TESTING SAMPLES OF THREE INDEPENDENT GROUPS

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Abstract

A method of testing triplesconsisting of samples of three independent groups such that no two samples of the same group are tested at the same time, no sample is tested with another one more than once and after eachtest one sample of the triple is included in the following triple is presented.

Keywords:

noise figure, noise parameters, noise matching

Introduction

When we test n-tuples of samples and each sample is to be tested more than once, it may be quite time-saving to keep after each test one of the samples in the testing environment and replace only the other ones. For instance, in cases when replacing a sample after the test takes long time because of necessity to clean the testing equipment or for another similar reason.

We present a method of testing triples consisting of samples of three independent groups such that no two samples of the same group are tested at the same time and no sample is tested with another one more than once. The number of samples of each group is constant.

The method is based on decompositions of group-divisible designs, namely 3-transversal designs, into isomorphic cycles.

A *group divisible design* k-GDD(n,r) is a triple (V,**G**,**B**) where V is a set of elements, **G** is a partition of V into r subsets $G_1, G_2, \ldots, G_r$ of the same cardnality n called groups and **B** is a collection of subsets of V of cardinality k called blocks such that $|G_1 \cap B| \leq 1$for any group $G_1 \in \mathbf{G}$ and any block $B \in \mathbf{B}$ and for any two elements x,y from distinct groups there is exactly one block containing both x and y. A *transversal design* k-TD(n) is a group divisible design k-GDD(n,k), i.e., $|G_l \cap B| = 1$ for any group $G_l \in \mathbf{G}$ and any block $B \in \mathbf{B}$. A *factor* E_i of a k-GDD(n,r) is a triple (V,**G**,$\mathbf{D}_i$) where **D** is a subset of **B**.

A *decomposition* of a k-GDD(n,r) is an m-tuple of factors E_i=(V,**G**,$\mathbf{D}_i$), i=1,2,... , m such that $\mathbf{D}_i \cap \mathbf{D}_j = 0$ and $\bigcup_{i=1}^{m} \mathbf{D}_i = \mathbf{B}$. Two factors E_i and E_j are *isomorphic* (denoted $E_i \cong E_j$) if there exists a one-to-one mapping $\phi_{ij}$of V onto itself such that $D' = \{\phi_{ij}(x_1), \phi_{ij}(x_2), \ldots, \phi_{ij}(x_k)\} \in \mathbf{D}_j$ if and only if $D = \{x_1, x_2, \ldots, x_k\} \in \mathbf{D}_i$. A decomposition is *isomorphic* if $E_i \cong E_j$ for every pair $1 \leq i \leq j \leq m$. A *path* of length q, P_q, is a sequence $x_0 - B_1 - x_1 - B_2 - \ldots - B_q - x_d$ of elements and blocks such that for each i =1,2,...,q the elements x_{i-1} and x_i belong to the block B_i and no block and no element appears more than once. Since each pair of elements in a k-GDD belongs to at most one block, the path is uniquely determined by the elements and we usually use the simpler notation $P_q = x_0 - x_1 - \ldots - x_q$. A *cycle* of length q, C_q, is a sequence $x_0 - B_1 - x_1 - B_2 - x_2 - \ldots - B_q - x_q$ (or simply x_0-x_1-...-x_q) of elements and blocks such that $x_0 = x_q$, for each i=1,2,... ,q the elements x_{i-1} and x_i belong to the block B_i and no block or element appears more than once.

One can see now that if we want to test triples of samples that belong to three different groups and each sample in the triple has to belong to another group, we can use a 3-TD(n) as a model. If we want to replace after each test just two of the samples, it is quite natural to search for "strings" like paths or cycles that are long enough to satisfy our requirements for multiple testing.

Hartman [2], Das and Rosa [1], and Phelps [3] studied decompositions of designs into two factors. We study decompositions of GDD's into smallest connected factors.

1. Smallest connected factors

It is not difficult toobserve that the smallest connected factor is acyclic. If a 3-TD(n) has such a factor $E^{(s)}$ with s blocks (i.e., triples consisting of one element of each group), it is obvious, at it contains $2s+1$ elements and therefore the number of elements of the 3-TD(n), $3n$, must be equal to $2s+1$. Hence $s = \frac{3n-1}{2}$ and n must be an odd number. So we can state the following simple observation.

Proposition 1

A 3-TD(n) has a connected acyclic factor only if n is odd.

Let us suppose now that n is odd, *say* $2m+1$.Then the number of blocks of the factor $E^{(s)}$is $s = \frac{3n-1}{2} = \frac{3(2m+1)-1}{2} = 3m + 1$. Since the number of blocks of the 3-TD($2m+1$) is $(2m+1)^2$, the 3-TD($2m+1$) is decomposable into connected acyclic factors only if $3m+1 \mid (2m+1)^2$. Suppose it is the case. Then there is a positive number k such that $(2m+1)^2 = k(3m+1)$. We can write $k = tm+1$, where $0 \le t \le Q$. Then we have $4m^2+4m+1 = (tm+1)(3m+1) = 3tm^2+(t+3)m+1$, which yields $4(m+1) = 3tm+t+3$. Hence $4m-3tm = t-1$ and $m = \frac{t-1}{4-3t}$. Since m is a non-negative integer andthe fraction is negative for all $t \neq 1$, we are left with $t=1$, which yields $m=0$. Then $n=1$ and the following holds.

Proposition 2

No 3-TD(n) with $n>1$ is decomposable into connected acyclic factors.

Let us consider now connected factors of $3-TD(2m+1)$'s with $3m+2$ blocks. The 3-TD(3) of the additive group Z_3 with groups $G_1 = \{0_1, 1_1, 2_1\}$, $G_2 = \{0_2, 1_2, 2_2\}$ and $G_3 = \{0_3, 1_3, 2_3\}$ and blocks $(0_1, 0_2, 0_3)$, $(0_1, 1_2, 1_3)$, $(0_1, 2_2, 2_3)$, $(1_1, 0_2, 1_3)$, $(1_1, 1_2, 2_3)$, $(1_1, 2_2, 0_3)$, $(2_1, 0_2, 2_3)$, $(2_1, 1_2, 0_3)$, $(2_1, 2_2, 1_3)$ has a connected factor $E_{(5)}$ with $3m+2$ blocks, e.g., $(0_1, 0_2, 0_3)$, $(0_1, 1_2, 1_3)$, $(1_1, 0_2, 1_3)$, $(2_1, 0_2, 2_3)$, $(2_1, 2_2, 1_3)$. The factor $E_{(5)}$ contains two cycles: $0_1 - (0_1, 0_2, 0_3) - 0_2 - (1_1, 0_2, 1_3) - 1_3 - (0_1, 1_2, 1_3) - 0_1$ and $2_1 - (2_1, 0_2, 2_3) - 0_2 - (1_1, 0_2, 1_3) - 1_3 - (2_1, 2_2, 1_3) - 2_1$, and is therefore not the "simplest possible", i.e., unicyclic. A necessary condition for decomposability into unicyclic factors follows.

Lemma 3

If a 3-TD(n) is decomposable into unicyclic factors, then $n \equiv 0 (mod\ 6)$.

Proof. Let $E_{(s)}$ be a unicyclic factor with s blocks. The shortest cycle, C_3 consists of 3 blocks that contain together 6 elements. Since every other block contributes 2 to the number of elements, we have $s = \frac{3n}{2}$. Therefore n must be even. On the other hand, the number of blocks of the factor must divide the number of blocks of the 3-TD(n), i.e., $3\frac{n}{2} \mid n^2$. This yields $3 \mid n$ and hence $n \equiv 0 (mod\ 6)$.

We show further that for every $n \equiv 0 (mod\ 6)$there is a decomposable 3-TD(n). We even show that the factors can be mutually isomorphic. But first we state the following.

Corollary 4

If a 3-TD(n) is decomposable into connected factors of size t, then $t \ge 3\frac{n}{2}$. The equality can hold only if $n \equiv 0 (mod\ 6)$.

2. Constructions

Now we present constructions of 3-TD's that are isodecomposable into unicyclic factors, namely cycles. We start with the case $n \equiv 6 (mod\ 12)$.

Construction 5

$n = 6(mod\ 12)$. Let $n = 12m+6$. First we construct a Latin square A of order $6m+3$ as follows. The first row is $1, 3m+3, 2, 3m+4, 3, \ldots, 3m+1, 6m+3, 3m+2$. An entry in i-th row and j-th column, $a^{i,j}$, is then equal to $a^{1,i+j-1}$. Then we construct a Latin square C of order $12m+6$ with entries $c^{i,j} = a^{i,j}$ for $1 \le i,j \le 6m+3$, $c^{i,j} = a^{i-6m-3,j}$ for $6m+4 \le i \le 12m+6, 1 \le j \le 6m+3$, $c^{i,j} = a^{i,j-6m-3}$ for $1 \le i \le 6m+3, 6m+4 \le j \le 12m+6$, and $c^{i,j} = a^{i-6m-3,j-6m-3}$ for $6m+4 \le i,j \le 12m+6$.

	1	2	3	4	5	6
1	1	3	2	4	6	5
2	3	2	1	6	5	4
3	2	1	3	5	4	6
4	4	6	5	1	3	2
5	6	5	4	3	2	1
6	5	4	6	2	1	3

Figure 1

The triples of the 3-TD($12m+6$) are then $(i_1, j_2, c_3^{i,j})$. One can notice that the Latin square C is a multiplication array of a commutative half-idempotent quasigroup. An example of the Latin square C is shown in Figure 1. Since the third element of a triple is determined uniquely, we usually write just (i_1, j_2, c_3).

The factor E_0 contains the blocks (i_1, i_2, c_3) for $i = 1, 2, \ldots, 12m+6$, the block $(1_1, (12m+6)_2, c_3)$ and the blocks $(j_1, (j+6m+2)_2, c_3)$ for $j = 2, 3, \ldots, 6m+3$. Then E_0 is the cycle $1_1 - (1_1, 1_2, 1_3) - 1_3 - ((6m+4)_1, (6m+4)_2, 1_3) - (6m+4)_2 - (2_1, (6m+4)_2, c_3) - 2_1 - (2_1, 2_2, 2_3) - 2_3 - \ldots - i_1 - (i_1, i_2, i_3) - i_3 - ((6m+3+i)_1, (6m+3+i)_2, i_3) - (6m+3+i)_2 - ((i+1)_1, (6m+3+i)_2, c_3) - (i+1)_1 - ((i+1)_1, (i+1)_2, (i+1)_3) - (i+1)_3 - \ldots - ((12m+6)_1, (12m+6)_2, (6m+3)_3) - (12m+6)_2 - (1_1, (12m+6)_2, c_3) - 1_1$.

The factor E_1 is deter- mined by the isomorphism $\psi_1 : E_0 \to E_1$ with $\psi_1(x_1) = x_1, \psi_1(y_2) = (y+6m+3)_2$, $\psi_1(z_3) = (z+6m+3)_3$.

E_2 is determined by $\psi_2 : E_0 \to E_2$, where $\psi_2(1_1) = (6m+3)_2, \psi_2(2_1) = 1_2, \psi_2(3_1) = 2_2, \ldots, \psi_2((6m+3)_1) = (6m+2)_2, \psi_2((6m+4)_1) = (12m+6)_2, \psi_2((6m+5)_1) = (6m+4)_2$, $\psi_2((6m+6)_1) = (6m+5)_2, \ldots, \psi_2((12m+6)_1) = (12m+5)_2, \psi_2(1_2) = 2_1, \psi_2(2_2) = 3_1, \psi_2(3_2) = 4_1, \ldots, \psi_2((6m+3)_2 = 1_1, \psi_2((6m+4)_2) = (6m+5)_1, \psi_2((6m+5)_2) = (6m+6)_1$, $\ldots, \psi_2((12m+6)_2) = (6m+4)_1, \psi_2(z_3) = z_3$

E_4 is determined by $\psi_4 : E_0 \to E_4$, where $\psi_4(1_1) = 4_1$, $\psi_4(2_1) = 5_1, \psi_4(3_1) = 6_1, \dots, \psi_4((6m+3)_1) = 3_1, \psi_4((6m+4)_1) = (6m+7)_1\ \psi_4((6m+5)_1) = (6m+8)_1, \dots, \psi_4((12m+6)_1) = (6m+6)_1$, $\psi_4(y_2) = y_2, \psi_4(1_3) = (3m+4)_3, \psi_4(2_3) = (3m+5)_3, \dots, \psi_4((6m+3)_3) = (3m+3)_3$, $\psi_4((6m+4)_3) = (9m+7)_3, \psi_4((6m+5)_3) = (9m+8)_3, \dots$, $\psi_4((12m+6)_3) = (9m+6)_3$.

In general, a factor E_t, where , $t = 4u + 2v + w, 1 \le t \le 12m + 5 = n - 1$ is determined by an isomorphism $\phi_t : E_0 \to E_t$, which is defined as the composition $\phi_t = \psi_4^u \circ \psi_2^v \circ \psi_1^w$, with $\psi_j^0 = id$.

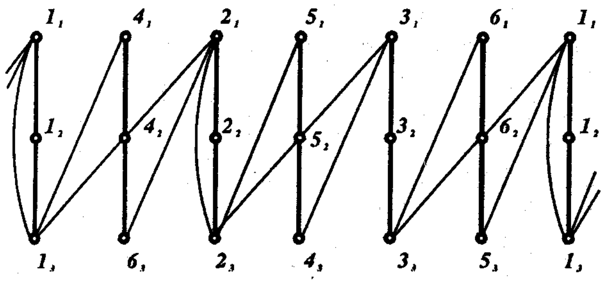

Figure 2

For n=6, the underlying factor $U(E_0)$ is shown in Figure 2 and the arrays corresponding to all factors are shown in Figure 3.

E_0	1	2	3	4	5	6
1	1					5
2		2		6		
3			3		4	
4				1		
5					2	
6						3

E_2	1	2	3	4	5	6
1		3				
2			1			
3	2					
4			5		3	
5	6					1
6		4		2		

E_1	1	2	3	4	5	6
1			2	4		
2	3				5	
3		1				6
4	4					
5		5				
6			6			

E_3	1	2	3	4	5	6
1					6	
2						4
3				5		
4		6				2
5			4	3		
6	5				1	

Figure 3

In the case $n \equiv 0(mod\ 12)$ we construct a Latin square corresponding to a non-commutative half-idempotent quasigroup.

Construction 6 $n \equiv 0(mod\ 12)$. Let $n = 12m$. First we construct an array B of order $6m$. The main diagonal is defined by $b^{i,i} = i, i = 1, 2, \dots, 6m$. The entries $b^{i,j}$, where $i - j \equiv 0(mod\ 2)$ are defined as follows. Let $2l = i - j(mod\ 6m)$, then $b^{i,j} = b^{l,l} + l$. To define the entries $b^{i,j}$, where $i - j \equiv 1(mod\ 2)$, we define $\hat{b}^{p,q}$ as the number of the set $\{1, 2, \dots, 6m\}$ such that $b^{p,q} \equiv \hat{b}^{p,q}(mod\ 6m)$. Then $b^{i,j} = \hat{b}^{i-1,j} + 6m$, i.e., $b^{p,q} \in \{6m+1, 6m+2, \dots, 12m\}$.

Then we construct a Latin square D of order $12m$ with entries $d^{i,j} = b^{i,j}$ for $1 \le i,j \le 6m$, $d^{i,j} = b^{i-6m,j}$ for $6m+1 \le i \le 12m, 1 \le j \le 6m$, $d^{i,j} = b^{i,j-6m}$ for $1 \le i \le 6m$, $6m+1 \le j \le 12m$, and $d^{i,j} = b^{i-6m,j-6m}$ for $6m+1 \le i,j \le 12m$. The triples of the 3-TD($12m$) are then $(i_1, j_2, d_3^{i,j})$. We again write usually just (i_1, j_2, d_3) instead of $(i_1, j_2, d_3^{i,j})$.

The factor E_0 contains the blocks (i_1, i_2, i_3) for $i = 1, 2, \dots, 12m$, the block $(1_1, (12m)_2, d_3)$ and the blocks $(j_1, (j-1+6m)_2, (j-1+6m)_3)$ for $j = 2, 3, \dots, 6m$. Then E_0 is the cycle $1_1 - (1_1, 1_2, 1_3) - 1_3 - ((6m+1)_1, (6m+1)_2, 1_3) - (6m+1)_2 - (2_1, (6m+1)_2, (6m+1)_3) - 2_1 - (2_1, 2_2, 2_3) - 2_3 - \dots - i_1 - (i_1, i_2, i_3) - i_3 - ((6m+i)_1$, $(6m+i)_2, i_3) - (6m+i)_2 - ((i+1)_1, (6m+i)_2, (6m+i)_3) - (i+1)_1 - ((i+1)_1, (i+1)_2, (i+1)_3) - (i+1)_3 - \dots -$ $((12m)_1, (12m)_2, (6m)_3) - (12m)_2 - (1_1, (12m)_2, (12m)_3) - 1_1$.

The other factors are defined similarly as in the case $n \equiv 6(mod\ 12)$. The factor E_1 is determined by the isomorphism $\psi_1 : E_0 \to E_1$ with $\psi_1(x_1) = x_1$, $\psi_1(y_2) = (y+6m)_2, \psi_1(z_3) = (z+6m)_3$

E_2 is determined by $\psi_2 : E_0 \to E_2$, where $\psi_2(1_1) = (6m)_2$, $\psi_2(2_1) = 1_2$, $\psi_2(3_1) = 2_2, \dots, \psi_2((6m)_1) = (6m-1)_2, \psi_2((6m+1)_1) = (12m)_2, \psi_2((6m+2)_1) = (6m+1)_2, \psi_2((6m+3)_1) = (6m+2)_2, \dots, \psi_2((12m)_1) = (12m-1)_2$, $(12m-1)_2, \psi_2(1_2) = 2_1, \psi_2(2_2) = 3_1, \psi_2(3_2) = 4_1, \dots, \psi_2((6m)_2) = 1_1$, $\psi_2((6m+1)_2) = (6m+2)_1$, $\psi_2((6m+2)_2) = (6m+3)_1, \dots, \psi_2((12m)_2) = (6m+1)_1, \psi_2(z_3) = z_3$.

E_4 is determined by $\psi_4 : E_0 \to E_4$, where $\psi_4(1_1) = 4_1, \psi_4(2_1) = 5_1$, $\psi_4(3_1) = 6_1, \dots$, $\psi_4((6m)_1) = 3_1$ $\psi_4((6m+1)_1) = (6m+4)_1$, $\psi_4((6m+2)_1) = (6m+5)_1, \dots$, $\psi_4((12m)_1) = (6m+3)_1, \psi_4(y_2) = y_2, \psi_4(1_3) = (9m+2)_3$, $\psi_4(2_3) = (2m+3)_3, \dots, \psi_4((6m)_3) = (9m+1)_3, \psi_4((6m+1)_3) = (6m)_3, \psi_4((6m+2)_3) = 1_3, \dots, \psi_4((12m)_3) = (6m-1)_3$.

In general, a factor E_t, where $t = 4u + 2v + w, 1 \le t \le 12m - 1 = n - 1$ is again determined by the isomorphism $\phi_t : E_0 \to E_t$, which is defined as the composition $\phi_t = \psi_4^u \circ \psi_2^v \circ \psi_1^w$, with $\psi_j^0 = id$.

Since we proved that for every $n \equiv 0(mod\ 6)$ there exists a 3-TD(n) which is isodecomposable into cycles, the complete characterization of *TD*'s that are isodecomposable into unicyclic factors follows immediately from the constructions and Lemma 3.

Theorem 7. A transversal design with group size n and block size 3 isodecomposable into unicyclic factors exists if and only if $n \equiv 0(mod\ 6)$. Moreover, for each such n there exists a 3-TD(n) isodecomposable into cycles.

3. Conclusion

It is an easy observation that in each cycle half of the elements of each group appear in one block and the other half in two blocks. Thus, having a transversal design decomposed into cycles, we can now combine two cycles to have each sample tested twice. Similarly, we can combine more cycles if we require another even nuber of tests for each sample. If we combine an odd number of cycles, then clearly half of the samples will be tested once more than the others.

Other desompositions of group divisible designs can indeed be constructed to satisfy specific needs of testing of several samples belonging to more than two different groups. Several methods for testing samples of one or two groups are already known.

The author believes that there are "real life" problems of similar kinds and hopes to receive a request to solve any specific problem on testing of *m*-tuples of samples. He also believes that a proper method will then be found.

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(Issued by the Faculty of Electrical Engineering and Informatics, Slovak Technical University, Bratislava).

(Only papers published in English are mentioned.)

No. 5 , 1995

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December 1995, Volume 4, Number 4