STABILITY OF NEUTRAL DELAY DIFFERENTIAL EQUATIONS AND THEIR DISCRETIZATIONS

STABILITA NEUTRÁLNÍCH DIFERENCIÁLNÍCH ROVNIC SE ZPOŽDĚNÍM A JEJICH DISKRETIZACÍ
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1. **INTRODUCTION**

Delay differential equations are widely used in science and engineering. They arise in modeling of problems where the rate of change of a time-dependent process is determined not only by its present state but also by a certain past state. Then, the use of ordinary differential equations turns out to be insufficient. Problems of this type occur in various fields such as biology, electrodynamics, medicine, economics and many others (various examples are presented in the book by Kolmanovskii and Myshkis [25]).

Unlike for the ordinary differential equations, for which we have several methods to obtain the analytical solution (e.g. separation of variables, variation of constant method etc.), there are no computational methods how to find the analytical solution of delay differential equations, not even in the linear case. Therefore, the qualitative and numerical analysis of these equations is of a great importance.

The basic numerical methods utilized to solve the delay differential equations originates from corresponding procedures for ordinary differential equations (with some additional requirements concerning the delayed terms). Although the methods are based on the same principal, their potential to preserve the qualitative behaviour of the analytic solution may be different.

One of the most important qualitative properties of differential equations is the asymptotic stability. Roughly speaking, this property describes a capability of the equation to eliminate possible errors in input data. The asymptotic stability for the linear delay differential equation can be defined as follows.

Consider the linear delay differential equation

$$x'(t) = ax(t) + b x(\xi(t)), \quad t \in (t_0, \infty), \quad (1.1)$$

where $a$, $b$ are real scalars and the function $\xi(t)$ is a continuous function satisfying $\xi(t) < t$ for all $t > t_0$ (some additional assumptions on $\xi(t)$ will be imposed throughout this thesis). Then (1.1) is called asymptotically stable if all its solutions $x(t)$ tend to zero as $t \to \infty$.

This notion can be introduced in the same sense also to other related linear equations such as differential equation with several delays

$$x'(t) = ax(t) + \sum_{i=1}^{r} b_i x(\xi_i(t)), \quad t \in (t_0, \infty), \quad (1.2)$$

or the delay differential equation of neutral type

$$x'(t) = ax(t) + b x(\xi(t)) + c x'(\xi(t)), \quad t \in (t_0, \infty). \quad (1.3)$$
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Since the resulting numerical formulae of numerical methods applied to (1.1)-(1.3) are difference equations, we are interested in their asymptotic stability, too. Analogously to the differential equation, we have the following definition of the asymptotic stability for a linear difference equation of the order $k$.

Consider the linear difference equation

$$y_n + \alpha_1 y_{n-1} + \alpha_2 y_{n-2} + \cdots + \alpha_k y_{n-k} = 0, \quad n = 0, 1, 2, \ldots,$$  \hspace{1cm} (1.4)

where $\alpha_i$ are real scalars and $k$ is a positive integer. Then (1.4) is called asymptotically stable if all its solutions $y_n$ tend to zero as $n \to \infty$.

The aim of this thesis is to investigate a potential of some numerical methods to retain the asymptotic stability of the linear delay differential equations. To do so, we have to determine conditions for coefficients $a$, $b$ under which (1.1) is asymptotically stable and compare them with those for the corresponding discretized equation. It assumes that both the conditions for the exact as well as discretized equation are strong enough in the sense that they are not only sufficient but also necessary.

The thesis is divided into two main parts according to type of the lag $\psi(t) = t - \xi(t)$ occurring in (1.1). In general, delay differential equations can be classified into two categories, namely those with finite time lag, i.e.

$$\limsup_{t \to \infty} \psi(t) < \infty$$

and those with infinite time lag, i.e.

$$\limsup_{t \to \infty} \psi(t) = \infty.$$

There are remarkable differences between these two categories of delay differential equations. Let us compare their typical representatives, which are the equations

$$x'(t) = a x(t) + b x(t - \tau), \quad t > 0 \hspace{1cm} (1.5)$$

and

$$x'(t) = a x(t) + b x(qt), \quad t > 0, \hspace{1cm} (1.6)$$

where $a$, $b$, $\tau > 0$ and $q \in (0, 1)$ are real numbers. Clearly, (1.5) has a finite lag and (1.6) belongs to the class of equations with the infinite lag. One of the differences between (1.5) and (1.6) consists in the decay rate of their solutions in the asymptotically stable case $a < -|b|$. While the solution of (1.5) decays exponentially, the solution of (1.6) decays algebraically. However, the most
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significant difference is in storage. In order to calculate all the future values of \( x(t) \) beyond some \( t^* > 0 \), we must remember all the past values in the interval \( \langle t^* - \psi(t^*), t^* \rangle \), which is bounded in case of (1.5), but unbounded in the case (1.6) as \( t^* \to \infty \) ([28]). This property plays a key role in their numerical discretization. For this reason, we treat them separately.

The thesis is organized as follows. Chapter 2 recalls some results on the asymptotic stability of the difference equations relevant to our further analysis.

Chapter 3 discusses stability of numerical methods for linear delay differential equations with constant lags. We consider (1.5) with \( a = 0 \) as the simplest case, and (1.3) with \( \xi(t) = t - \tau \) as the most general case. The necessary and sufficient conditions for the asymptotic stability of the exact and discretized equations are presented. Based on them, we describe some properties of the discretized equations. The short overview of an equation containing two constant delays is presented, too.

Chapter 4 is devoted to linear delay differential equations with infinite lags. Firstly, we consider the equation (1.6). We introduce a constrained mesh suitable for its discretization and recall results concerning the stability of some numerical formulae for (1.6). The asymptotic estimates of the exact and discretized equations are presented, too. Further, we generalize the results to the equation with a more general lag of the form (1.1). Lastly, the extension to the delay differential equation with several lags (1.2) is investigated. We present the sufficient conditions for the asymptotic stability of both the exact and discretized equations. The asymptotic estimate of the analytical solution is provided, too. Moreover, the necessary and sufficient conditions for the asymptotic stability as well as some asymptotic estimates are derived for the discretization of the differential equation with two iterated lags.

This thesis is based on the papers [5], [6], [21], [22] and [23]. Its short version presents and comments the most important results of the thesis (results and comments to Section 3.1, 3.2, 3.4, 4.1 and 4.3 are omitted because of limited range requirement).
2. SOME AUXILIARY RESULTS IN
DIFFERENCE EQUATIONS

In this chapter, we recall some results from the theory of difference equations. In the first part, we deal with difference equations with constant coefficients and we present necessary and sufficient criteria for their asymptotic stability. In the second part, we discuss difference equations with asymptotically constant coefficient. We recall some well-known results providing asymptotic description of the solutions.

2.1. Asymptotic stability of difference equations
with constant coefficients

Let us consider the following general linear difference equation of the order $k$

$$y_n + \alpha_1 y_{n-1} + \alpha_2 y_{n-2} + \cdots + \alpha_k y_{n-k} = 0, \quad n = 0, 1, 2, \ldots,$$

where $\alpha_i, i = 1, 2, \ldots, k$ are real scalars. It is well-known that the problem of its asymptotic stability is equivalent to the problem whether the characteristic polynomial

$$P(\lambda) = \lambda^k + \alpha_1 \lambda^{k-1} + \alpha_2 \lambda^{k-2} + \cdots + \alpha_k$$

(2.1)

is of a Schur type, i.e. whether all its zeros are located inside the open unit circle (see Elaydi [14]).

In general, this problem is solved by the Schur-Cohn criterion (see, e.g. Marden [29]), which yields necessary and sufficient conditions for fixed $\alpha_1$, $\alpha_2$, $\ldots$ $\alpha_k$ and $k$ ensuring the required zero property. However, the form of these conditions does not enable to formulate explicit description of the set of all $\alpha_1$, $\alpha_2$, $\ldots$ $\alpha_k$ and $k$ such that (2.1) is of a Schur type. The problem of such a description is solved only in some special cases of (2.1).

For the purposes of this work, we are mostly interested in analysis of the following difference equation

$$y_{n+1} + \alpha y_n + \beta y_{n-k+1} + \gamma y_{n-k} = 0, \quad n = 0, 1, 2, \ldots$$

(2.2)

Considering (2.2), the problem of explicit description of asymptotic stability conditions has been discussed in some particular cases. Recently, a system of explicit necessary and sufficient conditions for a general equation (2.2) has been found by Čermák et al. [8]. We note that it can be formulated
2. Some auxiliary results in difference equations

in a more compact form (see Theorem 3.1 and Theorem 3.2 of [8]), but for its easier treating in the asymptotic stability analysis we prefer the following one.

**Theorem 2.1.** Let \( \alpha, \beta \) and \( \gamma \) be real constants and \( k \) be a positive integer. Then \((2.2)\) is asymptotically stable if and only if one of the following conditions holds:

(C1) \( 1 + \alpha + \beta + \gamma > 0, \quad 1 + \alpha - \beta - \gamma > 0, \quad 1 - \alpha + \beta - \gamma > 0, \quad 1 - \alpha - \beta + \gamma > 0 \)
and \( k \) is any positive integer;

(C2) \( 1 + \alpha + \beta + \gamma > 0, \quad 1 + \alpha - \beta - \gamma = 0, \quad 1 - \alpha + \beta - \gamma > 0, \quad 1 - \alpha - \beta + \gamma > 0 \)
and \( k \) is any positive integer;

(C3) \( 1 + \alpha + \beta + \gamma > 0, \quad 1 + \alpha - \beta - \gamma > 0, \quad 1 - \alpha + \beta - \gamma = 0, \quad 1 - \alpha - \beta + \gamma > 0 \)
and \( k \) is any positive odd integer;

(C4) \( 1 + \alpha + \beta + \gamma > 0, \quad 1 + \alpha - \beta - \gamma > 0, \quad 1 - \alpha + \beta - \gamma > 0, \quad 1 - \alpha - \beta + \gamma = 0 \)
and \( k \) is any positive even integer;

(C5) \( 1 + \alpha + \beta + \gamma > 0, \quad 1 + \alpha - \beta - \gamma < 0, \quad 1 - \alpha + \beta - \gamma > 0, \quad 1 - \alpha - \beta + \gamma > 0 \)
and \( k \) is any positive integer such that

\[
k < \arccos \frac{\alpha^2 - \beta^2 + \gamma^2 - 1}{2|\alpha\gamma - \beta|} \bigg/ \arccos \frac{\alpha^2 - \beta^2 - \gamma^2 + 1}{2|\alpha - \beta\gamma|}; \quad (2.3)
\]

(C6) \( 1 + \alpha + \beta + \gamma > 0, \quad 1 + \alpha - \beta - \gamma > 0, \quad 1 - \alpha + \beta - \gamma < 0, \quad 1 - \alpha - \beta + \gamma > 0 \)
and \( k \) is any positive odd integer such that \((2.3)\) holds;

(C7) \( 1 + \alpha + \beta + \gamma > 0, \quad 1 + \alpha - \beta - \gamma > 0, \quad 1 - \alpha + \beta - \gamma > 0, \quad 1 - \alpha - \beta + \gamma < 0 \)
and \( k \) is any positive even integer such that \((2.3)\) holds.
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2.2. Asymptotic behaviour of difference equations with asymptotically constant coefficients

Let us consider the Poincaré difference equation of the form

\[ y_n + (\alpha_1 + \delta_{1,n})y_{n-1} + \cdots + (\alpha_k + \delta_{k,n})y_{n-k} = 0, \quad n = 0, 1, 2, \ldots, \tag{2.4} \]

where \( \alpha_k \neq 0, \alpha_j \in \mathbb{R}, 1 \leq j \leq k \) and

\[
\lim_{n \to \infty} \delta_{j,n} = 0, \quad 1 \leq j \leq k. \tag{2.5}
\]

The equation (2.4) can be regarded as a perturbation of the limiting constant coefficient difference equation

\[ y_n + \alpha_1 y_{n-1} + \cdots + \alpha_k y_{n-k} = 0, \quad n = 0, 1, 2, \ldots \tag{2.6} \]

having the characteristic polynomial

\[ P(\lambda) = \lambda^k + \alpha_1 \lambda^{k-1} + \alpha_2 \lambda^{k-2} + \cdots + \alpha_k. \tag{2.7} \]

It is natural to expect that the solutions of (2.4) retain some properties of the solutions of (2.6). This question has been studied by Poincaré and Perron, whose results can be summarized as follows (see e.g. [13]).

**Theorem 2.2.** Suppose (2.5) holds and let the zeros \( \lambda_j \) of characteristic polynomial (2.7) have distinct moduli. Then (2.4) has a fundamental set of solutions \( y_n^{(1)}, \ldots, y_n^{(k)} \) such that

\[
\lim_{n \to \infty} \frac{y_{n+1}^{(j)}}{y_n^{(j)}} = \lambda_j, \quad 1 \leq j \leq k.
\]

This result has been improved by Elaydi [13] who derived the asymptotic estimates of the fundamental set of solutions. We state here the results relevant to our further analysis.

**Theorem 2.3.** Suppose that the zeros \( \lambda_j \) of characteristic polynomial (2.7) are distinct. If

\[
\sum_{n=0}^{\infty} |\delta_{j,n}| < \infty, \quad 1 \leq j \leq k, \tag{2.8}
\]

then (2.4) has a fundamental set of solutions \( y_n^{(1)}, \ldots, y_n^{(k)} \) such that

\[ y_n^{(j)} = (c_j + o(1))\lambda_j^n, \quad c_j \neq 0, \quad 1 \leq j \leq k. \]
2. Some auxiliary results in difference equations

**Theorem 2.4.** Suppose that the characteristic polynomial (2.7) has a double multiple zero \( \lambda_1 = \lambda_2 \) and the remaining zeros \( \lambda_j \) are distinct. If

\[
\sum_{n=0}^{\infty} n|\delta_{j,n}| < \infty, \quad 1 \leq j \leq k,
\]

then (2.4) has a fundamental set of solutions \( y_n^{(1)}, \ldots, y_n^{(k)} \) such that

\[
y_n^{(1)} = (c_1 + o(1)) n \lambda_1^n, \quad c_1 \neq 0,
\]

\[
y_n^{(j)} = (c_j + o(1)) \lambda_j^n, \quad c_j \neq 0, \quad 2 \leq j \leq k.
\]

Another generalization has been provided by Pituk [32] who considered the case when (2.7) has one dominant zero.

**Theorem 2.5.** Suppose that the characteristic polynomial (2.7) has a simple dominant root \( \tilde{\lambda} \) and let (2.8) hold. Then

\[
\lim_{n \to \infty} ((\tilde{\lambda})^{-n} y_n) < \infty.
\]

Furthermore, Agarwal and Pituk also presented in [1] the following assertion comparing the growth rates of (2.4) and (2.6).

**Theorem 2.6.** Suppose (2.5) holds. If \( y_n \) is a solution of (2.4) then the quantity

\[
\rho = \rho(y) = \limsup_{n \to \infty} \sqrt[n]{|y_n|}
\]

is equal to the modulus of one of the characteristic zeros of (2.7).
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3.1. The equation $x'(t) = b x(t - \tau)$

See the full version of thesis.

3.2. The equation $x'(t) = a x(t) + b x(t - \tau)$

See the full version of thesis.

3.3. The equation $x'(t) = a x(t) + b x(t - \tau) + c x'(t - \tau)$

In this section, we deal with the neutral delay differential equation. Our main interest is the $N_\tau(0)$-stability of its $\Theta$-method discretization (a precise specification of this notion will be introduced later). It is known, that the $\Theta$-method discretization is not $N_\tau(0)$-stable for $0 \leq \Theta < 1/2$ and $1/2 < \Theta \leq 1$. The $N_\tau(0)$-stability of the trapezoidal rule discretization (the case $\Theta = 1/2$) has not been sufficiently clarified in the existing literature (see e.g. [17]), especially with respect to the so-called asymptotically critical case $|c| = 1$. Therefore, in this section we provide the necessary and sufficient conditions for the asymptotic stability of the differential equation as well as for its discretization. By their comparison, we resolve the question of the $N_\tau(0)$-stability for the trapezoidal rule and we mention some other consequences following from these conditions. These results have been presented in [6].

3.3.1. Asymptotic stability of the differential equation

We consider the neutral delay differential equation

$$x'(t) = a x(t) + b x(t - \tau) + c x'(t - \tau), \quad t > 0,$$  \hspace{1cm} (3.1)

where $a$, $b$, $c$ and $\tau > 0$ are real scalars. The asymptotic stability region $\Sigma^*_\tau$ for (3.1) is then defined as the set of all real triplets $(a, b, c)$ for which any solution $x(t)$ of (3.1) tends to zero as $t \to \infty$.

The standard way how to describe the asymptotic stability region for linear autonomous functional differential equations consists in analysis of
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zeros of the corresponding characteristic quasi-polynomial. In the case of (3.1) this quasi-polynomial becomes

\[ P(\lambda) \equiv \lambda - a - be^{-\lambda \tau} - c\lambda e^{-\lambda \tau}. \]  

(3.2)

This analysis was used in several papers to obtain asymptotic stability conditions for (3.1) in the pure delayed case \((c = 0)\) as well as in the neutral case \((c \neq 0)\). While the case \(|c| > 1\) easily implies instability of (3.1), the case \(|c| < 1\) is closely related to stability investigations of

\[ x'(t) = ax(t) + b x(t - \tau), \quad t > 0. \]  

(3.3)

It is well-known that the asymptotic stability property for (3.3) can be equivalently expressed as

\[ \Re(\lambda) \leq \delta < 0 \quad \text{for a real scalar } \delta \text{ and any zero } \lambda \text{ of } (3.2). \]  

(3.4)

We recall, that using this fact, descriptions of the exact stability region for (3.3) are known either in the form of parametric curves defining the stability boundary or directly in the form of explicit conditions on \(a, b\) and \(\tau\). For other types of stability conditions we refer to the recent paper by Huang [24]. The description of the stability boundary via parametric curves is convenient especially for two-parameter equations, because it enables to depict the stability picture in the plane of these parameters. However, considering a multi-parameter equation, an explicit system of conditions seems to be more useful. In this section, we utilize such a system.

The case \(|c| = 1\) turns out to be the most problematic in stability analysis of (3.1) (sometimes it is called the asymptotically critical case). More precisely, if \(|c| = 1\) and \(a + |b| < 0\) then all the zeros of (3.2) have negative real parts (see e.g. Freedman and Kuang [15]), but as observed by Gromova in [16], there always exists a sequence of zeros of (3.2) tending to the imaginary axis. Consequently, the condition (3.4), required in asymptotic stability analysis of the pure delayed case, is not satisfied. On this account, some authors involve into the exact stability set \(\Sigma^*_\tau\) only cases corresponding to the condition \(|c| < 1\), regarding it as the necessary condition for the asymptotic stability of (3.1) (see, e.g. Theorem 7.7.1 of [27] and its proof). However, such a description of \(\Sigma^*_\tau\) is not precise. Based on the paper by Freedman and Kuang [15], the stability set \(\Sigma^*_\tau\) can be described via the following necessary and sufficient conditions for the asymptotic stability of (3.1).
3. Delay differential equation with constant lag

**Theorem 3.1** (Theorem 2.1 in [6]). A triplet \((a, b, c)\) belongs to \(\Sigma^*\) if and only if either

\[ a \leq b < -a, \quad |c| < 1, \quad (3.5) \]

or

\[ a + |b| < 0, \quad |c| = 1, \quad (3.6) \]

or

\[ |a| + b < 0, \quad |c| < 1, \quad \tau < \tau^*, \quad (3.7) \]

where

\[ \tau^* = \left( \arccos \frac{a - bc}{ac - b} \right) / \left( \frac{b^2 - a^2}{1 - c^2} \right)^{1/2}. \quad (3.8) \]

**Remark 3.2.** (i) The conditions (3.5)–(3.6) describe the delay-independent stability region for (3.1), i.e. the set of all real triplets \((a, b, c)\) such that the solution \(x(t)\) of (3.1) tends to zero as \(t \to \infty\) for all lags \(\tau > 0\). We emphasize that this delay-independent stability region involves also the asymptotically critical case (3.6). One can observe its certain specific property in the frame of this region, namely the fact that the solution \(x(t)\) of (3.1) is no longer decaying exponentially due to the lack of (3.4).

(ii) The value \(\tau^*\) given by (3.8) defines the stability switch for (3.1), i.e. the critical value of a lag such that, assuming \(|a| + b < 0\) and \(|c| < 1\), the solution \(x(t)\) of (3.1) tends to zero as \(t \to \infty\) if and only if \(\tau < \tau^*\). The explicit expression of such a value is important for theoretical as well as practical reasons and it is a subject of current investigations also for other types of delay differential equations (see e.g. Matsunaga [30] and Matsunaga and Hashimoto [31]).

(iii) The problem of necessary and sufficient conditions for the asymptotic stability of (3.1) was discussed also by Ren [33]. The conclusions presented in this paper seem to be consistent with ours. Since its content is not generally accessible for language reasons (and also because of the above mentioned vagueness concerning the asymptotically critical case), we have preferred to discuss this matter in details.

**3.3.2. Discretization of the differential equation**

For the neutral delay differential equation (3.1), we consider the \(\Theta\)-method discretization. It yields the following recurrence

\[ y_{n+1} + \alpha y_n + \beta y_{n-k+1} + \gamma y_{n-k} = 0, \quad n = 0, 1, 2, \ldots \quad (3.9) \]
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with

\[ \begin{align*}
\alpha &= -\frac{1 + (1 - \Theta)ah}{1 - \Theta ah}, \\
\beta &= -\frac{\Theta bh + c}{1 - \Theta ah}, \\
\gamma &= -\frac{(1 - \Theta)bh - c}{1 - \Theta ah}, \\
k &= \frac{\tau}{h}. 
\end{align*} \]  

(3.10)

We assume \( \Theta ah \neq 1 \). By the asymptotic stability region \( \Sigma^{\Theta}_{\tau}(h) \) of the \( \Theta \)-method discretization of (3.1) we understand the set of all real triplets \((a, b, c)\) for which any solution \(y_n\) of (3.9), (3.10) tends to zero as \( n \to \infty \).

Further, we say that the \( \Theta \)-method for (3.1) is \( N\tau(0) \)-stable if

\[ \Sigma^{*}_{\tau} \subset \bigcap_{k=1}^{\infty} \Sigma^{\Theta}_{\tau}(h), \quad h = \frac{\tau}{k}. \]  

(3.11)

3.3.3. Numerical stability of the \( \Theta \)-method discretization and related issues

We begin this section by discussion on asymptotic stability of the trapezoidal rule discretization, which is the only \( \Theta \)-method of the order 2 with interesting stability properties within the considered class. Our first aim is to describe the stability region \( \Sigma^{1/2}_{\tau}(h) \) in the form of necessary and sufficient conditions imposed on \( a, b, c, \tau \) and \( h \). Having such a description, we can discuss not only some other significant properties of the trapezoidal rule, but also come back to the issue of its \( N\tau(0) \)-stability, especially with respect to the asymptotically critical case.

Further, we provide the necessary and sufficient conditions for the asymptotic stability of (3.9), (3.10) for \( \Theta \neq 1/2 \), too. Analogous to the trapezoidal rule, we mention also some consequences following from such a description. These consequences concern only the case \( \frac{1}{2} < \Theta \leq 1 \) because the case \( 0 \leq \Theta < \frac{1}{2} \) is not interesting from the stability viewpoint.

For given \( a, b, c \), we introduce the symbol

\[ \tilde{\tau}^{1/2}(h) = \left( h \arccos \frac{a - bc}{|ac - b|} \right) / \left( 2 \arctan \left( \frac{h}{2} \left( \frac{b^2 - a^2}{1 - c^2} \right)^{1/2} \right) \right), \]

where \( \omega = \text{sgn} \left( 1 - |c| \right) \).

Using this notation we have

**Theorem 3.3** (Theorem 3.2 in [6]). A triplet \((a, b, c)\) belongs to \( \Sigma^{1/2}_{\tau}(h) \) if and only if one of the following conditions holds:

\[ a \leq b < -a, \quad |c| < 1; \]
3. Delay differential equation with constant lag

\[ a + |b| < 0, \quad (-1)^{k+1}c = 1; \]
\[ |a| + b < 0, \quad |c| < 1, \quad \tau < \tilde{\tau}^{1/2}(h); \]
\[ -|a| + |b| < 0, \quad \text{sgn} (a)(-1)^k c > 1, \quad \tau < \tilde{\tau}^{1/2}(h). \]  

(3.12)

In Figure 3.1 and Figure 3.2 we illustrate \( \Sigma_{\tau}^{1/2}(h) \) for a fixed parameter \( c \). Figure 3.1 depicts this region in the case \( |c| > 1 \), which is described by (3.12). In the left part, we can see \( \Sigma_{1}^{1/2}(1/2) \) for \( c = -1.1 \), while the right part corresponds to \( \Sigma_{1}^{1/2}(1/2) \) for \( c = 1.1 \). Since the underlying differential equation is not asymptotically stable for any \( |c| > 1 \), these regions do not have their continuous counterparts. In Figure 3.2, we illustrate the case \( |c| < 1 \). The line \( a - b = 0 \) divides the delay-dependent and independent parts. While the delay-independent part (above this line) is common for both \( \Sigma_{\tau}^{*} \) and \( \Sigma_{\tau}^{1/2}(h) \), the delay-dependent part (below this line) is larger for \( \Sigma_{\tau}^{1/2}(h) \) (\( \Sigma_{\tau}^{*} \) is restricted by the dashed curve). The case \( |c| = 1 \) is discussed below.

Figure 3.1: Stability regions \( \Sigma_{1}^{1/2}(1/2) \) for \( c = -1.1 \) (the left part) and \( c = 1.1 \) (the right part)

**Remark 3.4.** Theorem 3.3 can be taken for a direct discrete counterpart to Theorem 3.1. In particular, the value \( \tilde{\tau}^{1/2}(h) \) defines the stability switch for (3.9), (3.10) with \( \Theta = 1/2 \) as a discrete analogue of the value \( \tilde{\tau}^{*} \) given by (3.8). One can easily check that

\[ \tilde{\tau}^{1/2}(h) \to \tilde{\tau}^{*} \quad \text{as} \quad h \to 0. \]

Later we show that this convergence is monotonous. Therefore, it might be natural to expect that \( \Sigma_{\tau}^{1/2}(h) \) becomes \( \Sigma_{\tau}^{*} \) as \( h \to 0 \). However, a more
detailed insight shows that this is not true. The problem appears at a part of the stability boundary corresponding to the asymptotically critical case $|c| = 1$. More precisely, triplets $(a, b, c)$ satisfying (3.6) belong to the exact stability region $\Sigma^*_r$ due to Theorem 3.1 but their involvement to $\Sigma^{1/2}_r(h)$ is restricted by the additional requirement $(-1)^{k+1}c = 1$ (see Theorem 3.3). Its fulfilment depends on parity of $k$, hence the limit as $h \to 0$ cannot be considered. Moreover, this fact has another consequence concerning the property (3.11) defining $N\tau(0)$-stability of the numerical method. In view of the previous discussion, this property cannot be obviously satisfied. It implies

**Corollary 3.5.** The trapezoidal rule is not $N\tau(0)$-stable.

This conclusion does not agree with the existing results on this topic (see e.g. Theorem 5.2 of [17]). The explanation of this discrepancy is clear. Assertions confirming $N\tau(0)$-stability of the trapezoidal rule utilize the description of the exact stability region $\Sigma^*_r$ involving only triplets $(a, b, c)$ with $|c| < 1$. However, as pointed out in Theorem 3.1, the triplets $(a, b, c)$ satisfying (3.6) belong to $\Sigma^*_r$ as well. Although they are lying on the boundary of $\Sigma^*_r$, a rigorous approach to the definition of $N\tau(0)$-stability yields the above mentioned conclusion. In this sense, the $\Theta$-method for the neutral equation (3.1) is not $N\tau(0)$-stable for any $0 \leq \Theta \leq 1$.

In this connection, one can observe the following interesting fact. The exact equation (3.1) is under the condition (3.6) asymptotically stable, but the decay rate of its solutions is only algebraic, not exponential. If we consider its trapezoidal rule discretization (3.9), (3.10) with $\Theta = 1/2$ under the condition (3.6), then considering only the fact that (3.9) is a linear homogeneous autonomous difference equation, we can expect just two possibilities:
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this discretization is either asymptotically stable with an exponential decay rate of its solutions (i.e. it is exponentially stable), or it is not asymptotically stable; nothing "between" like in the continuous case. Our analysis summarized in Theorem 3.3 shows that the discretization (3.9), (3.10) with \( \Theta = \frac{1}{2} \) in the asymptotically critical case respects, in a certain sense, this dilemma: it is asymptotically stable for \((a, b, c)\) satisfying (3.6) and \(k \in \mathbb{Z}^+\) if and only if \((-1)^{k+1}c = 1\). In such a case this stability is even exponential.

The following example illustrates this dependence of the asymptotic stability property on parity of \(k\) in the critical case \(c = 1\).

**Example 3.6.** We consider the equation

\[
x'(t) = ax(t) + x'(t - \tau), \quad t > 0, \\
x(t) = g(t), \quad -\tau \leq t \leq 0,
\]

\((3.13)\)

\(a < 0 < \tau\), whose thorough stability analysis was performed by Snow in [34]. This analysis revealed, among others, a rate of approach of the characteristic zeros to the imaginary axis and described an algebraic decay of the solutions \(x(t)\) to zero via the function \(t^{-\kappa}\) (as \(t \to \infty\)), where \(\kappa > 0\) depends upon the smoothness of \(g(t)\).

The following table presents the numerical solution \(y_n\) of (3.13) with \(a = -1\) and \(\tau = 1\) when applied the trapezoidal rule with \(k = 2, 3, \ldots, 7\). Since the characteristic equation of this discretization becomes

\[
\lambda^{k+1} + \frac{1 - 2k}{1 + 2k}\lambda^k - \frac{2k}{1 + 2k}\lambda + \frac{2k}{1 + 2k} = 0, 
\]

\((3.14)\)

one can check by use of \((C3)\) in Theorem 2.1 that all the zeros of \((3.14)\) are located inside the unit circle for \(k\) odd, but considering \(k\) even, a simple zero \(\lambda = -1\) appears. In our case, this zero is dominating (in the absolute value), hence the numerical solution \(y_n\) is eventually oscillatory and \(|y_n|\) tends to a non-zero finite limit. These observations correspond to the theoretical conclusions of Theorem 3.3 and are supported by the data in Table 3.1 (we set here \(y_0 = \cdots = y_{-k} = 1\)).

Notice also that an exponential rate of convergence of \(y_n\) to zero becomes smaller with respect to increasing (odd) \(k\) which corresponds to the fact that the decay of the exact solution \(x(t)\) is not exponential.

Theorem 3.3 implies other important properties of \(\Sigma_{1/2}(\tau)\). One of them describes an inclusion property of these stability regions with respect to changing \(\tau\). This property was shown in the delayed case \(c = 0\) and we confirm its validity (up to switches of parity of \(k\)) also in the neutral delay case \((c \neq 0)\).
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| $t_n$ | $|y_n|$ |
|-------|---------|
|       | $k = 2$ | $k = 3$ | $k = 4$ | $k = 5$ | $k = 6$ | $k = 7$ |
| 1000  | 5.882E-2| 1.499E-5| 1.539E-2| 9.653E-3| 1.877E-3| 1.046E-3|
| 5000  | 5.882E-2| 2.924E-11| 1.539E-2| 6.953E-3| 1.877E-3| 4.611E-3|
| 10000 | 5.882E-2| 4.348E-20| 1.539E-2| 6.897E-3| 1.877E-3| 3.526E-4|

Table 3.1: The values of $|y_n|$ for $a = -1$, $\tau = 1$

**Theorem 3.7.** Let $k_1 < k_2$ be arbitrary positive integers of the same parity and let $h_1 = \tau/k_1 > \tau/k_2 = h_2$ be corresponding stepsizes. Then

$$\sum_{\tau}^{1/2}(h_1) \supset \sum_{\tau}^{1/2}(h_2).$$

Now, we consider $\Theta \neq 1/2$. We state the necessary and sufficient conditions describing the asymptotic stability regions. Doing that, we introduce the symbols for given $a$, $b$, $c$ and $\Theta$

$$\tau_1^\Theta(h) = h \arctan \left( \frac{(b + a)(2(1 - c) + (1 - 2\Theta)(a + b)h)}{(b - a)(2(1 + c) + (1 - 2\Theta)(a - b)h)} \right)^{1/2} / \arctan \left( \frac{(b^2 - a^2)h^2}{(2(1 - c) + (1 - 2\Theta)(a + b)h)(2(1 + c) + (1 - 2\Theta)(a - b)h)} \right)^{1/2}$$

and

$$\tau_2^\Theta(h) = h \arctan \left( \frac{(b + a)(2(1 - c) + (1 - 2\Theta)(a + b)h)}{(b - a)(2(1 + c) + (1 - 2\Theta)(a - b)h)} \right)^{(-1)^{k/2}} / \arccot \left( \frac{(b^2 - a^2)h^2}{(2(1 - c) + (1 - 2\Theta)(a + b)h)(2(1 + c) + (1 - 2\Theta)(a - b)h)} \right)^{1/2}.$$
Remark 3.9.

Theorem 3.8. (a) Let $0 \leq \Theta < \frac{1}{2}$. A triplet $(a, b, c)$ belongs to $\Sigma_\tau^\Theta(h)$ if and only if one of the following conditions holds:

\[
\begin{align*}
& a \leq b < -a, \quad 2 + (1 - 2\Theta)ah > |2c - (1 - 2\Theta)bh|; \\
& a + |b| < 0, \quad 2 + (1 - 2\Theta)ah = (-1)^{k+1} (2c - (1 - 2\Theta)bh) > 0; \\
& |a| + b < 0, \quad 2 + (1 - 2\Theta)ah > |2c - (1 - 2\Theta)bh|, \quad \tau < \tilde{\tau}^\Theta_1(h); \\
& |a| - |b| > 0, \quad 2 + (1 - 2\Theta)ah < \text{sgn}(a) (-1)^{k} (2c - (1 - 2\Theta)bh), \quad \tau < \tilde{\tau}^\Theta_2(h).
\end{align*}
\]

(b) Let $\frac{1}{2} < \Theta \leq 1$. A triplet $(a, b, c)$ belongs to $\Sigma_\tau^\Theta(h)$ if and only if one of the following conditions holds:

\[
\begin{align*}
& a \leq b < -a, \quad 2 - (2\Theta - 1)ah > |2c + (2\Theta - 1)bh|; \\
& a \geq b > -a, \quad 2 - (2\Theta - 1)ah < -|2c + (2\Theta - 1)bh|; \\
& |a| - |b| > 0, \quad \text{sgn}(a) (2 - (2\Theta - 1)ah) = \text{sgn}(a) (-1)^{k+1} (2c + (2\Theta - 1)bh) < 0; \\
& |a| - |b| < 0, \quad \text{sgn}(-b) (2 - (2\Theta - 1)ah) > |2c + (2\Theta - 1)bh|, \quad \tau < \tilde{\tau}^\Theta_1(h); (3.15) \\
& |a| - |b| > 0, \quad 2 - (2\Theta - 1)ah < \text{sgn}(a) (-1)^{k} (2c + (2\Theta - 1)bh), \quad \tau < \tilde{\tau}^\Theta_2(h).
\end{align*}
\]

Remark 3.9. If $\frac{1}{2} < \Theta \leq 1$ then $\Sigma_\tau^\Theta(h)$ involves a subregion defined by the conditions (3.5) and (3.6) independently of $\tau$ and $h$. It particularly implies that, contrary to the trapezoidal rule case, if $(a, b)$ is a real couple satisfying $a + |b| < 0$, then $(a, b, 1) \in \Sigma_\tau^\Theta(h)$ as well as $(a, b, -1) \in \Sigma_\tau^\Theta(h)$ regardless of parity of $k$. In other words, (3.9), (3.10) is asymptotically stable in the critical case $|c| = 1$ for all possible stepsizes $h$ provided $a + |b| < 0$ and $\frac{1}{2} < \Theta \leq 1$.

In the asymptotically critical case $|c| = 1$, the stability properties of the $\Theta$-method (3.9), (3.10) are more favourable for $\Theta > 1/2$ than for $\Theta = 1/2$ (see Remark 3.9). If $|c| < 1$, the situation is different in the sense that the condition (3.11), defining $N\tau(0)$-stability of $\Theta$-methods, holds for $\Theta = 1/2$, but not for $\Theta > 1/2$. More precisely, a deeper analysis of the behaviour of transcendental curves forming a part of the true and numerical stability boundary reveals that there exist triplets $(a, b, c) \in \Sigma_\tau^*$ with $c$ close to $-1$ such that $(a, b, c) \not\in \Sigma_\tau^\Theta(\tau/2)$ for any $1/2 < \Theta < 1$ (see Theorem 4.3 of 17 and a related discussion).

As a consequence of Theorem 3.8, we can extend this result and specify such a neighbourhood of $c = -1$ with respect to the values of $\Theta$, $h$ and $\tau$. To make next steps as clear as possible, we use a simple geometrical argumentation. In particular, we avoid an analysis of the transcendental boundary curve $\tau = \tilde{\tau}^\Theta_1(h)$ and consider instead the first two inequalities of the condition (3.15). These inequalities guarantee the domain of $\tilde{\tau}^\Theta_1$, but also determine an area, where the corresponding curve $\tau = \tilde{\tau}^\Theta_1(h)$ is located.
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Let $\frac{1}{2} < \Theta \leq 1$. Since the delay-independent part of $\Sigma^*_\tau$ is involved in $\Sigma^\Theta_\tau(h)$ for any stepsize $h$ (see Remark $3.9$), we analyse the delay-dependent part. For a fixed $c \in (-1, 1)$ and a fixed $\tau > 0$, we consider the $(a, b)$-plane, where the delay-dependent part of $\Sigma^*_\tau$ is bounded above by the lines $a+b = 0$, $a - b = 0$ and below by the transcendental curve

$$\tau \left( \frac{b^2 - a^2}{1 - c^2} \right)^{1/2} - \arccos \frac{a - bc}{ac - b} = 0.$$  \hspace{1cm} (3.16)

Moreover, $P_1 = ((1 - c)/\tau, (c - 1)/\tau)$ is a double point for this stability boundary, i.e. the point, where the line $a + b = 0$ and the curve (3.16) intersect.

Now let $\Omega$ be a part of the $(a, b)$-plane bounded above by the lines $a + b = 0$, $a - b = 0$ and consider the delay-dependent part of $\Sigma^\Theta_\tau(h)$ restricted to $\Omega$. The analytical description of this area is given by (3.15). In particular, the second condition of (3.15) implies that such a delay-dependent part of $\Sigma^\Theta_\tau(h)$ is bounded below by the line

$$a - b - \frac{2(1 + c)}{(2\Theta - 1)h} = 0,$$  \hspace{1cm} (3.17)

which is parallel to $a - b = 0$ and orthogonal to $a + b = 0$. The lines (3.17) and $a + b = 0$ intersect at

$$P_2 = \left( \frac{(1 + c)}{(2\Theta - 1)h}, \frac{(1 + c)}{(2\Theta - 1)h} \right).$$

Comparing locations of $P_1$ and $P_2$ at the line $a + b = 0$, one can obtain an obvious geometrical conclusion: If $\Sigma^\Theta_\tau(h) \supset \Sigma^*_\tau$ then $P_1$ is located above $P_2$ or coincides with $P_2$ (equivalently, $h(1 - c) \leq (1 + c)\tau/(2\Theta - 1)$). In the opposite case, when $P_1$ is located below $P_2$, we can introduce a non-empty region $\tilde{\Sigma}^\Theta_\tau(h)$, bounded by the lines (3.17), $a + b = 0$ above and by the curve (3.16) below. Obviously, this region satisfies the property

$$\tilde{\Sigma}^\Theta_\tau(h) = \Sigma^*_\tau \setminus \Sigma^\Theta_\tau(h).$$  \hspace{1cm} (3.18)

Both these cases are depicted in Figure $3.3$ and Figure $3.4$, where Figure $3.3$ illustrates the case when $P_1$ is located below $P_2$ and Figure $3.4$ the opposite one. The dashed curve in both the figures indicates a part of the stability boundary of the underlying differential equation (see (3.16)) and the line (3.17) is denoted here as $p$.

Previous considerations can be summarized in the following...
3. Delay differential equation with constant lag

Figure 3.3: Delay-dependent parts of the stability regions $\Sigma^\alpha_{1.75}(1/2)$ and $\Sigma^*_1$ for $c = -0.9$

Figure 3.4: Delay-dependent parts of the stability regions $\Sigma^\alpha_{1.75}(1/2)$ and $\Sigma^*_1$ for $c = -0.3$

**Corollary 3.10.** Let $\frac{1}{2} < \Theta \leq 1$ and

\[
c < \frac{(1 - 2\Theta)h + \tau}{(1 - 2\Theta)h - \tau}
\]

for a given $h$ and $\tau$. Then the set $\tilde{\Sigma}^{\Theta}_\tau(h)$ is non-empty and, by \((3.18)\), for any $(a,b,c) \in \tilde{\Sigma}^{\Theta}_\tau(h)$ the exact equation \((3.1)\) is asymptotically stable, whereas its $\Theta$-method discretization \((3.9)\), \((3.10)\) is not asymptotically stable.
Remark 3.11. The condition (3.19) becomes more restrictive with increasing \( \Theta \) as well as with increasing \( h \). In particular, letting \( h \to 0 \) we can see that this critical value of \( c \) is tending to \(-1\), which corresponds to the stability condition (3.7) for the exact equation. On the other hand, in the case of the backward Euler method (\( \Theta = 1 \)), the inequality (3.19) can be read as

\[
\frac{h - \tau}{h + \tau} < c,
\]

which particularly implies that for any \( c < 0 \) one can find \((a, b, c) \in \Sigma^*_\tau\) and a stepsize \( h \) such that the corresponding backward Euler formula is not asymptotically stable, i.e. \((a, b, c) \notin \Sigma^1_\tau(h)\).

Our previous observations extend the discussion performed by Guglielmi in [17]. We recall that we did not employ here analysis of the transcendental curve \( \tau = \tilde{\tau}^\Theta_{\tau}(h) \), but only a region, where this curve is situated. On this account, the condition (3.19) is sufficient for the existence of a non-empty set \( \tilde{\Sigma}^\Theta_h(h) \), characterized by the property (3.18), but not necessary. Some additional calculations show the necessity of (3.19) when \( h = \tau \). More precisely, if we restrict to the delay-dependent case \( |a| + b < 0, |c| < 1 \), then \((a, b, c) \in \Sigma^\Theta_{\tau}(\tau)\) if and only if

\[
-2(1 + c) + (2\Theta - 1)(a - b)\tau < 0 < 1 - c + b\tau - \Theta(a + b)\tau. \tag{3.20}
\]

This condition follows either from (3.15) with \( h = \tau \), or it can be derived directly (the stability polynomial is now quadratic). To prove the necessity of (3.19) when \( h = \tau \), we assume that \( c \geq (\Theta - 1)/\Theta \), i.e. \( P_1 \) is above \( P_2 \) (see our geometrical argumentation in the \((a, b)\)-plane). In this case, the delay-dependent part of \( \tilde{\Sigma}^\Theta_h(\tau) \) is bounded above by the lines \( a + b = 0 \), \( a - b = 0 \), below by the line (3.17) and right by the line

\[
1 - c + b\tau - \Theta(a + b)\tau = 0 \tag{3.21}
\]

(see (3.20)). We recall that the exact stability set \( \Sigma^*_\tau \) is bounded below by the curve (3.16), hence it remains to compare the locations of this curve and the lines (3.17), (3.21). Straightforward calculations based on derivatives of the curve (3.16) and both the lines show that (3.16) is located above (3.17) and (3.21) in the investigated area (the only intersection of (3.16) and (3.21) is the point \( P_1 \) on the stability boundary). It implies

**Corollary 3.12.** Let \( \frac{1}{2} < \Theta \leq 1 \). Then

\[
\Sigma^*_\tau \subset \Sigma^\Theta_{\tau}(\tau) \iff c \geq (\Theta - 1)/\Theta.
\]
3. Delay differential equation with constant lag

3.4. The equation

\[ x'(t) = a x(t) + b_1 x(t - \tau_1) + b_2 x(t - \tau_2) \]

See the full version of thesis.
4. **Delay Differential Equation with Infinite Lag**

In this chapter, we study the delay differential equations with infinite lag. In the first section, we recall known results for the pantograph equation. The second section provides the generalization of these results for a delay differential equation with a general lag. Finally, in the third section we investigate the delay differential equations with several lags. The results of the second and third section originates from the paper which is now in a preparation.

4.1. The equation \( x'(t) = ax(t) + bx(qt) \)

See the full version of thesis.

4.2. The equation \( x'(t) = ax(t) + bx(\xi(t)) \)

Our next aim is to extend the results of the previous section to equations where the delayed argument is given by a general function \( \xi(t) \in C^1((t_0, \infty)) \) satisfying

\[
\begin{align*}
\xi(t_0) &= t_0, & \xi(t) &< t, & \text{for } t > t_0, & \lim_{t \to \infty} \xi(t) &= \infty, \\
\xi'(t_0) &< 1, & \xi'(t) &> 0, & \text{for } t \geq t_0, & \xi'(t) \text{ is non-increasing for } t \geq t_0. 
\end{align*}
\] (4.1)

We introduce a constrained mesh suitable for discretization of such an equation and determine the asymptotic stability of both the exact and discretized equation. Further, we provide and compare asymptotic estimates of their solutions.

4.2.1. Asymptotic stability of the differential equation

We consider the delay differential equation

\[
x'(t) = ax(t) + bx(\xi(t)), \quad t \in I = (t_0, \infty),
\] (4.2)

where \( a, b \) are real scalars and \( \xi(t) \) is a continuously differentiable function on \( I \) satisfying (4.1). The asymptotic stability region \( S^*_\xi \) for (4.2) is defined as the set of all real couples \((a, b)\) for which any solution \( x(t) \) of (4.2) tends to zero as \( t \to \infty \). To obtain its description, we first recall the result of
4. Delay differential equation with infinite lag

Heard [20] who studied the asymptotic behaviour of (4.2) using the Schröder’s equation. The Schröder’s equation is a functional equation

\[ \varphi(\xi(t)) = q \varphi(t), \quad t \in (t_0, \infty), \tag{4.3} \]

where \( q \in \mathbb{R} \) and \( \xi(t) \) is a given function. A study of this equation is given in the book Kuczma, Choczewski and Ger [26]. Here, we state the result relevant to our further analysis originating from Proposition 1 of Čermák [3].

**Proposition 4.1.** Let \( \xi(t) \) be a continuously differentiable function on \( (t_0, \infty) \) satisfying (4.1) and let \( q = \xi'(t_0) \). Then there exists a positive solution \( \varphi(t) \in C^1(I) \) of (4.3) with a positive and bounded derivative on \( I \) such that \( \lim_{t \to \infty} \varphi(t) = \infty \).

The asymptotic estimate derived by Heard [20] is as follows.

**Theorem 4.2.** Let \( a < 0 \) and \( q = \xi'(t_0) \). Then the solution \( x(t) \) of (4.2) satisfies

\[ x(t) = O\left((\varphi(t))^{-\log_q |b/a|}\right) \quad \text{as} \quad t \to \infty, \tag{4.4} \]

where \( \varphi(t) \) is a solution of the Schröder’s equation (4.3) with the properties described in Proposition 4.1.

Moreover, the analysis provided by Heard [20] implies the following necessary and sufficient condition describing \( S_{\xi}^* \).

**Theorem 4.3.** Let \( \xi(t) \) be a continuously differentiable function on \( I \) satisfying (4.1). A real couple \( (a, b) \) belongs to \( S_{\xi}^* \) if and only if

\[ |b| < -a. \]

4.2.2. Discretization of the differential equation

Similarly to the previous section, the Θ-method discretization of (4.2) on the uniform grid results in a difference equation of variable order with non-constant coefficients. Since analysis of such a difference equation is a difficult task, we introduce the following constrained grid proposed by Guglielmi and Zennaro [18], which is a generalization of a quasi-geometric mesh used for the discretization of the pantograph equation and ensures a fixed order of the resulting recurrence.

Let \( T_0 \) be a fixed positive number. We divide the interval \( (t_0, T_0) \) by \( p \) grid points

\[ t_0 < t_1 < t_2 < \cdots < t_p = T_0. \]
4. Delay differential equation with infinite lag

Then we build a primary mesh based on the following relation

\[ T_{k+1} = \xi^{-1}(T_k), \quad k = 0, 1, \ldots \]

Further, we evenly divide the first primary interval \( \langle T_0, T_1 \rangle \) into fixed number of \( m \) subintervals. The division of the subsequent primary intervals is then given by the relation

\[ t_n = \xi^{-1}(t_{n-m}), \quad n = p + m, p + m + 1, \ldots \]

Since \( \xi'(t) \leq \bar{q} = \xi'(t_0) < 1 \) on \( I \), then the stepsize satisfies

\[ h_n \geq \frac{h_{n-m}}{\bar{q}}, \quad n = p + m, p + m + 1, \ldots \] (4.5)

Therefore

\[ \lim_{n \to \infty} h_n = \infty \]

and, moreover, \( h_n \) increases exponentially. The resulting mesh is called an almost-geometric mesh, which is due to property (4.5).

The \( \Theta \)-method applied to (4.2) on the almost-geometric mesh is formally the same as for the proportional delay, i.e.

\[ y_{n+1} + \alpha y_n + \beta y_{n-m+1} + \gamma y_{n-m} = 0, \quad n = p + m, p + m + 1, \ldots \] (4.6)

with

\[ \alpha = -\frac{1 + (1 - \Theta)ah_n}{1 - \Theta ah_n}, \quad \beta = -\frac{\Theta bh_n}{1 - \Theta ah_n}, \quad \gamma = -\frac{(1 - \Theta)bh_n}{1 - \Theta ah_n}. \] (4.7)

We assume \( 1 - \Theta ah_n \neq 0 \), too. The difference with respect to the proportional case consists in a distinct growth of \( h_n \).

By the asymptotic stability region \( S^\Theta_{\xi,p}(m) \) of the \( \Theta \)-method discretization of (4.2) we understand the set of real couples \( (a,b) \) for which any solution \( y_n \) of (4.6), (4.7) tends to zero as \( n \to \infty \).

We say that the \( \Theta \)-method for (4.2) is asymptotically stable if it holds

\[ S^\ast_{\xi} \subset \bigcap_{m=1}^{\infty} S^\Theta_{\xi,p}(m). \]
4. Delay differential equation with infinite lag

4.2.3. Numerical stability of the \( \Theta \)-methods and related issues

The asymptotic stability of \( \Theta \)-methods for (4.2) follows directly from the analysis of more general equations provided by Guglielmi (see [18] and [19]) and it can be summarized as follows.

**Theorem 4.4.** The \( \Theta \)-method applied to (4.2) on the almost-geometric mesh is asymptotically stable if and only if \( 1/2 < \Theta \leq 1 \).

However, the precise description of the stability region \( S_{\xi,\rho}^\Theta(m) \) as well as the asymptotic properties of numerical solution have remained an unsolved problem. We provide the answer for both these issues. Firstly, we deal with the asymptotics of (4.6), (4.7).

**Theorem 4.5.** Let \( y_n \) be a solution of (4.6), (4.7), where \( a, b \neq 0 \) and \( 0 < \Theta \leq 1 \). Then we distinguish the following cases:

(a) Let \( |b|\Theta^m \geq |a|(1 - \Theta)^m \). If \( b\Theta^m + a(\Theta - 1)^m \neq 0 \), then

\[
y_n = O \left( |b/a|^{\frac{n}{m}} \right) \quad \text{as } n \to \infty.
\]

(4.8)

If \( b\Theta^m + a(\Theta - 1)^m = 0 \), then

\[
y_n = O \left( n|b/a|^{\frac{n}{m}} \right) \quad \text{as } n \to \infty.
\]

(b) Let \( |b|\Theta^m < |a|(1 - \Theta)^m \). Then there exists a constant \( \eta \) (depending on \( y_n \)) such that

\[
y_n = (\eta + o(1)) \left( \frac{\Theta - 1}{\Theta} \right)^n \quad \text{as } n \to \infty.
\]

**Theorem 4.6.** Let \( y_n \) be a solution of (4.6), (4.7), where \( a, b \neq 0 \), \( 1 + ah_n \neq 0 \) for all \( n \in \mathbb{Z}^+ \) and \( \Theta = 0 \). Then there exists a constant \( \nu \) (depending on \( y_n \)) such that

\[
y_n = (\nu + o(1)) \prod_{j=0}^{n-1} (1 + ah_j) \quad \text{as } n \to \infty.
\]

The proof of both these theorems is analogous to the proof of the corresponding theorems for the pantograph equation provided by Čermák in [4]. The analysis is based on the application of theory of Poincaré difference equations. More precisely, it utilizes Theorem 2.3 and Theorem 2.4 in the case (a), while Theorem 2.5 is employed in the case (b).
4. Delay differential equation with infinite lag

The following description of the asymptotic stability region $S_{\xi,p}^{\Theta}(m)$ follows from Theorem 4.5 and Theorem 4.6 due to the fact that their asymptotic estimates cannot be improved. More precisely, there exists solutions of (4.6), (4.7) asymptotically equivalent to $|b/a|^\frac{n}{m}$, $n|b/a|^\frac{n}{m}$, $(\Theta - 1)/\Theta$ and $\prod_{j=0}^{n-1}(1 + ah_j)$.

**Corollary 4.7.** Let $\xi(t)$ be a continuously differentiable function on $I$ satisfying (4.1). Then a real couple $(a,b)$ belongs to $S_{\xi,p}^{\Theta}(m)$ if and only if

$$ |b| < |a|, \quad \Theta > 1/2. $$

The remaining issue is the comparison of the asymptotic estimates of the exact and discretized equation. First, we observe that

$$ \varphi(t_n) = q^{-[(n-p)/m]}\varphi(\xi^{[-(n-p)/m]}(t_n)) = q^{-[(n-p)/m]}\varphi(t_{p+j}) $$

for some $j = 0, 1, \ldots, m - 1$. Thus,

$$ q^{\frac{p-m}{m}}\varphi(t_{p+j}) q^{-\frac{n}{m}} \leq \varphi(t_n) \leq q^{\frac{n}{m}}\varphi(t_{p+j}) q^{-\frac{n}{m}}. $$

Then, we can rewrite (4.8) as

$$ y_n = O\left(\left(q^{\log_q(|b/a|)}\right)^{\frac{n}{m}}\right) = O\left((q^{-\frac{n}{m}})^{-\log_q(|b/a|)}\right) = O\left((\varphi(t_n))^{-\log_q(|b/a|)}\right) \quad \text{as } n \to \infty. $$

We summarize previous considerations in the following

**Corollary 4.8.** Let $a, b \neq 0$, $0 < \Theta \leq 1$, $|b|\Theta^m \geq |a|(1 - \Theta)^m$ and $b\Theta^m + a(\Theta - 1)^m \neq 0$. Then the solution $y_n$ of (4.6), (4.7) satisfies

$$ y_n = O\left((\varphi(t_n))^{-\log_q(|b/a|)}\right) \quad \text{as } n \to \infty, $$

which presents exactly the same estimate of the numerical solution as (4.4) yields for the exact solution.

### 4.3. The equation $x'(t) = ax(t) + \sum_{i=1}^{r} b_i x(\xi_i(t))$

See the full version of thesis.
5. Conclusion

The doctoral thesis concerns with the qualitative and numerical analysis of the linear delay differential equations with constant as well as infinite lag.

In the first part of the thesis, we investigated the linear neutral delay differential equation

$$x'(t) = ax(t) + bx(t - \tau) + cx'(t - \tau), \quad t > 0,$$

where $a$, $b$, $c$ and $\tau > 0$ are real scalars. We discretized (5.1) using the $\Theta$-method and derived the necessary and sufficient conditions describing the stability region of both exact and discretized equations. Based on them, we concluded that the $\Theta$-method is not $N\tau(0)$-stable for any $0 \leq \Theta \leq 1$. Some properties of the discretized stability regions were mentioned, too. Further, we dealt with the particular cases of (5.1) where $c = 0$ and $c = a = 0$. The explicit conditions for the asymptotic stability regions of the $\Theta$-method discretization as well as the modified midpoint method were presented. We discussed some properties of the derived stability regions, mainly with respect to changing stepsize.

The second part of the thesis concerns the delay differential equations with infinite lag. We investigated the $\Theta$-method discretization on the constrained mesh and provided the description of the stability regions together with asymptotic estimates for the exact and numerical solution. The asymptotic stability of the equation with several infinite lags was also analysed. We derived the asymptotic estimate of its solution as well as the sufficient condition under which this equation is asymptotically stable. It was shown that for $0 \leq \Theta \leq 1/2$ the asymptotic stability region of $\Theta$-method discretization is an empty set, while for $1/2 < \Theta \leq 1$ it contains the presented stability region of the differential equation. The necessary and sufficient conditions and some asymptotic estimates were provided for the discretized equation with two delayed terms.

Finally, we mention some open problems and general remarks. We analysed separately the numerical stability of equations with constant and infinite lag, however the analysis of the equations with the infinite lag seems to be less complicated. As it was shown for differential equations with two lags, we are able to provide the necessary and sufficient conditions for discrete asymptotic stability of the equation with infinite lags but the analysis of the constant lag case is still an unsolved problem. The key role in the analysis of discretization of both kinds of delay differential equations plays the analysis of the corresponding delay difference equations. The delay difference equations do not have many original applications and therefore they have
5. Conclusion

not been widely studied. In fact, it is the numerical discretization which motivates further investigation of qualitative properties of different types of difference equations. In Table 5.1 we provide an overview of the difference equations which had to be analysed in order to discuss the asymptotic stability of numerical discretizations of the studied equations. We emphasize that our numerical analysis is based on results of Čermák et al. [7] and [8] who derived the necessary and sufficient conditions for the asymptotic stability of these difference equations in an explicit form.

<table>
<thead>
<tr>
<th>A numerical motivation</th>
<th>The resulting difference scheme</th>
</tr>
</thead>
<tbody>
<tr>
<td>Θ-method discretization for $x'(t) = ax(t) + bx(t - \tau) + cx'(t - \tau)$</td>
<td>$y_{n+1} + \alpha y_n + \beta y_{n-k+1} + \gamma y_{n-k} = 0$</td>
</tr>
<tr>
<td>Θ-method discretization for $x'(t) = ax(t) + b_1 x(\xi(t)) + b_2 x(\xi'(t))$</td>
<td>$y_{n+1} + \alpha y_n + \gamma y_{n-k} = 0$</td>
</tr>
<tr>
<td>trapezoidal rule for $x'(t) = ax(t) + bx(t - \tau)$</td>
<td>$y_{n+1} + \alpha y_n + \beta (y_{n-k+1} + y_{n-k}) = 0$</td>
</tr>
<tr>
<td>Θ-method discretization for $x'(t) = bx(t - \tau)$</td>
<td>$y_{n+1} - y_n + \beta y_{n-k+1} + \gamma y_{n-k} = 0$</td>
</tr>
<tr>
<td>trapezoidal rule for $x'(t) = bx(t - \tau)$</td>
<td>$y_{n+1} - y_n + \beta (y_{n-k+1} + y_{n-k}) = 0$</td>
</tr>
<tr>
<td>Euler methods for $x'(t) = bx(t - \tau)$</td>
<td>$y_{n+1} - y_n + \gamma y_{n-k} = 0$</td>
</tr>
<tr>
<td>modified midpoint method for $x'(t) = ax(t) + bx(t - \tau)$</td>
<td>$y_{n+2} + \mu y_n + \nu y_{n-k} = 0$</td>
</tr>
<tr>
<td>midpoint method for $x'(t) = bx(t - \tau)$</td>
<td>$y_{n+2} - y_n + \nu y_{n-k} = 0$</td>
</tr>
</tbody>
</table>

Table 5.1: The corresponding difference equations to the numerical methods for studied differential equations

The asymptotic stability analysis of the Θ-method for (5.1) is complete in the sense that the necessary and sufficient conditions describing its exact as well as discrete stability regions were derived. The open problem remains the investigation of the delay differential equation with two constant lags.

The presented results for the delay differential equation with the infinite lags are partial results of our current research and their generalization for
5. Conclusion

the neutral equation is one of its possible extensions. In particular, if we consider the neutral equation

\[ x'(t) = ax(t) + bx(\xi(t)) + c x'(\xi(t)), \quad t \in (t_0, \infty), \quad (5.2) \]

then our results on discretization of this equation in the pure delayed case \((c = 0)\) can be easily extended to the neutral case \((c \neq 0)\). However, appropriate stability and asymptotic results on the underlying differential equation \((5.2)\) are not known (a possible generalization of Theorem 4.2 to neutral equation \((5.2)\) can be the subject of the next research). Also, the asymptotic estimates for the delay differential equation with two infinite lags regardless of sign of its coefficient can be investigated.

The previous results, methods and problems become more complicated if we consider non-autonomous delay equations. In such a case, we probably cannot expect the optimal (i.e. necessary and sufficient) stability conditions, which is due to utilized techniques. For some related results on delay differential and difference equations we refer, e.g. to papers [2], [9], [10], [11] and [12], which may also serve as a motivation for future research.
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Abstract
The doctoral thesis discusses the asymptotic stability of delay differential equations and their discretizations. The linear delay differential equations with constant as well as infinite lag are considered. The necessary and sufficient conditions describing the asymptotic stability region of both exact and discretized linear neutral delay differential equation with constant lag are derived. We compare asymptotic stability domains of corresponding exact and discretized equations and discuss properties of derived stability regions with respect to a changing stepsize of the utilized discretization. Further, we investigate the linear delay differential equation with the infinite lag. We present the description of its exact and discrete asymptotic stability regions together with asymptotic estimates of its solutions. The linear delay differential equation with several infinite lags is discussed as well.