ANALYSIS OF CYLINDRICAL ANTENNAS BY MOMENT METHODS

Zbyněk RAIDA, Ivan HUDÁK
FEI VUT, ústav radioelektroniky
Purkyňova 1
612 00 BRNO
Phone 05 - 41149114
Fax 05 - 41149192
E-mail raida@urel.fee.vutbr.cz

Abstract

Presented submission reviews moment methods for the analysis of cylindrical antennas. By the presented methods, which differ in the way of evaluating singular integrals appearing during the analysis, current distribution and input impedance are computed. The methods are compared from the point of view of accuracy and computational requirements. Finally, results are confronted with King-Middleton theory.

Keywords

cylindrical antennas, moment methods

1. Introduction

In the previous paper [7], moment methods for the analysis of wire antennas, i.e. antennas the diameter of which is negligible in comparison with the wavelength, have been described.

The „wire antenna approach" provides simple algorithms the evaluation of which leads to current distributions and input impedances which well agree with experimental data.

If the diameter to wavelength ratio rises the idea of the concentration of current and charge on the wire axis is not valid more thanks to the skin effect[1] and the simple approach that does not exhibit any singularity has to be modified.

[1] In [3] pp.180-184, the „wire antenna approach" is pointed out not being valid even for wire antennas. This approach supposes slowly varying current distribution which is not fulfilled at the input and at the ends of the antenna. „It is crucious happening that in spite of this approximate solutions, analytic or numerical, of this nonvalid integral equation often gives results for the current distribution and input impedance that agrees very well with experimental data", writes author.

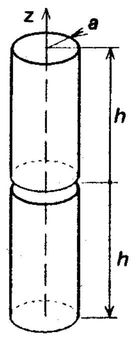

Fig.1 Cylindrical antenna

In all the paper, antennas are supposed being circular cylinders of radius a and length $2h$ which are situated to the axis z (fig.1) of the cylindrical coordinate system (r, ρ, z). Cylinders occur in the vacuum ($\mu = \mu_0$, $\varepsilon = \varepsilon_0$, $\sigma = 0$) and do not exhibit any losses.

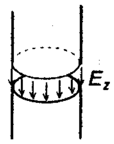

Fig.2 Excitating electrical field between antenna terminals

In the middle of the cylinder ($z = 0$), there is a short gap. In the gap, a hypothetical harmonic generator is assumed such that the excitating electrical field can be azimuthally symmetric. The voltage across the gap

$$V = -\int_{gap} E_z dz \qquad (1.1)$$

is supposed being 1V. In (1.1), E_z is the z-component of the excitating electrical field intensity on the interpolated antenna surface (fig.2). Outside the gap, E_z is zero because of the perfect conductivity of the cylinder.

In the next section, piece-wise constant basis functions combined with point match weighting and global cosine basis functions combined with Galerkin's method are described. In the conclusion, both the methods are compared.

All the theoretical conclusions are illustrated by results of computer simulations which have been performed in matlab 4.2.

2. Cylindrical antennas

If current and charges are considered on the surface of the antenna cylinder and rotationary symmetry of the problem is supposed then the initial set of equations is [7]

$$\frac{\partial I_z(z)}{\partial z}+j\omega\,\sigma(z)=0 \tag{2.1a}$$

$$A_z(z)=\frac{\mu}{4\pi}\int_{-h}^{h}\int_{0}^{2\pi}\frac{I_z(\xi)}{2\pi}\frac{e^{-jkR(z,\xi,\varphi,\varphi')}}{R(z,\xi,\varphi,\varphi')}d\varphi'\,d\xi \tag{2.1b}$$

$$\varphi(z)=\frac{1}{4\pi\varepsilon}\int_{-h}^{h}\int_{0}^{2\pi}\frac{\sigma(\xi)}{2\pi}\frac{e^{-jkR(z,\xi,\varphi,\varphi')}}{R(z,\xi,\varphi,\varphi')}d\varphi'\,d\xi \tag{2.1c}$$

$$-E_z^i(z)=-j\omega A_z(z)-\frac{\partial\varphi(z)}{\partial z}. \tag{2.1d}$$

Here, I_z and σ are current and charge density distributions, A_z and φ are vector and scalar potentials, ω is circular frequency and $R(z,\xi)$ is the distance between source and destination points (look at fig.3) (2.2)

$$R(z,\xi,\varphi,\varphi')=\sqrt{4a^2\sin^2((\varphi-\varphi')/2)+(z-\xi)^2}\,.$$

In this case, evaluation of integrals in (2.1b,c) is problematic because of the singularity at $z=\xi\cap\varphi=\varphi'$. Fortunately, the singularity is the logarithmic [6] and hence, the integrals exist.

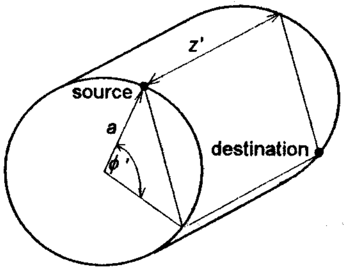

Fig.3 Segment of cylindrical antenna

For many years, various methods of computing the above singular integral have been developed. In the following paragraphs, two approaches are discussed: the first one uses piece-wise constant approximation and collocation and the second one is based on the global harmonic approximation and Galerkin's method.

2.1 Piece-wise constant approximation, point matching

If piece-wise constant approximation is used for solving (2.1) then computing the integral

$$\psi(z,n)=\frac{1}{2\pi}\int_{(n-0.5)\Delta}^{(n+0.5)\Delta}\int_{0}^{2\pi}\frac{e^{-jkR(z,\xi,\varphi,\varphi')}}{4\pi R(z,\xi,\varphi,\varphi')}d\varphi'\,d\xi \tag{2.3}$$

for $z=0$ and $n=0$ is the cardinal problem. Since the rotationary symmetry of the antenna is assumed the choice of φ plays no role; for the simpicity let $\varphi=0$. Since (2.3) is even function, the relation can be re-defined as

$$\psi=\frac{2}{2\pi}\int_{0}^{\Delta/2}\int_{0}^{2\pi}\frac{e^{-jkR(\xi,\varphi')}}{4\pi R(\xi,\varphi')}d\varphi'\,d\xi=2\Im \tag{2.4}$$

where

$$\Im=\frac{1}{2\pi}\int_{0}^{\Delta/2}\int_{0}^{2\pi}\frac{e^{-jkR}}{4\pi R}d\varphi'\,d\xi\,. \tag{2.5}$$

Now, the singular integral (2.5) has to be evaluated. Let $\Im$ be divided into two sub-integrals [5]

$$\Im=\Im_0+\Im_1 \tag{2.6a}$$

where

$$\Im_0=\frac{1}{2\pi}\int_{0}^{\Delta/2}\int_{0}^{2\pi}\frac{1}{4\pi R}d\varphi'\,d\xi \tag{2.6b}$$

keeps the singularity (hence, it requires additional handling) and

$$\Im_1=\frac{1}{2\pi}\int_{0}^{\Delta/2}\int_{0}^{2\pi}\frac{e^{-jkR}-1}{4\pi R}d\varphi'\,d\xi \tag{2.6c}$$

is a slowly varying function presenting no difficulties to numerical calculations.

Eqn. (2.6b) can be expanded into an infinite series with excelent convergence properties [5]

$$\Im_0=\frac{1}{4\pi}\sum_{n=0}^{\infty}A_n \tag{2.7}$$

where

$$A_n=\frac{2}{\pi}\frac{1}{(n!)^2}\left(\frac{a^2}{4}\right)^n\int_0^{\infty}k^{2n-1}K_0(ak)\sin(\Delta k/2)\,dk\,. \tag{2.8}$$

Here, K_0 is modified Bessel function of second kind. For the practical computing, (2.8) can be re-arranged

$$A_0=\ln\left[\frac{\Delta}{2a}+\sqrt{1+\left(\frac{\Delta}{2a}\right)^2}\right] \tag{2.9a}$$

$$A_1=\frac{1}{4}\left(\frac{\Delta}{2a}\right)\left[1+\left(\frac{\Delta}{2a}\right)^2\right]^{-\frac{3}{2}} \tag{2.9b}$$

$$A_2=\frac{5!!}{4^2(2!)^2}\left(\frac{\Delta}{2a}\right)\left[1+\left(\frac{\Delta}{2a}\right)^2\right]^{-\frac{7}{2}}-$$

$$-\frac{2.3!!}{4^2(2!)^2}\left(\frac{\Delta}{2a}\right)\left[1+\left(\frac{\Delta}{2a}\right)^2\right]^{-\frac{5}{2}} \tag{2.9c}$$

$$A_3=\frac{9!!}{4^3(3!)^2}\left(\frac{\Delta}{2a}\right)\left[1+\left(\frac{\Delta}{2a}\right)^2\right]^{-\frac{11}{2}}-\frac{8.7!!}{4^3(3!)^2}\left(\frac{\Delta}{2a}\right)\left[1+\left(\frac{\Delta}{2a}\right)^2\right]^{-\frac{9}{2}}$$

$$+\frac{8.5!!}{4^3(3!)^2}\left(\frac{\Delta}{2a}\right)\left[1+\left(\frac{\Delta}{2a}\right)^2\right]^{-\frac{7}{2}} \tag{2.9d}$$

etc.

Here, $n!!=n\,(n-2)\,(n-4)\ldots$.

The term A_0 is dominant; magnitudes of other terms rapidly fall down.

If point matching and delta function charge densities at the ends of segments are assumed then [7]

$$Z_{mn} = j\omega\mu\Delta\,\psi(m,n) + \frac{1}{j\omega\varepsilon\Delta}\left[\psi(m^+,n^+) - \psi(m^+,n^-)\right]$$

$$-\frac{1}{j\omega\varepsilon\Delta}\left[\psi(m^-,n^+) - \psi(m^-,n^-)\right] \qquad (2.10)$$

If $m=n$ then the singularity appears and ψ is computed according to the eqns.(2.5-9). In the opposite case, eqn. (2.4) is directly numerically evaluated.

The above described algorithm written in matlab synax follows. The program uses first 2 terms of (2.7) for computing.

First, all the non-singular integrals are computed:

```
for m = 1:(N+1)
 z = m*del;     % del denotes length of segment Δ
 psi(m+1) = quad8('g_int',-del/2,del/2,1e-5,0,
            z,a,k);
end
```

In the second step, singular integral is evaluated

```
rat  = 0.5*del/a;
A0   = ln( rat + sqrt( 1+rat^2);
A1   = 0.25*rat*sqrt((1+rat^2)^3);
psi(1)= 2*( A0 + A1)/(4*pi);
```

The integrated function `g_int` is defined as:

```
function out=g_int(ksi,z,a,k)
out=quad8('g',0,2*pi,1e-5,0,ksi,z,a,k)/(2*pi);

function out=g(fi,ksi,z,a,k)
R   = sqrt( (2*a*sin(fi))^2 + (z-ksi)^2);
out = exp(-j*k*R)/(4*pi*R);
```

Finally, the impedance matrix can be built up:

```
for m = 1:N
 for n = m:N
  dist = abs(m-n);   % source-destination distance
  hlp=2*psi(1+dist)-psi(1+abs(dist-1))-
      psi(1+abs(dist+1));
  Z(m,n)=j*omega*mi*delta*psi(1+dist)+
      hlp/(j*omega*epsilon*delta);
  Z(n,m) = Z(m,n);   % matrix is symmetrical
 end
end
```

2.2 Global cosine approximation, Galerkin's method

Initial equation we use for global approximation comes from the relation [4]

$$E_z = -\frac{j}{\omega\mu\varepsilon}\left[\frac{\partial^2 A_z}{\partial z^2} + k^2 A_z\right]. \qquad (2.11)$$

If infinitesimally narrow excitation gap and excitation voltage 1V are assumed then electrical intensity in the gap can be expressed as

$$E_z = -1\delta(z) \qquad |z| \le h. \qquad (2.12)$$

Substituting (2.12) to (2.11) yields

$$\frac{\partial^2 A_z}{\partial z^2} + k^2 A_z = \frac{\omega\mu\varepsilon}{j}\delta(z). \qquad (2.13)$$

Solution of differential equation (2.13) is of the form

$$A_z = C_1\cos(kz) + \frac{\omega\mu\varepsilon}{j2k}\sin(k|z|). \qquad (2.14)$$

Vectorial potential computed according to (2.14) has to be of the same value as this evaluated from (2.1b)

$$\int_{-h}^{+h} f(\xi)\,g(z-\xi)\,d\xi = C\cos(kz) + \frac{1}{2}\sin(k|z|). \qquad (2.15)$$

Here

$$C = \frac{j}{4\pi}\sqrt{\frac{\mu}{\varepsilon}} \qquad (2.16a)$$

$$f(\xi) = C\,I(\xi) \qquad (2.16b)$$

and

$$g(z-\xi) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{-jkR}}{R}\,d\varphi'. \qquad (2.16c)$$

In the solution process, such a value of the constant C has to be found to be fulfilled the condition $f(\pm h) = 0$.

If the function $f(\xi)$ corresponding to the unknown current distribution $I(\xi)$ is approximated by Fourier series of N terms

$$f_N(\xi) = \sum_{n=0}^{N} F_N\cos\left(n\pi\frac{\xi}{h}\right) \qquad (2.17)$$

then (2.15) comes to the linear equation of N+2 unknown coefficients F_0, F_1 ... F_N and C.

First, let's concentrate on computation of the kernel $g(z-\xi)$ of the integral equation (2.15). In [2], pp.3-18, it has been shown that $g(z-\xi)$ can be expanded to the infinite series (2.18)

$$g(z-\xi) = \frac{D_0}{2} + \sum_{m=1}^{\infty} D_m\cos\left(m\pi\frac{z-\xi}{2h}\right) \qquad 0 \le \xi \le 2h$$

where all the coefficients D_m can be evaluated according to the relation

$$D_m = \frac{1}{2h}\int_0^{2h} g(\varsigma)\cos\left(m\pi\frac{\varsigma}{2h}\right)d\varsigma. \qquad (2.19)$$

Since the kernel $g(\zeta)$ of (2.19) is in $\zeta = 0$ singular, $g(\zeta)$ is expressed as

$$g(\varsigma) = g_1(\varsigma) + g_2(\varsigma) \qquad (2.20)$$

where $g_1(\zeta) = g(\zeta)$ in the range $0 \le \zeta \le 2h$ and zero outside;
$g_0(\zeta) = g(\zeta)$ outside the range $0 \le \zeta \le 2h$ and zero inside.

Re-arranging (2.20) and applying cosine transform

$$G(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^{2h} g(\varsigma)\cos(\alpha\varsigma)\,d\varsigma \qquad (2.21a)$$

$$g(\varsigma)=\sqrt{\frac{\pi}{2}}\int_0^{\infty}G(\alpha)\cos(\alpha\varsigma)\,d\alpha \qquad (2.21b)$$

to both sides of (2.20) yields

$$G_1(\alpha)=G(\alpha)-G_2(\alpha) \qquad (2.22)$$

where

$$G(\alpha)=-j\sqrt{\frac{\pi}{2}}J_0(\beta a)H_0^{(2)}(\beta a) \qquad (2.23a)$$

and

$$G_2(\alpha)=\sqrt{\frac{2}{\pi}}\int_{2h}^{\infty}g(\varsigma)\cos(\alpha\varsigma)\,d\varsigma\,. \qquad (2.23b)$$

Comparison of (2.19) and (2.21a) shows that

$$D_m=\frac{1}{h}\sqrt{\frac{\pi}{2}}\left[G\left(\frac{m\pi}{2h}\right)-G_2\left(\frac{m\pi}{2h}\right)\right]. \qquad (2.24)$$

In the above relations,

$$\beta=\sqrt{k^2-\alpha^2}\,,\ \alpha=\frac{m\pi}{2h}\,,\ k=\frac{2\pi}{\lambda}.$$

There are no difficulties in evaluating $G_2(\alpha)$; it can be expanded in highly convergent infinite series and re-arranged to the form

$$G_2(\alpha<k)\cong-\sqrt{\frac{1}{2\pi}}\{Ci[(k+\alpha)2h]+Ci[(k-\alpha)2h]+j\pi-$$
$$-jSi[(k+\alpha)2h]-jSi[(k-\alpha)2h]\} \qquad (2.25a)$$

$$G_2(\alpha>k)\cong-\sqrt{\frac{1}{2\pi}}\{Ci[(\alpha+k)2h]+Ci[(\alpha-k)2h]-$$
$$-jSi[(\alpha+k)2h]+jSi[(\alpha-k)2h]\} \qquad (2.25b)$$

In (2.25),

$$Si(x)=\int_0^x\frac{\sin x}{x}dx=\frac{\pi}{2}-\int_x^{\infty}\frac{\sin x}{x}dx \qquad (2.26a)$$

$$Ci(x)=-\int_x^{\infty}\frac{\cos x}{x}dx\,. \qquad (2.26b)$$

It is convenient to rewrite $G(\alpha)$ from (2.23a) separately for the cases $\alpha<k$ and $\alpha>k$

$$G(\alpha<k)=-j\sqrt{\frac{\pi}{2}}J_0(ba)H_0^{(2)}(ba) \qquad (2.27a)$$

$$G(\alpha>k)=\sqrt{\frac{2}{\pi}}I_0(ba)K_0(ba)\,. \qquad (2.27b)$$

Here, J_0 and I_0 are Bessel and modified Bessel functions of first kind, K_0 is modified Bessel function of second kind and H_0 is Hankel function.

It is obvious, that all the functions (2.25) and (2.27) are singular for $\alpha=k$. Fortunately, it can be shown [2] that singularities subtract off if the difference $G(\alpha)$ - $G_2(\alpha)$ is computed

$$G_1(\alpha)\cong\frac{1}{\sqrt{2\pi}}\left\{\ln\frac{4h}{\gamma ka^2}+Ci(4kh)-jSi(4kh)\right\} \qquad (2.28)$$

where $\gamma=0.577215$.

Now, substituing (2.25), (2.27) and (2.28) to (2.24) enables evaluating coefficients D_m. If D_m are known, eqn. 2.18 can be completed

$$g(z-\xi)=\sum_{m=0}^{\infty}D_m\left[\cos\left(m\pi\frac{z}{2h}\right)\cos\left(m\pi\frac{\xi}{2h}\right)+\right.$$
$$\left.+\sin\left(m\pi\frac{z}{2h}\right)\sin\left(m\pi\frac{\xi}{2h}\right)\right] \qquad (2.29)$$

Since the current distribution is even function in our case sine terms of (2.29) are not needed

$$g(z-\xi)=\sum_{m=0}^{\infty}D_m\cos\left(m\pi\frac{z}{2h}\right)\cos\left(m\pi\frac{\xi}{2h}\right). \qquad (2.30)$$

Substituing the current approximation (2.17) and the kernel of integral equation (2.30) to the initial equation (2.15) yields (2.31a)

$$\sum_{n=0}^{N}\sum_{m=0}^{\infty}F_nD_m\gamma_{nm}\cos\left(m\pi\frac{z}{2h}\right)=C\cos(kz)+\frac{1}{2}\sin(k|z|)$$

where

$$\gamma_{nm}=2\int_0^h\cos\left(n\pi\frac{\xi}{h}\right)\cos\left(m\pi\frac{\xi}{2h}\right)d\xi\,. \qquad (2.31b)$$

An infinite set of simultaneous linear equations is obtained by multiplying both sides of (2.31) by the weighting function

$$w_p(z)=\cos\left[(2p+1)\pi\frac{z}{2h}\right] \qquad (2.32)$$

and by its integration from $-h$ to $+h$. The result is

$$\sum_{n=0}^{N}\sum_{m=0}^{\infty}F_nD_m\gamma_{nm}\beta_{pm}=Cr_pv_p, \qquad (2.33a)$$

where

$$\beta_{pm}=2\int_0^h\cos\left[(2p+1)\pi\frac{z}{2h}\right]\cos\left[m\pi\frac{z}{2h}\right]dz \qquad (2.33b)$$

$$r_p=2\int_0^h\cos(kz)\cos\left[(2p+1)\pi\frac{z}{2h}\right]dz \qquad (2.33c)$$

$$v_p=2\int_0^h\sin(kz)\cos\left[(2p+1)\pi\frac{z}{2h}\right]dz\,. \qquad (2.33d)$$

Equation (2.33a) can be rewritten

$$\sum_{n=0}^{N}\Gamma_{pn}F_n=Cr_p+v_p \qquad (2.34a)$$

where

$$\Gamma_{pn} = \sum_{m=0}^{\infty} D_m \gamma_{nm} \beta_{pm} . \qquad (2.34b)$$

Eqn.(2.34b) shows that Γ has to be evaluated as summ of infinite number of coefficients. Fortunately, thanks to the orthogonality of harmonic functions used for approximation and weighting, only two terms of this infinite summ are non-zero

$$\Gamma_{pn} = h\left[\beta_{p,2n} D_{2n} + \gamma_{n,2p+1} D_{2p+1}\right] . \qquad (2.35)$$

Set of equations generated by (2.34) can be rewritten to the matrix form

$$\Gamma F = Cr + v \qquad (2.36)$$

from which the vector of Fourier coefficients of searched current distribution can be obtained

$$F = c\Gamma^{-1} r + \Gamma^{-1} v . \qquad (2.37)$$

Now, such a value of the constant C has to be found to be fulfilled the condition of zero current at the ends of dipole. Since leading terms of the current approximation

$$f_N(z) = \sum_{n=1}^{N} F_n(C) \cos\left(n\pi \frac{z}{h}\right) \qquad (2.38)$$

are not good approximation to antenna current, the series (2.42) is slowly convergent. Hence, good low order approximation has to be chosen (e.g. sine current distribution) and expanded to the series

$$x(z) = \sum_{n=0}^{N} X_n \cos\left(n\pi \frac{z}{h}\right) \qquad (2.39)$$

for improving convergence of (2.38)

$$f_N(z) = x(z) + \sum_{n=0}^{N} \left[F_n(C) - X_n\right] \cos(n\pi z/h) . \qquad (2.40)$$

At the ends of dipole, all the cosines equal $(-1)^n$. Hence, boundary conditions are fulfilled if

$$\sum_{n=0}^{N} \left[F_n(C) - X_n\right](-1)^n = 0 . \qquad (2.41)$$

By letting N = 0, 1, ... N, (2.41) can be used to generate a sequence C_0, C_1, ... C_N. Extrapolating this sequence yields the coefficient C_∞ that gives the current approximation

$$f_N(z) = x(z) + \sum_{n=0}^{N} \left[F_n(C_\infty) - X_n\right] \cos\left(n\pi \frac{z}{h}\right) . \qquad (2.42)$$

Then, the input admitance can be computed according to the relation

$$Y_N = \frac{4\pi}{jZ_0} \left\{ x(0) + \sum_{n=0}^{n=N} \left[F_n(C_\infty) - X_n\right] \right\} . \qquad (2.43)$$

Since the algorithm based on the above description is rather complicated its list is not presented here. If the reader is interested in this list, he is kindly asked to contact authors.

3. Conclusions

Presented paper has reviewed two methods that can be used for numerical analysis of cylindrical antennas.

approximation	minimization	N = 16	N = 32
constant	collocation	84.6 + j40.6	86.5 + j43.2
cosine	Galerkin	85.1 + j43.3	84.7 + j42.3

Tab.1 Input impedance of symetrical dipole (a=0.001588λ, 2h=0.5λ) computed by discussed methods. Value taken from King-Middleton is (83.6 + j41.3)Ω.

By both methods, input impedance of the antenna h = 0.25 λ and a = 0.001588 λ has been computed (tab.1). It can be seen that no dramatical differences have appeared between the methods (tab.1). A bit better accuracy of the cosine approximation in combination with the Galerkin's method is payed by extremly high computational requirements and complexity of the algorithm.

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