BRNO UNIVERSITY OF TECHNOLOGY
VYSOKÉ UČENÍ TECHNICKÉ V BRNĚ

FACULTY OF ELECTRICAL ENGINEERING AND COMMUNICATION
FAKULTA ELEKTROTECHNIKY
A KOMUNIKAČNÍCH TECHNOLOGIÍ

DEPARTMENT OF CONTROL AND INSTRUMENTATION
ÚSTAV AUTOMATIZACE A MĚŘICÍ TECHNIKY

APPLICATION OF GENERALIZED LAGUERRE FUNCTIONS TO SYSTEM IDENTIFICATION AND MODELING
POUŽITÍ ZOBECNĚNÝCH LAGUERROVÝCH FUNKCÍ PRO IDENTIFIKACI A MODELOVÁNÍ

DOCTORAL THESIS
DIZERTAČNÍ PRÁCE

AUTHOR
AUTOR PRÁCE
Mgr. Martin Tůma

SUPERVISOR
ŠKOLITEL
prof. Ing. Pavel Jura, CSc.

BRNO 2017

Contents

Chapter 1

Introduction

Orthogonal functions are used in many different fields thanks to their properties suitable for finite precision computation. In this thesis, we will introduce the methods for continuous-time dynamical system identification and signal compression with the use of the generalized Laguerre functions. In the second chapter, we will show the definitions and basic properties of the generalized Laguerre polynomials and functions. The third chapter is devoted to the proposed method for system identification with the generalized Laguerre functions. After some motivational examples of continuous-time approach the least squares method for continuous-time identification will be described. Then the method for system identification with the Laguerre functions will be introduced. The proposed method was presented in [1, 2, 3] and in the upcoming publication from the conference ICNAAM 2016 "Continuous Time Models Identification with Laguerre Functions", which will be published in 2017. In the fourth chapter the question of the optimal choice of the generalization parameter α and the time-scale parameter p will be studied. It will be shown that increasing the value of time scale parameter p results in faster convergence to zero and increasing the value of α results in a more slowly starting weight function under which the generalized Laguerre polynomials are developed. In the fifth chapter we will take a look on the model reduction of the large-scale identified systems. The described identification method often produces systems with orders much higher than those of the original systems. We have to reduce the order of the model. In the sixth chapter the proposed identification method and the least squares method will be compared on the examples of the system identification. Seventh chapter will deal with the signal compression using the discrete cosine transform and discrete Laguerre transform. The basic definitions will be shown and the described transforms will be compared on the examples of the signal compression. The generalized discrete Laguerre transform will be proposed and compared with standard discrete Laguerre transform and discrete cosine transform. The work from seventh chapter was presented in [4, 5]. The contribution of this thesis is the introduction of the generalized Laguerre functions into the systems and signals modeling field. The usage of the generalized Laguerre function instead of the simple Laguerre functions in these fields and searching for the optimal parameters α and p is quite a new topic. There are some articles, mostly in the field of theoretical mathematics, which deal with the generalized Laguerre functions, but these functions definitely deserve more attention in the practical applications.

In the next chapter we will start with the basic definitions of generalized Laguerre polynomials and functions.

Chapter 2

Laguerre functions and their applications

2.1 Laguerre polynomials and functions

Historically, the Laguerre polynomials were introduced by Edmond Laguerre as polynomial solutions of the Laguerre differential equation in 1879 [6]; since then, they have been widely applied on various problems in mathematics, physics, and electrical engineering.

In the following, some basic definitions will be presented. The generalized Laguerre polynomials $l_n^{(\alpha)}(t)$ are the solution of the differential equation

$$ty^{''} + (\alpha + 1 - t)y^{'} + ny = 0, \quad n \in \mathbb{N}_0, \alpha \in (-1, \infty). \tag{2.1}$$

For $\alpha = 0$ the generalized Laguerre polynomials are often called simple Laguerre polynomials in the literature. The above differential equation (2.1) can be converted into the Sturm-Liouville form by multiplying $t^\alpha e^{-t}$

$$-\frac{d}{dt}(t^{\alpha+1}e^{-t}y^{'}) = nt^\alpha e^{-t}y. \tag{2.2}$$

Thus, the extensive theory concerning the Sturm-Liouville systems (see [7]) can be used for analyzing the properties of the solutions of the above equation, i.e., the generalized Laguerre polynomials.

One of the most important properties of the orthogonal polynomials is that they satisfy the 3-term recurrence relation; the generalized Laguerre polynomials then satisfy the following formulas:

$$(n+1)l_{n+1}^{(\alpha)}(t) = (2n + 1 + \alpha - t)l_n^{(\alpha)}(t) - (n + \alpha)l_{n-1}^{(\alpha)}(t), \tag{2.3}$$

$$tl_n^{\prime(\alpha)}(t) = nl_n^{(\alpha)}(t) - (n+\alpha)l_{n-1}^{(\alpha)}(t). \tag{2.4}$$

The above-shown relations are very important for practical computation.

Further, relevant literature refers to the simple Laguerre polynomials; these can be obtained simply by putting $\alpha = 0$:

$$l_n(t) \equiv l_n^{(0)}(t). \tag{2.5}$$

The orthonormalized Laguerre polynomials are denoted as the Laguerre functions $L_n^{(\alpha)}(t)$:

$$L_n^{(\alpha)}(t) = \sqrt{\frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}} e^{-t/2} t^{\alpha/2} l_n^{(\alpha)}(t). \tag{2.6}$$

The special case for $\alpha = 0$ is

$$L_n^{(0)}(t) = e^{-\frac{t}{2}} l_n^{(0)}(t). \tag{2.7}$$

These functions are often called the simple Laguerre functions (SLF).

The time-scale parameter p can be introduced into the definition of the Laguerre polynomials and functions, see [8]

$$l_n^{(\alpha)}(t,p) \equiv l_n^{\alpha}(2pt) = \frac{t^{-\alpha}e^{2pt}}{n!}\frac{d^n}{dt^n}(e^{-2pt}t^{n+\alpha}), \tag{2.8}$$

$$L_n^{(\alpha)}(t,p) = \sqrt{\frac{2p\Gamma(n+1)}{\Gamma(n+\alpha+1)}} e^{-pt}(2pt)^{\alpha/2} l_n^{\alpha}(t,p). \tag{2.9}$$

The choice of the time-scale parameter p and the generalization parameter α is crucial for the quality of the approximation of the system with the finite series of the Laguerre functions, this problem will be discussed below.

Chapter 3

The generalized Laguerre functions for continuous-time system identification

3.1 Identification of continuous-time system from sampled data

Let us assume a continuous time linear dynamical system described by the linear differential equation with constant coefficients. Let $u(t)$ be the input signal and $y(t)$ the output signal.

$$A(p)y(t) = B(p)u(t) + e(t), \tag{3.1}$$

$$y(t) = F(p)u(t) + \xi(t), \quad \xi(t) = \frac{1}{A(p)}e(t), \tag{3.2}$$

$$F(p) = \frac{B(p)}{A(p)} = \frac{b_0p^m + b_1p^{m-1} + \ldots + b_m}{p^n + a_1p^{n-1} + \ldots + a_n}, \quad n \geq m. \tag{3.3}$$

p is the time-domain differentiation operator, i.e

$$px(t) = \frac{dx(t)}{dt}, \tag{3.4}$$

and the additive terms $e(t), \xi(t)$ represents the noise.

Given this description, the identification problem is to determine a suitable model structure for (3.2) and then estimate the parameters that characterize this structure, based on the sampled input and output data $Z^N = \{u(t_k), y(t_k)\}_{k=1}^N$.

3.2 Description of the GLF method

The history of using the Laguerre orthonormal functions in system modeling and identification since their introduction in [9, 10] and [11] is rather long, with many papers documenting the differing theoretical approaches. In [12] the Laguerre functions were applied for the identification of the finite expansion of the transfer function. The approach in [12] was further developed in [13] with the use of the Kautz functions and in [14] with the generalized orthonormal basis functions. In [15] it was proved that nth order transfer function can be expanded into the ratio of two linear combinations of n Laguerre functions and the coefficients of these combinations were identified. This was an alternative approach to the transfer function approximated by a finite sum of orthonormal basis functions in [12], [13] and [14]. In [16] the signal transformation using finitely-supported filter kernels generated from Laguerre basis functions was proposed in order to avoid the calculation with infinite integral during the expansion of observation signals with Laguerre basis functions. In this section we will present new method for identification of the dynamical systems based on the transform of their inputs and outputs instead of the expansion of the transfer functions. The inputs and outputs will be expanded into the generalized Laguerre functions basis.

Let us assume that the input and output signals are square-integrable in the Lebesgue sense, i.e.,

$$u(t), y(t) \in L_2[0, \infty). \tag{3.5}$$

Thus, we can expand the input and output signals into the generalized Laguerre function series $L_n^{(\alpha)}(t, p)$

$$L_n^{(\alpha)}(t,p) = \sqrt{\frac{2p\Gamma(n+1)}{\Gamma(n+\alpha+1)}} e^{-pt}(2pt)^{\alpha/2} l_n^{\alpha}(t,p), \tag{3.6}$$

with the time-scale parameter p and the generalization parameter α

$$u(t) = \sum_{n=0}^{\infty} U_n(\alpha_1, p_1) L_n^{(\alpha_1)}(t, p_1), \tag{3.7}$$

$$y(t) = \sum_{n=0}^{\infty} Y_n(\alpha_2, p_2) L_n^{(\alpha_2)}(t, p_2). \tag{3.8}$$

The generalized Laguerre functions are orthonormal in the $[0, \infty)$, and therefore the coefficients $U_n(\alpha_1, p_1), Y_n(\alpha_2, p_2)$ can be expressed as

$$U_n(\alpha_1, p_1) = \int_0^{\infty} x(t) L_n^{(\alpha_1)}(t, p_1) dt, \tag{3.9}$$

$$Y_n(\alpha_2, p_2) = \int_0^\infty y(t) L_n^{(\alpha_2)}(t, p_2) dt. \tag{3.10}$$

For the Laplace images of the input and output of the system (3.2), we have the equation

$$\mathcal{L}\{y(t)\} = \frac{B(s)}{A(s)} \mathcal{L}\{u(t)\}, \tag{3.11}$$

where $\mathcal{L}$ is the symbol of the Laplace transform. Additional computation then yields

$$\sum_{n=0}^{\infty} Y_n(\alpha_2, p_2) \mathcal{L}\{L_n^{(\alpha_2)}(t, p_2)\} = \frac{B(s)}{A(s)} \sum_{n=0}^{\infty} U_n(\alpha_1, p_1) \mathcal{L}\{L_n^{(\alpha_1)}(t, p_1)\}. \tag{3.12}$$

For the Laplace transform of the generalized Laguerre function, we have the identity

$$\mathcal{L}\{L_n^{(\alpha)}(t, p)\} = \Phi(n, \alpha, p) \frac{P_n^{(\alpha)}(s)}{(s+p)^{n+1+\alpha/2}}, \tag{3.13}$$

$$\Phi(n, \alpha, p) = \sqrt{(2p)^{\alpha+1} \Gamma(n+1) \Gamma(n+\alpha+1)}, \tag{3.14}$$

$$P_n^{(\alpha)}(s) = \sum_{m=0}^{n} A_{n,m}^{(\alpha)} \sum_{i=0}^{n-m} \binom{n-m}{i} s^{n-m-i} p^i, \tag{3.15}$$

$$A_{n,m}^{(\alpha)} = \frac{(-1)^m (2p)^m \Gamma(m+\alpha/2+1)}{m!(n-m)!\Gamma(m+\alpha+1)}. \tag{3.16}$$

The above expression for the Laplace transform of the generalized Laguerre functions is another form of the identity derived in [17]. By additional editing of the previously introduced formula (3.12) we obtain

$$\sum_{n=0}^{\infty} Y_n(\alpha_2, p_2) \Phi(n, \alpha_2, p_2) \frac{P_n^{(\alpha_2)}(s)}{(s+p_2)^{n+1+\alpha_2/2}} = \\ = \frac{B(s)}{A(s)} \sum_{n=0}^{\infty} U_n(\alpha_1, p_1) \Phi(n, \alpha_1, p_1) \frac{P_n^{(\alpha_1)}(s)}{(s+p_1)^{n+1+\alpha_1/2}}. \tag{3.17}$$

This expression comprises only the powers of s; thus, it is sufficient to equate the coefficients of the same powers of s in order to get the coefficients of the polynomials $A(s), B(s)$, namely to obtain the unknown transfer function $F(s)$ as a fraction of two polynomials from the time progression of the input and output signals.

For practical computation, we can obtain only a finite number of terms N_1, N_2 in the Fourier expansion series for the input and output signals:

$$u_{N_1}(t) = \sum_{n=0}^{N_1} U_n(\alpha_1, p_1) L_n^{(\alpha_1)}(t, p_1), \tag{3.18}$$

$$y_{N_2}(t) = \sum_{n=0}^{N_2} Y_n(\alpha_2, p_2) L_n^{(\alpha_2)}(t, p_2). \tag{3.19}$$

It is also possible to measure the input and output signals only for the finite time T:

$$U_n^T(\alpha_1, p_1) = \int_0^T u(t) L_n^{(\alpha_1)}(t, p_1) dt, \tag{3.20}$$

$$Y_n^T(\alpha_2, p_2) = \int_0^T y(t) L_n^{(\alpha_2)}(t, p_2) dt. \tag{3.21}$$

We can write the above equation (3.17) in the form

$$\sum_{n=0}^{N_2} Y_n^T(\alpha_2, p_2)\Phi(n, \alpha_2, p_2) \frac{P_n^{(\alpha_2)}(s)}{(s+p_2)^{n+1+\alpha_2/2}} \approx$$
$$\approx \frac{B(s)}{A(s)} \sum_{n=0}^{N_1} U_n^T(\alpha_1, p_1)\Phi(n, \alpha_1, p_1) \frac{P_n^{(\alpha_1)}(s)}{(s+p_1)^{n+1+\alpha_1/2}}. \tag{3.22}$$

After multiplying both sides of the equation by the term

$$(s+p_2)^{N_2+1+\alpha_2/2}(s+p_1)^{N_1+1+\alpha_1/2} \tag{3.23}$$

and performing some computation, we obtain the following approximation of the transfer function:

$$\tilde{F}(s, N_1, N_2, T) \approx \frac{(s+p_1)^{N_1+1+\alpha_1/2}}{(s+p_2)^{N_2+1+\alpha_2/2}} *$$
$$* \frac{\sum_{n=0}^{N_2} Y_n^T(\alpha_2, p_2)\Phi(n, \alpha_2, p_2) P_n^{(\alpha_2)}(s)(s+p_2)^{N_2-n}}{\sum_{n=0}^{N_1} U_n^T(\alpha_1, p_1)\Phi(n, \alpha_1, p_1) P_n^{(\alpha_1)}(s)(s+p_1)^{N_1-n}}. \tag{3.24}$$

The following limit holds

$$\lim_{N_1\to\infty}\lim_{N_2\to\infty}\lim_{T\to\infty}\tilde{F}(s,N_1,N_2,T)=F(s). \tag{3.25}$$

The order of the above-approximated system (3.24) is

$$N=N_1+N_2+1+\max(\alpha_1/2,\alpha_2/2). \tag{3.26}$$

The quality of the approximation depends on the numbers of the Laguerre functions $N1$, $N2$ in the truncated expansions of the input and output signals, on the time-scale parameters p_1, p_2, and on the choice of the generalization parameters α_1 and α_2. The difference δ between the order of the numerator and the denominator in the transfer function approximation (3.24) is

$$\delta=\frac{\alpha_2-\alpha_1}{2}. \tag{3.27}$$

This can lead us to the non-integer transfer function approximation when the difference $\alpha_2-\alpha_1$ is not an even integer. In the next chapter we will show how to choose the optimal parameter α and the time-scale parameter p in the case of approximating the given signal

$$x(t)\in L_2[0,\infty) \tag{3.28}$$

by the truncated series of the GLF

$$x_N(t)=\sum_{n=0}^{N}X_n(\alpha,p)L_n^{(\alpha)}(t,p). \tag{3.29}$$

Chapter 4

Optimal parameters for the truncated GLF expansion

The choice of the optimal parameter α and the time-scale parameter p during the approximation of the given signal $x(t) \in L_2[0, \infty)$ by the truncated series of the GLF was studied in [17]. In this section we will present the results that will be used for the experiments with system identification in the chapter below.

Increasing the value of time-scale parameter p results in faster convergence to zero. Increasing the value of α results in a more slowly starting weight function under which the generalized Laguerre polynomials are developed. In turn, this leads to more slowly starting generalized Laguerre functions. Thus, with α, the center of energy of these functions can be shifted in time. Equivalently, a larger value for α results in more emphasis on the lower frequency components of the functions at the expense of the higher frequency components.

For functions $x(t) \in L_2(\mathbb{R}^+)$ we have that

$$(x(t), tx''(t)) = -(x(t), x'(t)) - (x'(t), tx'(t)). \tag{4.1}$$

This gives us

$$F(\alpha, p) = \frac{\alpha^2}{8p} m_{-1} - \frac{\alpha + 1}{2} m_0 + \frac{p}{2} m_1 + \frac{1}{2p} m_2, \tag{4.2}$$

where the moments m_{-1} to m_2 are defined as

$$m_{-1} = (x(t), \frac{1}{t} x(t)), \tag{4.3}$$

$$m_0 = (x(t), x(t)), \tag{4.4}$$

$$m_1 = (x(t), tx(t)), \tag{4.5}$$

$$m_2 = (x'(t), tx'(t)). \tag{4.6}$$

Setting the derivative $F(\alpha, p)$ with respect to p equal to zero yields a good time-scale parameter p for a given order of generalization α

$$p(\alpha) = \frac{\sqrt{\frac{\alpha^2 m_{-1} + 4m_2}{m_1}}}{2}. \tag{4.7}$$

Setting the derivatives of $F(\alpha, p)$ with respect to α and p both equal to zero yields one unique solution for a good time-scale parameter p and good order of generalization α

$$p = \sqrt{\frac{m_{-1} m_2}{|m_1 m_{-1} - m_0^2|}}, \tag{4.8}$$

$$\alpha = \frac{m_0}{m_{-1}} 2p. \tag{4.9}$$

The described procedure does not require complete knowledge of $x(t)$, only a few specific measurements of the function need to be known, and is thus useful in practical situations with experimental data. According to [17], the obtained time-scale parameter p and the order of generalization α are the best that can be found when only the basic measurements of the function $x(t)$ are available.

Chapter 5

Model reduction for GLF approximations

The described approximation method often produces systems with orders much higher than those of the original systems; we can therefore reduce the order of the model.

The representation of the transfer function (3.3) of the given system can be transformed into state space representation:

$$\sum = \begin{cases} \frac{dx(t)}{dt} &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t). \end{cases} \tag{5.1}$$

The equations above can be written in the following simple notation for the SISO LTI system:

$$\sum = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathbb{R}^{(N+1)\times(N+1)}. \tag{5.2}$$

We have to find the approximation $\hat{\sum}$ of $\sum$

$$\hat{\sum} = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right] \in \mathbb{R}^{(K+1)\times(K+1)}, \tag{5.3}$$

where $K \ll N$. This can be achieved via various methods, see [18, 19]. In this work, we will use the balanced truncation of the system based on omitting the part of the system corresponding to the $N - K$ smallest Hankel singular values in the Singular value decomposition of (5.2). This model reduction scheme is well grounded in theory and it is commonly used in practical computations. It was introduced in [20, 21] to the systems and control literature.

Chapter 6

Examples of system identification with GLF

6.1 Experiments with systems generated in Matlab

The MATLAB program was used to enable the practical implementation of the idea defined above. In the first section we will demonstrate the system identification with dynamical system which were generated in MATLAB. The input signals for the identification were chosen as $u(t) = e^{-t} = L_0^{(0)}(t, 1/2)$, $u(t) = te^{-t}$ and $u(t) = e^{-t}\cos(t)$. In the figures, the step input response of the reduced approximated systems of the order K with the simple Laguerre functions, with the generalized Laguerre functions and with LS method with SVF given by $L(s) = \frac{\lambda}{(s+\lambda)^n}$ is shown. The n is given by the order of the original system, λ is chosen larger than the guessed bandwidth (half of the sampling rate f_s), see [22]. The corresponding relative RMS errors (rRMSE)

$$rRMSE = \frac{RMS(y(t) - \hat{y}(t))}{RMS(y(t))} * 100\% \tag{6.1}$$

and graphs of relative approximation errors

$$\frac{y(t_k) - \hat{y}(t_k)}{y(t_k)} \tag{6.2}$$

are displayed, where $y(t)$ is step input response of the original system and $\hat{y}(t)$ is step input response of the approximated systems. The optimal parameters α and p are chosen for the GLF approximation according to the above chapter. The dominant

Hankel singular values of the approximated systems of the order N are visualized on a logarithmic scale. The number of dominant Hankel singular values can help us to find the order of the original system and to appropriately choose the order K of the reduced system. In this reduced version of Ph.D. thesis an example of one system is presented.

The 1st system is a first order dynamical system with one real pole:

$$F(s) = \frac{1}{5s+1}, \qquad u(t) = e^{-t}. \tag{6.3}$$

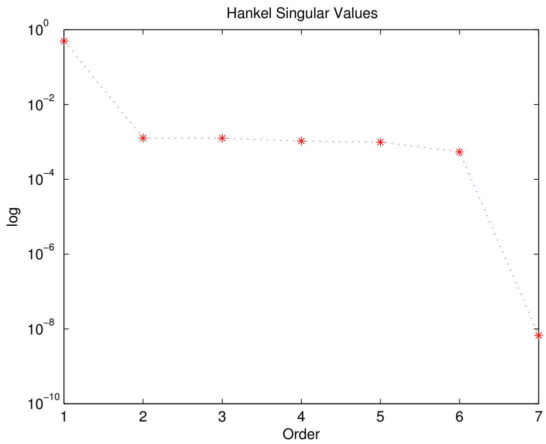


Figure 6.1.1: Hankel singular values of the GLF approximation of the 1st system

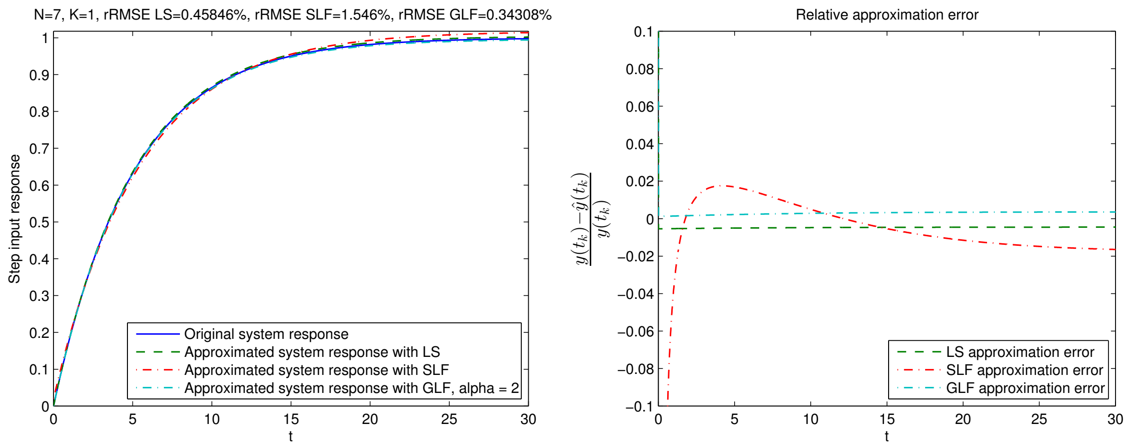


Figure 6.1.2: Step input response and relative approximation error of the 1st system

6.2 Experiments with real-life systems

For the experiments with real life data the universal modules FM2016 for realization of physical models of differential equations were made. The universal modules FM2016 were developed with the help of Czech Science Foundation under the project 16-08549S. Every equation is made from several modules. The universal module design was made. The parts of differential equations were modeled by plugging different components into the universal module. The universal module design is shown in the following figures. The input signals for identification were chosen as $u(t) = 10e^{-8t}$ and $u(t) = 10e^{-8t}cos(t)$. In the figures, the step input response of the reduced approximated systems of the order K with the simple Laguerre functions, with the generalized Laguerre functions and with LS method with SVF. The rRMSE is shown and the graphs of the approximation error and Hankel singular values for the GLF approximation are displayed. In this reduced version of Ph.D. thesis an example of one system is presented.

The 10^{th} system is a second order dynamical system with two real poles:

$$F(s) = \frac{0.5}{(0.492s+1)(0.267s+1)}, \qquad u(t) = 10e^{-8t}. \tag{6.4}$$

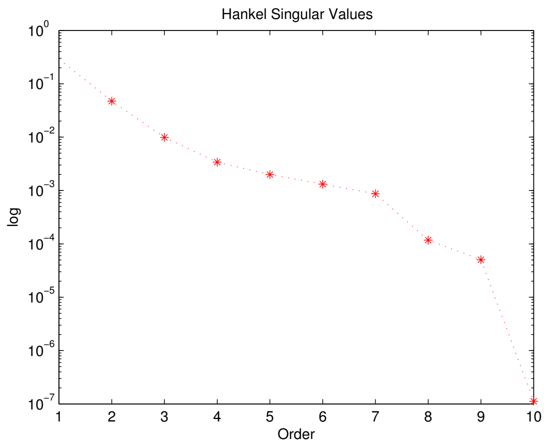


Figure 6.2.1: Hankel singular values of the GLF approximation of the 10^{th} system

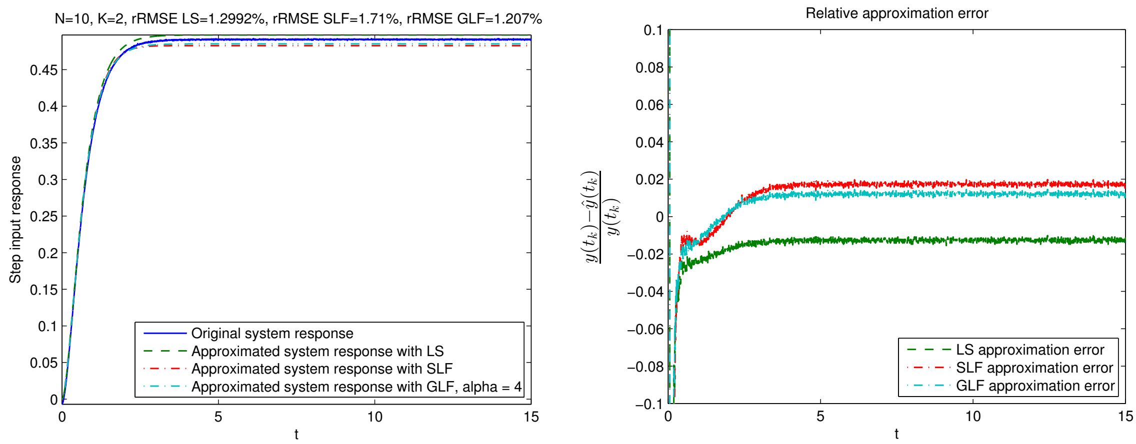


Figure 6.2.2: Step input response and relative approximation error of the 10^{th} system

Chapter 7

DLT for data compression

7.1 Discrete Laguerre and cosine transforms

In this chapter we will give the short introduction of the discrete orthogonal transforms. The comparison between the discrete Laguerre and cosine transforms (DLT, DCT) when applied on the data compression task will be presented. The impact of the choice of the optimal generalization parameter α during the discrete Laguerre transform with GLF basis functions will be presented.

The DCT was introduced in 1974 into electrical engineering literature by N. Ahmed, T. Natarajan and K.R. Rao in their article [23]. It is the real version of the discrete Fourier transform. Nowadays DCT and its modifications like the modified discrete cosine transform are the cores of many algorithms for data compression and signal processing. For example, DCT is used in the JPG and MP3 algorithms for image and sound processing. The main idea behind the use of the orthogonal transforms for data compression is their so-called “energy compaction property”, see [24]. It means that the most of the information is stored in the first few Fourier coefficients of the Fourier series for the original data.

A classical method for generating discrete orthogonal transforms is to start with an orthonormal set of polynomials and then use the Gauss-Jacobi procedure to generate the transform matrix. Using this procedure one can derive new transforms.

We will present short overview of the discrete orthogonal transforms, see [25] for more details. The coefficients of the discrete orthogonal transform of the following vector $z \in \mathbb{R}^{N+1}$ are defined by the following expression

$$c_n = \sum_{k=0}^{N} [w_k \phi_n(t_k)] z_k, \tag{7.1}$$

where $\phi_n(t)$ is set of orthogonal functions and t_k are roots of the following equation

$$\phi_{N+1}(t_k) = 0, \quad k = 0, 1, \ldots, N. \tag{7.2}$$

Coefficients w_k are weights of the Gauss quadrature theory, see [26] for the derivation of the weights w_k. The discrete orthogonal transform can be written in the following matrix form

$$c = Bz, \tag{7.3}$$

where

$$c = [c_0 c_1 \ldots c_N]^T, \tag{7.4}$$

$$z = [z_0 z_1 \ldots z_N]^T. \tag{7.5}$$

The matrix B is given by

$$B = \begin{bmatrix} w_0\phi_0(t_0) & w_1\phi_0(t_1) & \ldots & w_N\phi_0(t_N) \\ w_0\phi_1(t_0) & w_1\phi_1(t_1) & \ldots & w_N\phi_1(t_N) \\ \vdots & \vdots & \ddots & \vdots \\ w_0\phi_N(t_0) & w_1\phi_N(t_1) & \ldots & w_N\phi_N(t_N) \end{bmatrix}. \tag{7.6}$$

Matrix B can be written as a product of two matrices

$$B = \Phi^T W, \tag{7.7}$$

where

$$\Phi = \begin{bmatrix} \phi_0(t_0) & \phi_1(t_0) & \ldots & \phi_N(t_0) \\ \phi_0(t_1) & \phi_1(t_1) & \ldots & \phi_N(t_1) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_0(t_N) & \phi_1(t_N) & \ldots & \phi_N(t_N) \end{bmatrix}. \tag{7.8}$$

Matrix W is diagonal matrix with the weights w_k in the main diagonal, i.e.

$$W = \begin{bmatrix} w_0 & 0 & \ldots & 0 \\ 0 & w_1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & w_N \end{bmatrix}. \tag{7.9}$$

The inverse discrete orthogonal transform is given by the generalized Fourier series of the following form

$$z_k = \sum_{n=0}^{N} c_n \phi_n(t_k). \tag{7.10}$$

The above expression can be written in the matrix form

$$z = \Phi c. \tag{7.11}$$

Vectors $\{\phi_n(t_k)\}_{n=0}^{N}$ form orthogonal basis in $\mathbb{R}^{N+1}$, i.e.

$$\sum_{k=0}^{N} w_k \phi_m(t_k)\phi_n(t_k) = \delta_{mn}. \tag{7.12}$$

The above equation can be written in the following matrix form

$$\Phi B = I_{N+1}, \tag{7.13}$$

with the unity matrix I_{N+1}.

The DCT uses as basis functions ϕ_n the Chebyshev polynomials. Although there are many articles about the Laguerre polynomials and functions, the transform similar to DCT based on the Laguerre orthonormal functions wasn't introduced till 1995 when the article [26] appeared. In that article the DLT was defined with the help of Gauss-Laguerre integration in the similar way as the other finite orthonormal transforms. It was suggested, that this transform could lead to the better results in the data compression tasks than the DCT. It means that the DLT have the same energy compaction property as the DCT. This will work especially for the vectors, that decay exponentially to zero, i.e., that have the similar behavior as the Laguerre basis functions.

In [26] the coefficients w_k (7.9) were derived for simple Laguerre functions basis transform, i.e.

$$w_k = -\frac{a_{N+2}}{a_{N+1}} \frac{1}{L_{N+1}^{\prime(0)}(t_k, p) L_{N+2}^{(0)}(t_k, p)}, \qquad k = 0, 1, \ldots, N, \tag{7.14}$$

where t_k are roots of $L_{N+1}^{(0)}(t, p)$ and a_N are the coefficients of the terms x^N in $L_N^{(0)}(t, p)$,

$$\frac{a_{N+2}}{a_{N+1}} = -\frac{1}{N+2}. \tag{7.15}$$

The following recurrence equations hold (see (2.3) and (2.4))

$$2ptL'^{(0)}_{N+1}(t,p) = (N+1)L^{(0)}_{N+1}(t,p) - (N+1)L^{(0)}_{N}(t,p) = \tag{7.16}$$

$$= (2pt - N)L^{(0)}_{N+1}(t,p) + (N+2)L^{(0)}_{N+2}(t,p). \tag{7.17}$$

We can consider that t_k are roots of $L^{(0)}_{N+1}(t,p)$, so

$$(2pt_k - N)L^{(0)}_{N+1}(t_k,p) = 0. \tag{7.18}$$

It means that

$$2pt_kL'^{(0)}_{N+1}(t_k,p) = (N+2)L^{(0)}_{N+2}(t_k,p). \tag{7.19}$$

The above expression (7.14) for the weights w_k can be written in the following form

$$w_k = \frac{2pt_k}{(N+2)^2[L^{(0)}_{N+2}(p,t_k)]^2}. \tag{7.20}$$

Note that the derived weights w_k for DLT are the Gauss-Laguerre weights of the Gauss-Laguerre quadrature rule, see [27].

We can derive similar formula for generalized Laguerre functions basis. We can take the definition of the Gauss-Laguerre quadrature weights for generalized Laguerre polynomials basis (2.8) from [27], i.e.

$$w_k = \frac{\Gamma(\alpha+N+2)t_k}{(N+1)!(N+2)^2[l^{(\alpha)}_{N+2}(p,t_k)]^2(2pt_k)^{\alpha}e^{-2pt_k}}. \tag{7.21}$$

When we replace the generalized Laguerre polynomials (2.8) by the generalized Laguerre functions (2.9) in the above equation we can get the sought formula for the Gauss-Laguerre weights, i.e.

$$w_k = \frac{2pt_k}{(N+2)(\alpha+N+2)[L^{(\alpha)}_{N+2}(p,t_k)]^2}. \tag{7.22}$$

Note that, when

$$\alpha = 0 \tag{7.23}$$

the above formula reduces to the formula for the simple Laguerre function basis (7.20), which was derived in [26].

Since 1995 the DLT was used in the modeling only few times. The article [28] was published in 1995 after the original article about DLT. In [28] there was shown the application of the DLT to the speech coding. The DLT was compared to DCT in the classic speech coding algorithm [29]. It was shown, that it outperforms the DCT at low bitrates. In 2000 and 2001 the DLT was applied to the digital image watermarking by M.S.A. Gilani and A.N. Skodras in their articles [30, 31] and [32]. It was shown that the image quality is better with the use of the DLT instead of the classical approach with the DCT.

In the next section we will show the examples of the data compression with DLT similar to the examples of the data compression in the article [26]. We will demonstrate that with the proper choice of the generalization parameter α we can achieve even better results than in the [26]. We will use DLTopt: DLT based on the generalized Laguerre functions with the optimal choice of the parameter α, see (4.9) with the given time-scale parameter $p = \frac{1}{2}$. The examples of the data compression using DCT, DLT, DLTopt will be presented.

7.2 Examples of the data compression using DCT and DLT

Now the following data compression task for $z \in \mathbb{R}^{N+1}$ will be presented. Let's consider the generalized Fourier series for the vector z, i.e.,

$$z = \sum_{n=0}^{N} c_n \phi_n,$$

where $\{c_k\}$ are the Fourier coefficients for some orthonormal basis $\{\phi_n\}$ of $\mathbb{R}^{N+1}$. Now consider the truncated expansion for some $K \leq N$, i.e.,

$$\hat{z} = \sum_{n=0}^{K} c_n \phi_n.$$

The vector reconstruction $\hat{z}$ is the approximation of the vector z. This move from the vector z to the vector $\hat{z}$ is often called the compression of the vector z or simply the reduction of the model.

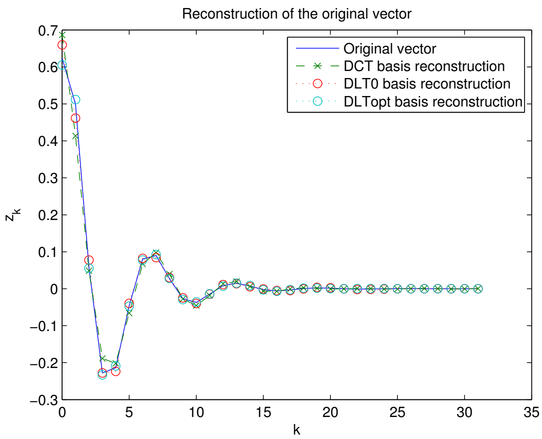
In the following there are the pictures of the vector of length 32 reconstruction for $K = 12, 16, 28$ using the discrete cosine basis (DCT), simple Laguerre functions basis (DLT) and generalized Laguerre functions basis (DLTopt) with the optimal choice of the parameter α. The graphs and tables of the relative compression error (rCE)

$$rCE = \frac{\|z - \hat{z}\|_2}{\|z\|_2} \tag{7.24}$$

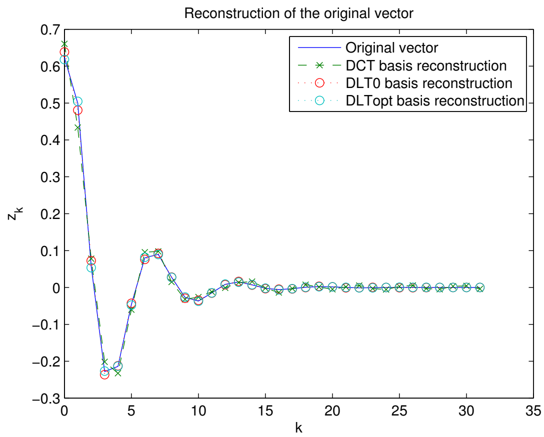
are shown with $K = 12, 16, 20, 24, 28$. All the experiments were done in MATLAB.

The 1. vector is sampled exponentially damped sine function. The results for this vector with DCT and DLT basis were presented in [26], p. 11.

$$z_k = e^{-0.3(k+1)} \sin(k+1), \qquad k = 0, 1, \ldots, 31.$$

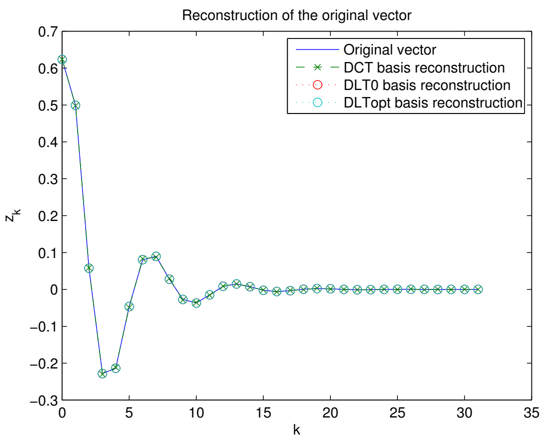


(a) Reconstruction for $K = 12$

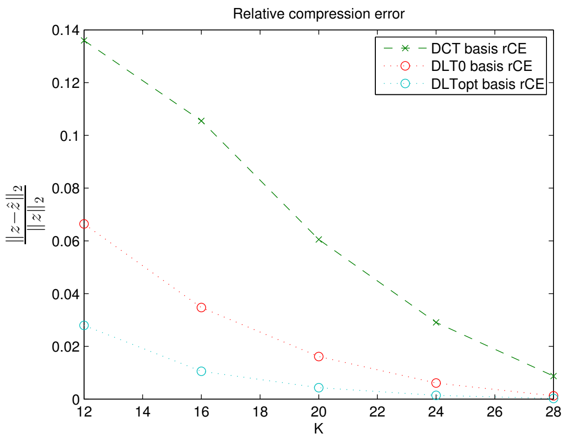


(b) Reconstruction for $K = 16$

Figure 7.2.1: Reconstruction of the 1. vector



(a) Reconstruction for $K = 28$



(b) Relative compression error

Figure 7.2.2: Reconstruction of the 1. vector

Chapter 8

Conclusion and future work

In this thesis the generalized Laguerre function were introduced and presented as useful tool in the identification and modeling field. The idea that the optimal choice of the time-scale parameter p and the generalization parameter α can bring better results than the usage of the simple Laguerre functions was presented in the two fields.

The GLF method for identification of continuous-time dynamical systems from sampled data was introduced in the chapter 3. The method for the optimal choice of the optimal time-scale parameter p and the generalization parameter α for the truncated GLF expansion was presented in the chapter 4. The GLF method was compared with SLF method and least squares SVF method on the continuous-time dynamical systems. The application of the generalized Laguerre functions in GLF method with appropriate choice of the parameters α and p can bring better results than the usage of the simple Laguerre functions in the SLF method and it is comparable with the traditional least squares SVF method. The number of dominant Hankel singular values in the GLF approximated system gives the idea of the order of the identified system. The results of the application of the generalized Laguerre functions were published in [1, 2, 3].

The introduction of discrete orthogonal transform using the generalized Laguerre functions was made in the chapter 7 It was shown not only that DLT can outperform the traditional approach with DCT but also that DLT with optimal choice of the generalization parameter α DLTopt can bring even better results. The work with DLT and DLTopt was published in [4, 5].

In the future work we will focus on the more precise choice of the optimal parameters α and p in approximation of the transfer function (3.24) without the substitution of α by the nearest even number. The substitution of the of the nearest even number is a must when we want to avoid the non-integer transfer function approximation (3.24).

The possible non-integer transfer function approximations with the optimal choice of parameters α and p will be examined. The fractional order transfer function can be described by the following formula, see [33]

$$a_0 D^{\alpha_n} y(t) + a_1 D^{\alpha_{n-1}} y(t) + \ldots + a_n D^{\alpha_0} y(t) = \tag{8.1}$$

$$= b_0 D^{\beta_m} u(t) + b_1 D^{\beta_{m-1}} u(t) + \ldots + b_m D^{\beta_0} u(t), \tag{8.2}$$

where D^{γ} is fractional derivative

$$D^{\gamma} f(t) = \lim_{h \to 0} h^{-\gamma} \sum_{j=0}^{\left[\frac{t}{h}\right]} (-1)^j \binom{\gamma}{j} f(t - jh). \tag{8.3}$$

Bibliography

[1] M. Tuma, “Laguerre functions in electrical engineering,” in *Student EEICT proceedings of the 18th conference*, pp. 303–307, 2012.

[2] M. Tuma and P. Jura, “Application of laguerre functions to system modeling,” in *MENDEL 2013*, pp. 403–408, 2013.

[3] M. Tuma and P. Jura, “Dynamical system identification with the generalized laguerre functions,” in *2015 7th International Congress on Ultra Modern Telecommunications and Control Systems and Workshops (ICUMT)*, pp. 220–225, Oct 2015.

[4] M. Tuma, “Application of laguerre functions to data compression,” in *GRANT Journal*, vol. 1, pp. 54–57, 2013.

[5] M. Tuma, “The choice of the optimal parameter in the data compression task using generalized laguerre functions,” in *GRANT Journal*, vol. 2, pp. 67–69, 2013.

[6] E. Laguerre, “Sur l'integrale $\int_x^\infty \frac{e^{-x}}{x} dx$,” *Bull. Soc. math. France*, vol. 7, pp. 72–81, 1879.

[7] W. N. Everitt, *A Catalogue of Sturm-Liouville Differential Equations*, pp. 271–331. Basel: Birkhäuser Basel, 2005.

[8] A. Back and A. Tsoi, “Nonlinear system identification using discrete laguerre functions,” *Journal of Systems Engineering*, vol. 6, 1996.

[9] Y. W. Lee, “Synthesis of electric networks by means of the fourier transforms of laguerre's functions,” *Journal of Mathematics and Physics*, vol. 11, no. 1-4, pp. 83–113, 1932.

[10] N. Wiener, *Extrapolation, Interpolation, and Smoothing of Stationary Time Series.* The MIT Press, 1964.

[11] Y. W. Lee, "Statistical theory of communication," *Science*, vol. 132, no. 3439, pp. 1546–1547, 1960.

[12] B. Wahlberg, "System identification using laguerre models," *IEEE Transactions on Automatic Control*, vol. 36, pp. 551–562, May 1991.

[13] B. Wahlberg, "System identification using kautz models," *IEEE Transactions on Automatic Control*, vol. 39, pp. 1276–1282, June 1994.

[14] P. V. D. Hof, P. Heuberger, and J. Bokor, "System identification with generalized orthonormal basis functions," *Automatica*, vol. 31, no. 12, pp. 1821–1834, 1995.

[15] C. T. Chou, M. Verhaegen, and R. Johansson, "Continuous-time identification of siso systems using laguerre functions," *IEEE Transactions on Signal Processing*, vol. 47, pp. 349–362, Feb 1999.

[16] Y. Kinoshita and Y. Ohta, "Continuous-time system identification using compactly-supported filter kernels generated from laguerre basis functions," in *49th IEEE Conference on Decision and Control (CDC)*, pp. 4461–4466, Dec 2010.

[17] H. J. W. Belt and A. C. den Brinker, "Optimal parametrization of truncated generalized laguerre series," in *Acoustics, Speech, and Signal Processing, 1997. ICASSP-97., 1997 IEEE International Conference on*, vol. 5, pp. 3805–3808, Apr 1997.

[18] A. C. Antoulas and D. C. Sorensen, "Approximation of large-scale dynamical systems: An overview," pp. 1093–1121, 2001.

[19] S. Gugercin and A. C. Antoulas, "A survey of model reduction by balanced truncation and some new results," *International Journal of Control*, vol. 77, no. 8, pp. 748–766, 2004.

[20] C. Mullis and R. Roberts, "Synthesis of minimum roundoff noise fixed point digital filters," *IEEE Transactions on Circuits and Systems*, vol. 23, pp. 551–562, Sep 1976.

[21] B. Moore, "Principal component analysis in linear systems: Controllability, observability, and model reduction," *IEEE Transactions on Automatic Control*, vol. 26, pp. 17–32, Feb 1981.

[22] L. Ljung, "Experiments with identification of continuous time models," 15th IFAC Symposium on System Identification, (Saint-Malo, France), 2009.

[23] N. Ahmed, T. Natarajan, and K. R. Rao, "Discrete cosine transform," *IEEE Transactions on Computers*, vol. C-23, pp. 90–93, Jan 1974.

[24] H. B. Kekre, S. Thepade, A. Athawale, A. Shah, P. Verlekar, and S. Shirke, *Performance evaluation of image retrieval using energy compaction and imagetiling over DCT row mean and DCT column mean*, pp. 158–167. New Delhi: Springer India, 2011.

[25] A. Bizopoulos, P. I. Lazaridis, T. Panagiotis, Z. Zaharias, G. Debarge, and P. Gallion, "Comparative study of dct and discrete legendre transform for image compression," in *ETAI 2011*, (OHRID Macedonia), Sept. 2011.

[26] G. Mandyam and N. Ahmed, "The discrete laguerre transform: derivation and applications," *IEEE Transactions on Signal Processing*, vol. 44, pp. 2925–2931, Dec 1996.

[27] M. Abramowitz, *Handbook of Mathematical Functions, With Formulas, Graphs, and Mathematical Tables,*. Dover Publications, Incorporated, 1974.

[28] G. Mandyam, N. Ahmed, and N. Magotra, "Application of the discrete laguerre transform to speech coding," in *Conference Record of The Twenty-Ninth Asilomar Conference on Signals, Systems and Computers*, vol. 2, pp. 1225–1228 vol.2, Oct 1995.

[29] R. Zelinski and P. Noll, "Adaptive transform coding of speech signals," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 25, pp. 299–309, Aug 1977.

[30] M. Gilani and A. Skodras, "Dlt-based digital image watermarking," in *First IEEE Balkan Conference on Signal Processing, Communications, Circuits and Systems*, Jun 2000.

[31] M. Gilani and A. Skodras, "Robust image watermarking by subband dlt," in *Int. Conf. on Media Future*, May 2001.

[32] M. Gilani and A. Skodras, "Watermarking of images in the dlt domain," tech. rep., Computer Technology Institute, Mar 2000.

[33] Y. Chen, I. Petras, and D. Xue, "Fractional order control - a tutorial," in *2009 American Control Conference*, pp. 1397–1411, June 2009.

[34] M. Tuma, "Reduction of a linear dynamical system and the problem of moments," in *STUDENT EEICT PROCEEDINGS OF THE 17th CONFERENCE*, vol. 3, pp. 396–400, 2011.

[35] M. Tuma, "Od problemu momentu k modernim iteracnim metodam," in *32. mezinarodni konference HISTORIE MATEMATIKY*, vol. 1, pp. 271–274, 2011.

[36] M. Tuma, "Interdisciplinary approach with the problem of moments," in *Sbornik prispevku Mezinarodni Masarykovy konference pro doktorandy a mlade vedecke pracovniky*, vol. 1, pp. 1701–1705, 2011.

[37] M. Tuma, "The problem of moments and its connections," in *Seminar of numerical analysis - Modeling and Simulation of Challenging Engineering Problems*, vol. 1, pp. 112–114, 2011.

Abstract: This thesis deals with the use of the Laguerre functions in system identification and modeling. After the short introduction the definition of the Laguerre polynomials and functions is given. The method for system identification of the continuous-time dynamical systems from discrete sampled data with generalized Laguerre functions is introduced and compared with the least squares method. The choice of the optimal parameters p and α for finite Fourier series with generalized Laguerre functions is discussed. The proposed identification method is compared with the least squares method on the examples. The discrete Laguerre transform with simple and generalized Laguerre functions is introduced. The application of the discrete Laguerre transform on the data compression is shown on the examples. It is pointed out that the discrete Laguerre transform can give better results than discrete cosine transform in the task of the data compression.

Keywords: generalized Laguerre function, compression, polynomial, identification, modeling

Abstrakt: Tato práce se zabývá použitím zobecněných Laguerrových funkcí pro modelování a identifikaci dynamických systémů. Po krátkém úvodu je uvedena definice zobecněných Laguerrových polynomů a funkcí a některé jejich vlastnosti. Je zavedena metoda pro identifikaci spojitých dynamických systémů z diskrétních nasamplovaných dat s použitím zobecněných Laguerrových funkcí a porovnána s metodou nejmenších čtverců. Je diskutována volba optimálních parametrů p a α pro konečné Fourierovy řady se zobecněnými Laguerrovými funkcemi. Na příkladech je porovnána navržená identifikační metoda s metodou nejmenších čtverců. Je zavedena diskrétní Laguerrova transformace a její rozšíření s použitím zobecněných Laguerrových funkcí. Na příkladech je předvedena aplikace diskrétní Laguerrovy transformace na kompresi dat. Je ukázáno, že použití diskrétní Laguerrovy transformace může vést k lepším výsledků při kompresi dat než tradiční postup s využitím diskrétní kosinové transformace.

Klíčová slova: zobecněná Laguerrova funkce, komprese, polynom, identifikace, modelování

TŮMA, M. Application of generalized Laguerre functions to system identification and modeling - short version. Brno: Vysoké učení technické v Brně, Fakulta elektrotechniky a komunikačních technologií, 2017. 34 s. Vedoucí dizertační práce prof. Ing. Pavel Jura, CSc..

Short curriculum vitae of the author

Name: Mgr. Martin Tůma

Date of birth: 20.1.1986

Address: Hlavní 21, Vojkovice, 66701

Education

2010–today: Doctoral degree study, Faculty of Electrical Engineering and Communication, Brno University of Technology

2008–2010: Mgr.(Msc) in Numerical Mathematics, Faculty of Mathematics and Physics, Charles University in Prague

2005–2008: Bc.(Bsc) in General Mathematics, Faculty of Mathematics and Physics, Charles University in Prague

Work experience

2013–today: Ph.D. student, Central European Institute of Technology - Brno University of Technology (CEITEC BUT)

2011–2012: OPVK CRR project assistant, Brno University of technology

2008–2010: Member of the research team Theory of Krylov subspace methods and its relationship to other mathematical disciplines, Academy of Sciences of the Czech Republic

Publications

- M. Tuma, “Reduction of a linear dynamical system and the problem of moments,” in *STUDENT EEICT PROCEEDINGS OF THE 17th CONFERENCE*, vol. 3, pp. 396–400, 2011
- M. Tuma, “Od problemu momentu k modernim iteracnim metodam,” in *32. mezinarodni konference HISTORIE MATEMATIKY*, vol. 1, pp. 271–274, 2011

- M. Tuma, “Interdisciplinary approach with the problem of moments,” in *Sbornik prispevku Mezinarodni Masarykovy konference pro doktorandy a mlade vedecke pracovniky*, vol. 1, pp. 1701–1705, 2011
- M. Tuma, “The problem of moments and its connections,” in *Seminar of numerical analysis - Modeling and Simulation of Challenging Engineering Problems*, vol. 1, pp. 112–114, 2011
- M. Tuma, “Laguerre functions in electrical engineering,” in *Student EEICT proceedings of the 18th conference*, pp. 303–307, 2012
- M. Tuma and P. Jura, “Application of laguerre functions to system modeling,” in *MENDEL 2013*, pp. 403–408, 2013
- M. Tuma, “Application of laguerre functions to data compression,” in *GRANT Journal*, vol. 1, pp. 54–57, 2013
- M. Tuma, “The choice of the optimal parameter in the data compression task using generalized laguerre functions,” in *GRANT Journal*, vol. 2, pp. 67–69, 2013
- M. Tuma and P. Jura, “Dynamical system identification with the generalized laguerre functions,” in *2015 7th International Congress on Ultra Modern Telecommunications and Control Systems and Workshops (ICUMT)*, pp. 220–225, Oct 2015