WEAKLY DELAYED PLANAR LINEAR DISCRETE SYSTEMS AND CONDITIONAL STABILITY

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Abstract: A discrete planar system

$$x(k+1) = Ax(k) + B_1x(k-m_1) + B_2x(k-m_2), \; k \geq 0$$

is analysed, where m_1, m_2 are constant integer delays, $0 < m_1 < m_2$, A, B_1, B_2 are constant 2×2 matrices, $A = (a_{ij})$, $B_l = (b^l_{ij})$, $i, j = 1,2$, $l = 1,2$ and $x\colon \{-m_2, -m_2+1, \dots\} \to R^2$. We get new results on conditional stability and asymptotic conditional stability.

Keywords: Conditional stability, conditional asymptotic stability, weakly delayed system, discrete system

1 INTRODUCTION

We investigate discrete planar systems

$$x(k+1) = Ax(k) + B_1x(k-m_1) + B_2x(k-m_2) \tag{1}$$

where m_1, m_2 are constant integer delays, $0 < m_1 < m_2$, $k \in Z_0^\infty$, A, B_1, B_2 are constant 2×2 matrices, $A = (a_{ij})$, $B_l = (b^l_{ij})$, $i, j = 1,2$, $l = 1,2$, $B_l \neq \Theta$, $l = 1,2$, Θ is 2×2 zero matrix and $x\colon Z^\infty_{-m_2} \to R^2$, $Z^q_s := \{s, s+1, \dots, q\}$. Consider initial problem

$$x(k) = \varphi(k) \tag{2}$$

for (1) where $k = -m_2, -m_2+1, \dots, 0$ with $\varphi\colon Z^0_{-m_2} \to R^2$. It is well-known that the initial problem (1), (2) has a unique solution on $Z^\infty_{-m_2}$.

Define a norm of a 2×2 matrix $A = \{a_{ij}\}^2_{i,j=1}$ as

$$\|A\| = \max\{|a_{11}| + |a_{12}|, |a_{21}| + |a_{22}|\}$$

and, for 2×1 vectors $x = (x_1, x_2)^T$, an vector norm

$$\|x\| = \max\{|x_1|, |x_2|\}.$$

For a discrete vector $\psi\colon Z^0_{-m_2} \to R^2$ we define

$$\|\psi\|_{m_2} := \max\{\|\psi(-m_2)\|, \|\psi(-m_2+1)\|, \dots, \|\psi(0)\|\}.$$

Definition 1 *The zero solution $x(k) = 0$, $k \in Z^\infty_{-m_2}$ of* (1) *is said to be*

- **a)** *Stable if, given $\varepsilon > 0$ and $k_0 \geq 0$, there exists $\delta = \delta(\varepsilon, k_0)$ such that $\varphi(k)$, $k \in Z^{k_0}_{k_0-m_2}$, $\|\varphi\|_{m_2} < \delta$ implies $\|x(k, k_0, \varphi)\| < \varepsilon$ for all $k \geq k_0$, uniformly stable if δ may be chosen independently of k_0, unstable if it is not stable;*

b) *Asymptotically stable if it is stable and* $\lim_{k\to\infty}\|x(k)\| = 0$*;*

c) *Conditionally stable (conditionally asymptotically stable) if it is stable (asymptotically stable) under the condition that a subspace P of the space all initial data with* $\dim P$ *satisfying*

$$1 < \dim P < 2(m_2+1)$$

is fixed.

The equation

$$D := \det\left(A+\lambda^{-m_1}B_1+\lambda^{-m_2}B_2-\lambda I\right) = 0 \tag{3}$$

where I is the unit 2×2 matrix, $\lambda \in C$ is characteristic equation to (1) and characteristic equation to

$$x(k+1) = Ax(k) \tag{4}$$

is

$$\det(A-\lambda I) = 0. \tag{5}$$

Definition 2 [*1*] *The system (1) is called a weakly delayed system if the characteristic equations (3), (5) corresponding to systems (1) and (4) are equal, i.e. if, for every* $\lambda \in C\setminus\{0\}$, $D = \det(A-\lambda I)$.

We consider a linear transformation $x(k) = \mathcal{S}y(k)$ with a nonsingular 2×2 matrix $\mathcal{S}$. Then, the discrete system for y is

$$y(k+1) = A_{\mathcal{S}}y(k) + B_{1\mathcal{S}}y(k-m_1) + B_{2\mathcal{S}}y(k-m_2) \tag{6}$$

with $A_{\mathcal{S}} = \mathcal{S}^{-1}A\mathcal{S}$, $B_{l\mathcal{S}} = \mathcal{S}^{-1}B_l\mathcal{S}$ where $l = 1,2$.

Lemma 1 [*1*] *If* (1) *is a weakly delayed system, then its arbitrary linear nonsingular transformation* $x(k) = \mathcal{S}y(k)$ *again leads to a weakly delayed system* (6)*.*

Following theorem is a criterion indicating whether a system is weakly delayed.

Theorem 1 [*1*] *System (1) is a weakly delayed system if and only if the following conditions hold simultaneously:*

$$b_{11}^l + b_{22}^l = 0, \quad \begin{vmatrix} b_{11}^l & b_{12}^l \\ b_{21}^l & b_{22}^l \end{vmatrix} = 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ b_{21}^l & b_{22}^l \end{vmatrix} + \begin{vmatrix} b_{11}^l & b_{12}^l \\ a_{21} & a_{22} \end{vmatrix} = 0, \quad l = 1,2, \tag{7}$$

$$\begin{vmatrix} b_{11}^1 & b_{12}^1 \\ b_{21}^2 & b_{22}^2 \end{vmatrix} + \begin{vmatrix} b_{11}^2 & b_{12}^2 \\ b_{21}^1 & b_{22}^1 \end{vmatrix} = 0. \tag{8}$$

For every matrix A there exists a nonsingular matrix S transforming it to the corresponding Jordan matrix form Λ, i.e. $\Lambda = S^{-1}AS$, where the form of Λ depends on the roots of the characteristic equation (5), i.e. on the roots of

$$\lambda^2 - (a_{11}+a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0. \tag{9}$$

It the following we will assume that (9) has two real distinct roots λ_1, λ_2. Then $\Lambda = \Lambda_1$ where

$$\Lambda_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \tag{10}$$

The transformation $y(k) = S^{-1}x(k)$ transforms (1) into a system

$$y(k+1) = \Lambda y(k) + B_1^* y(k-m_1) + B_2^* y(k-m_2), \; k \in Z_0^\infty \tag{11}$$

with $B_l^* = S^{-1}B_lS$, $B_l^* = (b_{ij}^{*l})$, $l = 1,2$, $i,j = 1,2$. The initial problem (2) transforms to

$$y(k) = \varphi^*(k),$$

$k \in Z_{-m_2}^0$, where $\varphi^*(k) = S^{-1}\varphi(k)$. Define $\Phi_1(k) := (0, \varphi_1^*(k))^T$, $\Phi_2(k) := (\varphi_2^*(k), 0)^T$, $k \in Z_{-m_2}^0$.

In the contribution we deal with what is called conditional stability and asymptotic conditional stability of linear weakly delayed discrete systems (1). We derive sufficient conditions for asymptotic conditional stability if $|\lambda_1| \le q < 1$ and $|\lambda_2| \ge 1$ or if $|\lambda_2| \le q < 1$ and $|\lambda_1| \ge 1$, and sufficient conditions for conditional stability if $|\lambda_1| = 1$ and $|\lambda_2| > 1$ or if $|\lambda_2| = 1$ and $|\lambda_1| > 1$. Obtained results on conditional stability are new and are given in Theorems 2–5. To prove them we use explicit analytic formulas, derived in [1].

2 CONDITIONAL STABILITY

Let $\Lambda = \Lambda_1$. From the necessary and sufficient conditions (7)–(8) for (11) it follows that (1) is weakly delayed if and only if either

$$I)\ b_{11}^{*l} = b_{22}^{*l} = b_{21}^{*l} = 0,\ b_{12}^{*l} \ne 0,\ l = 1,2,$$

or

$$II)\ b_{11}^{*l} = b_{22}^{*l} = b_{12}^{*l} = 0,\ b_{21}^{*l} \ne 0,\ l = 1,2.$$

Theorem 2 *If the case I) occurs, $|\lambda_1| \le q < 1$, $|\lambda_2| \ge 1$ and $\varphi_2^*(0) = 0$, then the zero solution of* (1) *is conditionally asymptotically stable.*

Proof: In this case, $\varphi^*(0) = (\varphi_1^*(0), 0)^T$ and $\Phi_2(0) = (\varphi_2^*(0), 0)^T = (0,0)$. As it follows from [1] the solution of the initial problem (1), (2) is $x(k) = Sy(k)$, $k \in Z_{-m_2}^{\infty}$ where

$$y(k) = \varphi^*(k) \quad \text{if} \quad k \in Z_{-m_2}^0,$$

$$y(k) = \Lambda_1^k \varphi^*(0) + \sum_{r=0}^{k-1} \lambda_1^{k-1-r} \left[b_{12}^{*1}\Phi_2(r-m_1) + b_{12}^{*2}\Phi_2(r-m_2) \right] \quad \text{if} \quad k \in Z_1^{m_1+1},$$

$$y(k) = \Lambda_1^k \varphi^*(0) + \sum_{r=0}^{k-1} \lambda_1^{k-1-r} \left[b_{12}^{*2}\Phi_2(r-m_1) \right] + b_{12}^{*1} \left[\sum_{r=0}^{m_1} \lambda_1^{k-1-r}\Phi_2(r-m_2) \right.$$
$$\left. + \Phi_2(0) \sum_{r=m_1+1}^{k-1} \lambda_1^{k-1-r}\lambda_2^{r-m_1} \right] \quad \text{if} \quad k \in Z_{m_1+2}^{m_2+1},$$

$$y(k) = \Lambda_1^k \varphi^*(0) + b_{12}^{*1} \left[\sum_{r=0}^{m_1} \lambda_1^{k-1-r}\Phi_2(r-m_1) + \Phi_2(0) \sum_{r=m_1+1}^{k-1} \lambda_1^{k-1-r}\lambda_2^{r-m_1} \right]$$
$$+ b_{12}^{*2} \left[\sum_{r=0}^{m_2} \lambda_1^{k-1-r}\Phi_2(r-m_2) + \Phi_2(0) \sum_{r=m_2+1}^{k-1} \lambda_1^{k-1-r}\lambda_2^{r-m_2} \right] \quad \text{if} \quad k \in Z_{m_2+2}^{\infty}.$$

For $k \in Z_{m_n+2}^{\infty}$, we get

$$\|y(k)\| \le \|\Lambda_1^k \varphi^*(0)\| + \left\| b_{12}^{*1} \left[\sum_{r=0}^{m_1} \lambda_1^{k-1-r}\Phi_2(r-m_1) + \Phi_2(0) \sum_{r=m_1+1}^{k-1} \lambda_1^{k-1-r}\lambda_2^{r-m_1} \right] \right\|$$
$$+ \left\| b_{12}^{*2} \left[\sum_{r=0}^{m_2} \lambda_1^{k-1-r}\Phi_2(r-m_2) + \Phi_2(0) \sum_{r=m_2+1}^{k-1} \lambda_1^{k-1-r}\lambda_2^{r-m_2} \right] \right\|$$

$$\leq \left\| \begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix} \begin{pmatrix} \varphi_1^*(0) \\ 0 \end{pmatrix} \right\| + |b_{12}^{*1}| \left[\sum_{r=0}^{m_1} |\lambda_1|^{k-1-r} \|\Phi_2(r-m_1)\| + \|\Phi_2(0)\| \sum_{r=m_1+1}^{k-1} |\lambda_1|^{k-1-r} |\lambda_2|^{r-m_1} \right]$$

$$+ |b_{12}^{*2}| \left[\sum_{r=0}^{m_2} |\lambda_1|^{k-1-r} \|\Phi_2(r-m_2)\| + \|\Phi_2(0)\| \sum_{r=m_2+1}^{k-1} |\lambda_1|^{k-1-r} |\lambda_2|^{r-m_2} \right]$$

$$\leq |\lambda_1|^k \|\varphi_1^*(0)\| + |b_{12}^{*1}| \left[\sum_{r=0}^{m_1} |\lambda_1|^{k-1-r} \|\Phi_2(r-m_1)\| \right] + |b_{12}^{*2}| \left[\sum_{r=0}^{m_2} |\lambda_1|^{k-1-r} \|\Phi_2(r-m_2)\| \right]$$

$$\leq q^k \|\varphi^*\|_{m_2} + |b_{12}^{*1}| \left[\sum_{r=0}^{m_1} q^{k-1-r} \|\varphi^*\|_{m_2} \right] + |b_{12}^{*2}| \left[\sum_{r=0}^{m_2} q^{k-1-r} \|\varphi^*\|_{m_2} \right]$$

$$\leq \left[q^k + |b_{12}^{*1}| \sum_{r=0}^{m_1} q^{k-1-r} + |b_{12}^{*2}| \sum_{r=0}^{m_2} q^{k-1-r} \right] \|\varphi^*\|_{m_2}$$

$$\leq \left[q^k + |b_{12}^{*1}| \left(q^{k-1-m_1} \frac{1-q^{m_1+1}}{1-q} \right) + |b_{12}^{*2}| \left(q^{k-1-m_2} \frac{1-q^{m_2+1}}{1-q} \right) \right] \|\varphi^*\|_{m_2}$$

$$= q^k \left[1 + |b_{12}^{*1}| \frac{q^{-m_1-1}-1}{1-q} + |b_{12}^{*2}| \frac{q^{-m_2-1}-1}{1-q} \right] \|\varphi^*\|_{m_2}.$$

Now, it is easy to see that

$$\lim_{k\to\infty} \|y(k)\| = 0.$$

Similarly can be proved the following theorem.

Theorem 3 *If the case II) occurs,* $|\lambda_2| \leq q < 1$, $|\lambda_1| \geq 1$ *and* $\varphi_1^*(0) = 0$*, then the zero solution of* (1) *is conditionally asymptotically stable.*

Theorem 4 *If the case I) occurs,* $|\lambda_1| = 1$, $|\lambda_2| > 1$ *and* $\varphi_2^*(0) = 0$*, then the zero solution of* (1) *is conditionally stable.*

Proof: We, perform the proof similarly to that of Theorem 2. We have, $\varphi^*(0) = (\varphi_1^*(0), 0)^T$ and $\Phi_2(0) = (\varphi_2^*(0), 0)^T = (0,0)$. For $k \in Z_{m_2+2}^{\infty}$, we get

$$\|y(k)\| \leq \|\Lambda_1^k \varphi^*(0)\| + \left\| b_{12}^{*1} \left[\sum_{r=0}^{m_1} \lambda_1^{k-1-r} \Phi_2(r-m_1) + \Phi_2(0) \sum_{r=m_1+1}^{k-1} \lambda_1^{k-1-r} \lambda_2^{r-m_1} \right] \right\|$$

$$+ \left\| b_{12}^{*2} \left[\sum_{r=0}^{m_2} \lambda_1^{k-1-r} \Phi_2(r-m_2) + \Phi_2(0) \sum_{r=m_2+1}^{k-1} \lambda_1^{k-1-r} \lambda_2^{r-m_2} \right] \right\|$$

$$\leq \left\| \begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix} \begin{pmatrix} \varphi_1^*(0) \\ 0 \end{pmatrix} \right\| + |b_{12}^{*1}| \left[\sum_{r=0}^{m_1} |\lambda_1|^{k-1-r} \|\Phi_2(r-m_1)\| + \|\Phi_2(0)\| \sum_{r=m_1+1}^{k-1} |\lambda_1|^{k-1-r} |\lambda_2|^{r-m_1} \right]$$

$$+ |b_{12}^{*2}| \left[\sum_{r=0}^{m_2} |\lambda_1|^{k-1-r} \|\Phi_2(r-m_2)\| + \|\Phi_2(0)\| \sum_{r=m_2+1}^{k-1} |\lambda_1|^{k-1-r} |\lambda_2|^{r-m_2} \right]$$

$$\leq |\lambda_1|^k \|\varphi_1^*(0)\| + |b_{12}^{*1}| \left[\sum_{r=0}^{m_1} |\lambda_1|^{k-1-r} \|\Phi_2(r-m_1)\| \right] + |b_{12}^{*2}| \left[\sum_{r=0}^{m_2} |\lambda_1|^{k-1-r} \|\Phi_2(r-m_2)\| \right]$$

$$\leq |\lambda_1|^k \|\varphi_1^*(0)\| + |b_{12}^{*1}| \left[\sum_{r=0}^{m_1} |\lambda_1|^{k-1-r} \|\Phi_2(r-m_1)\| \right] + |b_{12}^{*2}| \left[\sum_{r=0}^{m_2} |\lambda_1|^{k-1-r} \|\Phi_2(r-m_2)\| \right]$$

$$\leq \|\varphi^*\|_{m_2} + |b_{12}^{*1}| \left[\sum_{r=0}^{m_1} + |b_{12}^{*2}| \left[\sum_{r=0}^{m_2} \|\varphi^*\|_{m_2} \right] \right.$$

$$\leq \left[1 + |b_{12}^{*1}|(m_1+1) + |b_{12}^{*2}|(m_2+1) \right] \|\varphi^*\|_{m_2}.$$

We set

$$M := 1 + |b_{12}^{*1}|(m_1+1) + |b_{12}^{*2}|(m_2+1), \ \delta := \varepsilon/3.$$

This equality implies

$$\|y(k)\| \leq M\|\varphi^*\|_{m_2} < \varepsilon, k \in Z_{m_2+2}^{\infty}$$

if $\|\varphi^*\|_{m_2} < \delta$.

Theorem 5 *If the case II) occurs,* $|\lambda_2| = 1$, $|\lambda_1| > 1$ *and* $\varphi_1^*(0) = 0$, *then the zero solution of* (1) *is conditionally stable.*

The proof can be performed similarly to that of Theorem 4.

3 CONCLUSION

In the paper are derived sufficient conditions for conditional stability and asymptotic conditional stability of linear weakly delayed discrete systems (1) when the Jordan form of the matrix A is represented by the matrix Λ_1 defined by (10). For further results related to weakly delayed systems and representations of solutions of discrete systems we refer to [1]–[6] and to the references therein. Some stability results can be found, e.g. in [7].

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