varphi(-n) \quad \Delta \varphi(-\ell - 1) - \Delta \tilde{e}^{BC(-\ell+n-1)}_{mn} \varphi(-n) \\
&\quad - \sum_{s=-n}^{-\ell-m-2} \Delta \tilde{e}^{BC(-\ell-s-2)}_{mn} \Delta \varphi(s), \quad \ell \in \mathbb{Z}_{n-m}^{n-1}, \\
w_\ell &= \Delta \varphi(-\ell - 1), \quad \ell \in \mathbb{Z}_{n-m}^{n-1}, \\
w_n &= \varphi(-n).
\end{align*}
\]
5.2 Nonhomogeneous Initial Problem

We consider a nonhomogeneous initial Cauchy problem
\[ \Delta x(k) = Bx(k - m) +Cx(k - n) + f(k), \quad k \in \mathbb{Z}_0^\infty, \]
\[ x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-n}, \]  

(43) (44)

By the theory of linear equations, we can obtain its solution as the sum of a solution of adjoint homogeneous problem (41), (42) (satisfying the same initial data) and a particular solution of (43) being zero on an initial interval.

Let us, therefore, find such a particular solution \( x_p(k), k \in \mathbb{Z}_{-n}^\infty \) of the initial Cauchy problem
\[ \Delta x(k) = Bx(k - m) +Cx(k - n) + f(k), \quad k \in \mathbb{Z}_0^\infty, \]
\[ x(k) = 0, \quad k \in \mathbb{Z}_{-n}. \]  

(45) (46)

Theorem 5.3. The solution \( x = x_p(k) \) of the initial Cauchy problem (45), (46) can be represented on \( \mathbb{Z}_n^\infty \) in the form
\[ x_p(k) = \sum_{\ell=1}^k \tilde{e}_{mn}^{BC(k-\ell)} f(\ell - 1), \quad k \in \mathbb{Z}_n^\infty. \]

Combining the results of Theorems 5.1, 5.2 and 5.3, we get immediately the following two theorems, which describe the solution of (43), (44). The first theorem uses both discrete matrix delayed exponentials and the second one uses only the discrete matrix delayed exponential \( \tilde{e}_{mn}^{BCk} \).

Theorem 5.4. Let \( B, C \) be constant \( r \times r \) matrices such that
\[ BC = CB, \quad \det(B + C) \neq 0, \]
and let \( m, n \in \mathbb{N}, m < n \) be fixed integers. Then, the solution of the initial Cauchy problem (43), (44) can be expressed in the form:
\[ x(k) = \sum_{j=0}^n \tilde{e}_{mn}^{BC(k+j)} v_j + \sum_{\ell=1}^k \tilde{e}_{mn}^{BC(k-\ell)} f(\ell - 1) \]
where \( k \in \mathbb{Z}_n^\infty \) and
\[ v_0 = \varphi(-n) - \sum_{s=1}^n v_s, \]
\[ v_\ell = (B + C)^{-1} \left[ \Delta \varphi(-\ell) - \sum_{t=1}^{n-\ell} \Delta \tilde{e}_{mn}^{BCt} v_{t+\ell} \right], \quad \ell \in \mathbb{Z}_1^n. \]

Theorem 5.5. Let \( B, C \) be constant \( r \times r \) matrices with \( BC = CB \) and let \( m, n \in \mathbb{N}, m < n \) be fixed integers. Then, the solution of the initial Cauchy problem (43), (44) can be expressed in the form:
\[ x(k) = \sum_{j=0}^n \tilde{e}_{mn}^{BC(k+j)} w_j + \sum_{\ell=1}^k \tilde{e}_{mn}^{BC(k-\ell)} f(\ell - 1) \]
where \( k \in \mathbb{Z}_{-n}^\infty \) and

\[
w_\ell = \Delta \varphi(-\ell - 1) - \Delta e_{mn}^{BC(-\ell+n-1)} \varphi(-n) - \sum_{s=-n}^{-\ell-m-2} \Delta e_{mn}^{BC(-\ell-s-2)} \Delta \varphi(s),
\]

\( \ell \in \mathbb{Z}_0^{n-m-1} \),

\[
w_\ell = \Delta \varphi(-\ell - 1), \quad \ell \in \mathbb{Z}_{n-m}^{-1},
\]

\( w_n = \varphi(-n) \).

**Example 5.6.** Let us represent the solution of the scalar \((r = 1)\) problem (43), (44) where we put \( m = 2, n = 3, B = b, C = c, \varphi(-3) = 1, \varphi(-2) = 2, \varphi(-1) = 3, \varphi(0) = 4, f(k) = k + 1 \), using Theorem 5.5. Thus, we have

\[
\Delta x(k) = bx(k - 2) + cx(k - 3) + k + 1, \quad k \in \mathbb{Z}_0^\infty, \quad (47)
\]

\[
x(-3) = \varphi(-3) = 1,
\]

\[
x(-2) = \varphi(-2) = 2,
\]

\[
x(-1) = \varphi(-1) = 3,
\]

\[
x(0) = \varphi(0) = 4. \quad (48)
\]

By Theorem 5.5, the solution of problem (47), (48) is

\[
x(k) = \sum_{j=0}^{3} e_{2,3}^{bc(k+j)} w_j + \sum_{\ell=1}^{k} e_{2,3}^{bc(k-\ell)} \ell, \quad k \in \mathbb{Z}_{-3}^\infty
\]

where

\[
w_0 = \Delta \varphi(-1) - \Delta e_{2,3}^{bc2} \varphi(-3) - \sum_{s=-3}^{-4} \Delta e_{2,3}^{bc(-s-2)} \Delta \varphi(s) = 1 - (e_{2,3}^{bc3} - e_{2,3}^{bc2}) \cdot 1
\]

\[= 1 - (1 + b - 1) = 1 - b, \]

\[
w_1 = \Delta \varphi(-2) = 1,
\]

\[
w_2 = \Delta \varphi(-3) = 1,
\]

\[
w_3 = \varphi(-3) = 1.
\]

Thus, we get

\[
x(k) = e_{2,3}^{bc1}(1 - b) + e_{2,3}^{bc(k+1)} + e_{2,3}^{bc(k+2)} + \sum_{\ell=1}^{k} e_{2,3}^{bc(k-\ell)} \ell.
\]
6 Conclusion

Results presented in the thesis are important in two aspects. First, the known notion of discrete matrix delayed exponential function is used to get analytical representations of solutions to systems of linear discrete equations with a single delay and with impulses. Second, the definition of discrete matrix delayed exponential function is generalized to the case when linear systems contain two delays. A generalization is derived that copies, in a sense, the original definition of discrete matrix delayed exponential function. In addition to this, another definition of discrete matrix delayed exponential function is suggested. For both discrete matrix delayed exponentials, their main properties are discussed and they are used in formulas describing analytical solutions of linear discrete systems with two delays.

The future progress can be achieved by further generalizations of discrete matrix delayed exponential functions to the case when problems with multiple delays are considered. The results known in this field seem to be too cumbersome and so it may be expected that new results on the representation of solutions of linear problems with multiple delays will be very useful for applications.
References


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Abstract

The dissertation thesis is concerned with linear discrete systems with constant matrices of linear terms with a single or two delays. The main objective is to obtain formulas analytically describing exact solutions of initial Cauchy problems. To this end, some matrix special functions called discrete matrix delayed exponentials are defined and used. Their basic properties are proved. Such special matrix functions are used to derive analytical formulas representing the solutions of initial Cauchy problems.

First the initial problem
\[ \Delta x(k) = B(x(k - m) + f(k)), \quad k \in \mathbb{N} \cup \{0\}, \tag{a} \]
\[ x(k) = \varphi(k), \quad k = -m, -m + 1, \ldots, 0 \tag{b} \]
is discussed where \( B \) is a constant square matrix, \( f \) is a given nonhomogeneity, \( \varphi \) is an initial function and \( m \) is a positive integer. It is assumed that impulses are acting at some prescribed points and formulas describing the solutions of problem (a) and (b) are derived. Then, instead of the problem (a), (b), a generalized problem
\[ x(k + 1) = A x(k) + B x(k - m) + f(k), \quad k \in \mathbb{N} \cup \{0\}, \]
\[ x(k) = \varphi(k), \quad k = -m, -m + 1, \ldots, 0 \]
with impulses acting at each point is considered where \( A \) is a constant square matrix and \( B, f, \varphi \) and \( m \) are as above.

In the next part of the dissertation, two definitions of discrete matrix delayed exponentials for two delays are given and their basic properties are proved. Such discrete special matrix functions make it possible to find representations of solutions of linear systems with two delays. This is done in the last part of dissertation thesis. The below problem
\[ \Delta x(k) = B x(k - m) + C x(k - n) + f(k), \quad k \in \mathbb{N} \cup \{0\}, \tag{c} \]
\[ x(k) = \varphi(k), \quad k = -\max\{m, n\}, -\max\{m, n\} + 1, \ldots, 0 \tag{d} \]
is considered where \( C \) is a constant square matrix, \( n \) is a positive integer and \( B, f, \varphi \) and \( m \) are as above. Two different formulas giving the analytical solution of problem (c), (d) are derived.
Abstrakt

Dizertační práce se zabývá lineárními diskrétními systémy s konstantními maticemi a s jedním nebo dvěma zpožděními. Hlavním cílem je odvodit vzorce analyticky popisující řešení počátečních úloh. K tomu jsou definovány speciální maticové funkce zvané diskrétní maticové zpožděné exponentiály a je dokázána jejich základní vlastnost. Tyto speciální maticové funkce jsou základem analytických vzorců reprezentujících řešení počáteční úlohy.

Nejprve je uvažována počáteční úloha

\[ \Delta x(k) = Bx(k - m) + f(k), \quad k \in \mathbb{N} \cup \{0\}, \quad (a) \]
\[ x(k) = \varphi(k), \quad k = -m, -m + 1, \ldots, 0, \quad (b) \]

kde \( B \) je konstantní čtvercová matice, \( f \) je daná nehomogenita, \( \varphi \) je počáteční funkce a \( m \) je přirozené číslo. Dále předpokládáme, že v některých předepsaných bodech působí na řešení úlohy (a), (b) impulsy. Poté, kromě úlohy (a), (b), uvažujeme zobecněnou úlohu

\[ x(k + 1) = Ax(k) + Bx(k - m) + f(k), \quad k \in \mathbb{N} \cup \{0\}, \]
\[ x(k) = \varphi(k), \quad k = -m, -m + 1, \ldots, 0 \]

s impulsy působícími v každém bodě, kde \( A \) je konstantní čtvercová matice \( (B, f, \varphi \text{ a } m \text{ byly definovány výše}) \).

V další části dizertační práce jsou definovány dvě různé diskrétní maticové zpožděné exponentiály pro dvě zpoždění a jsou dokázány jejich základní vlastnosti. Tyto diskrétní maticové zpožděné exponentiály nám dávají možnost najít reprezentaci řešení lineárních systémů se dvěma zpožděními. Tato řešení jsou konstruována v poslední kapitole dizertační práce, kde je uvažován problém

\[ \Delta x(k) = Bx(k - m) + Cx(k - n) + f(k), \quad k \in \mathbb{N} \cup \{0\}, \quad (c) \]
\[ x(k) = \varphi(k), \quad k = -\max\{m, n\}, -\max\{m, n\} + 1, \ldots, 0 \quad (d) \]

\( C \) je konstantní čtvercová matice, \( n \) je přirozené číslo a \( B, f, \varphi, m \) byly definovány výše). Řešení problému (c), (d) je dáno pomocí dvou různých vzorců.