SYNTHESIS OF OPTIMIZED PIECEWISE-LINEAR SYSTEMS USING SIMILARITY TRANSFORMATION
PART I: BASIC PRINCIPLES

Zdeněk KOLKA
Dept. of Radio Electronics
Brno University of Technology
Purkyňova 118, 612 00 Brno
Czech Republic

Abstract

Practical realization of nonlinear dynamic systems based on their state models raises an issue of finding such a form with low sensitivity to changes of network parameters. Present paper deals with a method for sensitivity optimization that keeps qualitative character of the system dynamics and increases the robustness of practical realization.

Keywords

Dynamical systems, piecewise-linear systems, sensitivity, optimization

1. Introduction

A piecewise linear system of the third-order was found to be the simplest model of autonomous systems exhibiting interesting dynamic phenomena including chaos. The original circuit introduced by Chua [1] was later generalized into a Class *C* dimensionless system described by the ordinary differential equation of the form [3]

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{b}\,h\left(\mathbf{w}^T\mathbf{x}\right), \tag{1}$$

where $\mathbf{A} \in \mathfrak{R}^{nxn}$, $\mathbf{b} \in \mathfrak{R}^{n}$, $\mathbf{w} \in \mathfrak{R}^{n}$ (n=3), and $h(\,)$ is a scalar odd-symmetric piecewise linear (PWL) function (Fig. 1). It has been proved and practically demonstrated that the function can even be smooth, which significantly simplifies its practical realization (e.g. with diodes). From the theoretical point of view, the PWL approximation is much more suitable because it leads to the linear portions of the state space. The most frequently used form of $h(\,)$ introduced in Fig. 1 divides the space into three regions described by the state equations

$$\dot{\mathbf{x}} = \mathbf{A}_0\mathbf{x} \text{ (region } D_0), \;\; \dot{\mathbf{x}} = \mathbf{A}_1\mathbf{x} \pm \mathbf{b} \text{ (regions } D_{\pm 1}), \tag{2}$$

where $\mathbf{A}_0 = \mathbf{A}_1 + \mathbf{b}\,\mathbf{w}^T$.

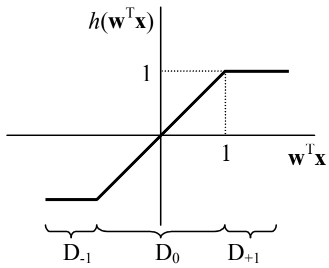

Fig. 1 Elementary form of *h( )* function.

In each region, the dynamic behavior is fully described by a set of eigenvalues of the corresponding linear system (2). The well-known theorem of linear system theory states, that two systems with identical eigenvalues exhibit qualitatively equivalent behavior. Their state trajectories are related by linear transformation. It has been shown in [2] this theorem applies also to the PWL system (1), where it ensures the global qualitative equivalence.

Let $\mathbf{T} \in \mathfrak{R}^{nxn}$ be a regular matrix and $\mathbf{y}$ be a state vector of a similar system so that $\mathbf{x} = \mathbf{T}\,\mathbf{y}$. Then the equations

$$\dot{\mathbf{y}} = \left(\mathbf{T}^{-1}\mathbf{A}_0\mathbf{T}\right)\,\mathbf{y}\,,\;\; \dot{\mathbf{y}} = \left(\mathbf{T}^{-1}\mathbf{A}_1\mathbf{T}\right)\,\mathbf{y} + \mathbf{T}^{-1}\mathbf{b} \tag{3}$$

describe the equivalent linear systems in each region of the state space. The whole PWL system exhibits similar behavior to the original one related by linear transformation [2]. The similarity transformation allows us to generate qualitatively equivalent PWL dynamical systems of Class *C*. A certain degree of freedom given by this transformation procedure can be used to improve sensitivity properties of the system.

2. Sensitivity criterion

As the dynamical behavior of a system is determined by the state matrix eigenvalues, it is desirable to find such a form of (3) which provides minimum sensitivity to the state matrix coefficients that somehow represent the parameters of the circuit elements. Let us suppose the system in each region of the state space is described by a simple linear ODE in the form

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x}\ . \tag{4}$$

The coefficients of the **A** matrix will be re-indexed for convenience as a_i , $i = 1 \ldots n^2$. We will assume all eigenvalues as simple, because multiple eigenvalue systems are obtained for low-dimensional sets of parameters and thus unattainable in practical realization. We propose to use the following definitions of relative sensitivities for the real and complex eigenvalues to be able to constrain tightly the eigenvalues with the high or low equivalent Q factor, i.e.

for real eigenvalues:

$$S_r(\lambda, a_i) = \frac{a_i}{\lambda}\frac{\partial \lambda}{\partial a_i} \quad , \tag{5a}$$

for complex eigenvalues:

$$S_{re}(\lambda, a_i) = \frac{a_i}{\mathrm{Re}(\lambda)}\mathrm{Re}\left(\frac{\partial \lambda}{\partial a_i}\right) \quad , \tag{5b}$$

$$S_{im}(\lambda, a_i) = \frac{a_i}{\mathrm{Im}(\lambda)}\mathrm{Im}\left(\frac{\partial \lambda}{\partial a_i}\right) \quad . \tag{5c}$$

Here the sensitivity invariants are in the form

$$\sum_{j=1}^{n^2} S_r(\lambda, a_j) = 1 \, , \; \sum_{j=1}^{n^2} S_{re}(\lambda, a_j) = 1 \, , \; \sum_{j=1}^{n^2} S_{im}(\lambda, a_j) = 1 \, . \tag{6}$$

The deviation of an eigenvalue is caused by the variations of coefficients of the state matrix **A** around their nominal values. Considering only the linear part of Taylor expansion we obtain

$$\frac{\Delta \lambda_i}{\lambda_i} = \sum_{j=1}^{n^2} S_r(\lambda_i, a_j)\frac{\Delta a_j}{a_j} \, , \tag{7}$$

and also similar expressions for the real and imaginary parts of the complex eigenvalues. The variations of all the eigenvalues can be written in matrix form as

$$\mathbf{D}_\lambda = \mathbf{S} \; \mathbf{D}_a \, , \tag{8}$$

where

$$\mathbf{D}_\lambda = \left[\frac{\Delta \lambda_1}{\lambda_1}, \ldots, \frac{\Delta \lambda_n}{\lambda_n}\right]^T , \; \mathbf{D}_a = \left[\frac{\Delta a_1}{a_1}, \ldots, \frac{\Delta a_{n^2}}{a_{n^2}}\right]^T ,$$

and $\mathbf{S} \in \Re^{nxn2}$ is a matrix of relative sensitivities, describing fully the quality of the system (4) realization. The complex conjugated eigenvalues are represented by two sensitivities for the real and imaginary parts and therefore the matrix **S** has n rows. In order to define an optimization task it is necessary to set up a reasonable norm of **S**.

Let us suppose the maximum relative deviations of all the parameters of **A** are the same. Since (8) is a linear relation, it is clear the maximum deviation of an eigenvalue occurs at the vertex of the n^2-dimensional cube of matrix parameters. We can conveniently use $\mathbf{D}_a = [\pm 1, \ldots, \pm 1]^T$ for the worst case. The relevant matrix norm is the so called "*infinity norm*" defined as

$$P_{WC} = \max_{i=1..n}\left(\sum_{j=1}^{n^2} \left|\mathbf{S}_{ij}\right|\right) \, . \tag{9}$$

From the sensitivity invariant (6) we obtain the condition

$$P_{WC} \geq 1 \, . \tag{10}$$

If all the dominant elements of **S** have the positive signs then $P_{WC} \approx 1$ and the worst-case deviation is invariant to the matrix changes. In reality, the coefficients of **A** are random numbers. The analysis in [4] shows, that in the case of uncorrelated deviations, the optimum sensitivity measure is a quadratic (Frobenius) norm

$$P_2 = \sqrt{\sum \mathbf{S}_{ij}^2} \, . \tag{11}$$

Both definitions (9) and (11) are in accordance, because a (theoretical) minimum is attained for $\mathbf{S}_{ij} = 1 / n^2$.

3. Symbolic solution

The similarity transformation allows an easy formulation of the optimization task. The system matrix **A** can be constructed from a given matrix **B** as

$$\mathbf{A} = \mathbf{T}^{-1}\mathbf{B}\mathbf{T} \, , \tag{12}$$

where coefficients of the regular matrix **T** are to be found.

Using the direct derivatives in (5) should be avoided to simplify a symbolic solution and to eliminate unnecessary errors in case of a numerical solution. Let us consider the definition eigenvalue formula

$$\det(\mathbf{1}\,\lambda - \mathbf{A}) = 0 \tag{13}$$

leading to the characteristic polynomial

$$C(\lambda) = c_n \lambda^n + \ldots + c_0 \quad \text{with} \quad c_n = 1 \, .$$

It is possible to divide C(λ) into the form

$$C(\lambda) = D(\lambda) - a_{ij}E_{(ij)}(\lambda) \, , \tag{14}$$

where $E_{(ij)}$ is the algebraic cofactor of the ($\mathbf{1}\lambda$ - **A**) matrix. From (14), taking into account that neither $D(\lambda)$ nor $E_{(ij)}(\lambda)$ contains a_{ij}, we can write

$$\frac{\partial \lambda}{\partial a_{ij}}D'(\lambda) - E_{(ij)}(\lambda) - a_{ij}E'_{(ij)}(\lambda)\frac{\partial \lambda}{\partial a_{ij}} = 0 \, , \tag{15}$$

and then

$$\frac{\partial \lambda}{\partial a_{ij}} = \frac{E_{(ij)}(\lambda)}{C'(\lambda)} \, , \tag{16}$$

where $C'(\lambda) = D'(\lambda) - a_{ij}\,E'_{(ij)}(\lambda)$ is the derivative of the characteristic polynomial with respect to λ. Due to the similarity transformation (12) the characteristic polynomial does not depend on **T**. To obtain the sensitivities defined in (5) it is necessary to consider the reindexing of a_{ij}.

Solution for n=2, real eigenvalues

Let us suppose the generating matrix **B** in the form

$$\mathbf{B} = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} . \qquad (17)$$

Using (5), (11), and (16) the quadratic sensitivity measure can be expressed by t_{ij} and λ_i . The global optimum of P_2 can be found by standard mathematical methods. It leads to the condition

$$t_{11}\, t_{22} = K\, t_{12}\, t_{21} \ . \qquad (18)$$

The coefficient K is

$$K = -1 \qquad \text{for } 0.48... \le c \le 2.08... \ , \qquad (19a)$$

$$K = \frac{2 - 3c - 3c^3 + 2c^4 \pm 2\sqrt{\left(1 + c^2\right)\left(1 - c\right)^2\left(1 - c - 2c^2 - c^3 + c^4\right)}}{c\left(1 - 4c + c^2\right)}$$

$$\text{otherwise.} \qquad (19b)$$

Here, $c = \lambda_2 / \lambda_1$. Substituting (18) into (12) we obtain

$$\mathbf{A}_{opt} = \begin{bmatrix} \dfrac{\lambda_1 (K - c)}{K - 1} & \dfrac{\lambda_1\, \psi\, K\, (c - 1)}{(K - 1)} \\ \dfrac{\lambda_1\, (1 - c)}{\psi (K - 1)} & \dfrac{\lambda_1 (cK - 1)}{K - 1} \end{bmatrix} , \qquad (20)$$

where ψ is an arbitrary nonzero number. Numerical experiments showed that the both values of coefficient K from (19b) satisfy the inequality $|K_1| << 1$ and $|K_2| >> 1$ for c outside the interval given in (19a). The matrix $\mathbf{A}_{opt}$ then converges to

$$\mathbf{A}_{opt\, K \to 0} = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}, \text{ and } \mathbf{A}_{opt\, K \to \pm\infty} = \begin{bmatrix} \lambda_1 & \psi \\ & \lambda_2 \end{bmatrix}, \qquad (21a)$$

where ψ is an arbitrary number.

Solution for n=2, complex conjugate eigenvalues

The generating matrix **B** has the form

$$\mathbf{B} = \begin{bmatrix} \lambda' & -\lambda'' \\ \lambda'' & \lambda' \end{bmatrix} , \qquad (22)$$

where λ' and λ'' are the real and imaginary parts. Global optimum of P_2 is given by the condition

$$t_{11}\, t_{21} = -t_{12}\, t_{22} \ . \qquad (23)$$

Substituting (23) into (12) we obtain

$$\mathbf{A}_{opt} = \begin{bmatrix} \lambda' & -\lambda''\psi \\ \dfrac{1}{\psi}\lambda'' & \lambda' \end{bmatrix} , \qquad (24)$$

where ψ is an arbitrary nonzero number.

The optimization process should be performed in all regions of the state space. The certain degree of freedom in the optimum forms (20) and (24) makes it possible.

4. Conclusion

The method described improves the performance of the circuits directly synthesized from Class C ordinary differential equation with respect to the non-conventional dynamical behavior. Decreasing eigenvalue sensitivity leads to more robust realization keeping the same qualitative character of the system.

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About author...

Zdeněk KOLKA was born in Brno, Czechoslovakia, in 1969. He received the M.S. (92) and Ph.D. (97) degrees in electrical engineering, both from the Faculty of Electrical Engineering and Computer Science, Brno University of Technology. At present he is an Assistant Professor at the Institute of Radio Electronics. He is interested in PWL modelling, circuit simulation, and nonlinear dynamical systems.