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Research Article

A Novel Iterative Scheme and Its Application to Differential Equations

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The purpose of this paper is to employ an alternative approach to reconstruct the standard variational iteration algorithm II proposed by He, including Lagrange multiplier, and to give a simpler formulation of Adomian decomposition and modified Adomian decomposition method in terms of newly proposed variational iteration method-II (VIM). Through careful investigation of the earlier variational iteration algorithm and Adomian decomposition method, we find unnecessary calculations for Lagrange multiplier and also repeated calculations involved in each iteration, respectively. Several examples are given to verify the reliability and efficiency of the method.

1. Introduction

Over the last few decades several analytical/approximate methods have been developed to solve nonlinear ordinary and partial differential equations. For initial and boundary-value problems in ordinary and partial differential equations, some of these techniques include the perturbation method [1], the variational iteration method [2–4], the decomposition method [5–8], and the homotopy methods [9–11].

The Adomian decomposition method [12–16] for solving differential and integral equations, linear or nonlinear, has been the subject of extensive analytical and numerical studies. The method, well addressed in [12–16], has a significant advantage in which it provides the solution in a rapid convergent series with elegantly computable components. In recent years, a large amount of literature has been developed concerning the application of Adomian decomposition method in applied sciences. In addition, the method reveals the analytical structure of the solution which is absent in numerical solutions.

He's variational iteration method [2–4] is based on a Lagrange multiplier technique developed by Inokuti et al. [17]. This method is, in fact, a modification of the general Lagrange multiplier method into an iteration method, which is called correction functional. The method has been shown to solve effectively, easily, and accurately a large class of nonlinear problems [18–23]. Generally, one or two iterations lead to high accurate solutions.

In the present study, we have linked up variational iteration method and Adomian decomposition method through Lagrange multiplier, which shows that VIM is another form of expressing ADM and vice versa. This study reveals that there is no need to integrate the differential equation again and again as we do in Adomian decomposition method. Advantage of new iterative scheme over the variational iteration method is that it avoids the unnecessary calculations and we can construct Lagrange multiplier very easily without construction of the correctional functional.

2. New Formulation for Adomian Decomposition Method and Variational Iteration Algorithm II

In order to elucidate the solution procedure, we consider the following nth order partial differential equation:

$$L^n f(x,t) = Rf(x,t) + Nf(x,t) + g(x,t), \quad t > 0,\ x \in L, \tag{1}$$

where $L^n = \partial^n/\partial t^n$, $n \geq 1$, R is a linear differential operator, N is a nonlinear differential operator, R and N are free of partial derivative with respect to variable t, and g is the source term. As we are familiar with the fact that in all kinds of iteration techniques, except the operator rest of the terms, are treated as a known function on the behalf of initial guess. In this present newly proposed idea, we have used the same concept. We have bound all terms in one function except operator. Consider

$$g + Nf + Rf = F\left(t, x, g, f, \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}, \ldots\right). \tag{2}$$

By incorporating (2) in (1), we get

$$L^n f = F\left(t, x, g, f, \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}, \ldots\right). \tag{3}$$

On integrating (3), we obtain

$$L^{(n-1)} f = \int_0^t F\left(\xi, x, g, f, \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}, \ldots\right) d\xi + c_1(x). \tag{4}$$

Again, by integrating (4), we have

$$\begin{aligned} L^{(n-2)} f = \int_0^t \int_0^\xi F\left(\tau, x, g, f, \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}, \ldots\right) d\tau d\xi + c_1(x)\, t \\ + c_2(x), \end{aligned} \tag{5}$$

since we know that multiple integral can be reduce to a single integral by using integral property. Hence, we can write (5) in the following form:

$$\begin{aligned} L^{(n-2)} f = \int_0^t (t-\xi) F\left(\xi, x, g, f, \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}, \ldots\right) d\xi + c_1(x)\, t \\ + c_2(x). \end{aligned} \tag{6}$$

If we continue this process of integration, we can get final form as follows:

$$\begin{aligned} f(x,t) = \int_0^t \frac{(t-\xi)^{n-1}}{(n-1)!} F\left(t, x, g, f, \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}, \ldots\right) d\xi \\ + \frac{c_1(x)\, t^{n-1}}{(n-1)!} + \frac{c_2(x)\, t^{n-2}}{(n-2)!} + \cdots c_n(x). \end{aligned} \tag{7}$$

By writing the constant of integration in the form $c_k(x) = (\partial f^{n-k}(x, 0^+))/\partial t^{n-k}$, $k = 1, \ldots, n$ and substituting (2) in (7) then (7), we have

$$\begin{aligned} f(x,t) = \sum_{k=0}^{n-1} \frac{\partial^k f(x, 0^+)}{\partial t^k} \frac{t^k}{k!} \\ + \int_0^t \frac{(t-\xi)^{n-1}}{(n-1)!} \left(Rf + Nf + g\right) d\xi. \end{aligned} \tag{8}$$

In iteration form (8), it can be written as follows:

$$\begin{aligned} f_{j+1}(x,t) = f_0(x,t) + \int_0^t \frac{(t-\xi)^{n-1}}{(n-1)!} \left(Rf_j + Nf_j + g\right) d\xi, \\ j = 0, 1, 2, \ldots, \end{aligned} \tag{9}$$

where $f_0(x,t) = \sum_{k=0}^{n-1} ((\partial^k f(x, 0^+))/\partial t^k)(t^k/k!)$.

In (9), $(t-\xi)^{n-1}/(n-1)!$ is Lagrange multiplier of He's variational iteration method, denoted by λ, if n is an odd integer, and (9) can be written in standard variational iteration algorithm II [3]

$$\begin{aligned} f_{j+1}(x,t) = f_0(x,t) + \int_0^t \lambda \left(Rf_j + Nf_j + g\right) d\xi, \\ f_0(x,t) = \sum_{k=0}^{n-1} \frac{\partial^k f(x, 0^+)}{\partial t^k} \frac{t^k}{k!}, \qquad \lambda = \frac{(t-\xi)^{n-1}}{(n-1)!}. \end{aligned} \tag{10}$$

Equation (10) is exactly the same as the standard He's variational iteration algorithm II [3]. Here is a point to be noted, if we change our initial guess by adding source term in it, the resulting formulation will give the results obtained by well-known Adomian decomposition method by decomposing the nonlinear term in (10). Consider

$$\begin{aligned} f_{j+1}(x,t) = \int_0^t \lambda \left(Rf_j + Nf_j\right) d\xi, \\ f_0(x,t) = H(x,t), \qquad \lambda = \frac{(t-\xi)^{n-1}}{(n-1)!}, \\ H(x,t) = \sum_{k=0}^{n-1} \frac{\partial f(x, 0^+)}{\partial t^k} \frac{t^k}{k!} + \int_0^t \lambda g(x, \xi)\, d\xi. \end{aligned} \tag{11}$$

Equation (11) is an alternative approach of Adomian decomposition method, where $H(x,t)$ is a term which arises from prescribed initial condition and source term. Furthermore, if we decompose the term $H(x,t)$ in (11) and write the resulting equation in the form

$$\begin{aligned} f_1(x,t) = H_1(x,t) + \int_0^t \lambda \left(Rf_0 + Nf_0\right) d\xi, \\ H(x,t) = \sum_{k=0}^{n-1} \frac{\partial f(x, 0^+)}{\partial t^k} \frac{t^k}{k!} + \int_0^t \lambda g(x, \xi)\, d\xi, \\ H(x,t) = H_0(x,t) + H_1(x,t), \qquad \lambda = \frac{(t-\xi)^{n-1}}{(n-1)!}, \end{aligned} \tag{12}$$

$$\begin{aligned} f_0(x,t) = H_0(x,t), \\ f_{j+1}(x,t) = \int_0^t \lambda \left(Rf_j + Nf_j\right) d\xi, \quad j \geq 1, \end{aligned} \tag{13}$$

equation (12) is an alternative form of modified Adomian decomposition method.

3. Illustrative Examples

In order to illustrate the solution procedure, we consider the following examples for ordinary and partial differential equations.

Example 1. Consider the Blasius equation

$$u'''(x) + \frac{1}{2}u(x)u''(x) = 0, \tag{14}$$

subject to the boundary conditions

$$u(0) = 0, \qquad u'(0) = 1, \quad u' \longrightarrow 0, \ x \longrightarrow \infty. \tag{15}$$

To solve the above given problem, we consider an extra initial condition; that is, $u''(0) = \alpha$. In order to solve (14) with this extra initial condition, we follow the formulation given in (10). Consider

$$\begin{aligned} u_{j+1}(x) &= u_0(x) - \int_0^x \frac{\lambda}{2}\left(u_j(\xi)\,u_j''(\xi)\right)d\xi, \\ u_0(x) &= u(0) + xu'(0) + \frac{x^2}{2!}u''(x) = x + \frac{x^2\alpha}{2!}, \end{aligned} \tag{16}$$

$$\lambda = \frac{(x-\xi)^2}{2!}. \tag{17}$$

By using (16), we obtain the following successive approximations:

$$\begin{aligned} u_1(x) &= x + \frac{\alpha x^2}{2} - \frac{\alpha x^4}{48} - \frac{\alpha^2 x^5}{240}, \\ u_2(x) &= x + \frac{\alpha x^2}{2} - \frac{\alpha x^4}{48} - \frac{\alpha^2 x^5}{240} + \frac{\alpha x^6}{960} + \frac{11\alpha^2 x^7}{20160} \\ &\quad + \frac{11\alpha^3 x^8}{161280} - \frac{\alpha^2 x^9}{193536} - \frac{\alpha^3 x^{10}}{518400} - \frac{\alpha^4 x^{11}}{5702400}, \\ &\ \vdots \end{aligned} \tag{18}$$

Equation (18) is the exactly the same as obtained by using classical VIM in [20] and one can find the value of α by using Padé approximant [21].

Example 2. Consider the nonhomogeneous wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2} + \eta(x,t), \tag{19}$$

where $\eta(x,t) = 2e^{-\pi t}\sin \pi x$, subject to the initial conditions

$$u(x,0) = \sin \pi x, \qquad u_t(x,0) = -\pi \sin \pi x, \tag{20}$$

whose exact solution is

$$u(x,t) = e^{-\pi t}\sin \pi x. \tag{21}$$

To solve (19), we follow the formulation, given in (11). Consider

$$\begin{aligned} u_{j+1}(x,t) &= \int_0^t \lambda\left(\frac{\partial^2 u_j}{\partial x^2}\right)d\xi, \\ u_0(x,t) &= H(x,t), \qquad \lambda = (t-\xi), \\ H(x,t) &= \sin \pi x - t\pi \sin \pi x \\ &\quad + \int_0^t (t-\xi)\left(2\pi^2 e^{-\pi\xi}\sin \pi x\right)d\xi, \\ u_0(x,t) &= H(x,t) = -\sin \pi x + t\pi \sin \pi x \\ &\quad + 2e^{-\pi t}\sin \pi x \\ u_{j+1}(x,t) &= \int_0^t (t-\xi)\left(\frac{\partial^2 u_j}{\partial x^2}\right)d\xi, \\ u_1(x,t) &= \left(2 - 2\pi t + \frac{\pi^2 t^2}{2!} - \frac{\pi^3 t^3}{3!}\right)\sin \pi x - 2e^{-\pi t}\sin \pi x, \\ u_2(x,t) &= \left(-2 + 2\pi t - \pi^2 t^2 + \frac{\pi^3 t^3}{3} - \frac{\pi^4 t^4}{4!} + \frac{\pi^5 t^5}{5!}\right)\sin \pi x \\ &\quad - 2e^{-\pi t}\sin \pi x, \\ u_3(x,t) &= \left(2 - 2\pi t + \pi^2 t^2 - \frac{\pi^3 t^3}{3} + \frac{\pi^4 t^4}{3(4)} \right. \\ &\quad \left. - \frac{\pi^5 t^5}{3(4)(5)} + \frac{\pi^6 t^6}{6!} - \frac{\pi^7 t^7}{7!}\right)\sin \pi x \\ &\quad - 2e^{-\pi t}\sin \pi x, \\ &\ \vdots \end{aligned} \tag{22}$$

Upon summing these iterations, we observe that

$$\begin{aligned} u(x,t) &= \left(1 - \pi t + \frac{\pi^2 t^2}{2!} - \frac{\pi^3 t^3}{3!} + \frac{\pi^4 t^4}{4!} - \frac{\pi^5 t^5}{5!} \right. \\ &\quad \left. + \frac{\pi^6 t^6}{6!} - \frac{\pi^7 t^7}{7!} + \cdots\right)\sin \pi x \approx e^{-\pi t}\sin \pi x. \end{aligned} \tag{23}$$

Solution (23) is exactly the same as obtained by using ADM in [22].

4. Conclusion

This paper helps us to gain insight into the idea of Adomian decomposition method and variational iteration method. By keeping in view both methods, we propose more simplified forms to calculate Lagrange multipliers. By introducing this Lagrange multiplier in ADM and VIM following the observations that have been made,

(i) there is no need to do integration process again and again like we do in Adomian decomposition method and one can get the same results of Adomian method.

(ii) It is easy to calculate the Lagrange multiplier of He's variational iteration method.

(iii) This new approach avoids the unnecessary calculations like we did in He's variational iteration method and Adomian decomposition method.

(iv) This study shows that VIM is another form of expressing ADM and vice versa.

So we can say that the present method is parallel form of ADM and can give good results of VIM with less effort.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contribution

The authors have made the same contribution. All authors read and approved the final paper.

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