Compensation-Based Game for Spectrum Sharing in the Gaussian Interference Channel

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Abstract. *This paper considers an optimization problem of sum-rate in the Gaussian frequency-selective channel. We construct a competitive game with an asymptotically optimal compensation to approximate the optimization problem of sum-rate. Once the game achieves the Nash equilibrium, all users in the game will operate at the optimal sum-rate boundary. The contributions of this paper are twofold. On the one hand, a distributed power allocation algorithm called iterative multiple waterlevels water-filling algorithm is proposed to efficiently achieve the Nash equilibrium of the game. On the other hand, we derive some sufficient conditions on the convergence of iterative multiple waterlevels water-filling algorithm in this paper. Through simulation, the proposed algorithm has a significant improvement of the performance over iterative water filling algorithm and achieves the close-to-optimal performance.*

Keywords

Interference channel, iterative water filling algorithm (IWFA), power allocation, game theory.

1. Introduction

The rate region of Gaussian interference channel (IC) is an open problem for almost thirty years in multiuser information theory. The best rate region up to date is the famous Han and Kobayashi (HK) region [1] proposed in 1981. The multiuser Gaussian frequency-selective IC is a special case of Gaussian IC which can be frequently encountered in the wireless/wire communication (e.g., OFDM and DSL). So the optimal code design is also a problem for the Gaussian frequency-selective IC. There have been many works devoted to the improvement of the rate region in the Gaussian frequency-selective IC. Literatures [2]-[5] discussed the use of cooperative game theory for analyzing the Gaussian frequency-selective IC with centralized control to compute the largest achievable rate region of the system (i.e., Nash bargain solution of the achievable rate-tuples). However, the coordination is usually unpractical in many scenarios.

The dynamic spectrum management is also a common approach to improve the sum-rate of the Gaussian frequency-selective IC. In 2002, Yu [6] proposed the iterative water-filling algorithm (IWFA) for computing power allocation (i.e., Nash equilibrium (NE)) which was one of the first dynamic spectrum management algorithms in the frequency-selective Gaussian IC. Due to the non-cooperative behavior and the selfish-optimum, it is a sub-optimal solution to the Gaussian frequency-selective IC. In 2004, Cendrillon proposed an optimal spectrum management [7] in the DSL scenario. However, it has a high complexity which makes it computationally intractable. Then, a simplified optimal spectrum balancing was derived in [8]. Several distributed algorithms [9]-[13] were proposed, whose performances are near-optimal.

In this paper, we consider the optimization problem of sum-rate in the Gaussian frequency-selective IC. We find that the optimization problem can be approximated as a competitive game with an asymptotically optimal compensation (called main game). The calculation of the optimal compensation is a big concern. In this paper, we model the compensation as a game model (called sub-game in this paper) and the NE of the sub-game can be computed by an iterative method. After the sub-game obtains the NE (i.e. the optimal compensation is reached), we start to calculate the NE of the main competitive game, at which the plays in the game can operate at the optimal rate frontier. The main contribution of the paper is the construction of the game model to solve the optimization problem of sum-rate. Furthermore, a distributed power allocation algorithm called iterative multiple waterlevels water-filling algorithm (IML-WFA) is proposed to achieve the NE of the main game and some sufficient conditions on the convergence of IML-WFA are derived in this paper. When compared with IWFA, the IML-WFA has a significant improvement of the performance and can achieve a close-to-optimal performance. Moreover, a simplified version of our algorithm is derived and it directly leads to an extended autonomous spectrum balancing (ASB) [10]-[12] with the multiple reference lines fashion.

The remainder of the paper is organized as follows. In Section 2, the system model for the Gaussian frequency-selective IC is formulated. In Section 3, we construct a new game model for the sum-rate maximum game. Then a distributed algorithm is derived for the dynamic power allocation. In Section 4, numerical examples are presented. Conclusions are drawn in Section 5.

2. System Model

This paper considers the Gaussian frequency-selective IC as applied to discrete multi-tone (DMT) technique [14]. Then, the Gaussian frequency-selective IC is changed to a set of parallel Gaussian ICs. Suppose the number of total users in the parallel Gaussian ICs is N and the number of subchannels of each user is K. Then, we index the users by $\mathrm{N} = \{1, 2,\ldots, N\}$and label these K subchannels by $\mathrm{K} = \{1, 2,\ldots, K\}$. So, the transmission at each tone can be modeled independently

$$\boldsymbol{y}(k)=\boldsymbol{C}(k)\boldsymbol{x}(k)+\boldsymbol{n}(k), k\in \mathrm{K} \tag{1}$$

where $\boldsymbol{x}(k) = [x_1(k),\ldots, x_N(k)]^T$ is the transmit signals at tone k. $x_i(k)$, $i \in \mathrm{N}$ is the signal transmit by user i at tone k. $\boldsymbol{y}(k) = [y_1(k),\ldots, y_N(k)]^T$ is the received signals at tone k. $y_i(k)$, $i \in \mathrm{N}$ is the signal received by user i at tone k. $\boldsymbol{n}(k) = [n_1(k),\ldots, n_N(k)]^T$ is the vector of additive noise at tone k. $n_i(k)$, $i \in \mathrm{N}$ is the additive noise with zero mean and variance $\sigma_i^2(k)$ seen by user i at tone k. $\boldsymbol{C}(k)$ is the $N\times N$ channel transfer matrix at tone k, whose (m,n)th entry is defined as $c_{mn}(k)$. $c_{mn}(k)$ denote the interference coefficient from user n to user m.

We define a matrix $\boldsymbol{A}(k)$ called channel gain matrix whose entry is the square of the corresponding entry in matrix $\boldsymbol{C}(k)$, i.e., $a_{mn}(k) = [c_{mn}(k)]^2$. We denote the vector containing allocated power of user $i \in \mathrm{N}$ at all tones as $\boldsymbol{p}_i = [p_i(1),\ldots, n_i(K)]^T$. The set of all feasible system power vectors of user i is defined as,

$$\mathrm{P}_i=\left\{\boldsymbol{p}_i \,\middle|\, \sum_{k=1}^{K} p_i(k)=P_i, p_i(k)\geq 0, k\in \mathrm{K}\right\}$$

where P_i is the total power constraint for user i.

Based on the above system model, we simply consider the interference as noise, then the achievable rate of user $i \in \mathrm{N}$ at tone k is,

$$r_i(k)=\log\left(1+\frac{a_{ii}(k)p_i(k)}{\sigma_i^2(k)+\sum_{j\neq i}a_{ij}(k)p_j(k)}\right). \tag{2}$$

Then, the achievable rate of user $i \in \mathrm{N}$ is,

$$R_i(\boldsymbol{p})=\sum_{k=1}^{K}r_i(k) \tag{3}$$

where $\boldsymbol{p} = [\boldsymbol{p}_1^T, \boldsymbol{p}_2^T,\ldots, \boldsymbol{p}_N^T]^T$.

Each user is subject to a total power constraint,

$$\sum_{k=1}^{K}p_i(k)\leq P_i, i\in \mathrm{N} \ . \tag{4}$$

3. Problem Formulation and Distributed Power Allocation Algorithm

The basic component of game theory is a game $\mathrm{G} = \{\mathrm{M}, \mathrm{S}, \{u_i, i \in \mathrm{M}\}\}$. M is the set of players. S_i is the set of strategy for player $i \in \mathrm{M}$, $\mathrm{S} = \mathrm{S}_1 \times \mathrm{S}_2 \times\ldots\mathrm{S}_N$ is the set of strategy profile of all players. u_i is the utility function which player i wishes to maximize. In our notion, M is equivalent to N. S_i is equivalent to P_i which consists of all feasible power vectors $\boldsymbol{p}_i$. u_i is equivalent to R_i.

From [7] and [8], the optimal spectrum management problem is defined as,

$$\begin{aligned}&\max \sum_{i\in \mathrm{N}} R_i(\boldsymbol{p})\\ &s.t.\ \ p_i(k)\geq 0, \forall i\in \mathrm{N}\ , \forall k\in \mathrm{K}\ .\\ &\sum_{k=1}^{K}p_i(k)=P_i, \forall i\in \mathrm{N}\end{aligned} \tag{5}$$

It is obvious that (5) can be regarded as a sum-rate maximum game among all users. Furthermore, (5) can be rewritten as follows,

$$\begin{aligned}&\max\left\{R_i(\boldsymbol{p})+\sum_{\substack{j\in \mathrm{N}\\ j\neq i}}R_j(\boldsymbol{p})\right\}\\ &s.t.\ \ p_i(k)\geq 0, \forall i\in \mathrm{N}\ , \forall k\in \mathrm{K}\\ &\sum_{k=1}^{K}p_i(k)=P_i, \forall i\in \mathrm{N}\end{aligned} \tag{6}$$

Then (6) can be considered as the competitive game (main game) with the compensation. The first term in (6) is the utility function user i wishes to maximize selfishly and the second term in (6) is the optimal compensation term for user i. Thus, we can generalize the following competitive game model with compensation.

$$\begin{aligned}&\max\left\{\tilde{R}_i(\boldsymbol{p})\right\}\\ &s.t.\ \ p_i(k)\geq 0, \forall i\in \mathrm{N}\ , \forall k\in \mathrm{K}\\ &\sum_{k=1}^{K}p_i(k)=P_i, \forall i\in \mathrm{N}\end{aligned} \tag{7}$$

where $\tilde{R}_i(\boldsymbol{p})=R_i(\boldsymbol{p})+\Gamma_i(\boldsymbol{p})$, $\Gamma_i(\boldsymbol{p})$ is the compensation function for user i.

Determining the compensation function is a big problem for us to construct the competitive game (7). If we choose the optimal compensation of (6), then the game turns back to the optimal spectrum management problem defined in (5). Therefore, we propose an approach to obtain the compensation by an asymptotically optimal fashion. The main thought is shown as follows.

We suppose the user i is the current user and fix the other users' transmit power allocation. Then, (7) can be modeled as,

$$\begin{aligned}&\max\left\{\tilde{R}_i(\boldsymbol{p}_i)\right\}\\ &s.t.\ \ p_i(k)\geq 0, \forall k\in \mathrm{K}\\ &\sum_{k=1}^{K}p_i(k)=P_i\end{aligned} \tag{8}$$

To obtain the asymptotically optimal compensation, we set the derivative of the optimal compensation term given in (6) with respect to $p_i(k)$,

$$\alpha_i\left(p_i(k)\right)=\frac{\partial \sum_{\substack{j\in \mathrm{N}\\ j\neq i}} R_j(\boldsymbol{p})}{\partial p_i(k)}$$

$$=\sum_{j\neq i}\frac{-a_{ji}(k)a_{jj}(k)p_j(k)}{\left(\sigma_j^2(k)+\sum_{m\neq j}a_{jm}(k)p_m(k)\right)^2+a_{jj}(k)p_j(k)\left(\sigma_j^2(k)+\sum_{m\neq j}a_{jm}(k)p_m(k)\right)} \quad (9)$$

We assume the difference between the current power allocation and the last power allocation is small, then we can replace the $p_i(k)$ with $\hat{p}_i(k)$ in (9), where $\hat{p}_i(k)$ denote the last power allocation for user i. Then (9) is a constant in the current time, and its integration is,

$$\Gamma_i\left(p_i(k)\right)=\alpha_i(k)p_i(k)\cong\Gamma_i(k) \quad (10)$$

where

$$\alpha_i(k)=\sum_{j\neq i}\frac{-a_{ji}(k)a_{jj}(k)p_j(k)}{\left(\sigma_j^2(k)+a_{ji}(k)\hat{p}_i(k)+\sum_{\substack{m\neq j\\ m\neq i}}a_{jm}(k)p_m(k)\right)^2+a_{jj}(k)p_j(k)\left(\sigma_j^2(k)+a_{ji}(k)\hat{p}_i(k)+\sum_{\substack{m\neq j\\ m\neq i}}a_{jm}(k)p_m(k)\right)} \quad (11)$$

From (8) and (10), we can formulate the following new game model (called sub-game in this paper) whose set of players is all tones of user i,

$$\begin{aligned}&\max\left\{\tilde{R}_i(\boldsymbol{p}_i)\right\}\\ &s.t.\ \ p_i(k)\geq 0,\forall k\in \mathrm{K}\\ &\sum_{k=1}^{K}p_i(k)=P_i\end{aligned} \quad (12)$$

where $\tilde{R}_i(\boldsymbol{p}_i)=R_i(\boldsymbol{p}_i)+\Gamma_i(k)$

It is obvious that (12) is a convex game. The NE to sub-game (12) is defined as follows, and by an iterative fashion such as the IWFA, the NE to (12) can be reached. Thus, the asymptotically optimal compensation $\Gamma_i(k)$ is also obtained.

Definition 1: Given an initial power allocation of all users and a channel gain matrix, the power allocation $\tilde{p}_i=[\tilde{p}_i(1),\tilde{p}_i(2),\ldots,\tilde{p}_i(K)]^T$ is a NE of the sub-game (12), if the following inequality holds,

$$\tilde{R}_i(\tilde{\boldsymbol{p}}_i)\geq\tilde{R}_i(\hat{\boldsymbol{p}}_i)$$

$$\hat{\boldsymbol{p}}_i=\left[\tilde{p}_i(1),\tilde{p}_i(2),\ldots\tilde{p}_i(m-1),p_i(m),\tilde{p}_i(m+1),\ldots,\tilde{p}_i(K)\right]\in \mathrm{P}_i$$

At each iteration, how can we obtain the maximum of (12)? The following theorem gives an approach to the problem.

Theorem 1: Given the power allocation of all other users, a channel gain matrix, and a sub-game defined in (12). The optimal power allocation of user i can be reached via multiple waterlevels water filling algorithm (ML-WFA).i.e.,

$$p_i(k)=\left(\mu_i(k)-\phi_i(k)\right)^+ \quad (13)$$

where

$$\mu_i(k)=\frac{1}{-\alpha_i(k)+\lambda_i} \quad (14)$$

which is referred to as the waterlevel.

$$\phi_i(k)=\left(\sigma_i^2(k)+\sum_{j\neq i}a_{ij}(k)p_j(k)\right)\Big/a_{ii}(k) \quad (15)$$

which is referred to as the normalized interference-plus-noise for user i at tone k.

The proof of **Theorem 1** is included in the Appendix.

Theorem 2: Given the power allocation of all other users, a link gain matrix and a sub-game model among all tones of user i defined in (12), the NE to the optimization problem (12) exists.

Proof From [15]–[17], if the following conditions are satisfied, in our notation, it says, if for $k = 1,\ldots, K$

i, P_i is compact and convex.

ii, $\tilde{R}_i$: $\mathrm{P}_i\rightarrow\mathbb{R}^+$ is a continuous function in $\boldsymbol{p}_i$.

iii, $\tilde{R}_i$ is convex function in $\boldsymbol{p}_i$.

Then the game is convex game and the NE exists.

The **Theorem 2** guarantees the existence of the NE in the sub-game (12).

Then we recall the main game model with a compensation defined in (7) and give the following NE [18] definition.

Definition 2: Given a link gain matrix, the power allocation $\boldsymbol{p}^*=[\boldsymbol{p}_1^*,\boldsymbol{p}_2^*,\ldots,\boldsymbol{p}_N^*]$ is a NE of the game defined in (7), if the following inequality holds:

$$\begin{aligned}&\tilde{R}_i\left(NE\left(\boldsymbol{p}_{-1}^*\right),NE\left(\boldsymbol{p}_{-2}^*\right),\ldots,NE\left(\boldsymbol{p}_{-N}^*\right)\right)\\ &\geq\tilde{R}_i\left(NE\left(\boldsymbol{p}_{-1}^*\right),NE\left(\boldsymbol{p}_{-2}^*\right),\ldots NE\left(\boldsymbol{p}_{-i+1}^*\right),NE\left(\boldsymbol{p}_{-i}\right),NE\left(\boldsymbol{p}_{-i-1}^*\right),\ldots NE\left(\boldsymbol{p}_{-N}^*\right)\right)\\ &\boldsymbol{p}^*,\boldsymbol{p}\in\mathrm{P}\end{aligned} \quad (16)$$

where $\boldsymbol{p}_{-i}=[\boldsymbol{p}_1,\ldots,\boldsymbol{p}_{i-1},\boldsymbol{p}_{i+1},\ldots,\boldsymbol{p}_N]^T$, $i\in\mathrm{N}$, $NE(\boldsymbol{p}_{-i})$ is the Nash equilibrium strategy of the player i if the remaining players chooses to play $\boldsymbol{p}_{-i}$ i.e., $NE(\boldsymbol{p}_{-i})=\boldsymbol{p}_i$ which can be obtained by sub-game (7).

$$\mathrm{P}=\left\{\boldsymbol{p}\in\square^{N\times K}\ \middle|\ \begin{aligned}&\boldsymbol{p}=\left[\boldsymbol{p}_1^T,\boldsymbol{p}_2^T,\ldots,\boldsymbol{p}_N^T\right]^T,\\ &\boldsymbol{p}_i=\left[p_{i,1},p_{i,2},\ldots,p_{i,K}\right]^T,i\in\mathrm{N}\\ &\sum_{k=1}^{K}p_i(k)=P_i,p_i(k)\geq 0,k\in\mathrm{K}\end{aligned}\right\}$$

is the set of all feasible system power vectors.

Obviously, the NE defined in **Definition 2** can also be reached by an iterative fashion. Now, we give a theorem to guarantee the existence of the NE of the game defined in (7).

Theorem 3: Given a power allocation of NE of user $i \in \mathrm{N}$, a link gain matrix and a competitive game model with a compensation defined in (7), the NE to the game (7) exists.

Proof The method of proof is similar to Theorem 2.

From above discussion, a distributed power algorithm (i.e., IML-WFA) is proposed to achieve the NE of the game defined in (7). The IML-WFA can be divided into two parts. One is the inner iteration (sub-game) and the other is the outer iteration (competitive game model with a compensation).The IML-WFA works as follows: with the total power constraint, the first user updates its power allocation and compensation function by deriving a ML-WFA solution (see **Theorem 1**) while considering the interference powers of itself and the other users as noises and the first user continues to do the multiple waterlevels water-filling algorithm until the process (called inner iteration) converges. The process is then successively applied to the second user, the third user, etc., until the total process (called outer iteration) converges. By definition, it has to converge to a NE of the game in (7). The algorithm is summarized in **Algorithm 1** and a simplified illustration is shown in Fig. 1.

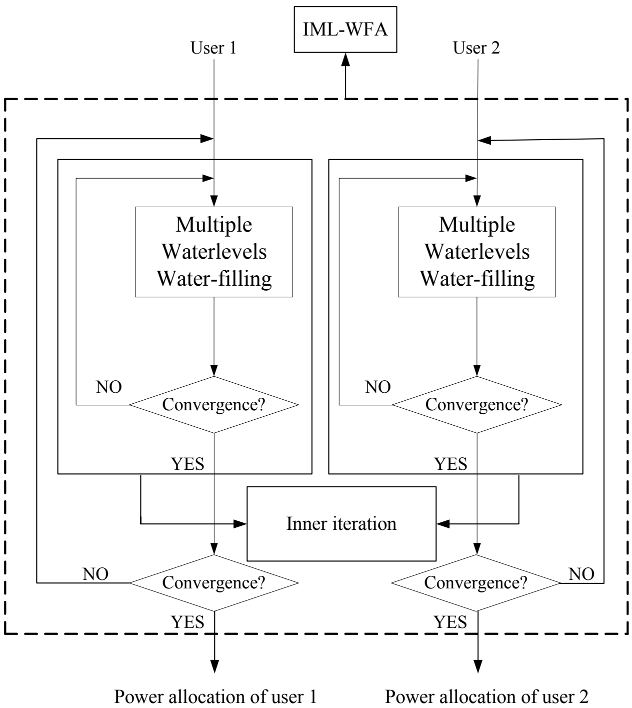

Fig. 1. A simplified illustration of IML-WFA (two-user case).

In **Algorithm 1,** *ML-WFA*(*,*,*) denotes the ML-WFA function which achieves the multiple levels water filling and returns the power allocation. The detail of ML-WFA function is shown in **Sub-Algorithm 1** and the derivation is included the proof of **Theorem 1** which is included in the Appendix.

Algorithm 1 (Distributed Power Allocation)

1 **Initialize** $p_i(k) = 0, \forall i \in \mathrm{N}, k \in \mathrm{K}$
and power constraint P_i
2 **Repeat**
3 **For** m=1 to N
4 **Repeat**
5 **For** n=1 to K
6 Compute $\phi_m(n)$ by (14)
7 Compute $\alpha_m(n)$ by (11)
8 **End**
9 $\boldsymbol{\phi}_m = \left[\phi_m(1), \phi_m(2), ..., \phi_m(K)\right]^T$,
$\boldsymbol{\alpha}_m = \left[\alpha_m(1), \alpha_m(2), ..., \alpha_m(K)\right]^T$
10 $\boldsymbol{p}_m = ML\text{-}WFA(\boldsymbol{\alpha}_m, \boldsymbol{\phi}_m, P_m)$
11 **Until convergence**
11 **End**
12**Until convergence**

The complexity of the proposed algorithms can be separated into two parts. One is the complexity of the inner iteration and the other is the complexity of the outer iteration.

Sub-Algorithm 1 (ML-WFA function)

$\boldsymbol{p}_m = ML\text{-}WFA(\boldsymbol{\alpha}_m, \boldsymbol{\phi}_m, P_m)$
1 Compute $\lambda_m^{ub} = \min\{\boldsymbol{\alpha}_m\}$
2 Compute $\lambda_m^{lb} = \max\{1/\boldsymbol{\phi}_m + \boldsymbol{\alpha}_m\}$
3 Compute $\lambda^0 = \left(\lambda_m^{ub} + \lambda_m^{lb}\right)/2$
4 **For** $n = 1$ *to* K
5 Compute $g(n) = 1/\left(-\alpha_m(n) + \lambda^0\right) - \phi_m(n)$
6 **If** $g(n) < 0$ **Then** $g(n) = 0$ **End**
7 **End**
8 Compute $g = g(1) + g(2) + .. + g(K) - P_m$
9 **While** $|g| > \varepsilon$ where ε is a small number
10 **If** $g > 0$ **Then** $\lambda_m^{lb} = \lambda^0$ **Else** $\lambda_m^{ub} = \lambda^0$ **End**
11 Update $\lambda^0 = \left(\lambda_m^{ub} + \lambda_m^{lb}\right)/2$
12 Update g by repeating step 4 to 8
13 **End**
14 Compute
$\boldsymbol{\mu}_m = 1/\left(-\boldsymbol{\alpha}_m + \lambda^0\right)$,
where $\boldsymbol{\mu}_m = \left[\mu_m(1), \mu_m(2), ..., \mu_m(K)\right]^T$
15 Compute $\boldsymbol{p}_m = \left(\boldsymbol{\mu}_m - \boldsymbol{\phi}_m\right)^+$

Since the inner iteration is the single-user water-filling process (which is a convex optimization process and has a computational complexity of order $O(K)$) and all the other users' compensations caused by current user at each tone in the water-filling process need to be computed, hence the inner iteration has a computational complexity of order $O(KN)$. Then the inner iteration process is applied to the N users resulting in an outer iteration process. Therefore the IML-WFA has a computational complexity of order $O(KN^2)$. The complexity of the proposed algorithms is equivalent to the complexity of iterative spectrum balancing (ISB) [9].

From the complexity analysis above, we can find that if there are many users in the Gaussian frequency-selective IC, the algorithm will be very complex because we have to compute all the other users' compensations at each tone. A natural idea for simplifying the algorithm is to use the best compensation to approximate the total compensation that the current user must receive at each tone. Then we have the following definition for the current user i,

$$\hat{a}_i(k) = \max_{m\in\mathcal{N}, m\neq i}\{a_{mi}(k)\}, k \in \mathrm{K}$$
$$w_k = \arg\left\{\max_{m\in\mathcal{N}, m\neq i}\{a_{mi}(k)\}\right\}, w_k \in N \quad (17)$$

Then, (10) can be approximated as follows,

$$\hat{\alpha}_i(k) = \frac{\hat{a}_i(k) a_{w_k w_k}(k) p_{w_k}(k)}{\left(\sigma^2_{w_k}(k) + \hat{a}_i(k) p_i(k)\right)^2 + a_{w_k w_k}(k) p_{w_k}(k)\left(\sigma^2_{w_k}(k) + \hat{a}_i(k) p_i(k)\right)}. \quad (18)$$

By the simplification of the compensations, the complexity is reduced to $O(KN)$ which is equivalent to the complexity of ASB [10]-[12]. The result given in (18) is similar with the result given in [11], [12, Section IV.*C*, Equ. (26)]. But the difference is obvious. In literature [11], [12], they choose a uniform reference line as the penalty the current user must suffer from and in this paper, we choose different reference line at different tone as the complemetarity the current user must pay for. So the result given in [11], [12] can be regard as the simplified form of our result given in (18) which can be considered as the extended ASB with the multiple reference lines fashion. Next, we will present some sufficient conditions for the convergence of the IML-WFA.

Theorem 4: If the channel gain matrices satisfy the condition,

$$\rho\left\{\max_{k\in\mathrm{K}}\{\boldsymbol{A}(k)\}\right\} < 1 \quad (19)$$

where $\rho(*)$ denotes the spectral radius of the matrix and the spectral radius is the maximal absolute value of eigenvalues. Then the distributed power algorithm (i.e., IML-WFA) converges.

The proof of **Theorem 4** is included in the Appendix.

Corollary 1: If the channel gain matrices satisfy any one of the following conditions,

i), $$\sum_{i\neq j}\max_{k\in\mathrm{K}}\left\{\left[\boldsymbol{A}(k)\right]_{ij}\right\} < 1, \quad (20)$$

ii), $$\sum_{j\neq i}\max_{k\in\mathrm{K}}\left\{\left[\boldsymbol{A}(k)\right]_{ij}\right\} < 1, \quad (21)$$

iii), $$\max_{k,i,j}\left\{\left[\boldsymbol{A}(k)\right]_{ij}\right\} < \frac{1}{N-1} \quad (22)$$

then the distributed power algorithm (i.e., IML-WFA) converges.

Proof From [21], we have the fact that,

$$\left\|\max_{k\in\mathrm{K}}\{\boldsymbol{A}(k)\}\right\| > \rho\left\{\max_{k\in\mathrm{K}}\{\boldsymbol{A}(k)\}\right\}$$

which holds for any matrix norm. So if $\left\|\max_{k\in\mathrm{K}}\{\boldsymbol{A}(k)\}\right\|_1 < 1$, then $\rho\left\{\max_{k\in\mathrm{K}}\{\boldsymbol{A}(k)\}\right\} < 1$.

Hence, $\left\|\max_{k\in\mathrm{K}}\{\boldsymbol{A}(k)\}\right\|_1 < 1$ then IML-WFA converges. It is equivalent to the following condition

$$\sum_{i\neq j}\max_{k\in\mathrm{K}}\left\{\left[\boldsymbol{A}(k)\right]_{ij}\right\} < 1$$

where $[*]_{ij}$ denote the (i,j) th entry in the matrix. Similarly, $\sum_{j\neq i}\max_{k\in\mathrm{K}}\left\{\left[\boldsymbol{A}(k)\right]_{ij}\right\} < 1$, then the IML-WFA converges. If we define a $N\times N$ matrix $\boldsymbol{B}$ whose diagonal entries are zero and off-diagonal entries are $1/(N-1)$. Then $\rho(\boldsymbol{B}) = 1$. If $[\boldsymbol{A}(k)]_{ij} < 1/(N-1)$, $\forall i,j,k$, then $\max_{k\in\mathrm{K}}\{\boldsymbol{A}(k)\} < \boldsymbol{B}$. It is obvious that $\rho\left\{\max_{k\in\mathrm{K}}\{\boldsymbol{A}(k)\}\right\} < \rho\{\boldsymbol{B}\} = 1$, then the IML-WFA converges.

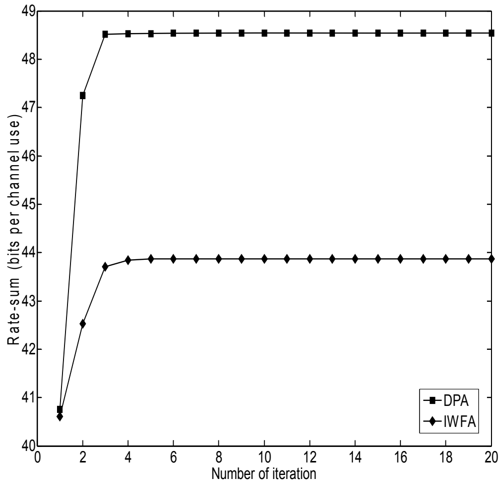

Fig. 2. Total throughput comparison of DPA and IWFA.

4. Numerical Result

In this section, we summarize the numerical examples comparing the performance of our distributed power allocation algorithm (DPA) with IWFA [6], autonomous spectrum balancing (ASB) [10]-[12] and optimal spectrum balancing (OSB) [8] in a four-user Gaussian frequency-selective IC. The channel is divided into 100 subchannels. The power constraint is $0 \le P_1, P_2 \le 20$. The total throughput is shown in Fig. 2. There is a significant improvement on sum-rate in DPA compared with IWFA. Fig. 3 illustrates the convergence of compensation function. It is clear that with the increasing of iterative number, the compensation function tends to a certain number which is the optimal compensation for the utility function of the competitive game. From Fig. 2, we find that the IML-WFA converges at the fourth iteration. So in Fig. 3, the compensation function is almost a horizontal line. By running through all possible profiles of power, we get the achievable rate region as shown in Fig. 4, which shows that DPA achieves near optimal performance similar as OSB and ASB, and significant gains over IWFA. This simulation coincides with the analysis in (18) that our algorithm is the extended version of ASB. We also perform many simulations with more than two users with different scenarios. We find that there are more users in the scenario, then the algorithm gains more performance improvement over IWFA.

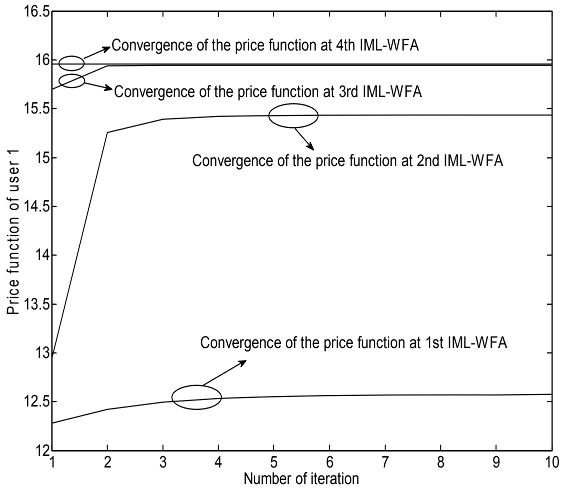

Fig. 3. Convergence of price function.

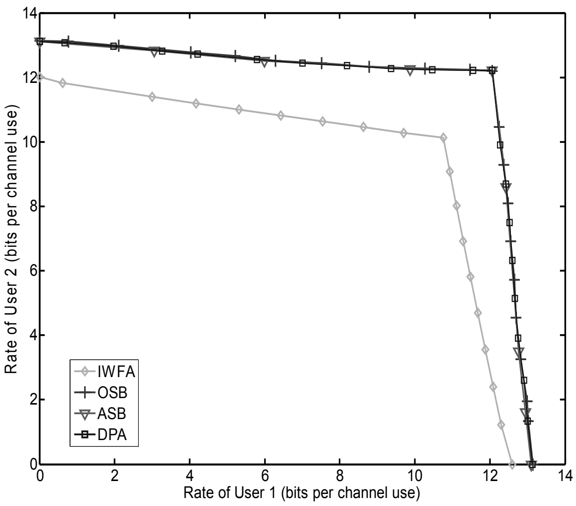

Fig. 4. Rate Region.

5. Conclusion

This paper presents the distributed power allocation algorithm based on the competitive game with an asymptotically optimal compensation. Through an iterative method, we can calculate the compensation with an asymptotically optimal fashion. Then a distributed power allocation algorithm is proposed to reach the NE of the main competitive game, which can be implemented in low complexity in the Gaussian frequency-selective IC. We also find that the algorithm extends the ASB to the multiple reference lines fashion. By simulation, DPA exhibits the close-to-optimal performance and works near the rate region frontier. Because it is a distributed algorithm, it can be used in the Gaussian frequency-selective IC scenario (e.g. OFDM, DSL and ad-hoc network) with less or without the help of centralized control.

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Appendix

Proof of **Theorem 1**

Proof The Lagrangian function of (12) is defined as follows,

$$\tilde{L}_i(\boldsymbol{p}_i) = -\tilde{R}_i + \lambda_i\left(\sum_{k=1}^{K} p_i(k) - P_i\right). \quad (23)$$

We define,

$$\phi_i(k) = \left(\sigma_i^2(k) + \sum_{j\neq i} a_{ij}(k)\, p_j(k)\right)\Big/ a_{ii}(k). \quad (24)$$

The Karush-Kuhn-Tucker (KKT) conditions are given by,

$$-\frac{\partial \tilde{R}_i}{\partial p_i(k)} + \lambda_i = 0\,, \quad (25)$$

$$-\frac{\partial\left\{\sum_k \log\left(1+\frac{p_i(k)}{\phi_i(k)}\right) + \alpha_i(k)\, p_i(k)\right\}}{\partial p_i(k)} + \lambda_i = 0, \quad (26)$$

$$\frac{-1}{\phi_i(k)+p_i(k)} - \alpha_i(k) + \lambda_i = 0\,, \quad (27)$$

$$p_i(k) = \left(\frac{1}{-\alpha_i(k)+\lambda_i} - \phi_i(k)\right)^+, \quad (28)$$

Let
$$\mu_i(k) = \frac{1}{-\alpha_i(k)+\lambda_i} \quad (29)$$

which is called waterlevel for user i at tone k.

Then (28) can be rewritten as,

$$p_i(k) = \left(\mu_i(k) - \phi_i(k)\right)^+ \quad (30)$$

It is a multiple waterlevels water filling algorithm (ML-WFA). Then we have proven the **Theorem 2.**

Next, we pay our attention to the efficient implementation of ML-WFA. From [19], it is obvious that two constraint functions impose on the waterlevel

$$\mu_i(k), k \in \mathrm{K}.$$

One is the single waterlevel constrain function which comes from (29),

$$f_k(\mu_i(k)) = \frac{1}{\mu_i(k)} + \alpha_i(k) = \lambda_i, k \in \mathrm{K}. \tag{31}$$

Using the total power constraint, the other constraint function is obtained,

$$g(\{\mu_i(k), k \in \mathcal{K}\}) = \sum_{k\in\mathcal{K}} \mu_i(k) - \sum_{k\in\mathcal{K}} \phi_i(k) - P_i = 0. \tag{32}$$

Since $p_i(k) \geq 0$, then we bound the waterlevel as follows,

$$\phi_i^{max} \leq \mu_i(k) < \infty, \forall k \in \mathrm{K} \tag{33}$$

where $\phi_i^{max} = \max_{k\in\mathrm{K}} \{\phi_i(k)\}$.

Due to the monotonicity of (31), we can bound the λ_i by (33),

$$\lambda_i^{lb} = \max_{k\in\mathrm{K}} \{f_k(\phi_i^{max})\} \leq \lambda_i \leq \min_{k\in\mathrm{K}} \{f_k(\infty)\} = \lambda_i^{ub}. \tag{34}$$

Equation (32) can be rewritten as follows,

$$g(\{\mu_i(k), k \in \mathrm{K}\}) = \sum_{k\in\mathrm{K}} \mu_i(k) - \sum_{k\in\mathrm{K}} \phi_i(k) - P_i = 0$$

$$\Rightarrow g(\lambda_i) = \sum_{k\in\mathrm{K}} \left(\frac{1}{\alpha_i(k) + \lambda_i} \right) - \sum_{k\in\mathrm{K}} \phi_i(k) - P_i = 0. \tag{35}$$

So, the rest work is to find the root of (35) between λ_i^{lb} and λ_i^{ub}. This can be easily implemented via the bisection method and the ML-WFA is summarized in **Sub-Algorithm 1**.

Proof of **Theorem 4**

Proof The utility function for the user i is,

$$\begin{aligned} \tilde{R}_i(\boldsymbol{p}) &= \sum_k \log\left(1 + \frac{p_i(k)}{\phi_i(k)}\right) - \alpha_i(k) p_i(k) \\ s.t.\ & p_i(k) \geq 0, \forall i \in \mathrm{N}, \forall k \in \mathrm{K} \\ & \sum_{k=1}^{K} p_i(k) = P_i, \forall i \in \mathrm{N} \end{aligned} \tag{36}$$

where $$\phi_i(k) = \left(\sigma_i^2(k) + \sum_{j\neq i} a_{ij}(k) p_j(k) \right) \Big/ a_{ii}(k). \tag{37}$$

Equation (37) can be expressed in matrix form [20] as follows

$$\boldsymbol{\phi} = \boldsymbol{G}\boldsymbol{p} + \boldsymbol{\sigma} \tag{38}$$

where $\boldsymbol{\phi} = [\boldsymbol{\phi}_1^T, \boldsymbol{\phi}_2^T, \ldots, \boldsymbol{\phi}_N^T]^T$ is an M dimensional column vector ($M = K \times N$), $\boldsymbol{\phi}_i = [\phi_i(1), \phi_i(2), \ldots, \phi_i(K)]^T$, $i \in \mathrm{N}$ is a K dimensional column vector. $\boldsymbol{\sigma}$ is an M-dimensional column vector. $\boldsymbol{p} = [\boldsymbol{p}_1^T, \boldsymbol{p}_2^T, \ldots, \boldsymbol{p}_N^T]^T$ is an M dimensional column vector. $\boldsymbol{p}_i = [\boldsymbol{p}_i(1), \boldsymbol{p}_i^T(2), \ldots, \boldsymbol{p}_i(K)]^T$ is a K dimensional column vector and $\boldsymbol{G}$ is an $M \times M$ matrix.

In general, $\boldsymbol{G}$ is a partitioned matrix with zero diagonal blocks. The (i,j) th block is a $K \times K$ diagonal matrix.

From (29), we mark the maximum of (29) as

$$\bar{\mu}_i = \max_{k\in\mathrm{K}} \{\mu_i(k)\}. \tag{39}$$

Then, the $\mu_i(k)$ can be formulated as

$$\bar{\mu}_i = \mu_i(k) - \delta_i(k), k \in \mathrm{K} \tag{40}$$

where $\delta_i(k)$ is defined as the adjustment variable for waterlevel at each tone. Then, it is straightforward to check that,

$$p_i(k) = \bar{\mu}_i - \phi_i(k) - \delta_i(k), \tag{41}$$

$$\bar{\mu}_i = \left(P_i + \sum_{k=1}^{K} (\phi_i(k) + \delta_i(k)) \right) \Big/ K. \tag{42}$$

Then, (41) can be expressed as,

$$p_i(k) = \frac{1}{K}\sum_{k=1}^{K} \phi_i(k) - \phi_i(k) + \frac{P_i}{K} + \frac{1}{K}\sum_{k=1}^{K} \delta_i(k) - \delta_i(k), k \in \mathrm{K}. \tag{43}$$

Equation (43) can be formulated as matrix form,

$$\boldsymbol{p}_i = \boldsymbol{Q}_i \boldsymbol{\phi}_i + \boldsymbol{d} \tag{44}$$

where $\boldsymbol{Q}_i \in \mathbb{R}^{K\times K}$,

$$[\boldsymbol{Q}_i]_{kl} = \begin{cases} \frac{1}{K} & k \neq l \\ \frac{1}{K} - 1 & k = l \end{cases} \tag{45}$$

$\boldsymbol{d}$ is an K-dimensional column vector whose entries can be obtained from (43).

From (44), we obtain the following relation,

$$\boldsymbol{p} = \boldsymbol{V}\boldsymbol{\phi} + \bar{\boldsymbol{d}} \tag{46}$$

where, $\boldsymbol{V} = diag(\boldsymbol{Q}_1, \boldsymbol{Q}_2, \ldots, \boldsymbol{Q}_N)$, $\bar{\boldsymbol{d}}$ is an M dimensional column vector.

Combining (38) with (46), the following mapping relation can be obtained,

$$\boldsymbol{p} = \boldsymbol{V}(\boldsymbol{G}\boldsymbol{p} + \boldsymbol{\sigma}) + \bar{\boldsymbol{d}} = \boldsymbol{V}\boldsymbol{G}\boldsymbol{p} + \boldsymbol{h} = \boldsymbol{T}\boldsymbol{p} + \boldsymbol{h} \tag{47}$$

where $\boldsymbol{T} = \boldsymbol{V}\boldsymbol{G}$, $\boldsymbol{h} = \boldsymbol{V}\boldsymbol{\sigma} + \bar{\boldsymbol{d}}$. From (47), we find that the next power allocation is a mapping transformation of the current power allocation.

Hence, we rewrite the (47) as follows,

$$\boldsymbol{p}^{(n+s+1)} = \boldsymbol{T}^n \boldsymbol{p}^{(s+1)} + \boldsymbol{h} \tag{48}$$

where, $\boldsymbol{p}^{(n)}$ denotes the power allocation at the nth iteration.

It is straightforward to check that,

$$\left\| \boldsymbol{p}^{(n+s+1)} - \boldsymbol{p}^{(n+s)} \right\| = \left\| \boldsymbol{T}^{n} \left(\boldsymbol{p}^{(s+1)} - \boldsymbol{p}^{(s)} \right) \right\|. \quad (49)$$

Then,

$$\left\| \boldsymbol{p}^{(n+s+1)} - \boldsymbol{p}^{(n+s)} \right\| \le \left\| \boldsymbol{T} \right\|^{n} \left\| \left(\boldsymbol{p}^{(s+1)} - \boldsymbol{p}^{(s)} \right) \right\|. \quad (50)$$

It is obvious that if $||\boldsymbol{T}|| < 1$, then IML-WFA converges. Since $\boldsymbol{T} = \boldsymbol{VG}$, then $||\boldsymbol{T}|| \le ||\boldsymbol{V}||\,||\boldsymbol{G}||$. $\boldsymbol{V}$ is block diagonal. Each diagonal element $\boldsymbol{Q}_i$ has the special structure as shown in (45), then we can bound the norm of $\boldsymbol{V}$ by bounding the norm of $\boldsymbol{Q}_i$. We choose L_1-norm [21] to bound the norm of $\boldsymbol{Q}_i$. From (45), it is obvious that,

$$\left\| \boldsymbol{Q}_i \right\|_1 < \frac{2(K-1)}{K}. \quad (51)$$

Then,

$$\left\| \boldsymbol{V} \right\|_1 < \frac{2(K-1)}{K}, \quad (52)$$

$$\left\| \boldsymbol{V} \right\|_1 \left\| \boldsymbol{G} \right\|_1 < 1 \Rightarrow \left\| \boldsymbol{G} \right\|_1 < \frac{K}{2(K-1)}. \quad (53)$$

Similarly, L_2-norm and L_∞-norm can be use to bound the norm of $\boldsymbol{Q}_i$, then the following result can be obtained,

$$\left\| \boldsymbol{G} \right\|_\infty < \frac{K}{2(K-1)}, \quad (54)$$

$$\left\| \boldsymbol{G} \right\|_2 < 1. \quad (55)$$

So, if the $\boldsymbol{G}$ satisfies (53), (54) or (55), then the IML-WFA converges.

If a permutation of row-column imposes on the $\boldsymbol{G}$, then we obtain the following diagonal matrix,

$$\boldsymbol{G} = diag\left(\boldsymbol{A}(1), \boldsymbol{A}(2), ..., \boldsymbol{A}(K) \right) \quad (56)$$

where $\boldsymbol{A}(k), k \in \mathcal{K}$ is the channel gain matrix defined in Section 2. As the permutation does not change the norm, we get

$$\left\| \boldsymbol{G} \right\|_p = \left\| \max_{k \in \mathcal{K}} \left\{ \boldsymbol{A}(k) \right\} \right\|_p, \; p = 1, 2, \infty. \quad (57)$$

Let $p = 2$, then we get,

$$\left\| \max_{k \in \mathcal{K}} \left\{ \boldsymbol{A}(k) \right\} \right\|_2 < 1 \quad (58)$$

then the IML-WFA converges.

Equation (58) is equivalent to the following inequation,

$$\rho \left\{ \max_{k \in \mathcal{K}} \left\{ \boldsymbol{A}(k) \right\} \right\} < 1. \quad (59)$$

So if (59) is satisfied, then the IML-WFA converges.

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