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HOMOGENIZATION IN PERFORATED DOMAINS

HOMOGENIZACE NA OBLASTECH S DÍRAMI

DISERTAČNÍ PRÁCE

DOCTORAL THESIS

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Abstrakt

Numerické řešení matematických modelů popisujících chování materiálů s jemnou strukturou (kompozitní materiály, jemně perforované materiály, atp.) obvykle vyžaduje velký výpočetní výkon. Proto se při numerickém modelování původní materiál nahrazuje ekvivalentním materiálem homogenním.

V této práci je k nalezení homogenizovaného materiálu použita dvojškálová konvergence založena na tzv. rozvinovacím operátoru (anglicky unfolding operator). Tento operátor poprvé použil J. Casado-Díaz. V disertační práci je operátor definován jiným způsobem, než jak uvádí původní autor. To dovoluje pro něj dokázat některé nové vlastnosti. Analogicky je definován operátor pro funkce definované na perforovaných oblastech a jsou dokázány jeho vlastnosti. Na závěr je rozvinovací operátor použit k nalezení homogenizovaného řešení speciální skupiny diferenciálních problémů s integrální okrajovou podmínkou. Odvozené homogenizované řešení je ilustrováno na numerických experimentech.

Summary

The numerical solving of mathematical models describing the mechanical behavior of materials with a fine structure (composite materials, finely perforated materials etc.) usually requires huge computational performance. Hence in numerical modeling the original material is replaced by an equivalent homogeneous one.

In this work a two-scale convergence based on a periodical unfolding operator is used to find the homogenized material. The operator was for the first time used by J. Casado-Díaz. In this Ph.D. thesis, the operator is defined in a slightly different way which allows us to prove some of its new properties. The unfolding operator for functions defined on a perforated domain is defined analogically and its properties are proved. Finally, this operator is used to find the homogenized solution of a special family of problems with an integral boundary condition; some numerical results are presented.

Klíčová slova

Homogenizace, perforovaná oblast, dvojškálová konvergence, rozvinovací operátor.

Keywords

Homogenization, perforated domain, two-scale convergence, unfolding operator.

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Prohlašuji, že jsem svou disertační práci *Homogenizace na oblastech s dírami* vypracovala samostatně pod vedením školitele prof. RNDr. Jana Franců, CSc., s použitím materiálů uvedených v seznamu literatury.

I declare that this Ph.D thesis entitled *Homogenization in perforated domains* is the result of my own autonomous work under the guidance of prof. RNDr. Jan Franců, CSc. and I used materials that have been included in the bibliographic references.

Ing. Petra Rozehnalová

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Ing. Petra Rozehnalová

List of used symbols and abbreviations

N	dimension of a space
$\mathbb{R},\ \mathbb{R}^N$	real numbers, N-dimensional real vector space
$\mathbb{Z},\ \mathbb{Z}^N$	integer numbers, N-dimensional integer vector space
c, C	constants
$A \times B$	Cartesian product of sets A, B
1_A	characteristic function of a set A, $1_A(x) = 1$ for $x \in A$, otherwise 0
Ω	bounded domain (open connected set) in $\mathbb{R}^N$ with Lipschitz boundary
$\overline{\Omega},\ \partial\Omega$	closure and boundary of Ω
$u \cdot v$	scalar product of two vectors u and v in $\mathbb{R}^N$
$V,\ V'$	normed linear space and its dual space
$\Vert\cdot\Vert_V,\ \vert.\vert_V$	norm and seminorm on V
$(u, v)_V$	scalar product of u and v on a linear space V
$\langle\cdot,\cdot\rangle_{V',V}$	duality pairing between V' and V
$E = \{\varepsilon_k\}_{k=0}^{\infty}$	scale; descending sequence of positive numbers, such that $\varepsilon_k \searrow 0$ as $k \to \infty$
$\{u_\varepsilon\}$	sequence of functions $\{u_{\varepsilon_k}\}_{k=0}^{\infty}$
$u_n \to u$	sequence $\{u_n\}$ converges strongly to u
$u_n \rightharpoonup u$	sequence $\{u_n\}$ converges weakly to u
$\mathcal{C}^\infty(\Omega)$	space of infinitely differentiable functions $u : \Omega \to \mathbb{R}$
$\mathcal{D}(\Omega)$	space of functions from $\mathcal{C}^\infty(\Omega)$ with compact support
$L^p(\Omega)$	Lebesgue space, see Definition 2.16
$W^{1,p}(\Omega)$	Sobolev space, see Definition 2.27
$H^1(\Omega)$	Hilbert space $W^{1,2}(\Omega)$
$H^1_0(\Omega)$	Hilbert space $H^1(\Omega)$ with zero trace on $\partial\Omega$
∇u	gradient of function u
$\nabla_y u$	gradient of function $u = u(x_1, x_2, \ldots, u_N, y_1, y_2 \ldots, y_N)$ with respect to y-variable, i.e. $(\frac{\partial u}{\partial y_1}, \frac{\partial u}{\partial y_2}, \ldots, \frac{\partial u}{\partial y_N})$

Y	reference cell; N-dimensional interval $\langle 0, l_1) \times \langle 0, l_2) \times \cdots \times \langle 0, l_N)$, where $l_1, \ldots, l_N$ are fixed positive numbers
Y_ε^k	ε-scaled system of the cells $Y_\varepsilon^k = \varepsilon(Y + k)$, $k \in \mathcal{K} = \left\{ k \in \mathbb{R}^N \mid k = \xi \cdot (l_1, l_2, \ldots, l_N), \xi \in \mathbb{Z}^N \right\}$
$\widehat{\Omega}_\varepsilon$	cells inside Ω, i.e. $\left(\bigcup_{k \in \Xi^\varepsilon} Y_\varepsilon^k \right) \cap \Omega$, where $\Xi^\varepsilon = \left\{ k \in \mathbb{R}^N \text{ s.t. } Y_\varepsilon^k \subset \overline{\Omega} \right\}$
Λ_ε	cells crossing boundary $\partial\Omega$, i.e. $\Omega \setminus \widehat{\Omega}_\varepsilon$
$\mathcal{T}_\varepsilon$	periodic unfolding operator, see Definition 3.2
T	reference hole, open bounded set in $\mathbb{R}^N$ with a smooth boundary
T_ε^j	hole, see Section 4.1
Y^*	perforated reference cell, i.e. $Y \setminus \overline{T}$
Ω_ε^*	part of Ω occupied by material, see (29)
$T_{\text{int},\,\varepsilon}^i$	sets T_ε^j which are completely inside Ω and do not intersect the boundary $\partial\Omega$, i.e. the sets $\overline{T_\varepsilon^j} \subset \Omega$
$T_{\text{int},\,\varepsilon}$	interior holes; i.e. $\bigcup_{i=1}^{m(\varepsilon)} T_{\text{int},\,\varepsilon}^i$
$T_{\text{ext},\,\varepsilon}$	holes crossing the boundary $\partial\Omega$; i.e. $\left(T_\varepsilon \setminus T_{\text{int},\,\varepsilon} \right) \cap \Omega$,
$\partial_{\text{ext}}\Omega_\varepsilon^*$	exterior boundary of Ω_ε^*, i.e. $\partial_{\text{ext}}\Omega_\varepsilon^* = \partial\Omega_\varepsilon^* \setminus \partial T_{\text{int},\,\varepsilon}$
$\widehat{\Omega}_\varepsilon^*$	$\widehat{\Omega}_\varepsilon \setminus T_{\text{int},\,\varepsilon}$
Λ_ε^*	$\Omega_\varepsilon^* \setminus \widehat{\Omega}_\varepsilon^*$
$\mathcal{T}_\varepsilon^*$	periodic unfolding operator for perforated domains, see Definition 4.1
$\tilde{u}$	extension by zero of finction $u : \Omega_\varepsilon^* \to \mathbb{R}$ into Ω
$\mathcal{M}_\Omega$	mean value operator over Ω, i.e. $\mathcal{M}_\Omega(u) = \fint_\Omega u(x)\,\mathrm{d}x$
$\mathcal{M}_\varepsilon$	local average operator, see Definition 3.9

Contents

1. Introduction

Theory of homogenization was developed for modeling media with a fine periodical structure. In a physical setting, homogenization means replacing a heterogeneous material by an equivalent homogeneous one, in mathematical setting it means approximating equations with highly oscillating coefficients by equations with constant ones.

The mathematical approach consists of considering a sequence of problems with a material with a more and more refined structure. Hence, we get a sequence of solutions. The principal question is: How does the sequence behave? Does the limit, the so called homogenized solution, exists? If so, how can it be characterized? This approach was first introduced by J.B. Keller (1973) and developed by I. Babuška (1975). More about the homogenization can be found in the monograph [BLP78] or in the textbook [CD99].

Other problems for which a similar approach can be used are problems defined on periodically perforated domains. Let Ω be a domain in $\mathbb{R}^N$ and let it be periodically perforated by holes. We shall construct a sequence of domains with an increasing number of holes and decreasing their volume. Again, we are interested in a behavior of the limit solution.

When we try to find the homogenized solution several difficulties occur. Some of them are common for the case with and without holes. The following problem can illustrate the typical situation in the setting with no holes.

For $\varepsilon = 1, {}^1\!/_2, {}^1\!/_3, \dots$, let us assume a sequence of solutions $\{u_\varepsilon\}$ to a problem

$$\left\{ \begin{array}{rcll} -\nabla \cdot (A_\varepsilon \nabla u_\varepsilon) & = & f & \text{in} \quad \Omega, \\ u_\varepsilon & = & 0 & \text{on} \quad \partial\Omega, \end{array} \right. \tag{1}$$

where $A_\varepsilon(x) = A\left(\frac{x}{\varepsilon}\right)$ and $A(y)$ is a Y-periodic function satisfying $0 < \alpha \le A(y) \le \beta$.

Weak formulation of this problem is:

$$\left\{ \begin{array}{l} \text{Find } u_\varepsilon \in H_0^1(\Omega) \text{ such that} \\ \displaystyle\int_\Omega A_\varepsilon(x)\, \nabla u_\varepsilon(x) \cdot \nabla v(x)\, \mathrm{d}x = \int_\Omega f(x)\, v(x)\, \mathrm{d}x, \qquad \forall v \in H_0^1(\Omega). \end{array} \right. \tag{2}$$

For $A_\varepsilon \in L^\infty(\Omega)$, the domain Ω with a "good" boundary and $f \in L^2(\Omega)$, the unique weak solution u_ε exists and satisfies $\|u_\varepsilon\|_{H_0^1(\Omega)} \le C$. Since the sequence $\{u_\varepsilon\}$ is bounded in $H_0^1(\Omega)$, it contains a weakly converging subsequence of gradients $\{\nabla u_\varepsilon\}$.

When we are tending to the limit, it turns out that the left-hand side of (2) contains a product of two weakly converging sequences, $\{A_\varepsilon\}$ and $\{\nabla u_\varepsilon\}$. In this case it is not possible to reach to the limit directly, since a limit of product need not to be a product of two weakly converging sequences.

In the past, several approaches to overcome this problem were developed.

- *Multiple-scale method* is summarized in monograph by A. Bensoussan, J.-L. Lions and G. Papanicolaou [BLP78]. The method uses the asymptotic expansion of the solution u_ε to find the homogenized one.

- *Local energy method* (called also the oscillating test function method) was introduced by L. Tartar [Tar97] in the years 1977 and 1978. The method is based on a special choice of oscillating test functions in the weak formulation of the problem.

- *Two-scale convergence method* introduced by G. Nguetseng [Ngu89] in 1989 and developed by G. Allaire [All92] in 1992. In this method a new type of convergence is defined. The limit of two-scale convergent sequence has two variables, the second one describes local behavior. This method requires introducing a special space for test functions.

- *Periodic unfolding method* is an alternative approach to the two-scale convergence. It was introduced by J. Casado-Díaz [CD00] in 2000 and D. Cioránescu, A. Damlamian and G. Griso [CDG02], L. Nechvátal [Nec04] and J. Franců [Fra10]. It removes problems with the choice of space for test functions, therefore it is more natural. A comprehensive survey of the application of this method to the problems in domains with holes is described by Cioránescu, Damlamian, Donato, Griso and Zaki [Cio+12].

Let us turn our attention back to the problems defined on the domain with holes. In this case, one more problem arises. Let Ω_ε^* denotes a periodically perforated domain with period εY. For $\varepsilon \searrow 0$ the period is smaller and smaller and the domain is perforated by more and finer holes.

A model situation looks as follows: For $\varepsilon = 1, {}^1\!/_2, {}^1\!/_3, \ldots$, let us assume a sequence $\{u_\varepsilon\}$, where u_ε is a solution of the problem

$$\left\{ \begin{array}{rcll} -\Delta u_\varepsilon &=& f & \text{in} \quad \Omega_\varepsilon^*, \\ u_\varepsilon &=& 0 & \text{on} \quad \partial\Omega_\varepsilon^*. \end{array} \right. \tag{3}$$

A weak formulation of the problem (3) is:

$$\left\{ \begin{array}{l} \text{Find } u_\varepsilon \in H_0^1(\Omega_\varepsilon^*) \text{ such that} \\ \displaystyle\int_{\Omega_\varepsilon^*} \nabla u_\varepsilon(x) \cdot \nabla v_\varepsilon(x)\,\mathrm{d}x = \int_{\Omega_\varepsilon^*} f(x)\, v_\varepsilon(x)\,\mathrm{d}x, \qquad \forall v_\varepsilon \in H_0^1(\Omega_\varepsilon^*). \end{array} \right. \tag{4}$$

The problem is that each solution u_ε of problem (4) is defined on a different domain Ω_ε^*. Hence, it is not clear in which sense the convergence of the sequence $\{u_\varepsilon\}$ can be understood. Even if there existed some u_0 for which $\|u_\varepsilon - u_0\|_{H_0^1(\Omega_\varepsilon^*)} \to 0$, as $\varepsilon \searrow 0$, one could not speak about "convergence" (in a strong or weak sense) of the sequence $\{u_\varepsilon\}$.

Several methods to avoid this issue have been developed over time:

- Quite an intuitive approach is a construction of an *uniformly bounded extension operator* P_ε from $H_0^1(\Omega_\varepsilon^*)$ to $H_0^1(\Omega)$. Then, we can transform our problem of finding a "limit" of $\{u_\varepsilon\}$ by another one: Find a limit of the sequence $\{P_\varepsilon(u_\varepsilon)\}$ in the fixed space $H_0^1(\Omega)$.

This approach has a limitation. The existence of operator P_ε depends on the boundary conditions of the problem (in the case that they are more complicated than in our model example) and also on the shape of the holes (for example they should have a sufficiently smooth boundary and should not intersect the boundary of Ω).

- Another approach is to use an *unfolding operator* to transform functions u_ε, resp. ∇u_ε defined on Ω_ε^* to the fixed domain $\Omega \times Y$.

As we can see the *periodic unfolding method* is the technique which solves both problems mentioned above. This is the reason why the method is so suitable for problems defined on perforated domains.

Goal and contribution of the thesis

Let Ω be a bounded set, and Y a reference cell in $\mathbb{R}^N$. The unfolding operator $\mathcal{T}_\varepsilon$ associates to any function in $L^p(\Omega)$ with a function in $L^p(\Omega \times Y)$.

The main disadvantage of an unfolding operator introduced in [CD00], [CDG02] is that it does not conserve integrals. It means that in general for $u \in L^\infty(\Omega)$

$$\int_\Omega u(x)\,\mathrm{d}x \neq \frac{1}{|Y|} \iint_{\Omega\times Y} \mathcal{T}_\varepsilon(u)(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{5}$$

It can be shown that the left-hand side of (5), for $u \geq 0$, is always grater or equal than its right-hand side. The equality holds only in limit, i.e. for $\varepsilon \to 0$.

This issue was removed by redefining this operator. The operator was improved by J. Franců and N.Svanstedt in [FS12]. This change simplifies the proofs and removes several difficulties and necessity of introducing "unfolding criterion for integrals" (see e.g. [CDG08]).

This thesis aims to prove properties of this improved unfolding operator, mainly the convergence for the sequence of gradients and applying an analogical approach to perforated domains. Finally, our purpose is to use this new operator to find a homogenized solution of the special family of the problems with an integral boundary condition and present some numerical results.

The thesis intents to be self-contained work suitable as the first reading for engineers and applied mathematicians. It is organized as follows:

In Section 2 we review some results and concepts of functional analysis and variational elliptic problems that will be used in the sequel.

Section 3 introduces the notation, defines the improved unfolding operator for the fixed domain (without holes), the definition is the same as in [FS12]. The properties are proved in detail. In the end of the chapter an important result for applications is shown - a convergence for sequences of gradients. The proofs in this section are new. Although they follow similar reasoning as the ones in [CD00] or [CDG02], they make use of conservation of integrals which makes them simpler and more transparent.

Section 4 is devoted to unfolding for perforated domains. There is a new definition of an unfolding operator for perforated domains. The operator is defined in such manner that it conserves integrals and it transforms functions from perforated domains Ω_ε^* to the fixed domain $\Omega \times Y$ (which does not depends on ε). Finally, the properties of the operator are proved.

In Section 5 we apply the periodic unfolding method to a boundary value problem on a perforated domain which arises from the study of a torsion of an elastic bar or a distribution of an electric field (we will call it Torsion boundary value problem). The problem is derived in [FNJ12] and [FR15]. Homogenization of the torsion problem has been studied by Rauch and Taylor [RT75] and Ciorănescu and Paulin [CP79], but the usage of the periodic unfolding method to find a homogenized solution is new.

Section 6 is a continuation of the previous one and it contains a numerical examples.

Appendix describes some computational aspects of solving the homogenized problem and problem on perforated domain.

Related works

A homogenization on a periodically perforated domain for miscellaneous boundary value problems was treated by numerous authors. Let us mention some milestones in this area.

The Laplace equation with a homogeneous Dirichlet condition in the domain where the holes are regularly distributed and the size of the holes decreases when the number of the holes increases was studied by Murat and Ciorănescu [MC97]. They showed that even in this problem an interesting behavior of the limit solution occurs.

In this problem we can identify three different situations. *The first situation* is when the size of holes decreases too quickly - quicker than the size of the cell period. Then u^ε converges to the solution of the Dirichlet problem in Ω. *The second situation* is when the size of holes decreases too slowly. Then u^ε converges to the zero function. *Between these two cases* there is the third one when the size of holes is critical, in that case an additional zero order term appears in the right-hand side of the limit equation.

In [MC97] there are quite strict assumptions on the distribution and shape of the holes. This limitation has been removed by Dal Maso and Garroni [MG94]. This break through made possible the solving the general case of homogeneous Dirichlet problems without any geometrical assumptions.

A problem with homogeneous Neumann boundary condition with some geometrical assumptions on holes was studied by Hruslov [Hru79].

Some assumptions on the size and shape of holes which are admissible for a periodic homogenization with Neumann boundary condition are given by Damlamian and Donato [DD02].

A classical situation is when the holes are distributed periodically and the ratio of material volume to the period volume is constant. This situation with a different type of boundary conditions has been described in numerous papers. Laplace equation with homogeneous mixed (Dirichlet and Neumann) boundary conditions was studied by Cardone, D'Apice and Maio [CDM02], elliptic equations with linear Robin resp. with non-linear conditions were studied by Cioránescu, Donato and Zaki in [CDZ06] resp. in [CDZ07], elliptic equations with non-homogeneous mixed boundary conditions were studied by Esposito, D'Apice and Gaudiello [EDG02].

A problem on domains with holes which are distributed periodically and their size is diminishing with respect to the period (the so called *small holes*) was studied by Murat and Cioránescu in [MC97] (homogeneous Dirichlet boundary conditions), and also by Conca and Donato in [CD88] (non-homogeneous Neumann boundary condition), by Cioránescu and Ould Hammouda in [COH08] (elliptic equations with a non-homogeneous mixed boundary conditions), by Ould Hammouda in [OH11] (elliptic equations with non-homogeneous Neumann boundary).

A non-periodical behavior of the holes has been studied by Nguetseng in [Ngu04].

2. Preliminaries

In this section we give a survey of some results and concepts of functional analysis that will be used in the sequel. Namely, we recall main properties of Banach and Hilbert spaces, especially Lebesgue and Sobolev spaces and weak convergence in them. In the end of the section we summarize the main results of elliptic problems and conditions under which these problems have a unique solution.

All functional spaces are considered to be real.

2.1. Banach and Hilbert spaces

Let us begin by recalling the notations of Banach and a Hilbert spaces which are the functional spaces we work with.

Let V be a linear space. A mapping $\|\cdot\|_V : V \to \mathbb{R}_0^+$ is called a *norm* on a linear space V if it satisfies the three following properties:

(i) separates points, i.e. $\|u\|_V = 0 \Leftrightarrow u = 0$,

(ii) absolute homogeneity, i.e. $\|\alpha\, u\|_V = |\alpha|\, \|u\|_V$,

(iii) triangle inequality, i.e. $\|u_1 + u_2\|_V \leq \|u_1\|_V + \|u_2\|_V$.

A *seminorm* on V is a mapping $|\cdot|_V : V \to \mathbb{R}_0^+$, which satisfies only properties (ii) and (iii).

The linear space V is called a *Banach space*, if it is endowed with the norm and it is complete in this norm.

A mapping $(\cdot,\cdot)_V : V \times V \to \mathbb{R}$ is called scalar product on V if it satisfies the following properties:

(i) symmetry, i.e. $(u_1, u_2)_V = (u_2, u_1)_V$,

(ii) linearity in the first component, i.e. $(\alpha_1 u_1 + \alpha_2 u_2, u)_V = \alpha_1 (u_1, u)_V + \alpha_2 (u_2, u)_V$,

(iii) $(u,u)_V \geq 0$ and $(u,u)_V = 0 \Leftrightarrow u = 0$.

A complete linear space V with scalar product is called a *Hilbert space*. Each Hilbert space is also a Banach space with the norm associated to this scalar product:

$$\|u\|_V = \sqrt{(u,u)_V}.$$

Definition 2.1 (Bounded linear operator on Banach spaces)**.** Let V, W be two Banach spaces. The operator $A : V \to W$ is said to be *linear*, if $u_1, u_2 \in V$ and $\alpha \in \mathbb{R}$ satisfies $A(u_1 + u_2) = A(u_1) + A(u_2)$ and $A(\alpha u_1) = \alpha A(u_1)$. The operator A is *bounded* if there exists a constant $C > 0$ such that

$$\|A(u)\|_W \leq C \|u\|_V \quad \forall u \in V.$$

Proposition 2.2. *Let A be a linear operator from V to W, then the following statements are equivalent:*

1. *A is continuous at point $u_0 \in V$, i.e. $\forall u_n \in V,\ u_n \to u_0 \Rightarrow A(u_n) \to A(u_0)$,*
2. *A is continuous, i.e. $\forall u_n, u \in V,\ u_n \to u \Rightarrow A(u_n) \to A(u)$,*
3. *A is bounded, i.e. $\exists C > 0$ such that $\|A(u)\|_W \le C\|u\|_V$ $\forall u \in V$.*

For proof see [Rud91] p. 24-25.

Proposition 2.3. *Let V, W be a Banach spaces. The set of all continuous linear operators from V into W, denoted by $\mathcal{L}(V, W)$, with a norm*

$$\|A\|_{\mathcal{L}(V,W)} = \sup_{u \in V \setminus \{0\}} \frac{\|A(u)\|_W}{\|u\|_V} \quad \forall A \in \mathcal{L}(V, W)$$

is a Banach space.

For proof see [Rud91] p. 92-93 or [Yos65] p. 111-112. From definition of the norm on $\mathcal{L}(V, W)$ one gets

$$\|A(u)\|_W \le \|A\|_{\mathcal{L}(V,W)} \|u\|_V \quad \forall u \in V, A \in V'.$$

Moreover, the linearity of A implies

$$\|A\|_{\mathcal{L}(V,W)} = \sup_{u \in V \setminus \{0\}} \frac{\|A(u)\|_W}{\|u\|_V} = \sup_{u \in V, \|u\|=1} \|A(u)\|_W.$$

Definition 2.4 (Dual space)**.** Let V be a Banach space. The set $\mathcal{L}(V, \mathbb{R})$ of all linear continuous functionals from V into $\mathbb{R}$ is called the *dual space* of V and is denoted by V'. For $F \in V'$, the image $F(u)$ of $u \in V$ is denoted by $\langle F, u \rangle_{V',V}$. The bracket $\langle \cdot, \cdot \rangle_{V',V}$ is called *duality pairing* between V' and V.

Remark. Since $\mathbb{R}$ is complete the dual space is a Banach space with the norm

$$\|F\|_{V'} = \sup_{u \in V \setminus \{0\}} \frac{\left|\langle F, u \rangle_{V',V}\right|}{\|u\|_V} \quad \forall F \in V'.$$

Moreover, one has

$$\left|\langle F, u \rangle_{V',V}\right| \le \|F\|_{V'} \|u\|_V \quad \forall u \in V.$$

The dual space $V'' = (V')'$ of the V' is called *bidual* or *second dual* and it is also a Banach space.

Proposition 2.5. *Let V be a Banach space and let $J : V \to V''$ be the linear mapping defined by*

$$\langle J(u), u' \rangle_{V'',V'} = \langle u', u \rangle_{V',V} \quad \forall u \in V, \forall u' \in V'.$$

Then, J is an isometry, i.e.:

$$\|J(u)\|_{V''} = \|u\|_V.$$

For proof see [Rud91] p. 95 or [Yos65], p. 113. Thanks to this result V can be identified with a subspace $J(V) \subset V''$.

Definition 2.6 (Reflexive Banach space)**.** Let V be a Banach space and J be the map defined by Proposition 2.5. V is said to be *reflexive* iff $J(V) = V''$.

If V is reflexive, we identify V and V''.

Proposition 2.7 (Riesz representation theorem)**.** *Let V be a Hilbert space. For each $u' \in V'$ there exists a unique $u \in V$ such that*

$$\langle u', v \rangle_{V',V} = (u, v)_V \quad \forall v \in V.$$

Moreover the mapping $u' \in V' \mapsto u \in V$ is an isometric isomorphism.

For proof see [Yos65], p. 90.

Proposition 2.8. *Hilbert spaces are reflexive.*

Let us recall that a set S in a topological space V is called *dense* in a set M if the closure of S contains M. In other words for each $u \in M$ there exists a sequence $\{u_n\} \in S$ such that $u_n \to u$.

Topological space having a countable dense subset is called a *separable space.*

2.2. Weak convergence

Definition 2.9 (Weak convergence)**.** Let V be a Banach space and V' its dual space. The sequence $\{u_n\}$ in V is said to *weakly converge* to $u \in V$ if

$$\langle u', u_n \rangle_{V',V} \to \langle u', u \rangle_{V',V}, \quad \forall u' \in V'.$$

Weak convergence will be denoted by

$$u_n \rightharpoonup u \quad \text{weakly in } V.$$

Proposition 2.10. *The limit of weak convergence is unique.*

Proposition 2.11. *Strong convergence implies weak convergence.*

For proof see [Yos65], p. 120.

Proposition 2.12. *Every weakly converging sequence $\{u_n\}$ is bounded in V, i.e. there exists a constant C such that*

$$\|u_n\|_V \le C, \quad \forall n \in \mathbb{N}.$$

For proof see [Yos65], p. 120, or [KF75], p. 219-220.

Proposition 2.13 (Compactness, Eberlein–Šmulian)**.** *Let V be a reflexive Banach space. Then every bounded sequence $\{u_n\}$ in V contains a weakly convergent subsequence, i.e. there exists a subsequence $\{u_{n_k}\} \subset \{u_n\}$ and $u \in V$, such that $u_{n_k} \rightharpoonup u$ weakly in V.*

For proof see [Yos65], p. 126.

Proposition 2.14 (Eberlein–Šmulian)**.** *Let V be a reflexive Banach space. If each weakly convergent subsequence of $\{u_n\}$ in V has the same limit u, then the whole sequence $\{u_n\}$ weakly converges to u.*

For proof see [Yos65], p. 124.

Proposition 2.15. *Let $\{u_n\}$ be a sequence in V, and $\{v_n\}$ be a sequence in V' such that*

$$\begin{aligned} v_n &\to v \quad \text{strongly in } V', \\ u_n &\rightharpoonup u \quad \text{weakly in } V. \end{aligned} \tag{6}$$

Then

$$\lim_{n\to\infty} \langle v_n, u_n \rangle_{V',V} = \langle v, u \rangle_{V',V}.$$

Proof. From the Remark below Definition 2.4 we get

$$\begin{aligned} \lim_{n\to\infty} \left| \langle v_n, u_n \rangle_{V',V} - \langle v, u \rangle_{V',V} \right| &= \lim_{n\to\infty} \left| \langle v_n - v, u_n \rangle_{V',V} + \langle v, u_n - u \rangle_{V',V} \right| \le \\ &\le \lim_{n\to\infty} \|v_n - v\|_{V'} \|u_n\|_V + \lim_{n\to\infty} \left| \langle v, u_n - u \rangle_{V',V} \right|. \end{aligned}$$

To pass to the limit in the first term we use the following: By assumptions, sequence $\{u_n\}$ weakly converges, hence the sequence is bounded (see Proposition 2.12). Sequence $\{v_n\}$ strongly converges. Thus

$$\lim_{n\to\infty} \|v_n - v\|_{V'} \|u_n\|_V = 0. \tag{7}$$

Both terms tend to the zero. Indeed, to pass to the limit in the second term we use definition 2.9. If $u_n \rightharpoonup u$ weakly converges in V then, by the definition,

$$\lim_{n\to\infty} \left| \langle v, u_n - u \rangle_{V',V} \right| = 0. \tag{8}$$

Summing up (7) with (8) we get the result. □

2.3. Lebesgue spaces

We shell work with integrable functions.

Definition 2.16 (Lebesgue spaces, L^p spaces)**.** Let Ω be an open bounded set in $\mathbb{R}^N$, $u : \Omega \to \mathbb{R}$ be a measurable function on Ω and $p \in \langle 1, \infty \rangle$. Let us denote

$$\|u\|_{L^p(\Omega)} = \left[\int_\Omega |u(x)|^p \, \mathrm{d}x \right]^{\frac{1}{p}} \quad \text{for } \; p \in \langle 1, \infty)$$

and for $p = \infty$

$$\|u\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega} |u(x)|.$$

The spaces of integrable functions on Ω are called *Lebesgue spaces* and are denoted by $L^p(\Omega)$, i.e.

$$L^p(\Omega) = \Big\{ u \mid u : \Omega \to \mathbb{R}, u \text{ is measurable on } \Omega \text{ and } \|u\|_{L^p(\Omega)} < \infty \Big\}.$$

More precisely elements of Lebesgue spaces are the classes of functions which differ on at most zero measure set.

Proposition 2.17. *Let Ω be an open bounded set in $\mathbb{R}^N$ and $p \in \langle 1, \infty \rangle$. The set $L^p(\Omega)$ equipped with the norm $\|u\|_{L^p(\Omega)}$ is a Banach space.*

Moreover, the space $L^2(\Omega)$ is a Hilbert space with the scalar product

$$(u, v)_{L^2(\Omega)} = \int_\Omega u(x)\, v(x)\, \mathrm{d}x.$$

For proof see [AF03], Theorem 2.15, p. 29.

Proposition 2.18. *The space $L^p(\Omega)$ is separable for $p \in \langle 1, \infty)$ and reflexive for $p \in (1, \infty)$.*

For the proof of separability see [AF03], Theorem 2.21, p. 32, and for reflexivity see [AF03], Theorem 2.46, p. 49.

Proposition 2.19 (Hölder inequality)**.** *Let u be in $L^p(\Omega)$ and v in $L^{p'}(\Omega)$, where $p \in \langle 1, \infty \rangle$ and p' is its conjugate, i.e.*

$$\begin{array}{lllll} p' & = & \frac{p}{p-1} & \textit{for} & p \in (1, \infty), \\ p' & = & 1 & \textit{for} & p = \infty, \\ p' & = & \infty & \textit{for} & p = 1. \end{array} \tag{9}$$

Then,

$$\int_\Omega u(x)\, v(x)\, \mathrm{d}x = \|u\, v\|_{L^1(\Omega)} \le \|u\|_{L^p(\Omega)} \|v\|_{L^{p'}(\Omega)}.$$

For $p = 2$ the inequality is called *Cauchy-Schwartz inequality.*

For proof see [Yos65], p. 33, or [AF03], Theorem 2.4, p. 24.

Definition 2.20. Let u be function $\Omega \to \mathbb{R}$ the *support* of u, denoted by $\operatorname{supp} u$

$$\operatorname{supp} u = \overline{\{x \in \Omega \mid u(x) \neq 0\}}.$$

We denote by $\mathcal{D}(\Omega)$ the set of infinitely differentiable functions whose support is a compact set contained in Ω.

Proposition 2.21 (Approximation by compactly supported smooth functions)**.** *For $p \in \langle 1, \infty)$, the space $\mathcal{D}(\Omega)$ is dense in $L^p(\Omega)$.*

For proof see [KJF77], Theorem 2.6.1, p. 73.

Proposition 2.22 (Riesz Representation Theorem for $L^p(\Omega)$)**.** *Let $p \in (1, \infty)$ and p' be its conjugate. Further let F be a linear continuous functional on $L^p(\Omega)$ (i.e. it belongs to the $[L^p(\Omega)]'$ - dual space of $L^p(\Omega)$). Then for each F there exists unique $f \in L^{p'}(\Omega)$ such that*

$$\langle F, u \rangle_{[L^p(\Omega)]', L^p(\Omega)} = \int_\Omega f(x)\, u(x) \,\mathrm{d}x \quad \forall u \in L^p(\Omega).$$

Moreover

$$\|F\|_{[L^p(\Omega)]'} = \|f\|_{L^{p'}(\Omega)}.$$

For proof see [AF03], Theorem 2.44, p. 47.

Remark. Due Riesz Representation Theorem, the space $[L^p(\Omega)]'$ can be identified with $L^{p'}(\Omega)$ for $p \in (1, \infty)$.

In L^p spaces the weak convergence is defined as follows. It is the special case of Definition 2.9.

Definition 2.23 (Weak convergence in L^p spaces)**.** Let $\{u_n\}$ be a sequence in $L^p(\Omega)$ with $p \in \langle 1, \infty)$. The sequence $\{u_n\}$ *weakly converges* to u in $L^p(\Omega)$, i.e.

$$u_n \rightharpoonup u \quad \text{weakly in } L^p(\Omega)$$

iff

$$\int_\Omega u_n(x)\, v(x) \,\mathrm{d}x \to \int_\Omega u(x)\, v(x) \,\mathrm{d}x \quad \forall v \in L^{p'}(\Omega),$$

where p, p' are conjugate exponents.

Proposition 2.24. *Let $\{u_n\}$ be a sequence in $L^p(\Omega)$ and $u \in L^p(\Omega)$, $1 < p < \infty$. Further let $S(\Omega)$ be a dense subspace of $L^{p'}(\Omega)$, with ${}^1\!/_p + {}^1\!/_{p'} = 1$. Then the following properties are equivalent:*

(a) $u_n \rightharpoonup u$ *weakly in* $L^p(\Omega)$.

(b) *(i)* $\{u_n\}$ *is bounded in* $L^p(\Omega)$, *i.e.* $\|u_n\|_{L^p(\Omega)} < C$ *independently of* n,

(ii) $\displaystyle\int_\Omega \Big(u_n(x) - u(x)\Big)\, \varphi(x) \,\mathrm{d}x \to 0 \quad \forall \varphi \in S(\Omega)$.

Proof. Suppose that (a) holds then (i) follows from Proposition 2.12 and (ii) is obtained by testing the weak convergence for the function $\varphi \in S(\Omega) \subset L^{p'}(\Omega)$.

Assume now that (b) holds. Let $\psi \in L^{p'}(\Omega)$. Since $S(\Omega)$ is dense subspace of $L^{p'}(\Omega)$, for any positive ν there exists a function $\varphi_\nu \in S(\Omega)$ such that

$$\|\psi - \varphi_\nu\|_{L^{p'}(\Omega)} \le \nu.$$

Then,

$$\int_\Omega \Big(u_n(x) - u(x)\Big)\psi(x)\,\mathrm{d}x =$$
$$= \int_\Omega \Big(u_n(x) - u(x)\Big)\varphi_\nu(x)\,\mathrm{d}x + \int_\Omega \Big(u_n(x) - u(x)\Big)\Big(\psi(x) - \varphi_\nu(x)\Big)\,\mathrm{d}x \quad (10)$$

Due to condition (ii) the first term converges to zero. From the (i), the definition of φ_ν and the Hölder inequality, we derive

$$\int_\Omega \Big(u_n(x) - u(x)\Big)\Big(\psi(x) - \varphi_\nu(x)\Big)\,\mathrm{d}x \le \overline{C}\,\nu.$$

Since $\|\psi - \varphi_\nu\|_{L^{p'}(\Omega)}$ can be chosen arbitrary small, the property (a) follows from (10). $\square$

Proposition 2.25. *Let $\{u_n\}$, $\{v_n\}$ be sequences in $L^2(\Omega)$ such that*

$$u_n \to u \quad \textit{strongly in } L^2(\Omega),$$
$$v_n \rightharpoonup v \quad \textit{weakly in } L^2(\Omega).$$

and further let $\{u_n\, v_n\}$ be bounded in $L^2(\Omega)$. Then,

$$u_n\, v_n \rightharpoonup u\, v \quad \textit{weakly in } L^2(\Omega).$$

The proposition follows from the previous proposition and Proposition 2.15.

Lemma 2.26 (The fundamental lemma of the calculus of variations, Du Bois-Reymond's lemma, Testing lemma)**.** *Let $u \in L^1(\Omega)$ and satisfy*

$$\int_\Omega u(x)\,\varphi(x)\,\mathrm{d}x = 0 \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Then, $u(x) = 0$ almost everywhere in Ω.

For proof see [AF03], Lemma 3.31, p. 74.

2.4. Sobolev spaces

In this part we give a short presentation of $W^{1,p}$, H^1 and H^1_0 spaces. Let Ω be a domain in $\mathbb{R}^N$ with Lipschitz continuous boundary.

Definition 2.27 (Sobolev spaces $W^{1,p}$ and H^1)**.** Let $p \in \langle 1, \infty\rangle$. The *Sobolev space* $W^{1,p}(\Omega)$ is defined by

$$W^{1,p}(\Omega) = \left\{ u \mid u,\ \frac{\partial u}{\partial x_i} \in L^p(\Omega),\ i = 1, \dots, N \right\},$$

where $\frac{\partial u}{\partial x_i}$ are taken in the sense of distribution.

For $p = 2$ the space $W^{1,2}(\Omega)$ is denoted by $H^1(\Omega)$, i.e.

$$H^1(\Omega) = \left\{ u \mid u,\ \frac{\partial u}{\partial x_i} \in L^2(\Omega),\ i = 1, \dots, N \right\}.$$

Proposition 2.28. *The Sobolev space $W^{1,p}(\Omega)$ with the norm*

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)}$$

is a Banach space. For $p \in \langle 1, \infty)$, this norm is equivalent to the following one

$$\|u\|_{W^{1,p}(\Omega)} = \left(\int_\Omega |u|^p \,\mathrm{d}x + \sum_{i=1}^{N} \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^p \mathrm{d}x \right)^{\frac{1}{p}} = \left(\|u\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{[L^p(\Omega)]^N} \right)^{\frac{1}{p}},$$

where

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N} \right)$$

and

$$\|\nabla u\|_{[L^p(\Omega)]^N} = \left(\sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}$$

Moreover, the space $H^1(\Omega)$ is a Hilbert space with the scalar product

$$(u, v)_{H^1(\Omega)} = (u, v)_{L^2(\Omega)} + \sum_{i=1}^{N} \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right)_{L^2(\Omega)}.$$

For proof see [AF03], Theorem 3.3, p. 60 and Theorem 3.6, p. 61.

Sobolev space with zero trace on $\partial\Omega$, $H^1_0(\Omega)$, is the closure of $C^\infty_0(\Omega)$ in $H^1(\Omega)$.

Let us denote by $|v|_{H^1(\Omega)}$ a semi-norm on the space $H^1(\Omega)$

$$|v|_{H^1(\Omega)} = \left(\sum_{i=1}^{N} \left\| \frac{\partial v}{\partial x_i} \right\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}.$$

Proposition 2.29 (Approximation by smooth functions)**.** *For $p \in \langle 1, \infty)$, the space $\mathcal{C}^\infty(\Omega) \cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$.*

For proof see [AF03] p. 65.

Proposition 2.30. *$W^{1,p}(\Omega)$ is separable for $p \in \langle 1, \infty)$ and reflexive for $p \in (1, \infty)$.*

For proof see [AF03] Theorem 3.6, p. 61.

Proposition 2.31 (Poincaré inequality)**.** *For the domain Ω there exists $c = c(\Omega) > 0$ such that*

$$c \|u\|_{L^2(\Omega)} \le \|\nabla u\|_{[L^2(\Omega)]^N} = |u|_{H^1(\Omega)}, \quad \forall u \in H^1_0(\Omega).$$

For proof see [AF03], Theorem 6.30, p.183.

The last theorem implies the following one.

Proposition 2.32. *On $H_0^1(\Omega)$ the seminorm $|u|_{H^1(\Omega)}$ is equivalent to the norm $\|u\|_{H^1(\Omega)}$, i.e. there exists a constant $c = c(\Omega)$ such that*

$$|u|_{H^1(\Omega)} \leq \|u\|_{H^1(\Omega)} \leq c\,|u|_{H^1(\Omega)} \quad \forall u \in H_0^1(\Omega).$$

For that reason we set

$$\|u\|_{H_0^1(\Omega)} = |u|_{H^1(\Omega)}.$$

Notation 2.33 (Mean value operator over Ω)**.** The mean value operator over Ω is denoted by $\mathcal{M}_\Omega(u)$ and defined by

$$\mathcal{M}_\Omega(u) = \int_\Omega u(x)\,\mathrm{d}x.$$

Proposition 2.34 (Poincaré-Wirtinger inequality)**.** *For a bounded domain Ω and $p \in \langle 1, \infty\rangle$ there exists a constant $C = C(p, \Omega) > 0$ such that*

$$\|u - \mathcal{M}_\Omega(u)\|_{L^p(\Omega)} \leq C\,\|\nabla u\|_{[L^p(\Omega)]^N} \quad \forall u \in W^{1,p}(\Omega).$$

For proof see [Eva98], p. 275.

Functions in $W^{1,p}(\Omega)$ are not in general continuous and are defined "only" almost everywhere in Ω. Since $\partial\Omega$ has zero measure, "u restricted to $\partial\Omega$" is not defined. The notion of a *trace operator* resolves this problem.

Proposition 2.35 (Trace theorem)**.** *Let $p \in (1, \infty)$. Then, there exists unique linear continuous operator*

$$T : W^{1,p}(\Omega) \to L^p(\partial\Omega),$$

such that

$$T(u) = u|_{\partial\Omega} \quad \textit{for} \quad \forall u \in \mathcal{C}(\bar{\Omega}).$$

Furthermore there exists a constant $C = C(p, \Omega) > 0$ such that

$$\|T(u)\|_{L^p(\partial\Omega)} \leq C\,\|u\|_{W^{1,p}(\Omega)} \quad \textit{for} \quad u \in W^{1,p}(\Omega).$$

The operator T is called trace operator and $T(u)$ is called the trace of u on $\partial\Omega$.

For a proof of a slightly more general case of the trace theorem see [Eva98], p. 258.

By Riesz representation theorem (Theorem 2.7), abstract definition of the weak convergence leads to the weak convergence in H^1 spaces.

Definition 2.36 (Weak convergence in H^1 spaces)**.** Let $\{u_n\}$ be a sequence in $H^1(\Omega)$. The sequence $\{u_n\}$ *weakly converges* to u, i.e.

$$u_n \rightharpoonup u \quad \text{weakly in } H^1(\Omega)$$

if and only if $\{u_n\}$ satisfies

$$\int_\Omega \left(u_n(x)\,v(x) + \sum_{i=1}^N \frac{\partial u_n(x)}{\partial x_i}\,\frac{\partial v(x)}{\partial x_i} \right) \mathrm{d}x \to \int_\Omega \left(u(x)\,v(x) + \sum_{i=1}^N \frac{\partial u(x)}{\partial x_i}\,\frac{\partial v(x)}{\partial x_i} \right) \mathrm{d}x \quad \forall v \in H^1(\Omega).$$

Theorem 2.37. *Let $\{u_n\}$ be a sequence in $H^1(\Omega)$, such that*

$$u_n \rightharpoonup u \qquad \text{weakly in } H^1(\Omega).$$

Then,

$$\begin{aligned} u_n &\to u && \text{strongly in } L^2(\Omega), \\ \frac{\partial u_n}{\partial x_i} &\rightharpoonup \frac{\partial u}{\partial x_i} && \text{weakly in } L^2(\Omega). \end{aligned}$$

Proof of the strong convergence is based on the Rellich-Kondrachov compact embedding theorem concerning Sobolev spaces (see [Eva98], §5.8.1). Proof of the weak convergence of derivatives follows form the definition of weak convergence in the space $H^1(\Omega)$.

2.5. Abstract linear problems

In this section we consider an abstract linear problem which is a typical model for many applications.

Definition 2.38 (Bilinear form)**.** Let V be a Hilbert space and A be a mapping, $A : V \times V \to \mathbb{R}$. A is called the *bilinear form* on V if it is linear in both variables, i.e.: for any $\alpha_1, \alpha_2 \in \mathbb{R}$ and $u_1, u_2, u, v_1, v_2, v \in V$ there is

$$\begin{aligned} A(\alpha_1 u_1 + \alpha_2 u_2, v) &= \alpha_1 A(u_1, v) + \alpha_2 A(u_2, v), \\ A(u, \alpha_1 v_1 + \alpha_2 v_2) &= \alpha_1 A(u, v_1) + \alpha_2 A(u, v_2). \end{aligned}$$

Definition 2.39. Let A be a bilinear form on V. Then A is bounded on V if there exists $C > 0$, such that

$$|A(u,v)| \le C \|u\|_V \|v\|_V, \quad \forall u, v \in V.$$

Proposition 2.40. *Let A be a bilinear form on V. Then A is bounded if and only if A is continuous on $V \times V$.*

Consider the abstract problem: Let V be a Hilbert space, A be continuous bilinear form on V, b be continuous linear form on V.

$$\begin{cases} \text{Find } u \in V \text{ such that} \\ A(u,v) = b(v), \quad \forall v \in V. \end{cases} \tag{11}$$

The following theorem provides conditions under which the problem (11) admits unique solution and this solution has a stability property, namely the solution is controlled by the data.

Proposition 2.41 (Lax-Milgram lemma)**.** *Let V be a Hilbert space with a scalar product $(\cdot,\cdot)$ and a norm $\|\cdot\|_V$. Let A be a continuous bilinear form on $V \times V$ such that A is V-elliptic, i.e. there exists a constant $\alpha > 0$, such that*

$$A(u,u) \ge \alpha \|u\|_V^2, \quad \forall u \in V.$$

Further, let b be a bounded linear functional on V, i.e. there exists a constant $\beta > 0$, such that

$$|b(u)| \leq \beta \left\| u \right\|_V, \quad \forall u \in V.$$

Then, the problem (11) has one and only one solution which satisfies a priori estimate

$$\left\| u \right\|_V \leq \frac{\beta}{\alpha}.$$

For proof see [Eva98], p. 297-299.

3. Periodic unfolding

Let us introduce a notation and conventions which are used in homogenization, two-scale convergence and periodic unfolding.

Definition 3.1 (Scale)**.** A descending sequence $E = \{\varepsilon_k\}_{k=0}^{\infty}$ of positive numbers, such that $\varepsilon_k \searrow 0$ as $k \to \infty$, is called the *scale*.

In the following, as it is usual in the homogenization, all sequences will be denoted by the subscript ε_k, for example $\{a_{\varepsilon_k}\}$, or very often even only by the subscript ε, for example $\{a_\varepsilon\}$.

In the periodic homogenization, Y denotes a *reference cell* in $\mathbb{R}^N$. Here, we will define it as the N-dimensional interval

$$Y = \langle 0, l_1) \times \langle 0, l_2) \times \cdots \times \langle 0, l_N), \tag{12}$$

where $l_1, \ldots, l_N$ are fixed positive numbers.

Space $\mathbb{R}^N$ can be written as a union of the disjoint cells $Y_k = Y + k$, which are the cell Y shifted by vectors k, i.e.

$$\mathbb{R}^N = \bigcup_{k\in\mathcal{K}} (Y + k), \quad \mathcal{K} = \Big\{ k \in \mathbb{R}^N \mid k = (\xi_1\, l_1, \xi_2\, l_2, \ldots, \xi_N\, l_N), \xi \in \mathbb{Z}^N \Big\}$$

Periodic unfolding has appeared in [ADH90] and [CD00]. First of all we define splitting of each point in $\mathbb{R}^N$ in two parts. The idea is analogical to the following one: each real number x can be uniquely split to the integer part $[x]$ and the fractional part $\{x\} \in \langle 0, 1)$. Since the disjoint cells Y_k cover whole $\mathbb{R}^N$, for each point $x \in \mathbb{R}^N$ it holds $x = [x]_Y + \{x\}_Y$, where $[x]_Y$ denotes the shift of the cell Y_k containing x, and $\{x\}_Y$ stands for the relative position of x with respect to the cell Y_k, i.e. $[x]_Y \in \mathcal{K}$ and it is such that $x - [x]_Y$ belongs to Y. Set $\{x\}_Y = x - [x]_Y$. See Figure 1.

Let Ω be a bounded domain in $\mathbb{R}^N$ with a Lipschitz boundary $\partial\Omega$ and let ε be a positive real number. Using ε-scaled system of the cells $Y_\varepsilon^k = \varepsilon(Y + k), k \in \mathcal{K}$, the domain Ω can be split into two parts: $\widehat{\Omega}_\varepsilon$ and Λ_ε.

The set $\widehat{\Omega}_\varepsilon$ contains cells Y_ε^k lying inside Ω, while the set Λ_ε is a strip on the boundary composed of cells Y_ε^k intersecting the boundary $\partial\Omega$, see Figure 2. More precisely:

$$\begin{gathered} \Xi^\varepsilon = \Big\{ k \in \mathbb{R}^N \text{ s.t. } Y_\varepsilon^k \subset \overline{\Omega} \Big\}, \quad \widehat{\Omega}_\varepsilon = \left(\bigcup_{k\in\Xi^\varepsilon} Y_\varepsilon^k \right) \cap \Omega, \\ \Lambda_\varepsilon = \Omega \setminus \widehat{\Omega}_\varepsilon, \quad \text{so} \quad \Omega = \widehat{\Omega}_\varepsilon \cup \Lambda_\varepsilon\ . \end{gathered} \tag{13}$$

Now, we define the unfolding operator.

Definition 3.2 (Unfolding operator)**.** For each function $u : \Omega \to \mathbb{R}$ and $\varepsilon > 0$, the unfolding operator $\mathcal{T}_\varepsilon$ is defined as follows:

$$\mathcal{T}_\varepsilon(u)(x, y) = \begin{cases} u\Big(\varepsilon\big[\frac{x}{\varepsilon}\big]_Y + \varepsilon y\Big) & \text{for } (x, y) \in \widehat{\Omega}_\varepsilon \times Y, \\ u(x) & \text{for } (x, y) \in \Lambda_\varepsilon \times Y. \end{cases} \tag{14}$$

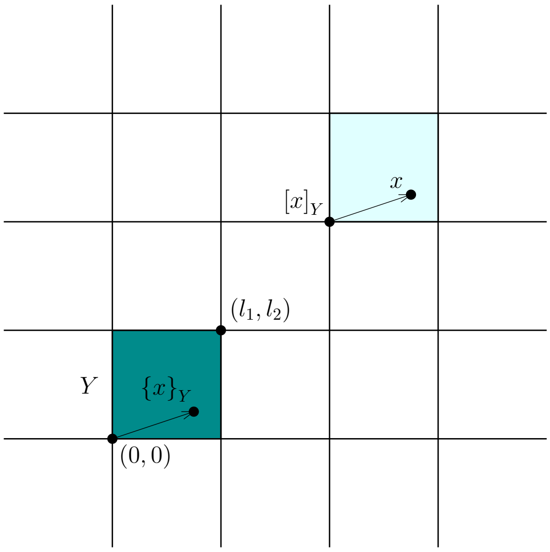

Figure 1: Decomposition $x = [x]_Y + \{x\}_Y$.

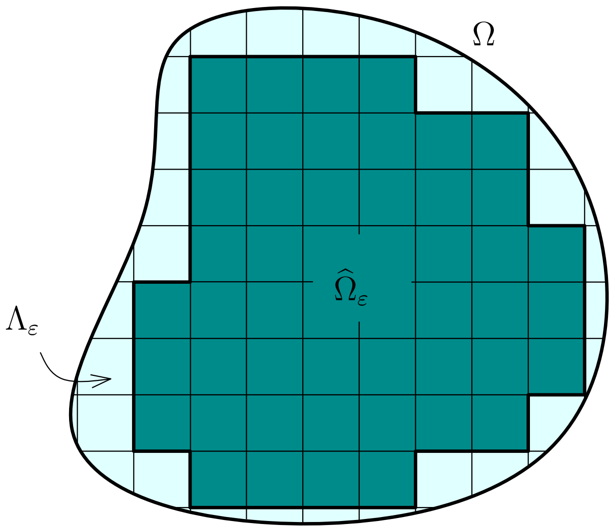

Figure 2: Domain Ω *is split into two disjoint parts:* Λ_ε *(light) and* $\widehat{\Omega}_\varepsilon$ *(dark).*

This definition was firstly used by Franců in [Fra07] and [Fra10]. The unfolding operator introduced in [CDG02], [Dam06] and [CDG08] is defined in a different way. It differs in values for points $[x, y] \in \Lambda_\varepsilon \times Y$ (incomplete cells), where in their definition $\mathcal{T}_\varepsilon(u)(x, y) = 0$.

Our approach conserves integrals, see Theorem 3.3 (iii), which simplifies proofs, and removes several difficulties (for example introducing “unfolding criterion for integrals”, see [CDG08], Proposition 2.6).

An example of unfolded functions is on Figures 4.

3.1. Properties of unfolding operator

Let us survey properties of the unfolding operator.

Theorem 3.3. *Let $\mathcal{T}_\varepsilon$ be the unfolding operator and $\varepsilon > 0$. Then*

(i) The operator $\mathcal{T}_\varepsilon$ is multiplicative, i.e. for all $u, v : \Omega \to \mathbb{R}$ it holds

$$\mathcal{T}_\varepsilon(u\,v) = \mathcal{T}_\varepsilon(u)\,\mathcal{T}_\varepsilon(v).$$

(ii) The unfolding operator $\mathcal{T}_\varepsilon$ is linear, i.e. for all $\alpha, \beta \in \mathbb{R}$ and $u, v : \Omega \to \mathbb{R}$ we have

$$\mathcal{T}_\varepsilon(\alpha\,u + \beta\,v) = \alpha\,\mathcal{T}_\varepsilon(u) + \beta\,\mathcal{T}_\varepsilon(v).$$

(iii) The unfolding operator $\mathcal{T}_\varepsilon$ conserves the integral, i.e. for all $u \in L^1(\Omega)$ we have

$$\iint\limits_{\Omega\times Y} \mathcal{T}_\varepsilon(u)(x,y)\,\mathrm{d}x\,\mathrm{d}y = |Y| \int\limits_\Omega u(x)\,\mathrm{d}x. \tag{15}$$

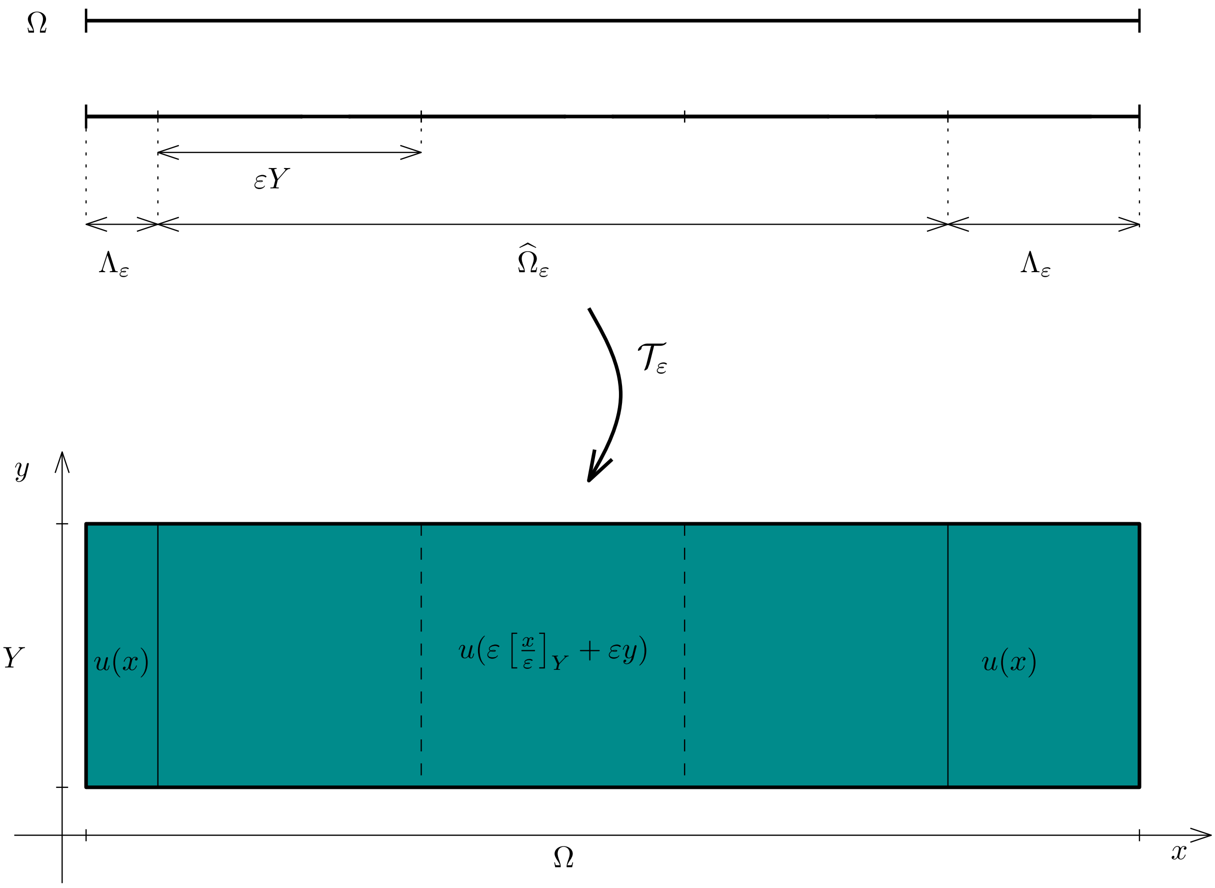

Figure 3: Example of the unfolding of a function $u(x)$ defined on domain Ω.

(iv) *The unfolding operator $\mathcal{T}_\varepsilon$, conserves the norm, i.e. for every $u \in L^p(\Omega)$, $p \in \langle 1, \infty)$ it holds*

$$\|\mathcal{T}_\varepsilon(u)\|_{L^p(\Omega\times Y)} = |Y|^{\frac{1}{p}}\, \|u\|_{L^p(\Omega)}.$$

Thus the operator $\mathcal{T}_\varepsilon$ is bounded and its norm satisfies

$$\|\mathcal{T}_\varepsilon\|_{\mathcal{L}(L^p(\Omega),L^p(\Omega\times Y))} = |Y|^{\frac{1}{p}}.$$

(v) *$\mathcal{T}_\varepsilon$ is a continuous operator from $L^p(\Omega)$ to $L^p(\Omega \times Y)$, where $p \in \langle 1, \infty)$.*

Proof. (i) The property follows directly from the Definition 3.2.

(ii) The linearity of unfolding operator is obvious.

(iii) From the Definition 3.2 one gets

$$I = \iint\limits_{\Omega\times Y} \mathcal{T}_\varepsilon(u)(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint\limits_{\widehat{\Omega}_\varepsilon\times Y} \mathcal{T}_\varepsilon(u)(x,y)\,\mathrm{d}x\,\mathrm{d}y + \iint\limits_{\Lambda_\varepsilon\times Y} \mathcal{T}_\varepsilon(u)(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

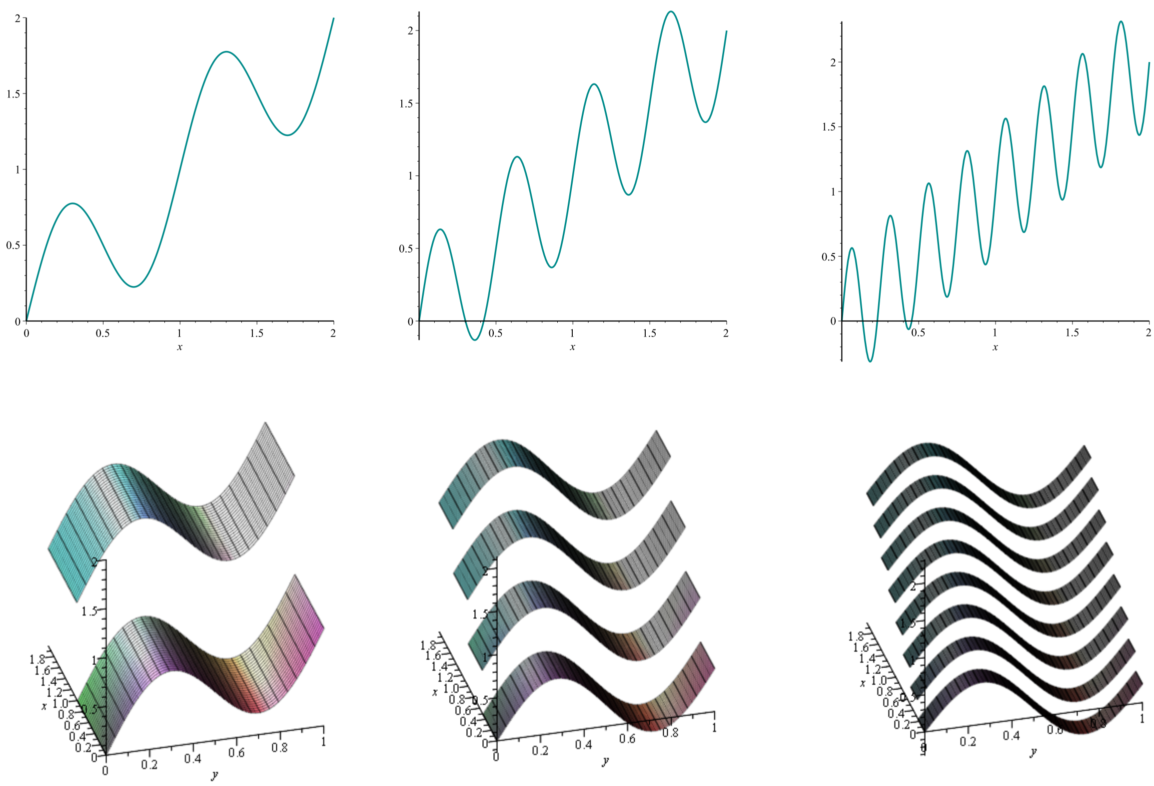

Figure 4: Functions $u_\varepsilon(x) = \frac{1}{2}\sin(2\pi\,\frac{x}{\varepsilon}) + x$ and its unfolding $\mathcal{T}_\varepsilon(u_\varepsilon)$, for $\varepsilon = 1, {}^1\!/_2, {}^1\!/_4$, domain $\Omega = (0,2)$ and reference cell $Y = \langle 0, 1)$.

Using definition (13) of the $\widehat{\Omega}_\varepsilon$ the first integral can be split. Thus

$$I = \sum_{k \in \Xi^\varepsilon} \iint\limits_{\varepsilon(Y+k)\times Y} \mathcal{T}_\varepsilon(u)(x,y)\,\mathrm{d}x\,\mathrm{d}y + |Y| \int\limits_{\Lambda_\varepsilon} u(x)\,\mathrm{d}x.$$

The unfolded function $\mathcal{T}_\varepsilon(u)(x,y)$ is constant in x on each $\varepsilon(Y+k)\times Y$. This yields:

$$I = \sum_{k \in \Xi^\varepsilon} |\varepsilon(Y+k)| \int\limits_Y u(\varepsilon(y+k))\,\mathrm{d}y + |Y| \int\limits_{\Lambda_\varepsilon} u(x)\,\mathrm{d}x =$$

$$= \varepsilon^N |Y| \sum_{k \in \Xi^\varepsilon} \int\limits_Y u(\varepsilon(y+k))\,\mathrm{d}y + |Y| \int\limits_{\Lambda_\varepsilon} u(x)\,\mathrm{d}x.$$

After the change of variable $\varepsilon(y+k) = x$ in the integral and simple calculations we get

$$I = |Y| \sum_{k \in \Xi^\varepsilon} \int\limits_{\varepsilon(Y+k)} u(x)\,\mathrm{d}x + |Y| \int\limits_{\Lambda_\varepsilon} u(x,y)\,\mathrm{d}x = |Y| \int\limits_\Omega u(x)\,\mathrm{d}x.$$

(iv) Let us show that $\|\mathcal{T}_\varepsilon(u)\|_{L^p(\Omega\times Y)}$ is equal to $|Y|^{\frac{1}{p}} \|u\|_{L^p(\Omega)}$.
It follows directly from the property (iii). Indeed,

$$\|\mathcal{T}_\varepsilon(u)\|^p_{L^p(\Omega\times Y)} = \iint\limits_{\Omega\times Y} \mathcal{T}^p_\varepsilon(u)(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint\limits_{\Omega\times Y} \mathcal{T}_\varepsilon(u^p)(x,y)\,\mathrm{d}x\,\mathrm{d}y = |Y| \int\limits_\Omega u^p(x)\,\mathrm{d}x.$$

Hence

$$\|\mathcal{T}_\varepsilon(u)\|_{L^p(\Omega\times Y)} = \Bigg(|Y| \int\limits_{\Omega\times Y} u^p(x,y)\,\mathrm{d}x\,\mathrm{d}y\Bigg)^{\frac{1}{p}} = |Y|^{\frac{1}{p}} \|u\|_{L^p(\Omega)}.$$

Then the boundedness of linear operator is straightforward.

$$\|\mathcal{T}_\varepsilon\|_{\mathcal{L}(L^p(\Omega),L^p(\Omega\times Y))} = \sup_{u \in L^p(\Omega)\setminus\{0\}} \frac{\|\mathcal{T}_\varepsilon(u)\|_{L^p(\Omega\times Y)}}{\|u\|_{L^p(\Omega)}} = |Y|^{\frac{1}{p}} < \infty.$$

(v) The continuity of linear operators is equivalent to its boundedness.

□

3.2. Two-scale convergence

There exist two different ways for defining the two-scale convergence. The earliest approach was introduced in [Ngu89] (for $L^2(\Omega)$-space) and more developed in [All92]. It was generalized for $L^p(\Omega)$-space, $p \in (1,\infty)$, in [LNW02]. The unfolding operator defined by the Definition 3.2 enables us to introduce a new definition. Comparison of both approaches can be found in [FS12].

Definition 3.4 (Two-scale convergence)**.** Let $\mathcal{T}_\varepsilon$ be the unfolding operator, $E = \{\varepsilon\}$ be a scale, $\{u_\varepsilon\}$ be a sequence in $L^p(\Omega)$ and $u_0 \in L^p(\Omega \times Y)$, $p \in \langle 1, \infty)$.

A sequence $\{u_\varepsilon\}$ is said to *strongly two-scale converge* to u_0 in $L^p(\Omega)$ with respect to the scale E if the sequence $\{\mathcal{T}_\varepsilon(u_\varepsilon)\}$ converges to u_0 strongly in $L^p(\Omega \times Y)$.

A sequence $\{u_\varepsilon\}$ is said to *weakly two-scale converge* to u_0 in $L^p(\Omega)$ with respect to the scale E if the sequence $\{\mathcal{T}_\varepsilon(u_\varepsilon)\}$ converges to u_0 weakly in $L^p(\Omega \times Y)$.

Remark. In the definition above, the weak two-scale convergence in $L^p(\Omega)$ is transformed to the weak convergence in $L^p(\Omega \times Y)$ of unfolded sequence. To check the weak convergence in the space $L^p(\Omega \times Y)$ one has to use test functions from the dual space $L^{p'}(\Omega \times Y)$. Moreover for bounded sequence in $L^p(\Omega \times Y)$ it is sufficient, due to the density property, to check this convergence only by smooth functions from $\mathcal{D}(\Omega \times Y)$.

Now we investigate convergence properties related to the unfolding operator. The following results follow directly from the definition and also from the theory of L^p-spaces.

Theorem 3.5. *Let $\{u_\varepsilon\}$ be a sequence in $L^p(\Omega)$ and $u_0 \in L^p(\Omega \times Y)$, $p \in \langle 1, \infty)$. Then*

(i) Any constant sequence $\{u\} \in L^p(\Omega)$ strongly two-scale converges to itself,

$$\mathcal{T}_\varepsilon(u) \to u_0 \quad \textit{strongly in } L^p(\Omega \times Y),$$

where $u_0(x, y) = u(x)$.

(ii) Any sequence $\{u_\varepsilon\}$ two-scale converging (strongly or weakly) in $L^p(\Omega)$ is bounded in $L^p(\Omega)$, i.e. $\|u_\varepsilon\|_{L^p(\Omega)} \leq C$.

(iii) If a two-scale limit u_0 exists, then it is unique as an element of L^p-spaces.

(iv) If $\{u_\varepsilon\}$ strongly converges to u^*, i.e.

$$u_\varepsilon \to u^* \quad \textit{strongly in } L^p(\Omega).$$

Then it strongly two-scale converges to $u_0(x, y) = u^*(x)$, i.e.

$$\mathcal{T}_\varepsilon(u_\varepsilon) \to u_0 \quad \textit{strongly in } L^p(\Omega \times Y).$$

(v) If $\{u_\varepsilon\}$ strongly two-scale converges to u_0,

$$\mathcal{T}_\varepsilon(u_\varepsilon) \to u_0 \quad \textit{strongly in } L^p(\Omega \times Y),$$

Then it weakly two-scale converges to the same limit

$$\mathcal{T}_\varepsilon(u_\varepsilon) \rightharpoonup u_0 \quad \textit{weakly in } L^p(\Omega \times Y).$$

(vi) For $p \in (1, \infty)$. If $\{u_\varepsilon\}$ weakly two-scale converges to u_0,

$$\mathcal{T}_\varepsilon(u_\varepsilon) \rightharpoonup u_0 \quad \textit{weakly in } L^p(\Omega \times Y),$$

Then it converges weakly

$$u_\varepsilon \rightharpoonup u^* \quad \textit{weakly in } L^p(\Omega),$$

where $u^*(x) = \frac{1}{|Y|} \int_Y u_0(x, y)\, \mathrm{d}y = \mathcal{M}_Y(u_0)(x)$.

Proof. (i) First of all, let us show that for $\varphi \in \mathcal{D}(\Omega)$

$$\lim_{\varepsilon \to 0} \mathcal{T}_\varepsilon(\varphi)(x) = \varphi(x). \tag{16}$$

From the definition it follows $\mathcal{T}_\varepsilon(\varphi)(x) = \varphi(x)$, for every ε, on the boundary strip Λ_ε.

Since the term $\left\{\frac{x}{\varepsilon}\right\}_Y$ is non-negative and bounded and $\varepsilon \searrow 0$, we get

$$\lim_{\varepsilon \to 0} \varepsilon \left\{\frac{x}{\varepsilon}\right\}_Y = 0.$$

Using this result we derive

$$\lim_{\varepsilon \to 0} \varepsilon \left[\frac{x}{\varepsilon}\right]_Y = \lim_{\varepsilon \to 0} \varepsilon \left(\frac{x}{\varepsilon} - \left\{\frac{x}{\varepsilon}\right\}_Y\right) = \lim_{\varepsilon \to 0} \varepsilon \, \frac{x}{\varepsilon} = x.$$

On $\widehat{\Omega}_\varepsilon$, finally,

$$\lim_{\varepsilon \to 0} \varphi\Big(\varepsilon \left[\frac{x}{\varepsilon}\right]_Y + \varepsilon\, y\Big) = \varphi(x).$$

And thus we get the limit (16).

Now, let $\varphi \in \mathcal{D}(\Omega)$ and $u \in L^p(\Omega)$. Adding and subtracting $(\mathcal{T}_\varepsilon(\varphi) - \varphi)$, using the Theorem 3.3 (ii), (iv) and the triangle inequality, one gets:

$$\begin{aligned}
\|\mathcal{T}_\varepsilon(u) - u\|_{L^p(\Omega\times Y)} &= \|\mathcal{T}_\varepsilon(u-\varphi) + (\mathcal{T}_\varepsilon(\varphi) - \varphi) + (\varphi - u)\|_{L^p(\Omega\times Y)} \leq \\
&\leq |Y|^{\frac{1}{p}}\, \|u-\varphi\|_{L^p(\Omega)} + |Y|^{\frac{1}{p}}\, \|u-\varphi\|_{L^p(\Omega)} + \|\mathcal{T}_\varepsilon(\varphi) - \varphi\|_{L^p(\Omega\times Y)} \leq \\
&\leq 2\, |Y|^{\frac{1}{p}}\, \|u-\varphi\|_{L^p(\Omega)} + \|\mathcal{T}_\varepsilon(\varphi) - \varphi\|_{L^p(\Omega\times Y)}.
\end{aligned}$$

The space $\mathcal{D}(\Omega)$ is dense in $L^p(\Omega)$, so for each $\varepsilon > 0$ there exists φ such that $\|u-\varphi\|_{L^p(\Omega)} < \varepsilon$.

Using (16), we conclude

$$\begin{aligned}
0 \leq \lim_{\varepsilon \to 0} \|\mathcal{T}_\varepsilon(u) - u\|_{L^p(\Omega\times Y)} &= \lim_{\varepsilon \to 0}\Big(2\, |Y|^{\frac{1}{p}}\, \|u-\varphi\|_{L^p(\Omega)} + \|\mathcal{T}_\varepsilon(\varphi) - \varphi\|_{L^p(\Omega\times Y)}\Big) \leq \\
&\leq \lim_{\varepsilon \to 0}\Big(2\, |Y|^{\frac{1}{p}}\, \varepsilon + \|\mathcal{T}_\varepsilon(\varphi) - \varphi\|_{L^p(\Omega\times Y)}\Big) = 0.
\end{aligned}$$

(ii),(iii) The properties follow from the fact that the weak two-scale convergence is by its definition equivalent to the weak convergence in $L^p(\Omega \times Y)$. By the Proposition 2.10 and 2.12 these two properties hold for the weak convergence in any Banach space.

(iv) Adding and subtracting $\mathcal{T}_\varepsilon(u^*)$, using Theorem 3.3 (ii) and (iv), we get

$$\begin{aligned}
\|\mathcal{T}_\varepsilon(u_\varepsilon) - u_0\|_{L^p(\Omega\times Y)} &= \|\mathcal{T}_\varepsilon(u_\varepsilon) - u_0 + \mathcal{T}_\varepsilon(u^*) - \mathcal{T}_\varepsilon(u^*)\|_{L^p(\Omega\times Y)} \leq \\
&\leq \|\mathcal{T}_\varepsilon(u_\varepsilon) - \mathcal{T}_\varepsilon(u^*)\|_{L^p(\Omega\times Y)} + \|\mathcal{T}_\varepsilon(u^*) - u_0\|_{L^p(\Omega\times Y)} = \\
&= \|\mathcal{T}_\varepsilon(u_\varepsilon - u^*)\|_{L^p(\Omega\times Y)} + \|\mathcal{T}_\varepsilon(u^*) - u_0\|_{L^p(\Omega\times Y)} = \\
&= |Y|^{\frac{1}{p}} \|u_\varepsilon - u^*\|_{L^p(\Omega)} + \|\mathcal{T}_\varepsilon(u^*) - u_0\|_{L^p(\Omega\times Y)}.
\end{aligned} \tag{17}$$

Now, we can pass to the limit. Since the sequence $\{u_\varepsilon\}$ converges to u^* strongly in $L^p(\Omega)$, the first expression on the last line converges to zero.

From the property (i) it follows that

$$\mathcal{T}_\varepsilon(u^*) \to u_0 \quad \text{strongly in } L^p(\Omega \times Y).$$

Thus, the second expression on the last line in (17) also converges to zero.

Adding this together, we get

$$\|\mathcal{T}_\varepsilon(u_\varepsilon) - u_0\|_{L^p(\Omega\times Y)} \to 0.$$

(v) Let $\varphi \in L^{p'}(\Omega \times Y)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Using the Proposition 2.19, the result is straightforward.

$$\iint_{\Omega\times Y} (\mathcal{T}_\varepsilon(u_\varepsilon)(x,y) - u_0(x,y))\, \varphi(x,y)\,\mathrm{d}x\,\mathrm{d}y \leq$$
$$\leq \|\varphi\|_{L^{p'}(\Omega\times Y)} \|\mathcal{T}_\varepsilon(u_\varepsilon)(x,y) - u_0\|_{L^p(\Omega\times Y)} \to 0.$$

(vi) Let $\varphi \in L^{p'}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. From the Theorem 3.3 (i), (iii) we obtain:

$$\int_\Omega u_\varepsilon(x)\ \varphi(x)\,\mathrm{d}x = \frac{1}{|Y|}\iint_{\Omega\times Y} \mathcal{T}_\varepsilon(u_\varepsilon)(x,y)\,\mathcal{T}_\varepsilon(\varphi)(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

By the assumption, $\{\mathcal{T}_\varepsilon(u_\varepsilon)\}$ converges weakly, and according (i), $\{\mathcal{T}_\varepsilon(\varphi)\}$ converges strongly. Thus, using the Proposition 2.15, we get

$$\frac{1}{|Y|}\iint_{\Omega\times Y} \mathcal{T}_\varepsilon(u_\varepsilon)(x,y)\mathcal{T}_\varepsilon(\varphi)(x,y)\,\mathrm{d}x\,\mathrm{d}y \to \frac{1}{|Y|}\iint_{\Omega\times Y} u_0(x,y)\varphi(x)\,\mathrm{d}x\,\mathrm{d}y =$$
$$= \int_\Omega \left(\frac{1}{|Y|}\int_Y u_0(x,y)\,\mathrm{d}y\right)\varphi(x)\,\mathrm{d}x = \int_\Omega u^*(x)\ \varphi(x)\,\mathrm{d}x.$$

□

Relations between convergences above can be expressed by the following diagram:

strong $\Rightarrow$ two-scale strong $\Rightarrow$ two-scale weak $\Rightarrow$ weak.

Examples Let us assume a domain $\Omega = (0,1)$, a reference cell $Y = \langle 0,1)$ and scales $E = \{\varepsilon\} = \{1, {}^1\!/_2, {}^1\!/_3, {}^1\!/_4, \dots\}$. Then the set Λ_ε defined by (13) is empty for every ε and hence $\widehat{\Omega}_\varepsilon = \Omega$ for every ε.

Let $f, g \in L^p(\Omega)$ and $\psi \in L^\infty_{\mathrm{per}}(Y)$, such that $\int_Y \psi(x)dx = 0$.

1. Let us assume a sequence $\{u_\varepsilon\}$, where
$$u_\varepsilon(x) = f(x)\,\psi\Big(\frac{x}{\varepsilon}\Big) + g(x).$$
Then the sequence $\{u_\varepsilon\}$ converges to g weakly in $L^p(\Omega)$, but not strongly (unless $f(x) \equiv 0$ or $\psi(x) \equiv 0$).
The unfolding $\mathcal{T}_\varepsilon(u_\varepsilon)$ is
$$\mathcal{T}_\varepsilon(u_\varepsilon)(x,y) = \mathcal{T}_\varepsilon(f)(x,y)\,\psi(y) + \mathcal{T}_\varepsilon(g)(x,y) \quad \text{for } (x,y) \in \Omega \times Y.$$
Due to the Theorem 3.5 (i), the sequence $\{\mathcal{T}_\varepsilon(u_\varepsilon)\}$ strongly two-scale converges with respect to the scale E in $L^p(\Omega)$ to
$$u_0(x,y) = f(x)\,\psi(y) + g(x).$$
The example shows that the local oscillations of u_ε, which are lost in the weak limit, are conserved in the strong two-scale limit.

2. Modifying the function u_ε to
$$u_\varepsilon(x) = f(x)\,\psi\Big(\frac{2x}{\varepsilon}\Big) + g(x),$$
we get a sequence which converges also two-scale strongly with respect to E but the limit is
$$u_0(x,y) = f(x)\,\psi(2y) + g(x).$$
The weak limit is unchanged.

3. Let us make another modification of the function u_ε,
$$u_\varepsilon(x) = f(x)\,\psi\Big(\frac{x}{\varepsilon^2}\Big) + g(x).$$
The sequence $\{u_\varepsilon\}$ again converges to g weakly in $L^p(\Omega)$. But its unfolding
$$\mathcal{T}_\varepsilon(u_\varepsilon)(x,y) = \mathcal{T}_\varepsilon(f)(x,y)\,\psi\Big(\frac{y}{\varepsilon}\Big) + \mathcal{T}_\varepsilon(g)(x,y) \quad \text{for } (x,y) \in \Omega \times Y.$$
converge only two-scale weakly with respect to E to $u_0(x,y) = g(x)$ in $L^p(\Omega)$. In the limit the local oscillations are lost.

4. Let us assume a function
$$u_\varepsilon(x) = f(x)\,\psi\Big(\frac{\sqrt{2}x}{\varepsilon}\Big) + g(x).$$
In general case the function $\psi\big(\frac{\sqrt{2}x}{\varepsilon}\big)$ does not belong to $L^\infty_{\mathrm{per}}(Y)$. The sequence $\{u_\varepsilon\}$ converges only weakly in $L^p(\Omega)$ and neither converges two-scale strongly nor two-scale weakly with respect to E in $L^p(\Omega)$.

Theorem 3.6 (Compactness)**.** *Let $p \in \langle 1, \infty)$ and $\{u_\varepsilon\}$ be a bounded sequence in $L^p(\Omega)$. Then there exists a subscale $E' = \{\varepsilon'\}$, a subsequence $\{u_{\varepsilon'}\}$ in $L^p(\Omega)$ and $u_0 \in L^p(\Omega \times Y)$ such that $\{u_{\varepsilon'}\}$ two-scale weakly converges with respect to subscale E' to u_0 in $L^p(\Omega)$, i.e.*

$$\mathcal{T}_{\varepsilon'}(u_{\varepsilon'}) \rightharpoonup u_0 \quad \textit{weakly in } L^p(\Omega \times Y).$$

Proof. If $\{u_\varepsilon\}$ is bounded in $L^p(\Omega)$ then, thanks to Theorem 3.3 (ii), $\{\mathcal{T}_\varepsilon(u_\varepsilon)\}$ is bounded in $L^p(\Omega \times Y)$. Thus there exists subscale E', subsequence $\{\mathcal{T}_{\varepsilon'}(u_{\varepsilon'})\}$ and $u_0 \in L^p(\Omega \times Y)$ such that $\{\mathcal{T}_{\varepsilon'}(u_{\varepsilon'})\}$ weakly converges to u_0 in $L^p(\Omega \times Y)$, which is equivalent to the two-scale weak convergence with respect to E' in $L^p(\Omega)$. □

We will finish this section with two results which are fundamental in applications and homogenization theory.

Theorem 3.7 (Limit of product of the sequences)**.** *Let $p \in (1, \infty)$. Assume a scale $E = \{\varepsilon\}$, a sequence $\{u_\varepsilon\} \in L^p(\Omega)$ and $\{v_\varepsilon\} \in L^q(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} < 1$, such that $\{u_\varepsilon\}$ converges two-scale strongly to u_0 in $L^p(\Omega)$ and $\{v_\varepsilon\}$ converges two-scale weakly to v_0 in $L^q(\Omega)$, both with respect to E. Then the product $\{u_\varepsilon\, v_\varepsilon\}$ converges to the limit $u_0\, v_0$ two-scale weakly in $L^r(\Omega)$.*

Proof. For any $\varphi \in L^{r'}(\Omega \times Y)$, where $\frac{1}{r} + \frac{1}{r'} = 1$, we have

$$L \;=\; \iint\limits_{\Omega\times Y} \mathcal{T}_\varepsilon(u_\varepsilon\, v_\varepsilon)(x,y)\ \varphi(x,y)\, \mathrm{d}x\, \mathrm{d}y \;=\; \iint\limits_{\Omega\times Y} \mathcal{T}_\varepsilon(u_\varepsilon)(x,y)\ \mathcal{T}_\varepsilon(v_\varepsilon)(x,y)\ \varphi(x,y)\, \mathrm{d}x\, \mathrm{d}y.$$

Adding and subtracting the term $(u_0\ \mathcal{T}_\varepsilon(v_\varepsilon)\ \varphi)$ in the integrand leads to

$$L = \iint\limits_{\Omega\times Y} (\mathcal{T}_\varepsilon(u_\varepsilon) - u_0)(x,y)\ \mathcal{T}_\varepsilon(v_\varepsilon)(x,y)\ \varphi(x,y)\, \mathrm{d}x\, \mathrm{d}y \;+ \\ + \iint\limits_{\Omega\times Y} u_0(x,y)\ \mathcal{T}_\varepsilon(v_\varepsilon)(x,y)\ \varphi(x,y)\, \mathrm{d}x\, \mathrm{d}y. \quad (18)$$

For the first integral in (18) we have the estimate

$$\iint_{\Omega\times Y} (\mathcal{T}_\varepsilon(u_\varepsilon) - u_0)(x,y)\ \mathcal{T}_\varepsilon(v_\varepsilon)(x,y)\ \varphi(x)\, \mathrm{d}x\, \mathrm{d}y \le \\ \le \|\mathcal{T}_\varepsilon(u_\varepsilon) - u_0\|_{L^p(\Omega\times Y)}\, \|\mathcal{T}_\varepsilon(v_\varepsilon)\|_{L^q(\Omega\times Y)}\, \|\varphi\|_{L^{r'}(\Omega\times Y)}.$$

As a weakly convergent sequence $\{\mathcal{T}_\varepsilon(v_\varepsilon)\}$ is bounded and $\{\mathcal{T}_\varepsilon(u_\varepsilon)\}$ converges to u_0 strongly, thus the integral tends to zero.

Since, by assumption, $\{\mathcal{T}_\varepsilon(v_\varepsilon)\}$ weakly converges in $L^q(\Omega \times Y)$, for the second integral in (18) we have

$$\iint\limits_{\Omega\times Y} u_0(x,y)\ \mathcal{T}_\varepsilon(v_\varepsilon)(x,y)\ \varphi(x,y)\, \mathrm{d}x\, \mathrm{d}y \to \iint\limits_{\Omega\times Y} u_0(x,y)\ v_0(x,y)\ \varphi(x,y)\, \mathrm{d}x\, \mathrm{d}y.$$

□

Theorem 3.8 (Limit of product of the sequences)**.** *Let $p \in (1, \infty)$. Assume a scale $E = \{\varepsilon\}$, a sequence $\{u_\varepsilon\} \in L^p(\Omega)$ and $\{v_\varepsilon\} \in L^{p'}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$, such that $\{u_\varepsilon\}$ converges two-scale strongly to u_0 in $L^p(\Omega)$ and $\{v_\varepsilon\}$ converges two-scale weakly to v_0 in $L^{p'}(\Omega)$, both with respect to E.*

Then,

$$\int_\Omega u_\varepsilon(x)\ v_\varepsilon(x)\,\mathrm{d}x \to \frac{1}{|Y|}\iint_{\Omega\times Y} u_0(x,y)\ v_0(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Proof. Proof is analogical to the proof of the Theorem 3.7. □

3.3. Two-scale convergence and gradients

In many applications a sequence of gradients $\{\nabla u_\varepsilon\}$ appears. From the definition of unfolding operator we derive, for $u \in W^{1,p}(\Omega)$,

$$\mathcal{T}_\varepsilon(\nabla u) = \begin{cases} \frac{1}{\varepsilon}\nabla_y \mathcal{T}_\varepsilon(u) & \text{on} \quad \widehat{\Omega}_\varepsilon \times Y, \\ \nabla u = \nabla_x \mathcal{T}_\varepsilon(u) & \text{on} \quad \Lambda_\varepsilon \times Y. \end{cases} \tag{19}$$

The equality can be rewritten by means of the characteristic function 1_{Λ_ε} of a set Λ_ε,

$$\mathcal{T}_\varepsilon(\nabla u) = \frac{1}{\varepsilon}\nabla_y \mathcal{T}_\varepsilon(u) + \nabla_x \mathcal{T}_\varepsilon(u)\, 1_{\Lambda_\varepsilon}.$$

The main result shown in this part is: if a sequence $\{u_\varepsilon\}$ converges weakly in $W^{1,p}(\Omega)$, then the sequence $\{\nabla u_\varepsilon\}$ converges two-scale weakly in $[L^p(\Omega)]^N$, see Theorem 3.11.

The proof is based on a suitable splitting of the function u_ε into two parts: $u_\varepsilon = u_\varepsilon^1 + \varepsilon u_\varepsilon^2$. The function u_ε^2 is designed to capture oscillations and in such a way that $u_\varepsilon^2 = 0$ on Λ_ε.

In the first step, for a well chosen function u_ε^1, we show that $\{\nabla u_\varepsilon^1\}$ converges two-scale weakly in $[L^p(\Omega)]^N$.

In the second step we prove that $\{\mathcal{T}_\varepsilon(u_\varepsilon^2)\}$ converges weakly in $L^p(\Omega, W^{1,p}(Y))$. This and the equality

$$\nabla_y \mathcal{T}_\varepsilon(u_\varepsilon^2) = \varepsilon\,\mathcal{T}_\varepsilon(\nabla(u_\varepsilon^2)) = \mathcal{T}_\varepsilon(\nabla(\varepsilon\, u_\varepsilon^2)) \tag{20}$$

implies that the terms $\nabla_y \mathcal{T}_\varepsilon(u_\varepsilon^2) = \mathcal{T}_\varepsilon(\nabla(\varepsilon u_\varepsilon^2))$ converge weakly in $[L^p(\Omega \times Y)]^N$.

Finally, combining these two results gives $\mathcal{T}_\varepsilon(\nabla u_\varepsilon^1) + \mathcal{T}_\varepsilon(\nabla(\varepsilon u_\varepsilon^2)) = \mathcal{T}_\varepsilon(\nabla u_\varepsilon)$ converge weakly in $[L^p(\Omega \times Y)]^N$ and thus $\{\nabla u_\varepsilon\}$ converges two-scale weakly in $[L^p(\Omega)]^N$.

The crucial part of the proof is the way of choosing the functions u_ε^1 and u_ε^2. For that reason let us introduce a local average operator $\mathcal{M}_\varepsilon$ and its properties.

Definition 3.9 (Local average operator $\mathcal{M}_\varepsilon$)**.** The local average operator $\mathcal{M}_\varepsilon : L^p(\Omega) \to L^p(\Omega)$ for $p \geq 1$ is defined by

$$\mathcal{M}_\varepsilon(u)(x) = \begin{cases} \dfrac{1}{\varepsilon^N |Y|}\displaystyle\int_{\varepsilon\left(\left[\frac{x}{\varepsilon}\right]_Y + Y\right)} u(t)\,\mathrm{d}t & \text{for} \quad x \in \widehat{\Omega}_\varepsilon, \\ u(x) & \text{for} \quad x \in \Lambda_\varepsilon. \end{cases}$$

Let us remind that $\mathcal{M}_Y(u)(x) = \frac{1}{|Y|}\int_Y u(x,y)\,\mathrm{d}y$.

Proposition 3.10 (Properties of the local average operator $\mathcal{M}_\varepsilon$)**.** *For any $u \in L^p(\Omega)$, where $p \geq 1$, it holds:*

(i)

$$\mathcal{M}_\varepsilon(u)(x) = \mathcal{M}_Y\Big(\mathcal{T}_\varepsilon(u)\Big)(x).$$

(ii)

$$\mathcal{T}_\varepsilon\Big(\mathcal{M}_\varepsilon(u)\Big)(x,y) = \mathcal{M}_\varepsilon(u)(x,y) = \mathcal{M}_\varepsilon(u)(x).$$

(iii) Let $v \in L^{p'}(\Omega)$, then

$$\int_\Omega \mathcal{M}_\varepsilon(u)(x)\ v(x)\,\mathrm{d}x = \int_\Omega \mathcal{M}_\varepsilon(u)(x)\ \mathcal{M}_\varepsilon(v)(x)\,\mathrm{d}x = \int_\Omega u(x)\ \mathcal{M}_\varepsilon(v)(x)\,\mathrm{d}x.$$

(iv) Let $\{u_\varepsilon\}$ be a sequence in $L^p(\Omega)$ for $p \in (1,\infty)$ such that $u_\varepsilon \rightharpoonup u_0$ weakly in $L^p(\Omega)$. Then

$$\mathcal{M}_\varepsilon(u_\varepsilon) \rightharpoonup u_0 \quad \text{weakly in } L^p(\Omega).$$

Proof. (i) We prove the result separately on the domains $\widehat{\Omega}_\varepsilon$ and Λ_ε.

On $\widehat{\Omega}_\varepsilon$ by the usual change of variable cell by cell one obtains

$$\mathcal{M}_\varepsilon(u)(x) = \frac{1}{|Y|}\int_Y \mathcal{T}_\varepsilon(u)(x,y)\,\mathrm{d}y = \mathcal{M}_Y\Big(\mathcal{T}_\varepsilon(u)\Big)(x).$$

On Λ_ε we use the fact that $\mathcal{T}_\varepsilon(u)(x,y) = u(x,y) = u(x)$. Hence

$$\begin{aligned}\mathcal{M}_Y\Big(\mathcal{T}_\varepsilon(u)\Big)(x) = \mathcal{M}_Y(u)(x) = \frac{1}{|Y|}\ \int_Y u(x)\,\mathrm{d}y = \frac{1}{|Y|}\ u(x)\int_Y 1\,\mathrm{d}y = \\ = \frac{1}{|Y|}\ u(x)\ |Y| = u(x) = \mathcal{M}_\varepsilon(u)(x).\end{aligned}$$

(ii) It follows from the fact that $\mathcal{M}_\varepsilon(u)(x)$ is piecewise constant in $\widehat{\Omega}_\varepsilon$.

(iii) Proof is obvious.

(iv) Let $\varphi \in L^{p'}(\Omega)$. Adding and subtracting $\mathcal{M}_\varepsilon(u_0)$ we get:

$$\begin{aligned}&\int_\Omega [\mathcal{M}_\varepsilon(u_\varepsilon)(x) - u_0(x)]\,\varphi(x)\,\mathrm{d}x = \\ &\quad = \int_\Omega [\mathcal{M}_\varepsilon(u_\varepsilon)(x) - \mathcal{M}_\varepsilon(u_0)(x)]\,\varphi(x)\,\mathrm{d}x + \int_\Omega [\mathcal{M}_\varepsilon(u_0)(x) - u_0(x)]\,\varphi(x)\,\mathrm{d}x. \quad (21)\end{aligned}$$

Let us show that the first integral on the previous line converges to zero. Using the linearity of operator $\mathcal{M}_\varepsilon$ (which follows directly from its definition), the property (iii) and the weak convergence of the $\{u_\varepsilon\}$ leads to:

$$I_1 = \int\limits_\Omega [\mathcal{M}_\varepsilon(u_\varepsilon)(x) - \mathcal{M}_\varepsilon(u_0)(x)]\,\varphi(x)\,\mathrm{d}x = \int\limits_\Omega \mathcal{M}_\varepsilon(u_\varepsilon - u_0)(x)\,\varphi(x)\,\mathrm{d}x =$$
$$= \int\limits_\Omega [u_\varepsilon(x) - u_0(x)]\,\mathcal{M}_\varepsilon(\varphi)(x)\,\mathrm{d}x \to 0\ . \quad (22)$$

The second integral on the last line of (21) also converge to the zero. Indeed, by using property (i) and Theorem 3.5 (i) we get:

$$I_2 = \int\limits_\Omega [\mathcal{M}_\varepsilon(u_0)(x) - u_0(x)]\,\varphi(x)\,\mathrm{d}x = \int\limits_\Omega \Big[\mathcal{M}_Y\Big(\mathcal{T}_\varepsilon(u_0)\Big)(x) - u_0(x)\Big]\,\varphi(x)\,\mathrm{d}x \to$$
$$\to \int\limits_\Omega \Big[\mathcal{M}_Y\Big(u_0\Big)(x) - u_0(x)\Big]\,\varphi(x)\,\mathrm{d}x = \int\limits_\Omega [u_0(x) - u_0(x)]\,\varphi(x)\,\mathrm{d}x = 0\ . \quad (23)$$

Summing up (22) and (23) provides the result.

□

Theorem 3.11. *Let a sequence $\{u_\varepsilon\}$ be bounded in $W^{1,p}(\Omega)$, for $p \in (1,\infty)$. i.e.*

$$\|u_\varepsilon\|_{W^{1,p}(\Omega)} \le C.$$

Then there exists a subsequence (still denoted $\{u_\varepsilon\}$) and functions $u_0 \in W^{1,p}(\Omega)$ and $u_0^* \in L^p(\Omega; W^{1,p}_{per}(Y))$ such that

(i) $\mathcal{T}_\varepsilon(u_\varepsilon) \rightharpoonup u_0$ *weakly in* $L^p(\Omega; W^{1,p}(Y))$,

(ii) $\mathcal{T}_\varepsilon(\nabla u_\varepsilon) \rightharpoonup \nabla u_0 + \nabla_y u_0^*$ *weakly in* $[L^p(\Omega \times Y)]^N$, *i.e.* $\{\nabla u_\varepsilon\}$ *converges two-scale weakly in* $[L^p(\Omega)]^N$.

Moreover, $\mathcal{M}_Y(u_0^*) = 0$.

Proof. Property (i) - To prove that there exists a subsequence of $\{\mathcal{T}_\varepsilon(u_\varepsilon)\}$ weakly converging in $L^p(\Omega; W^{1,p}(Y))$ it is enough to show that the sequence $\{\mathcal{T}_\varepsilon(u_\varepsilon)\}$ is bounded in the same space (see Theorem 2.13).

Using definitions of the norms of the spaces $L^p(\Omega; W^{1,p}(Y))$ and $L^p(\Omega \times Y)$ and in the last step equality (19) leads to

$$\|\mathcal{T}_\varepsilon(u_\varepsilon)\|^p_{L^p(\Omega;W^{1,p}(Y))} = \int_\Omega \left(\left(\|\mathcal{T}_\varepsilon(u_\varepsilon)\|^p_{L^p(Y)} + \|\nabla_y \mathcal{T}_\varepsilon(u_\varepsilon)\|^p_{[L^p(Y)]^N} \right)^{\frac{1}{p}} \right)^p \mathrm{d}x =$$
$$= \|\mathcal{T}_\varepsilon(u_\varepsilon)\|_{L^p(\Omega\times Y)} + \|\nabla_y \mathcal{T}_\varepsilon(u_\varepsilon)\|_{[L^p(\Omega\times Y)]^N} = \|\mathcal{T}_\varepsilon(u_\varepsilon)\|_{L^p(\Omega\times Y)} + \|\varepsilon \mathcal{T}_\varepsilon(\nabla u_\varepsilon)\|_{\left[L^p(\widehat{\Omega}_\varepsilon\times Y)\right]^N}.$$

Using the Theorem 3.3, property (iv), gives us

$$\|\mathcal{T}_\varepsilon(u_\varepsilon)\|_{L^p(\Omega\times Y)} + \|\varepsilon\,\mathcal{T}_\varepsilon(\nabla u_\varepsilon)\|_{\left[L^p(\widehat{\Omega}_\varepsilon\times Y)\right]^N} \le |Y|^{\frac{1}{p}}\,\|u_\varepsilon\|_{L^p(\Omega)} + |Y|^{\frac{1}{p}}\,\varepsilon\,\|\nabla u_\varepsilon\|_{[L^p(\Omega)]^N}.$$

The last term is bounded for all ε, since according to the assumptions the sequence $\{u_\varepsilon\}$ is bounded in $W^{1,p}(\Omega)$ and $\varepsilon \searrow 0$.

Property (ii) - The proof is carried out in several steps.

First step - splitting the function u_ε. Let us split the function u_ε. Set $u_\varepsilon = u_\varepsilon^1 + \varepsilon\, u_\varepsilon^2$, where

$$u_\varepsilon^1 = \mathcal{M}_\varepsilon(u_\varepsilon) \quad \text{and thus} \quad u_\varepsilon^2 = \frac{1}{\varepsilon}\,[u_\varepsilon - \mathcal{M}_\varepsilon(u_\varepsilon)].$$

Since u_ε^1 is piecewise constant in $\widehat{\Omega}_\varepsilon$, and using the definition 3.2 of unfolding operator, we get

$$\mathcal{T}_\varepsilon(\nabla u_\varepsilon^1)(x,y) = \begin{cases} 0 & \text{for} \quad [x,y] \in \widehat{\Omega}_\varepsilon \times Y, \\ \nabla u_\varepsilon(x) & \text{for} \quad [x,y] \in \Lambda_\varepsilon \times Y. \end{cases}$$

Using (19) leads to

$$\nabla_y \mathcal{T}_\varepsilon(u_\varepsilon^2) = \varepsilon\,\mathcal{T}_\varepsilon(\nabla(u_\varepsilon^2)) = \mathcal{T}_\varepsilon(\nabla(\varepsilon u_\varepsilon^2)).$$

Summing up these results we get

$$\mathcal{T}_\varepsilon(\nabla u_\varepsilon) = \mathcal{T}_\varepsilon(\nabla u_\varepsilon^1) + \mathcal{T}_\varepsilon(\varepsilon\,\nabla u_\varepsilon^2) = \nabla u_\varepsilon\,1_{\Lambda_\varepsilon} + \mathcal{T}_\varepsilon(\nabla u_\varepsilon\,1_{\widehat{\Omega}_\varepsilon}) = \nabla u_\varepsilon\,1_{\Lambda_\varepsilon} + \nabla_y \mathcal{T}_\varepsilon(u_\varepsilon^2). \tag{24}$$

Second step - convergence of $\mathcal{T}_\varepsilon(\nabla u_\varepsilon^1)$. For $\varepsilon \to 0$ the terms $\nabla u_\varepsilon\,1_{\Lambda_\varepsilon}$ converge to zero because $\{\nabla u_\varepsilon\}$ is bounded in $[L^p(\Omega)]^N$ and $|\Lambda_\varepsilon| \to 0$, i.e.

$$\nabla u_\varepsilon\,1_{\Lambda_\varepsilon} \to 0 \quad \text{strongly in } [L^p(\Omega)]^N. \tag{25}$$

Third step - convergence of $\mathcal{T}_\varepsilon(\nabla u_\varepsilon^2)$. For u_ε^2 we get

$$\mathcal{T}_\varepsilon(u_\varepsilon^2)(x,y) = \begin{cases} \frac{1}{\varepsilon}\,[\mathcal{T}_\varepsilon(u_\varepsilon)(x,y) - \mathcal{M}_\varepsilon(u_\varepsilon)(x,y)] & \text{for} \quad [x,y] \in \widehat{\Omega}_\varepsilon \times Y, \\ 0 & \text{for} \quad [x,y] \in \Lambda_\varepsilon \times Y. \end{cases}$$

Let us denote by y^c the vector function

$$y^c = \left(y_1 - \frac{l_1}{2}, y_2 - \frac{l_2}{2}, \ldots, y_N - \frac{l_N}{2}\right),$$

where $l_1, l_2, \ldots, l_N$ are dimensions of the reference cell Y, see (12).

The function $\mathcal{T}_\varepsilon(u_\varepsilon^2) - y^c \cdot \nabla u_0$ has mean value equals to zero. Indeed,

$$\mathcal{M}_Y\Big(\mathcal{T}_\varepsilon(u_\varepsilon^2) - y^c \cdot \nabla u_0\Big) = \mathcal{M}_Y\left[\frac{1}{\varepsilon}\Big(\mathcal{T}_\varepsilon(u_\varepsilon) - \mathcal{T}_\varepsilon(\mathcal{M}_\varepsilon(u_\varepsilon))\Big)\right] - \mathcal{M}_Y(y^c \cdot \nabla u_0) =$$
$$= \mathcal{M}_Y\left[\frac{1}{\varepsilon}\Big(\mathcal{T}_\varepsilon(u_\varepsilon) - \mathcal{M}_\varepsilon(u_\varepsilon)\Big)\right] - 0 = \frac{1}{\varepsilon}\Big[\mathcal{M}_Y\Big(\mathcal{T}_\varepsilon(u_\varepsilon)\Big) - \mathcal{M}_Y\Big(\mathcal{M}_\varepsilon(u_\varepsilon)\Big)\Big].$$

On $\widehat{\Omega}_\varepsilon \times Y$ it is equal to

$$\frac{1}{\varepsilon}\Big[\mathcal{M}_Y\Big(\mathcal{T}_\varepsilon(u_\varepsilon)\Big) - \mathcal{M}_Y\Big(\mathcal{M}_\varepsilon(u_\varepsilon)\Big)\Big] = \frac{1}{\varepsilon}\left[\mathcal{M}_\varepsilon(u_\varepsilon) - \mathcal{M}_\varepsilon(u_\varepsilon)\right] = 0$$

and on $\Lambda_\varepsilon \times Y$ it gives

$$\frac{1}{\varepsilon}\Big[\mathcal{M}_Y\Big(\mathcal{T}_\varepsilon(u_\varepsilon)\Big) - \mathcal{M}_Y\Big(\mathcal{M}_\varepsilon(u_\varepsilon)\Big)\Big] = \frac{1}{\varepsilon}\left[\mathcal{M}_Y(u_\varepsilon) - \mathcal{M}_Y(u_\varepsilon)\right] = 0.$$

Applying the Poincaré-Wirtinger inequality (see Proposition 2.34) in Y to the function $\mathcal{T}_\varepsilon(u_\varepsilon^2) - y^c \cdot \nabla u_0$ we get

$$\Big\|\mathcal{T}_\varepsilon(u_\varepsilon^2) - y^c \cdot \nabla u_0\Big\|_{L^p(\Omega\times Y)} \leq C\Big\|\nabla_y \mathcal{T}_\varepsilon(u_\varepsilon^2) - \nabla u_0\Big\|_{[L^p(\Omega\times Y)]^N}.$$

Since, due to assumptions, the term $\|\nabla_y \mathcal{T}_\varepsilon(u_\varepsilon^2) - \nabla u_0\|_{[L^p(\Omega\times Y)]^N}$ is bounded, the inequality above implies boundedness of $\{\mathcal{T}_\varepsilon(u_\varepsilon^2) - y^c \cdot \nabla u_0\}$. Terefore there exists u_0^* in $L^p(\Omega; W^{1,p}(Y))$ such that, up to a subsequence,

$$\mathcal{T}_\varepsilon(u_\varepsilon^2) - y^c \cdot \nabla u_0 \rightharpoonup u_0^* \quad \text{weakly in } L^p(\Omega, W^{1,p}(Y)). \tag{26}$$

In other words

$$\mathcal{T}_\varepsilon(u_\varepsilon^2) \rightharpoonup y^c \cdot \nabla u_0 + u_0^* \quad \text{weakly in } L^p(\Omega, W^{1,p}(Y)).$$

And thus

$$\nabla_y \mathcal{T}_\varepsilon(u_\varepsilon^2) = \mathcal{T}_\varepsilon(\nabla u_\varepsilon) \rightharpoonup \nabla u_0 + \nabla_y u_0^* \quad \text{weakly in } [L^p(\Omega \times Y)]^N. \tag{27}$$

From (24),(25) and (27) follows that $\{\nabla u_\varepsilon\}$ converges two-scale weakly in $[L^p(\Omega)]^N$.

Fourth step - average of the function u_0^*. Since for the expression on the left-hand side in (26) holds $\mathcal{M}_Y\Big(\mathcal{T}_\varepsilon(u_\varepsilon^2) - y^c \cdot \nabla u_0\Big) = 0$, the same holds for the right-hand side, i.e.

$$\mathcal{M}_Y(u_0^*) = 0.$$

Fifth step - Y-periodicity of u_0^*. Since reference cell Y is a N-dimensional cube, u_0^* must satisfies, in the sense of traces,

$$u_0^*(x, y_1, \dots, y_{i-1}, 0, y_{i+1}, \dots, y_N) = u_0^*(x, y_1, \dots, y_{i-1}, l_i, y_{i+1}, \dots, y_N) \quad \text{for } i = 1, \dots, N.$$

Without lost of generality, assume $i = N$. Set $y' = (y_1, \dots, y_{N-1})$ and $e_N = (0, \dots, 0, 1)$. For any $\psi \in \mathcal{D}(\Omega \times Y')$ we have:

$$I_\varepsilon = \iint\limits_{\Omega\times Y'} \Big(\mathcal{T}_\varepsilon(u_\varepsilon^2)(x, (y', l_N)) - \mathcal{T}_\varepsilon(u_\varepsilon^2)(x, (y', 0))\Big)\, \psi(x, y')\, \mathrm{d}x\, \mathrm{d}y' =$$

$$= \iint\limits_{\widehat{\Omega}_\varepsilon\times Y'} \frac{1}{\varepsilon}\left[u_\varepsilon\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big]_Y + \varepsilon(y', l_N)\Big) - u_\varepsilon\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big]_Y + \varepsilon(y', 0)\Big)\right]\, \psi(x, y')\, \mathrm{d}x\, \mathrm{d}y'.$$

By a change of variable one gets

$$I_\varepsilon = \iint\limits_{\left(\widehat{\Omega}_\varepsilon - \varepsilon l_N e_N\right)\times Y'} u_\varepsilon\Big(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon\,(y',0)\Big)\,\frac{1}{\varepsilon}\,\psi(x-\varepsilon\, l_N e_N, y')\,\mathrm{d}x\,\mathrm{d}y' -$$

$$- \iint\limits_{\widehat{\Omega}_\varepsilon\times Y'} u_\varepsilon\Big(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon\,(y',0)\Big)\,\frac{1}{\varepsilon}\,\psi(x,y')\,\mathrm{d}x\,\mathrm{d}y' =$$

$$= \iint\limits_{\left(\widehat{\Omega}_\varepsilon - \varepsilon\, l_N e_N\right)\times Y'} \mathcal{T}_\varepsilon(u_\varepsilon)(x,(y',0))\,\frac{1}{\varepsilon}\,\psi(x-\varepsilon\, l_N e_N, y')\,\mathrm{d}x\,\mathrm{d}y' -$$

$$- \iint\limits_{\widehat{\Omega}_\varepsilon\times Y'} \mathcal{T}_\varepsilon(u_\varepsilon)(x,(y',0))\,\frac{1}{\varepsilon}\,\psi(x,y')\,\mathrm{d}x\,\mathrm{d}y'.$$

The sequence $\{\mathcal{T}_\varepsilon(u_\varepsilon)\}$ converges weakly in $L^p(\Omega; W^{1,p}(Y))$. By the trace Theorem 2.35, the trace of $\mathcal{T}_\varepsilon(u_\varepsilon)$ on $\Omega\times Y'$ converges weakly to u_0 in $L^p(\Omega\times Y')$. Hence $\{I_\varepsilon\}$ converges to

$$- \iint\limits_{\Omega\times Y'} u_0(x)\,\frac{\partial\psi}{\partial x_n}(x,y')\,\mathrm{d}x\,\mathrm{d}y. \tag{28}$$

By similar arguments together with the fact that

$$(y^c\cdot\nabla u_0)(x,(y',l_N)) - (y^c\cdot\nabla u_0)(x,(y',0)) = \frac{\partial u_0}{\partial x_N}(x)$$

we obtain

$$\iint\limits_{\Omega\times Y'} [(y^c\cdot\nabla u_0)(x,(y',l_N)) - (y^c\cdot\nabla u_0)(x,(y',0))]\,\psi(x,y')\,\mathrm{d}x\,\mathrm{d}y' =$$

$$= \iint\limits_{\Omega\times Y'} \frac{\partial u_0}{\partial x_N}(x)\,\psi(x,y')\,\mathrm{d}x\,\mathrm{d}y' = - \iint\limits_{\Omega\times Y'} u_0(x)\,\frac{\partial\psi}{\partial x_N}(x,y')\,\mathrm{d}x\,\mathrm{d}y.$$

This together with (28) yields

$$\iint\limits_{\Omega\times Y'} \Big(u_0^*(x,(y',l_N)) - u_0^*(x,(y',0))\Big)\,\psi(x,y')\,\mathrm{d}x\,\mathrm{d}y' = 0 \quad \text{for } \psi\in\mathcal{D}(\Omega\times Y').$$

By lemma 2.26, $u_0^*(x,(y',l_N)) = u_0^*(x,(y',0))$ in the sense of traces, and thus u_0^* is y_N-periodic. The same holds in the others directions. □

Remark. The vector function $y^c = (y_1^c, y_2^c, \ldots, y_N^c)$ is designed in such a way to satisfy, for $i = 1\ldots N$:

1. $\dfrac{\partial y_i^c}{\partial y_i} = 1$ and
2. $\mathcal{M}_Y(y_i^c) = 0.$

To fulfill the first condition we suggest its components in the form $y_i^c = y_i - c_i$, $c_i \in \mathbb{R}$. The constants c_i are estimated from the second condition

$$\mathcal{M}_Y(y_i^c) = \int_Y y_i - c_i \,\mathrm{d}y = \int_Y y_i \,\mathrm{d}y - c_i \int_Y 1 \,\mathrm{d}y = \int_Y y_i \,\mathrm{d}y - c_i|Y|.$$

So

$$c_i = \frac{\int_Y y_i \,\mathrm{d}y}{|Y|},$$

which is the i-th coordinate of the centroid of the reference cell Y. For the cell Y defined in (12) we conclude that $y^c = \left(y_1 - \frac{l_1}{2}, y_2 - \frac{l_2}{2}, \dots, y_N - \frac{l_N}{2}\right)$.

4. Periodic unfolding for perforated domains

4.1. Domain with holes

The aim of this chapter is to redefine the unfolding operator in such a way that it is suitable for periodically perforated domains. Let us begin with defining a domain with holes.

As in the previous chapter, let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary. We consider scales $E = \{\varepsilon_k\}$, defined by the Definition 3.1. For each ε let $\{T_\varepsilon^j\}_{j=1}^{n(\varepsilon)}$ be a system of disjoint bounded domains in $\mathbb{R}^N$ representing the "holes". Let us suppose that they have Lipschitz boundary.

Let Ω_ε^* denote the part of Ω occupied by material. It is defined as Ω without holes T_ε^j, i.e.

$$\Omega_\varepsilon^* = \Omega \setminus T_\varepsilon, \quad \text{where} \quad T_\varepsilon = \bigcup_{j=1}^{n(\varepsilon)} \overline{T_\varepsilon^j}. \tag{29}$$

We assume that Ω_ε^* is a multiply connected set. Furthermore we denote by $T_{\mathrm{int},\,\varepsilon}^i$, $i = 1, \ldots, m(\varepsilon)$, the "interior holes", they are such sets T_ε^j which are completely inside Ω and do not intersect the boundary $\partial\Omega$, i.e. the sets $\overline{T_\varepsilon^j} \subset \Omega$. Their union is denoted by $T_{\mathrm{int},\,\varepsilon}$,

$$T_{\mathrm{int},\,\varepsilon} = \bigcup_{i=1}^{m(\varepsilon)} T_{\mathrm{int},\,\varepsilon}^i.$$

Let the sets T_ε^j which intersect the boundary be denoted by $T_{\mathrm{ext},\,\varepsilon}$, i.e.

$$T_{\mathrm{ext},\,\varepsilon} = \Big(T_\varepsilon \setminus T_{\mathrm{int},\,\varepsilon}\Big) \cap \Omega,$$

and $\partial_{\mathrm{ext}}\Omega_\varepsilon^*$ denote the exterior boundary of Ω_ε^*,

$$\partial_{\mathrm{ext}}\Omega_\varepsilon^* = \partial\Omega_\varepsilon^* \setminus \partial T_{\mathrm{int},\,\varepsilon}.$$

4.1.1. Periodically perforated domain

Till now, the holes T_ε^j were distributed in a very general manner. For $\varepsilon_i \neq \varepsilon_j$ there was not, in general, any connection between T_{ε_i} and T_{ε_j}.

In sequel, we define periodically distributed holes. In this case, for $\varepsilon \searrow 0$, there are more and more holes with a smaller and smaller volume.

As in the previous chapter, let *reference cell* Y in $\mathbb{R}^N$ be N-dimensional interval defined by (12).

Let $T \subset Y$ be an open bounded set with a smooth boundary. This set represents reference holes in Y. The part of the reference cell Y occupied by a material is denoted by Y^*:

$$Y^* = Y \setminus \overline{T}.$$

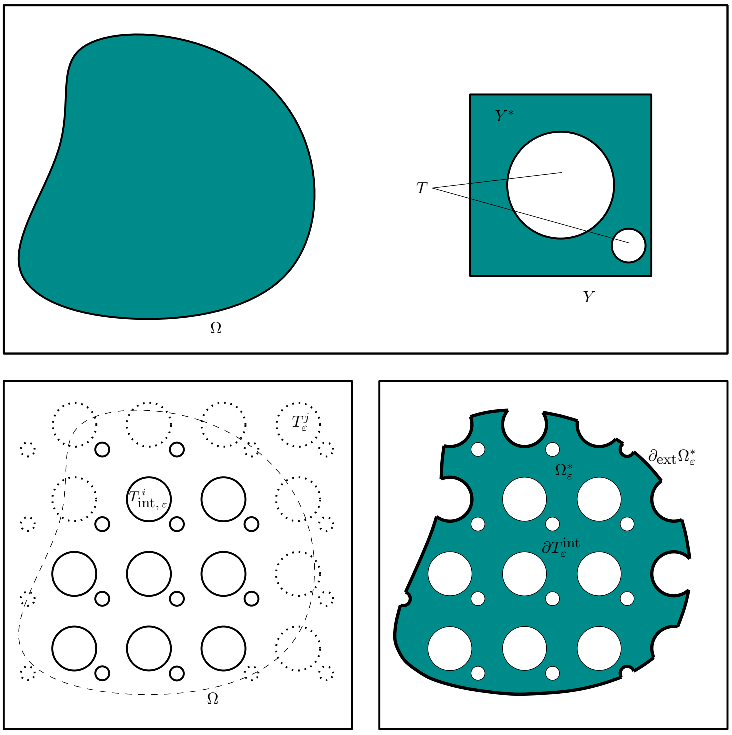

Figure 5: Periodically perforated domain. Upper: domain Ω and reference cell; lower left: inner holes $T^i_{\mathrm{int},\varepsilon}$; lower right: part of Ω occupied by material Ω_ε^* (marked by cyan), with its exterior boundary $\partial_{\mathrm{ext}}\Omega_\varepsilon^*$ and interior boundary $\partial T_{\mathrm{int},\varepsilon}$.

Let us introduce function r, which determines how fast the shrinking of holes is. Let r be a positive increasing function, such that

$$\lim_{\varepsilon\to 0} r(\varepsilon) = 0.$$

Then the sets T_ε (used in (29) to construct the perforated domain Ω_ε^*) is defined as a translates and scaled images of T, so

$$T_\varepsilon = \bigcup_{k\in\mathcal{K}} (r(\varepsilon)(T+k)),$$

where $\mathcal{K} = \left\{ k \in \mathbb{R}^N \mid k = (\xi_1 l_1, \xi_2 l_2, \ldots, \xi_N l_N), \xi \in \mathbb{Z}^N \right\}$.

It is necessary to choose the function r in such a way that ensures that the holes are always inside the cells, i.e.

$$r(\varepsilon)\, T \subset \varepsilon\, Y \quad \forall \varepsilon.$$

Furthermore, we suppose that, if the set T consists of more connected disjoint sets then these sets remain disjoint for all ε.

We can distinguish three typical kinds of behaviors of the holes. For that reason let us denote by θ_ε the ratio of the volume of material in cell and the volume of cell, i.e.

$$\theta_\varepsilon = \frac{|\varepsilon Y - r(\varepsilon) T|}{|\varepsilon Y|}.$$

The case when $r(\varepsilon) = \varepsilon$ is very classical, the ratio θ_ε is constant for all ε . A case when $\frac{r(\varepsilon)}{\varepsilon} \to 0$ as $\varepsilon \to 0$ is called *small holes*. In such case the volume of holes goes to zero quicker than the volume of material in the cell, i.e. $\theta_\varepsilon \to 1$ as $\varepsilon \to 0$. In the last case $\theta_\varepsilon \to 0$, which means that the shrinking of the holes is slower than the shrinking of the cells. An example of these three cases is on the Figure 6.

4.2. Unfolding operator $\mathcal{T}_\varepsilon^*$ in perforated domains

Analogically as for fixed domains, let us split the domain Ω_ε^* in two parts. We define (see figure 7)

$$\widehat{\Omega}_\varepsilon^* = \widehat{\Omega}_\varepsilon \setminus T_{\text{int},\,\varepsilon} \quad \text{and} \quad \Lambda_\varepsilon^* = \Omega_\varepsilon^* \setminus \widehat{\Omega}_\varepsilon^*, \tag{30}$$

where $\widehat{\Omega}_\varepsilon$ is defined by (13).

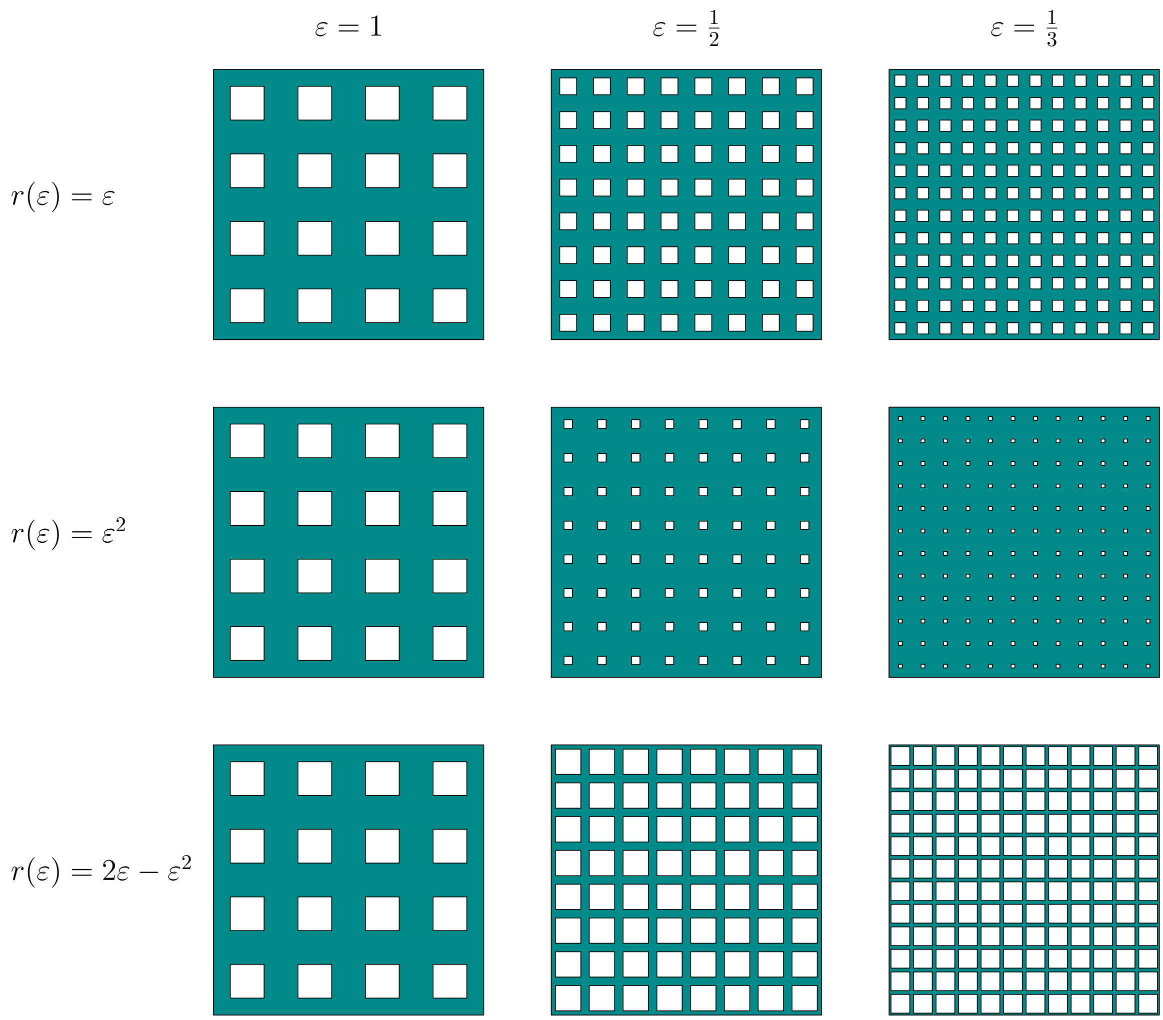

Figure 6: Example of three different behaviors of the holes depending on the choice of function r. A case on the middle line belongs to the cases called small holes.

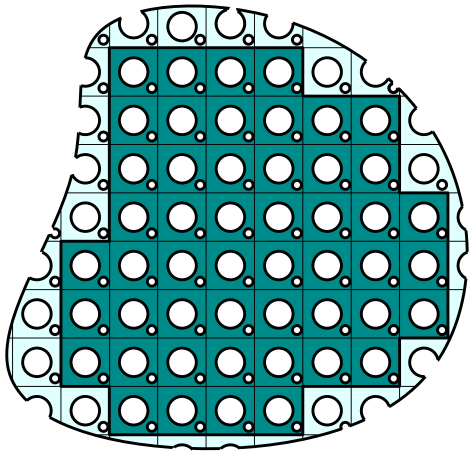

Figure 7: Domains Λ_ε^* (light) and $\widehat{\Omega}_\varepsilon^*$ (dark).

In the following part, we introduce an unfolding operator $\mathcal{T}_\varepsilon^*$ for perforated domains and we follow the same ideas as in Section 3. In sequel, we cover the case where the ratio of the volume of material to the volume of cell is constant for all ε, i.e. the function $r(\varepsilon) = \varepsilon$.

Definition 4.1 (Unfolding operator for perforated domains)**.** An operator $\mathcal{T}_\varepsilon^*$ maps a function $u : \Omega_\varepsilon^* \to \mathbb{R}$ to $\mathcal{T}_\varepsilon^*(u) : \Omega \times Y \to \mathbb{R}$, and is defined as follows:

$$\mathcal{T}_\varepsilon^*(u)(x,y) = \begin{cases} u\Big(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\Big) & \text{for } (x,y) \in \widehat{\Omega}_\varepsilon \times Y^*, \\ u(x) & \text{for } (x,y) \in \Lambda_\varepsilon^* \times Y, \\ 0 & \text{otherwise.} \end{cases} \tag{31}$$

For u defined on Ω_ε^* we denote its extension by zero into Ω by $\tilde{u}$. The same notation will be used for functions defined on $\Omega \times Y^*$ extended by zero into $\Omega \times Y$. The relationship between $\mathcal{T}_\varepsilon^*$ and $\mathcal{T}_\varepsilon$ is given by

$$\mathcal{T}_\varepsilon^*(u) = \mathcal{T}_\varepsilon(\tilde{u}). \tag{32}$$

Theorem 4.2 (Properties of the unfolding operator for perforated domain)**.** *Let $\mathcal{T}_\varepsilon^*$ be the unfolding operator for perforated domains defined by (31). Then for all $\varepsilon \in E$ we have:*

(i) The operator $\mathcal{T}_\varepsilon^*$ is multiplicative, i.e. for all $u, v : \Omega_\varepsilon^* \to \mathbb{R}$ we have

$$\mathcal{T}_\varepsilon^*(u\,v) = \mathcal{T}_\varepsilon^*(u)\,\mathcal{T}_\varepsilon^*(v).$$

(ii) The unfolding operator $\mathcal{T}_\varepsilon^*$ is linear, i.e. for all $\alpha, \beta \in \mathbb{R}$ and $u, v : \Omega_\varepsilon^* \to \mathbb{R}$,

$$\mathcal{T}_\varepsilon^*(\alpha\,u + \beta\,v) = \alpha\,\mathcal{T}_\varepsilon^*(u) + \beta\,\mathcal{T}_\varepsilon^*(v).$$

(iii) The unfolding operator $\mathcal{T}_\varepsilon^*$ conserves the integral, i.e. for all $u \in L^1(\Omega_\varepsilon^*)$ one has

$$\iint\limits_{\Omega\times Y} \mathcal{T}_\varepsilon^*(u)(x,y)\,\mathrm{d}x\,\mathrm{d}y = |Y| \int\limits_{\Omega_\varepsilon^*} u(x)\,\mathrm{d}x.$$

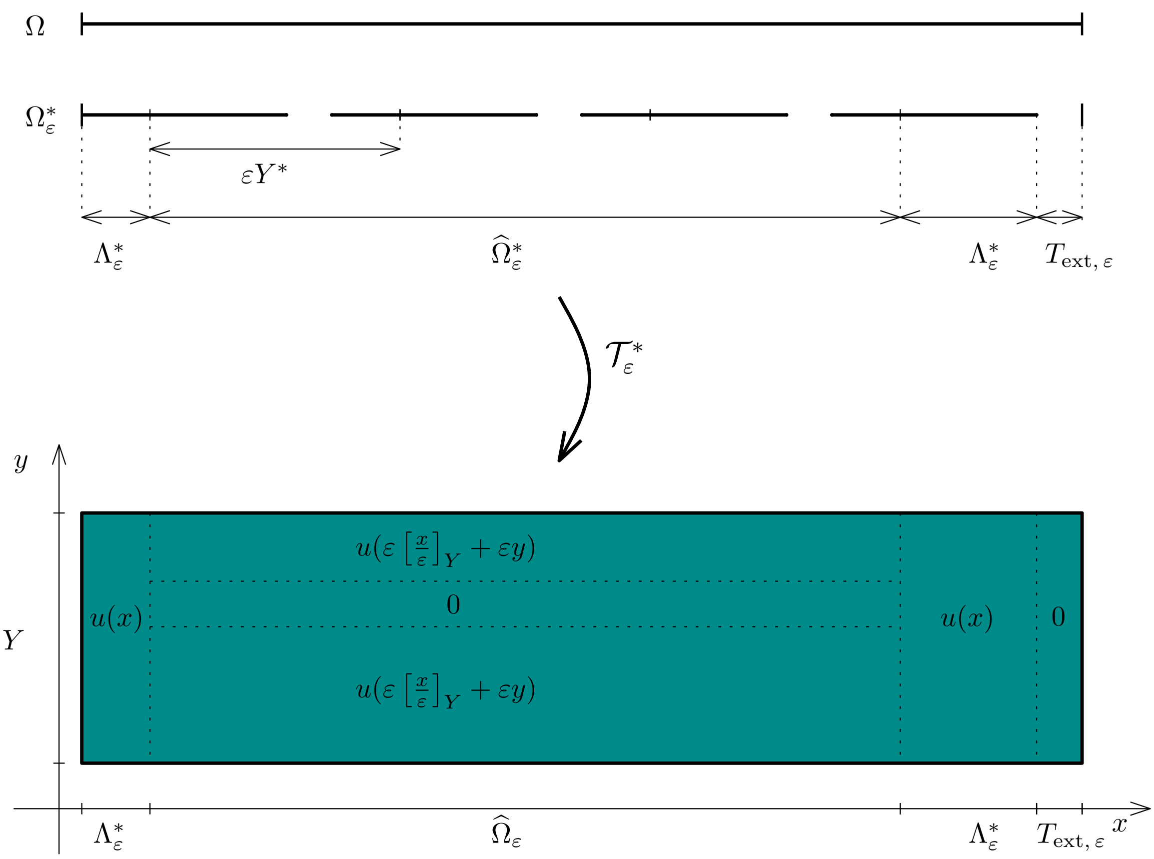

Figure 8: Example of the unfolding of a function $u(x)$ defined on periodically perforated domain Ω_ε^*.

(iv) *The unfolding operator $\mathcal{T}_\varepsilon^*$ conserves the norm in the sense that for every $u \in L^p(\Omega_\varepsilon^*)$, $p \in \langle 1, \infty)$, it holds*

$$\|\mathcal{T}_\varepsilon^*(u)\|_{L^p(\Omega\times Y)} = |Y|^{\frac{1}{p}}\,\|u\|_{L^p(\Omega_\varepsilon^*)}.$$

Thus $\mathcal{T}_\varepsilon^*$ is bounded and its norm satisfies:

$$\|\mathcal{T}_\varepsilon^*\|_{\mathcal{L}(L^p(\Omega_\varepsilon^*),L^p(\Omega\times Y))} = |Y|^{\frac{1}{p}}.$$

(v) *$\mathcal{T}_\varepsilon^*$ is continuous operator for $L^p(\Omega_\varepsilon^*)$ to $L^p(\Omega\times Y)$, where $p \in \langle 1, \infty)$.*

Proof. (i) It follows directly from the Definition 4.1.

(ii) The Linearity of operator $\mathcal{T}_\varepsilon^*$ is obvious.

(iii) From the Definition 4.1 one gets

$$I = \iint\limits_{\Omega\times Y} \mathcal{T}_\varepsilon^*(u)(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint\limits_{\widehat{\Omega}_\varepsilon\times Y^*} \mathcal{T}_\varepsilon^*(u)(x,y)\,\mathrm{d}x\,\mathrm{d}y + \iint\limits_{\Lambda_\varepsilon^*\times Y} \mathcal{T}_\varepsilon^*(u)(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Using (13), the first integral can be split. In the second integral we use the equality $\mathcal{T}_\varepsilon^*(u)(x,y) = u(x)$ on $\Lambda_\varepsilon^* \times Y$. Thus

$$I = \sum_{k\in\Xi^\varepsilon} \iint\limits_{\varepsilon(Y+k)\times Y^*} \mathcal{T}_\varepsilon^*(u)(x,y)\,\mathrm{d}x\,\mathrm{d}y + |Y| \int\limits_{\Lambda_\varepsilon^*} u(x)\,\mathrm{d}x,$$

where Ξ^ε is given by (13).

The unfolded function $\mathcal{T}_\varepsilon^*(u)(x,y)$ is constant in x on each $\varepsilon(Y+k)\times Y^*$. This yields:

$$I = \sum_{k\in\Xi^\varepsilon} |\varepsilon(Y+k)| \int\limits_{Y^*} u(\varepsilon(y+k))\,\mathrm{d}y + |Y| \int\limits_{\Lambda_\varepsilon^*} u(x)\,\mathrm{d}x =$$

$$= \varepsilon^N |Y| \sum_{k\in\Xi^\varepsilon} \int\limits_{Y^*} u(\varepsilon(y+k))\,\mathrm{d}y + |Y| \int\limits_{\Lambda_\varepsilon^*} u(x)\,\mathrm{d}x.$$

The change of variable $\varepsilon(y+k) = x$ and simple calculations yields the result.

$$I = |Y| \sum_{k\in\Xi^\varepsilon} \int\limits_{\varepsilon(Y^*+k)} u(x)\,\mathrm{d}x + |Y| \int\limits_{\Lambda_\varepsilon^*} u(x,y)\,\mathrm{d}x = |Y| \int\limits_{\Omega_\varepsilon^*} u(x)\,\mathrm{d}x.$$

(iv) Because the equality (32) holds, the operator $\mathcal{T}_\varepsilon^*$ possesses properties which follow directly from the Theorem 3.3.

Let us show, that the unfolding operator for perforated domains is bounded. For the norm of an unfolded function u it holds:

$$\|\mathcal{T}_\varepsilon^*(u)\|_{L^p(\Omega\times Y)} = \|\mathcal{T}_\varepsilon(\tilde{u})\|_{L^p(\Omega\times Y)} = |Y|^{\frac{1}{p}}\, \|\tilde{u}\|_{L^p(\Omega)} = |Y|^{\frac{1}{p}}\, \|u\|_{L^p(\Omega_\varepsilon^*)}.$$

(v) The continuity of linear operator is equivalent to its boundedness.

□

Definition 4.3 (Two-scale convergence for perforated domains)**.** Let $\mathcal{T}_\varepsilon^*$ be the unfolding operator for perforated domains defined by (31), $E = \{\varepsilon\}$ be a scale, $\{u_\varepsilon\}$ be a sequence in $L^p(\Omega_\varepsilon^*)$ and $u_0 \in L^p(\Omega\times Y)$, $p \in \langle 1,\infty)$.

A sequence $\{u_\varepsilon\}$ is said to *strongly two-scale converge* to u_0 in $L^p(\Omega)$ with respect to the scale E if the sequence $\{\mathcal{T}_\varepsilon^*(u_\varepsilon)\}$ converges to u_0 strongly in $L^p(\Omega\times Y)$.

A sequence $\{u_\varepsilon\}$ is said to *weakly two-scale converge* to u_0 in $L^p(\Omega)$ with respect to the scale E if the sequence $\{\mathcal{T}_\varepsilon^*(u_\varepsilon)\}$ converges to u_0 weakly in $L^p(\Omega\times Y)$.

Theorem 4.4. *Let $\{u_\varepsilon\}$ be a sequence in $L^p(\Omega_\varepsilon^*)$ and $u_0 \in L^p(\Omega\times Y)$, $p \in \langle 1,\infty)$. Then*

(i) Any constant sequence $\{u\} \in L^p(\Omega)$ strongly two-scale converges to itself, i.e.

$$\mathcal{T}_\varepsilon^*(u) \to u_0 \quad \textit{strongly in } L^p(\Omega\times Y),$$

where

$$u_0(x,y) = \begin{cases} u(x) & [x,y] \in \Omega\times Y^*, \\ 0 & \textit{otherwise.} \end{cases}$$

(ii) *Any sequence* $\{u_\varepsilon\} \in L^p(\Omega_\varepsilon^*)$ *two-scale converging (strongly or weakly) in* $L^p(\Omega)$ *is bounded in* $L^p(\Omega_\varepsilon^*)$*, i.e.*

$$\|u_\varepsilon\|_{L^p(\Omega_\varepsilon^*)} \leq C.$$

(iii) *If a two-scale limit* u_0 *exists, then it is unique as an element of* L^p*-spaces.*

(iv) *If* $\{u_\varepsilon\}$ *strongly two-scale converges to* u_0 *in* $L^p(\Omega)$*, i.e.*

$$\mathcal{T}_\varepsilon^*(u_\varepsilon) \to u_0 \quad \textit{strongly in } L^p(\Omega \times Y).$$

Then it weakly two-scale converges to the same limit

$$\mathcal{T}_\varepsilon^*(u_\varepsilon) \rightharpoonup u_0 \quad \textit{weakly in } L^p(\Omega \times Y).$$

(v) *For* $p \in (1, \infty)$*, if* $\{u_\varepsilon\}$ *weakly two-scale converges to* u_0 *in* $L^p(\Omega)$*,*

$$\mathcal{T}_\varepsilon^*(u_\varepsilon) \rightharpoonup u_0 \quad \textit{weakly in } L^p(\Omega \times Y).$$

Then its extension by zero converges weakly

$$\widetilde{u_\varepsilon} \rightharpoonup u^* \quad \textit{weakly in } L^p(\Omega),$$

where $u^*(x) = \frac{1}{|Y|} \int_{Y^*} u_0(x,y)\,\mathrm{d}y = \frac{|Y^*|}{|Y|} \mathcal{M}_{Y^*}(u_0)(x)$*.*

Proof.

(i)-(iv) The proof is analogical to the proof of the Theorem 3.3.

(v) Let $\varphi \in L^{p'}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. From the Theorem 4.2 (i) and the equality (iii) we obtain:

$$\int_\Omega \widetilde{u_\varepsilon}(x)\ \varphi(x)\,\mathrm{d}x = \frac{1}{|Y|} \iint_{\Omega \times Y} \mathcal{T}_\varepsilon(\widetilde{u_\varepsilon})(x,y) \mathcal{T}_\varepsilon(\varphi)(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Now, we can tend to the limit. Using the Theorem 4.4 (i) we obtain the result.

$$\int_\Omega \widetilde{u_\varepsilon}(x)\ \varphi(x)\,\mathrm{d}x \to \frac{1}{|Y|} \iint_{\Omega \times Y^*} u_0(x,y)\varphi(x)\,\mathrm{d}x\,\mathrm{d}y =$$
$$= \int_\Omega \left(\frac{1}{|Y|} \int_{Y^*} u_0(x,y)\,\mathrm{d}y \right) \varphi(x)\,\mathrm{d}x = \int_\Omega u^*(x)\ \varphi(x)\,\mathrm{d}x.$$

□

4.3. Unfolding operator $\mathcal{T}_\varepsilon^*$ and gradients

Consider a function $u \in W^{1,p}(\Omega_\varepsilon^*)$. As in the case without holes, it is straightforward that

$$\mathcal{T}_\varepsilon^*(\nabla u) = \begin{cases} \frac{1}{\varepsilon}\nabla_y \mathcal{T}_\varepsilon^*(u) & \text{on } \widehat{\Omega}_\varepsilon \times Y^*, \\ \nabla u = \nabla \mathcal{T}_\varepsilon^*(u) & \text{on } \Lambda_\varepsilon^* \times Y, \\ 0 & \text{otherwise.} \end{cases} \tag{33}$$

Now we will state the main result about the convergence of an unfolded sequence of gradients $\{\mathcal{T}_\varepsilon^*(\nabla u_\varepsilon)\}$ in the same spirit as that of the Theorem 3.11.

Theorem 4.5. *Let a sequence $\{u_\varepsilon\}$ be bounded in $W^{1,p}(\Omega_\varepsilon^*)$, for $p \in (1, \infty)$. i.e.*

$$\|u_\varepsilon\|_{W^{1,p}(\Omega_\varepsilon^*)} \leq C.$$

Then, there exists a subsequence (still denoted $\{u_\varepsilon\}$) and functions $u_0 \in W^{1,p}(\Omega)$ and $u_0^* \in L^p(\Omega; W^{1,p}_{per}(Y))$ such that

(i) $\mathcal{T}_\varepsilon^*(u_\varepsilon) \rightharpoonup u$ weakly in $L^p(\Omega; W^{1,p}(Y))$, where

$$u(x,y) = \begin{cases} u_0(x) & [x,y] \in \Omega \times Y^*, \\ 0 & \textit{otherwise.} \end{cases}$$

(ii) $\mathcal{T}_\varepsilon^*(\nabla u_\varepsilon) \rightharpoonup \nabla u_0 + \nabla_y u_0^*$ weakly in $[L^p(\Omega \times Y)]^N$, i.e. $\{\nabla u_\varepsilon\}$ converges two-scale weakly in $[L^p(\Omega)]^N$.

Moreover, $\mathcal{M}_Y(u_0^*) = 0$ and $u_0^* = -y^c \cdot \nabla u_0 \quad on \quad \Omega \times T$.

Proof. The first two properties follow from the Theorem 3.11 and property of the unfolding operator on perforated domains, namely that $\mathcal{T}_\varepsilon^*(u) = \mathcal{T}_\varepsilon(\tilde{u})$.

The proof is analogical to the one of the Theorem 3.11. Instead of $\mathcal{T}_\varepsilon$, resp. $\mathcal{M}_\varepsilon$, we use $\mathcal{T}_\varepsilon^*$, resp. $\mathcal{M}_\varepsilon^*$, where

$$\mathcal{M}_\varepsilon^*(\varphi)(x) = \begin{cases} \dfrac{1}{\varepsilon^N |Y^*|} \displaystyle\int_{\varepsilon\left(\left[\frac{x}{\varepsilon}\right]_Y + Y^*\right)} \varphi(t)\,\mathrm{d}t & \text{for } \; x \in \widehat{\Omega}_\varepsilon^*, \\ \varphi(x) & \text{for } \; x \in \Lambda_\varepsilon^*. \end{cases}$$

It remains to prove that $u_0^* = -y^c \cdot \nabla u_0$ on $\Omega \times T$. By the same reasoning as in the proof of the Theorem 3.11 we can show that

$$\mathcal{T}_\varepsilon^*(u_\varepsilon^2) \rightharpoonup y^c \cdot \nabla u_0 + u_0^* \quad \text{weakly in } L^p(\Omega, W^{1,p}(Y)).$$

Further from the definition of the unfolding operator for the perforated domains it follows that

$$\mathcal{T}_\varepsilon^*(u_\varepsilon^2) = 0 \quad \text{on} \quad \Omega \times T, \quad \forall \varepsilon.$$

We can conclude that

$$y^c \cdot \nabla u_0 + u_0^* \; = 0 \quad \text{on} \quad \Omega \times T.$$

□

5. Application

5.1. Torsion problem

Study of elastic torsion of a bar leads to a problem described in [FNJ12; FR15]. Here, a more general problem is studied and the case of elastic torsion is obtained as an application.

Let us start with a definition:

Definition 5.1. Let $\alpha, \beta \in \mathbb{R}$, such that $0 < \alpha < \beta$. We say that a matrix function $A(x) = \Big(a_{ij}^{\varepsilon}(x)\Big) \in [L^{\infty}(\Omega)]^{N\times N}$ belongs to a set $M(\alpha, \beta, \Omega)$ if and only if

$$\begin{array}{llcll} (i) & (A(x)\lambda, \lambda) & \geq & \alpha|\lambda|^2, & \text{(ellipticity)}, \\ (ii) & |A(x)\lambda| & \leq & \beta|\lambda|, & \text{(boundedness)}. \end{array} \tag{34}$$

$\forall \lambda \in \mathbb{R}^N,$ a.e. in Ω.

Now we can state a boundary problem:

$$\left.\begin{array}{rcll} -\nabla \cdot (A^{\varepsilon}\nabla u_{\varepsilon}) & = & f & \text{in } \ \Omega_{\varepsilon}^{*}, \\ u_{\varepsilon} & = & 0 & \text{on } \ \partial_{\text{ext}}\Omega_{\varepsilon}^{*}, \\ u_{\varepsilon} & = & \text{const.} & \text{on } \ \partial T_{\text{int},\,\varepsilon}^{i}; \quad i = 1, \ldots, m(\varepsilon), \\ \displaystyle\int\limits_{\partial T_{\text{int},\,\varepsilon}^{i}} A^{\varepsilon}(x)\,\frac{\partial u_{\varepsilon}}{\partial n}(x)\,\mathrm{d}x & = & \displaystyle\int\limits_{T_{\text{int},\,\varepsilon}^{i}} f(x)\,\mathrm{d}x & \end{array}\right\} \tag{35}$$

where:

- Ω_{ε}^{*}, $\partial_{\text{ext}}\Omega_{\varepsilon}^{*}$, $T_{\text{int},\,\varepsilon}^{i}$, etc. are defined in the beginning of the Section 4.1.
- $f \in L^2(\Omega)$,
- $A^{\varepsilon}(x) = \Big(a_{ij}^{\varepsilon}(x)\Big)_{i,j=1\ldots N}$ is a matrix function from the set $M(\alpha, \beta, \Omega_{\varepsilon}^{*})$,
- n is the outward-pointing unite normal (i.e. on the inner boundary, n is directed inward to the holes),
- $m(\varepsilon)$ denotes number of interior holes.

For $f(x) = -2$ and $N = 2$ we get a torsion problem derived in [FR15].

Let us introduce the linear space

$$\mathcal{S}_{\varepsilon}(\Omega) = \Big\{v \in H_0^1(\Omega), \ \text{s.t. } v = 0 \text{ in } \overline{T_{\text{ext},\,\varepsilon}}, \ v = \text{const. in } \overline{T_{\text{int},\,\varepsilon}^{i}}, \ i = 1, \ldots, m(\varepsilon)\Big\}.$$

with the norm

$$\|v\|_{\mathcal{S}_\varepsilon(\Omega)} = \|\nabla v\|_{[L^2(\Omega_\varepsilon^*)]^N}. \tag{36}$$

Let us define the weak formulation of the problem (35):

$$\left.\begin{aligned} &\text{Find } u_\varepsilon \in \mathcal{S}_\varepsilon(\Omega) \text{ such that} \\ &\int_{\Omega_\varepsilon^*} A^\varepsilon(x)\,\nabla u_\varepsilon(x)\cdot\nabla v(x)\,\mathrm{d}x = \int_\Omega f(x)\,v(x)\,\mathrm{d}x, \qquad \forall v \in \mathcal{S}_\varepsilon(\Omega). \end{aligned}\right\} \tag{37}$$

Proposition 5.2. *Let $A^\varepsilon\,\nabla u_\varepsilon \in [\mathcal{C}^1(\Omega_\varepsilon^*)]^N$, $u_\varepsilon \in \mathcal{C}^1(\Omega_\varepsilon^*)$ and u_ε solves the problem (37), then it also solves the boundary problem (35).*

Proof. Let us suppose that (37) holds and let us choose $v \in \mathcal{C}^1(\Omega_\varepsilon^*) \cap \mathcal{S}_\varepsilon(\Omega)$.

Using integration by parts of the left-hand side of (37) we get

$$\begin{aligned} LHS = &\int_{\Omega_\varepsilon^*} A^\varepsilon(x)\,\nabla u_\varepsilon(x)\cdot\nabla v(x)\,\mathrm{d}x = \\ &= -\int_{\Omega_\varepsilon^*} \nabla\cdot\Big(A^\varepsilon(x)\,\nabla u_\varepsilon(x)\Big)\,v(x)\,\mathrm{d}x + \int_{\partial\Omega_\varepsilon^*} A^\varepsilon(x)\,\nabla u_\varepsilon(x)\cdot n\,v(x)\,\mathrm{d}x. \end{aligned}$$

Since v is equal to zero on the exterior boundary $\partial_{\mathrm{ext}}\Omega_\varepsilon^*$ (it follows from the properties of the space $\mathcal{S}_\varepsilon(\Omega)$), it results in

$$LHS = -\int_{\Omega_\varepsilon^*} \nabla\cdot\Big(A^\varepsilon(x)\,\nabla u_\varepsilon(x)\Big)\,v(x)\,\mathrm{d}x + \sum_{i=1}^{m(\varepsilon)} \int_{\partial T^i_{\mathrm{int},\,\varepsilon}} A^\varepsilon(x)\,\nabla u_\varepsilon(x)\cdot n\,v(x)\,\mathrm{d}x. \tag{38}$$

Since v is equal to zero in $\overline{T_{\mathrm{ext},\,\varepsilon}}$, right-hand side of (37) can be rewritten to the form

$$RHS = \int_\Omega f(x)\,v(x)\,\mathrm{d}x = \int_{\Omega_\varepsilon^*} f(x)\,v(x)\,\mathrm{d}x + \sum_{i=1}^{m(\varepsilon)} \int_{T^i_{\mathrm{int},\,\varepsilon}} f(x)\,v(x)\,\mathrm{d}x. \tag{39}$$

Let us choose in (38) and (39) $v \in \mathcal{D}(\Omega_\varepsilon^*)$ extended by zero to Ω. These functions are in $\mathcal{S}_\varepsilon(\Omega)$ and hence they can be used as a test functions and are equal to zero on all boundary $\partial\Omega_\varepsilon^*$. Then, from (37) it follows

$$-\int_{\Omega_\varepsilon^*} \nabla\cdot\Big(A^\varepsilon(x)\,\nabla u_\varepsilon(x)\Big)\,v(x)\,\mathrm{d}x - \int_{\Omega_\varepsilon^*} f(x)\,v(x)\,\mathrm{d}x = 0. \tag{40}$$

By using the Lemma 2.26, from the equation (40) we get

$$-\nabla\cdot\Big(A^\varepsilon\,\nabla u_\varepsilon\Big) - f = 0 \qquad \text{a. e. in } \Omega_\varepsilon^*. \tag{41}$$

Now, let us choose v such that $v = \text{const.}$ on hole $T^i_{\text{int},\,\varepsilon}$ (and let us denote this constant by c_i) and $v = 0$ on all other holes (i.e. on Ω^j_ε, for $i \neq j$) Then, from (37), (38), (39) and (41), it follows

$$c_i \int\limits_{\partial T^i_{\text{int},\,\varepsilon}} A^\varepsilon(x)\,\nabla u_\varepsilon(x)\cdot n\,\mathrm{d}x = c_i \int\limits_{T^i_{\text{int},\,\varepsilon}} f(x)\,\mathrm{d}x \qquad \text{for } i = 1,\dots,m(\varepsilon). \tag{42}$$

Finally, from (40) and (42) together with the properties of the space $\mathcal{S}_\varepsilon(\Omega)$ it follows that u_ε fulfill the problem (35). □

To find the homogenized solution to problem (37) we will use as test functions rapidly oscillating functions. The following result concerns their two-scale convergence.

Proposition 5.3 (Unfolding of rapidly oscillating function on perforated domain)**.** *Let $v \in L^p_{\text{per}}(Y^*)$, $p \in \langle 1, \infty)$. Furthermore, let $\{v_\varepsilon\}$ be a sequence defined by*

$$v_\varepsilon(x) = v\Big(\frac{x}{\varepsilon}\Big) \quad \forall x \in \Omega^*_\varepsilon.$$

Then,

$$\mathcal{T}^*_\varepsilon(v_\varepsilon)(x,y) = \begin{cases} v(y) & \text{for } (x,y) \in \widehat{\Omega}_\varepsilon \times Y^*, \\ v(x) & \text{for } (x,y) \in \Lambda^*_\varepsilon \times Y, \\ 0 & \text{otherwise.} \end{cases}$$

and the sequence $\{v_\varepsilon\}$ strongly two-scale converges in $L^p(\Omega)$, i.e.

$$\mathcal{T}^*_\varepsilon(v_\varepsilon) \;\to\; v_0 \quad \text{ strongly in } L^p(\Omega\times Y), \tag{43}$$

where $v_0(x,y) = \tilde{v}(y)$.

Proof. The form of unfolded function $\mathcal{T}^*_\varepsilon(v_\varepsilon)$ follows directly from its definition.

The convergence (43) can be deduced from the following

$$\|\mathcal{T}^*_\varepsilon(v_\varepsilon) - v_0\|^p_{L^p(\Omega\times Y)} = \iint\limits_{\Omega\times Y} \Big(\mathcal{T}^*_\varepsilon(v_\varepsilon)(x,y) - v_0(x,y)\Big)^p\,\mathrm{d}x\,\mathrm{d}y =$$
$$= \iint\limits_{\widehat{\Omega}_\varepsilon\times Y^*} \Big(v(y) - v(y)\Big)^p\,\mathrm{d}x\,\mathrm{d}y \;+ \iint\limits_{\Lambda^*_\varepsilon\times Y} \Big(v(x) - v(x)\Big)^p\,\mathrm{d}x\,\mathrm{d}y \;+ \iint\limits_{T_{\text{ext},\,\varepsilon}\times Y^*} \Big(0 - v(y)\Big)^p\,\mathrm{d}x\,\mathrm{d}y.$$

The first and the second integral is equal to zero. The third one converges to zero since $|T_{\text{ext},\,\varepsilon}\times Y^*| \to 0$ as $\varepsilon \to 0$ and by assumption $\int_{Y^*} u^p(y)\,\mathrm{d}y < \infty$. □

Theorem 5.4. *Let u_ε be the solution of the problem (37). Assume that*

$$\mathcal{T}^*_\varepsilon(A^\varepsilon) \;\to\; A \quad \text{a.e. in } \Omega\times Y \tag{44}$$

for a matrix $A = A(x,y)$ such that

$$A = (a_{ij})_{i,j=1\dots N} \in M(\alpha,\beta,\Omega\times Y).$$

Then, there exists $u_0 \in H_0^1(\Omega)$ *and* $u_0^* \in L^2(\Omega, H^1_{\mathrm{per}}(Y))$ *such that*

$$\|u_\varepsilon - u_0\|_{L^2(\Omega_\varepsilon^*)} \to 0,$$

$$\mathcal{T}_\varepsilon^*(u_\varepsilon) \rightharpoonup u \quad \textit{weakly in } L^2(\Omega, H^1(Y)), \ \textit{where}$$

$$u(x,y) = \begin{cases} u_0(x) & [x,y] \in \Omega \times Y^*, \\ 0 & \textit{otherwise.} \end{cases} \tag{45}$$

$$\mathcal{T}_\varepsilon^*(\nabla u_\varepsilon) \rightharpoonup \nabla u_0 + \nabla_y u_0^* \quad \textit{weakly in } [L^2(\Omega \times Y)]^N, \ \textit{where}$$

$$\mathcal{M}_Y(u_0^*) = 0 \quad \textit{and} \quad u_0^* = -y^c \cdot \nabla u_0 \quad \textit{on} \quad \Omega \times T.$$

The pair (u_0, u_0^*) *is the unique solution of the problem:*

$$\left.\begin{array}{l} \textit{Find } u_0 \in H_0^1(\Omega) \textit{ and } u_0^* \in L^2(\Omega, H^1_{\mathrm{per}}(Y)) \textit{ such that} \\ \displaystyle\frac{1}{|Y|} \iint\limits_{\Omega \times Y^*} A(x,y)\,[\nabla u_0(x) + \nabla_y u_0^*(x,y)] \cdot [\nabla \Psi(x) + \nabla_y \Phi(x,y)]\,\mathrm{d}x\,\mathrm{d}y = \int\limits_\Omega f(x)\,\Psi(x)\,\mathrm{d}x, \\ \forall \Psi \in H_0^1(\Omega), \\ \forall \Phi \in L^2(\Omega, H^1_{\mathrm{per}}(Y)), \textit{ such that } \Phi + y_c \cdot \nabla \Psi \textit{ is constant in } y \textit{ on } \Omega \times T. \end{array}\right\} \tag{46}$$

Proof. The proof is divided into 3 steps.

First step - existence and uniqueness of the homogenized solution. By Lax-Milgram lemma, Problem (46) has a unique solution. Choosing $v = u_\varepsilon$ and using (34)(i), (37) and the Hölder inequality we can get the following estimate:

$$\|u_\varepsilon\|^2_{\mathcal{S}_\varepsilon(\Omega)} = \|\nabla u_\varepsilon\|^2_{[L^2(\Omega_\varepsilon^*)]^N} = \int_{L^2(\Omega_\varepsilon^*)} \nabla u_\varepsilon(x) \cdot \nabla u_\varepsilon(x)\,\mathrm{d}x \leq$$
$$\leq \frac{1}{\alpha} \int_{\Omega_\varepsilon^*} A^\varepsilon(x)\,\nabla u_\varepsilon(x) \cdot \nabla u_\varepsilon(x)\,\mathrm{d}x = \frac{1}{\alpha}\int_\Omega f(x)\,u_\varepsilon(x)\,\mathrm{d}x \leq \frac{1}{\alpha}\|f\|_{L^2(\Omega)}\|u_\varepsilon\|_{L^2(\Omega)}.$$

Therefore

$$\|u_\varepsilon\|_{\mathcal{S}_\varepsilon(\Omega)} \leq \frac{1}{\alpha}\|f\|_{L^2(\Omega)}. \tag{47}$$

As seen above, $\{u_\varepsilon\}$ is bounded in $\mathcal{S}_\varepsilon(\Omega)$. Then the Theorem 4.5 implies convergences (45) at least for subsequences.

Second step - identification of the limit. Let $\Psi \in \mathcal{D}(\Omega)$ and $\varphi_1 \in \mathcal{D}(Y)$ be periodically extended to $\mathbb{R}^N$, such that $\varphi_1 \equiv 1$ on T. We choose in (37) the test function

$$v_\varepsilon = \begin{cases} \Psi\,[1 - \varphi_{1\,\varepsilon}] + \mathcal{M}_\varepsilon(\Psi)\,\varphi_{1\,\varepsilon} & \text{in} \quad \widehat{\Omega}_\varepsilon, \\ \\ \Psi\,[1 - \varphi_{1\,\varepsilon}] & \text{in} \quad \Lambda_\varepsilon, \end{cases}$$

where $\varphi_{1\,\varepsilon}(x) = \varphi_1\Big(\frac{x}{\varepsilon}\Big)$. The function v_ε belongs to $\mathcal{S}_\varepsilon(\Omega)$ since φ_1 vanishes on ∂Y and is constant on the holes, $\mathcal{M}_\varepsilon(\Psi)$ is piecewise constant and Ψ vanishes on $\partial\Omega$.

For such function v_ε it holds:

$$\mathcal{T}^*_\varepsilon(v_\varepsilon)(x,y) = \begin{cases} \mathcal{T}^*_\varepsilon(\Psi)(x,y)\,[1-\varphi_1(y)] + \mathcal{T}^*_\varepsilon(\mathcal{M}_\varepsilon(\Psi))(x,y)\,\varphi_1(y) & \text{in} \quad \widehat{\Omega}_\varepsilon \times Y^*, \\ \Psi(x)\,[1-\varphi_{1\,\varepsilon}(x)] & \text{in} \quad \Lambda^*_\varepsilon \times Y, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\Big(\mathcal{T}^*_\varepsilon(\mathcal{M}_\varepsilon(\Psi)) - \mathcal{T}^*_\varepsilon(\Psi)\Big)\,1_{\widehat{\Omega}_\varepsilon \times Y^*} \to 0$ uniformly on $\Omega \times Y$ and $|\Lambda^*_\varepsilon \times Y| \to 0$ we conclude

$$\mathcal{T}^*_\varepsilon(v_\varepsilon) \to \Psi_0 \quad \text{strongly in } L^2(\Omega \times Y), \quad \text{where} \quad \Psi_0(x,y) = \begin{cases} \Psi(x) & [x,y] \in \Omega \times Y^*, \\ 0 & \text{otherwise.} \end{cases}$$

For the gradient of function v_ε it holds

$$\mathcal{T}^*_\varepsilon(\nabla v_\varepsilon)(x,y) = \begin{cases} \mathcal{T}^*_\varepsilon(\nabla\Psi)(x,y)\,[1-\varphi_1(y)] + \\ \qquad + \frac{1}{\varepsilon}\,\nabla_y\varphi_1(y)\,\mathcal{T}^*_\varepsilon(\mathcal{M}_\varepsilon(\Psi) - \Psi)(x,y) & \text{in} \quad \widehat{\Omega}_\varepsilon \times Y^*, \\ \nabla\Psi(x)\,[1-\varphi_{1\,\varepsilon}(x)] - \frac{1}{\varepsilon}\,\Psi(x)\,(\nabla\varphi_1)\Big(\frac{x}{\varepsilon}\Big) & \text{in} \quad \Lambda^*_\varepsilon \times Y, \\ 0 & \text{otherwise.} \end{cases}$$

Now, our goal is to find the strong limit of $\{\mathcal{T}^*_\varepsilon(\nabla v_\varepsilon)\}$ in $[L^2(\Omega \times Y)]^N$. First, we show that $\Big\{\frac{1}{\varepsilon}\,\mathcal{T}^*_\varepsilon(\mathcal{M}_\varepsilon(\Psi) - \Psi)\Big\}$ converges to $\nabla\Psi \cdot y_c$. Indeed,

$$\mathcal{M}_Y\bigg(\Big[\frac{1}{\varepsilon}\,\mathcal{T}^*_\varepsilon(\mathcal{M}_\varepsilon(\Psi) - \Psi) - \nabla\Psi \cdot y_c\Big]\,1_{\widehat{\Omega}_\varepsilon \times Y^*}\bigg) = 0,$$

and

$$\begin{aligned} \nabla_y\bigg(\Big[\frac{1}{\varepsilon}\,\mathcal{T}^*_\varepsilon(\mathcal{M}_\varepsilon(\Psi) - \Psi) - \nabla\Psi \cdot y_c\Big]\,1_{\widehat{\Omega}_\varepsilon \times Y^*}\bigg) &= \\ = [\mathcal{T}^*_\varepsilon(\nabla\Psi) - \nabla\Psi]\,1_{\widehat{\Omega}_\varepsilon \times Y^*} \to 0 \quad &\text{strongly in } \Big[L^2(\Omega \times Y)\Big]^N. \end{aligned}$$

Hence, by Poincaré-Wirtinger inequality,

$$[\mathcal{T}^*_\varepsilon(\nabla\Psi) - \nabla\Psi]\,1_{\widehat{\Omega}_\varepsilon \times Y^*} \to \nabla\Psi \cdot y_c \quad \text{strongly in } \Big[L^2(\Omega \times Y)\Big]^N.$$

Proving strong convergence on $\Lambda^*_\varepsilon \times Y$ is not straightforward. Although $|\Lambda^*_\varepsilon \times Y| \to 0$, $\frac{1}{\varepsilon}\,\Psi\,(\nabla\varphi_1)\Big(\frac{\cdot}{\varepsilon}\Big)$ is not bounded on $\Lambda^*_\varepsilon \times Y$. Let $\Sigma_\varepsilon = \Big\{k \in \mathbb{R}^N, \text{ s.t. } \partial\Omega \in Y^\varepsilon_k\Big\}$. By change of variable $\frac{x}{\varepsilon} - k = t$, we derive

$$\left\| \frac{1}{\varepsilon}\, \Psi\, (\nabla\varphi_1)\Big(\frac{\cdot}{\varepsilon}\Big) \right\|^2_{[L^2(\Lambda^*_\varepsilon\times Y)]^N} = \iint\limits_{\Lambda^*_\varepsilon\times Y} \Big(\frac{1}{\varepsilon}\, \Psi(x)\, (\nabla\varphi_1)\Big(\frac{x}{\varepsilon}\Big)\Big)^2 \,\mathrm{d}x\,\mathrm{d}y =$$

$$= |Y| \sum_{k\in\Sigma_\varepsilon} \int\limits_{\varepsilon(Y+k)} \Big(\frac{1}{\varepsilon}\, \Psi(x)\, (\nabla\varphi_1)\Big(\frac{x}{\varepsilon}\Big)\Big)^2 \,\mathrm{d}x =$$

$$= |Y| \sum_{k\in\Sigma_\varepsilon} \int\limits_{Y} \Big(\frac{\varepsilon^N}{\varepsilon}\, \Psi(\varepsilon(t+k))\, (\nabla\varphi_1)(t+k)\Big)^2 \,\mathrm{d}t.$$

The function $\nabla\varphi_1$ is Y-periodic and bounded and since $\Psi \in \mathcal{D}(\Omega)$, $\Psi(\varepsilon(t+k)) \to 0$ uniformly on Y. Thus, we get

$$\frac{1}{\varepsilon}\, \Psi\, (\nabla\varphi_1)\Big(\frac{\cdot}{\varepsilon}\Big)\, 1_{\Lambda^*_\varepsilon\times Y} \to 0 \quad \text{strongly in } \Big[L^2(\Omega\times Y)\Big]^N.$$

Using all this results above, we have

$$\mathcal{T}^*_\varepsilon(\nabla v_\varepsilon) \to \nabla\Psi - \nabla_y(\nabla\Psi\cdot y_c\,\varphi_1) \quad \text{strongly in } \Big[L^2(\Omega\times Y)\Big]^N.$$

Using v_ε as a test function in (37) we obtain for the left-hand side

$$LHS = \int\limits_{\Omega^*_\varepsilon} A^\varepsilon(x)\, \nabla u_\varepsilon(x)\cdot\nabla v_\varepsilon(x)\,\mathrm{d}x =$$

$$= \frac{1}{|Y|} \iint\limits_{\Omega\times Y} \mathcal{T}^*_\varepsilon(A^\varepsilon)(x,y)\; \mathcal{T}^*_\varepsilon(\nabla u_\varepsilon)(x,y)\cdot \mathcal{T}^*_\varepsilon(\nabla v_\varepsilon)(x,y)\,\mathrm{d}x\,\mathrm{d}y \to$$

$$\to \frac{1}{|Y|} \iint\limits_{\Omega\times Y} A(x,y)\; [\nabla u_0(x) + \nabla_y u^*_0(x,y)]\cdot \Big[\nabla\Psi(x) - \nabla_y\Big(\varphi_1(y)\, y_c\cdot\nabla\Psi(x)\Big)\Big]\,\mathrm{d}x\,\mathrm{d}y, \quad (48)$$

and for the right-hand side

$$RHS = \int\limits_{\Omega} f(x)\, v_\varepsilon(x)\,\mathrm{d}x = \frac{1}{|Y|} \iint\limits_{\Omega\times Y} \mathcal{T}_\varepsilon(f)(x,y)\, \mathcal{T}_\varepsilon(v_\varepsilon)(x,y)\,\mathrm{d}x\,\mathrm{d}y \to$$

$$\to \frac{1}{|Y|} \iint\limits_{\Omega\times Y} f(x)\,\Psi(x)\,\mathrm{d}x = \int\limits_{\Omega} f(x)\,\Psi(x)\,\mathrm{d}x. \quad (49)$$

Now, taking in (37) as a test function

$$v_\varepsilon = \begin{cases} \varepsilon\, \Big(\Psi_1\, \psi_\varepsilon + \mathcal{M}_\varepsilon(\Psi_1)\, \varphi_{c\,\varepsilon}\Big) & \text{in } \; \widehat{\Omega}_\varepsilon, \\ \varepsilon\, \Psi_1\, \psi_\varepsilon & \text{in } \; \Lambda_\varepsilon, \end{cases}$$

where $\Psi_1 \in \mathcal{D}(\Omega)$, $\psi \in \mathcal{C}^\infty_{\mathrm{per}}(Y)$ $\psi \in \mathcal{C}^\infty_{\mathrm{per}}(Y)$, such that $\psi \equiv 0$ on T, and $\varphi \in \mathcal{D}(Y)$, φ_c is constant on T and $\psi_\varepsilon(x) = \psi\big(\frac{x}{\varepsilon}\big)$ resp. $\varphi_{c\,\varepsilon}(x) = \varphi_c\big(\frac{x}{\varepsilon}\big)$.

The unfolding of gradient of v_ε has a form

$$\mathcal{T}^*_\varepsilon(\nabla v_\varepsilon)(x,y) = \begin{cases} \varepsilon\, \mathcal{T}^*_\varepsilon(\nabla \Psi_1)(x,y)\, \psi(y) + \mathcal{T}^*_\varepsilon(\Psi_1)(x,y)\, \nabla_y \psi(y) + & \\ \qquad + \mathcal{T}^*_\varepsilon(\mathcal{M}_\varepsilon(\Psi_1))(x,y)\, \nabla_y \varphi_c(y) & \text{in} \quad \widehat{\Omega}^*_\varepsilon \times Y^*, \\ \varepsilon\, \nabla \Psi_1(x)\, \psi_\varepsilon(x) + \Psi_1(x)\, (\nabla \psi)\Big(\frac{x}{\varepsilon}\Big) & \text{in} \quad \Lambda^*_\varepsilon \times Y, \\ 0 & \text{otherwise.} \end{cases}$$

Since $|\Lambda^*_\varepsilon \times Y| \to 0$, and further

$$\varepsilon\, \mathcal{T}^*_\varepsilon(\nabla \Psi_1)(x,y)\, \psi(y)\, 1_{\widehat{\Omega}^*_\varepsilon \times Y^*} \to 0 \quad \text{strongly in } \Big[L^2(\Omega \times Y)\Big]^N,$$

$$\mathcal{T}^*_\varepsilon(\Psi_1)\, 1_{\widehat{\Omega}^*_\varepsilon \times Y^*} \to \Psi_1 \quad \text{strongly in } L^2(\Omega \times Y),$$

and also

$$\mathcal{T}^*_\varepsilon(\mathcal{M}_\varepsilon(\Psi_1))\, 1_{\widehat{\Omega}^*_\varepsilon \times Y^*} \to \Psi_1 \quad \text{strongly in } L^2(\Omega \times Y),$$

we conclude that

$$\mathcal{T}^*_\varepsilon(\nabla v_\varepsilon) \to \Psi_1\, \nabla_y(\psi + \varphi_c) \quad \text{strongly in } \Big[L^2(\Omega \times Y)\Big]^N.$$

Using v_ε as a test function in (37) we get

$$\begin{aligned} LHS &= \int_{\Omega^*_\varepsilon} A^\varepsilon(x)\, \nabla u_\varepsilon(x) \cdot \nabla v_\varepsilon(x)\, \mathrm{d}x = \\ &= \frac{1}{|Y|} \iint_{\Omega \times Y} \mathcal{T}^*_\varepsilon(A^\varepsilon)(x,y)\ \mathcal{T}^*_\varepsilon(\nabla u_\varepsilon)(x,y) \cdot \mathcal{T}^*_\varepsilon(\nabla v_\varepsilon)(x,y)\, \mathrm{d}x\, \mathrm{d}y \to \\ &\to \frac{1}{|Y|} \iint_{\Omega \times Y} A(x,y)\ [\nabla u_0(x) + \nabla_y u^*_0(x,y)] \cdot \Psi_1(x)\, \nabla_y[\psi(y) + \varphi_c(y)]\, \mathrm{d}x\, \mathrm{d}y, \end{aligned} \tag{50}$$

and

$$RHS = \int_\Omega f(x)\, v_\varepsilon(x)\, \mathrm{d}x \to 0. \tag{51}$$

Third step - Conclusion. Let us denote by Φ the function

$$\Phi(x,y) = \Psi_1(x)\, [\psi(y) + \varphi_c(y)] - \varphi_1(y)\, y_c \cdot \nabla \Psi(x).$$

The results (48), (49), (50) and (51) imply

$$\frac{1}{|Y|} \iint_{\Omega \times Y} A(x,y)\ [\nabla u_0(x) + \nabla_y u^*_0(x,y)] \cdot [\nabla \Psi(x) + \nabla_y \Phi(x,y)]\, \mathrm{d}x\, \mathrm{d}y = \int_\Omega f(x)\, \Psi(x). \tag{52}$$

Now, every function, which belongs to the space $\mathcal{D}(\Omega) \otimes \mathcal{C}^\infty_{\mathrm{per}}(Y)$ and is constant in y on T can be written as a product $\Psi_1(x)\, [\psi(y) + \varphi_c(y)]$, furthermore $\varphi_1(y)\, y_c \cdot \nabla \Psi(x) \in \mathcal{D}(\Omega) \otimes \mathcal{C}^\infty_{\mathrm{per}}(Y)$. By the density of $\mathcal{D}(\Omega) \otimes \mathcal{C}^\infty_{\mathrm{per}}(Y)$ in $L^2(\Omega, H^1_{\mathrm{per}}(Y))$ the results (52) is valid for every $\Psi \in H^1_0(\Omega)$ and $\Phi \in L^2(\Omega, H^1_{\mathrm{per}}(Y))$, such that $\Phi + y_c \cdot \nabla \Psi$ is constant in y on $\Omega \times T$. $\square$

6. Numerical examples

We present numerical example for dimension $N = 2$.

Let $x = (x_1, x_2) \in \Omega$ and $y = (y_1, y_2) \in Y$, where Ω is a simple domain in $\mathbb{R}^2$ and $Y = \langle 0, l_1) \times \langle 0, l_2)$, l_1, l_2 are real positive numbers. Vector function y^c has the form $y^c = (\, y_1^c,\, y_2^c\,)$. Furthermore, let us suppose that A is a function only in variable y, i.e. $A(x, y) = A(y)$.

We would like to solve the problem, derived in the Theorem 5.4:

$$\begin{cases} \text{Find } u_0 \in H_0^1(\Omega) \text{ and } u_0^* \in L^2(\Omega, H^1_{\mathrm{per}}(Y)) \text{ such that} \\ \dfrac{1}{|Y|} \displaystyle\iint\limits_{\Omega\times Y^*} A(y)\,[\nabla u_0(x) + \nabla_y u_0^*(x,y)] \cdot [\nabla \Psi(x) + \nabla_y \Phi(x,y)]\,\mathrm{d}x\,\mathrm{d}y = \int\limits_{\Omega} f(x)\,\Psi(x)\,\mathrm{d}x, \\ \forall \Psi \in H_0^1(\Omega),\ \forall \Phi \in L^2(\Omega, H^1_{\mathrm{per}}(Y)),\ \text{s. t. } \Phi + y^c \cdot \nabla \Psi \text{ is constant in } y \text{ on } \Omega \times T, \\ \mathcal{M}_Y(u_0^*) = 0, \\ u_0^* = -y^c \cdot \nabla u_0 \quad \text{on} \quad \Omega \times T. \end{cases} \tag{53}$$

We will look for u_0, u_0^* in two steps. At first, we will compute auxiliary functions denoted $\hat{\chi}_1, \hat{\chi}_2$ and subsequently, using them, we will find homogenized solutions u_0, u_0^*.

Let us choose $\Psi(x) \equiv 0$ as a test function in (53). We suggest function u_0^* in the form

$$u_0^*(x,y) = -\hat{\chi}_1(y)\,\frac{\partial u_0}{\partial x_1}(x) - \hat{\chi}_2(y)\,\frac{\partial u_0}{\partial x_2}(x).$$

Then, (53) takes the form

$$\iint\limits_{\Omega\times Y^*} A \left[\frac{\partial u_0}{\partial x_1}\,\frac{\partial \Phi}{\partial y_1} + \frac{\partial u_0}{\partial x_2}\,\frac{\partial \Phi}{\partial y_2} \right] \mathrm{d}x\,\mathrm{d}y =$$
$$= \iint\limits_{\Omega\times Y^*} A \left[\frac{\partial \hat{\chi}_1}{\partial y_1}\,\frac{\partial u_0}{\partial x_1}\,\frac{\partial \Phi}{\partial y_1} + \frac{\partial \hat{\chi}_2}{\partial y_1}\,\frac{\partial u_0}{\partial x_2}\,\frac{\partial \Phi}{\partial y_1} + \frac{\partial \hat{\chi}_1}{\partial y_2}\,\frac{\partial u_0}{\partial x_1}\,\frac{\partial \Phi}{\partial y_2} + \frac{\partial \hat{\chi}_2}{\partial y_2}\,\frac{\partial u_0}{\partial x_2}\,\frac{\partial \Phi}{\partial y_2} \right] \mathrm{d}x\,\mathrm{d}y.$$

From this we see that the problem (53) is fulfilled when the auxiliary function $\hat{\chi}_i$, $i = 1, 2$, satisfies

$$\int\limits_{Y^*} A\,\frac{\partial \Phi}{\partial y_i}\,\mathrm{d}y = \int\limits_{Y^*} A\,\nabla \hat{\chi}_i \cdot \nabla \Phi\,\mathrm{d}y, \quad \forall\ \Phi \in L^2(\Omega, H^1_{\mathrm{per}}(Y)),\ \text{s. t. } \Phi \text{ is constant in } y \text{ on } T.$$

Rewriting this, we derive the following problem

$$\begin{cases} \text{Find } \hat{\chi}_i \in H^1_{\text{per}}(Y) \text{ such that} \\ \int\limits_{Y^*} A\,\nabla(\hat{\chi}_i - y_i)\cdot\nabla\Phi\,\mathrm{d}y = 0, \quad \forall\, \Phi \in L^2(\Omega, H^1_{\text{per}}(Y)), \text{ s. t. } \Phi \text{ is constant in } y \text{ on } T, \\ \mathcal{M}_Y(\hat{\chi}_i) = 0, \\ \hat{\chi}_i = -y_i^c \text{ on } T. \end{cases} \tag{54}$$

Now, let us choose as a test function in (53) a function

$$\Phi(x,y) = -\varphi_1(y)\,y^c\cdot\Psi(x),$$

where $\Psi \in \mathcal{D}(\Omega)$ and φ is Y-periodic function which $\varphi_1|_Y \in \mathcal{D}(Y)$, $\varphi_1 \equiv 1$ on T.

Then, (53) takes the form

$$\frac{1}{|Y|}\iint\limits_{\Omega\times Y^*} A\left[\nabla u_0 + \nabla_y\left(-\hat{\chi}_1\frac{\partial u_0}{\partial x_1} - \hat{\chi}_2\frac{\partial u_0}{\partial x_2}\right)\right]\cdot\left[\nabla\Psi - \nabla_y(\varphi_1\,y^c\cdot\Psi)\right]\mathrm{d}x\,\mathrm{d}y = \int\limits_\Omega f\,\Psi\,\mathrm{d}x.$$

Simple computations yield the problem

$$\begin{cases} \text{Find } u_0 \in H^1_0(\Omega) \text{ such that} \\ \dfrac{|Y^*|}{|Y|}\displaystyle\int\limits_\Omega \mathcal{A}\,\nabla u_0\cdot\nabla\Psi\,\mathrm{d}x\,\mathrm{d}y = \int\limits_\Omega f\,\Psi\,\mathrm{d}x, \quad \forall\Psi \in H^1_0(\Omega). \end{cases} \tag{55}$$

Where matrix $\mathcal{A}$ is given by $\mathcal{A} = (a_{ij})_{i,j=1,2}$

$$a_{11} = \int\limits_{Y^*} A\left[\left(1-\frac{\partial\hat{\chi}_1}{\partial y_1}\right)\left(1-\frac{\partial(y_1^c\,\varphi_1)}{\partial y_1}\right) + \frac{\partial\hat{\chi}_1}{\partial y_2}\,\frac{\partial(y_1^c\,\varphi_1)}{\partial y_2}\right]\mathrm{d}y, \tag{56}$$

$$a_{12} = -\int\limits_{Y^*} A\left[\left(1-\frac{\partial\hat{\chi}_1}{\partial y_1}\right)\frac{\partial(y_2^c\,\varphi_1)}{\partial y_1} + \frac{\partial\hat{\chi}_1}{\partial y_2}\left(1-\frac{\partial(y_2^c\,\varphi_1)}{\partial y_2}\right)\right]\mathrm{d}y, \tag{57}$$

$$a_{21} = -\int\limits_{Y^*} A\left[\left(1-\frac{\partial\hat{\chi}_2}{\partial y_2}\right)\frac{\partial(y_1^c\,\varphi_1)}{\partial y_2} + \frac{\partial\hat{\chi}_2}{\partial y_1}\left(1-\frac{\partial(y_1^c\,\varphi_1)}{\partial y_1}\right)\right]\mathrm{d}y, \tag{58}$$

$$a_{22} = \int\limits_{Y^*} A\left[\left(1-\frac{\partial\hat{\chi}_2}{\partial y_2}\right)\left(1-\frac{\partial(y_2^c\,\varphi_1)}{\partial y_2}\right) + \frac{\partial\hat{\chi}_2}{\partial y_1}\,\frac{\partial(y_2^c\,\varphi_1)}{\partial y_1}\right]\mathrm{d}y. \tag{59}$$

In the sequel, we present results of the homogenization of torsion problem derived in [FR15]. We assume $\Omega = (0,1)\times(0,1)$, reference cell $Y = \langle 0,1)\times\langle 0,1)$, reference hole $T = \left(\frac{1}{4},\frac{3}{4}\right)\times\left(\frac{1}{4},\frac{3}{4}\right)$. Torsion problem is obtained for $A(y) = 1$, $f(x) = -2$.

According to the behavior of the holes, we distinguish three cases. They were described in Section 4.1.1.

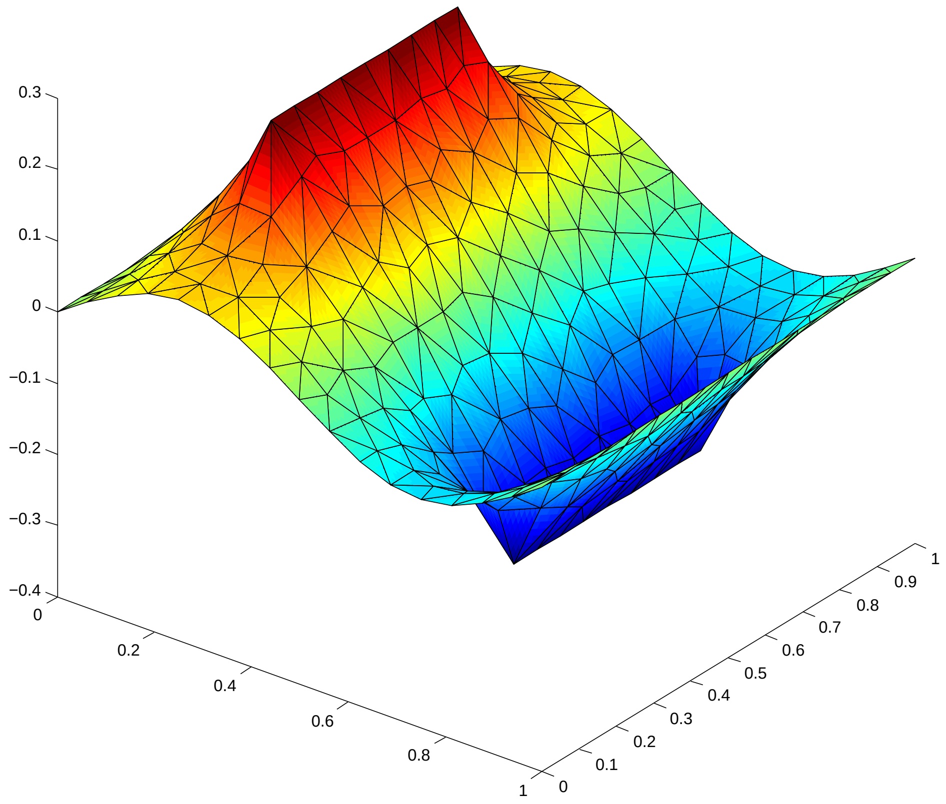

Figure 9: Auxiliary function $\hat{\chi}_1$.

- First, let us present results for $r(\varepsilon) = \varepsilon$, as for this case the Theorem 5.4 and all results in this chapter were derived. The sequence of domains is shown on the upper line on Figure 6. In the first step, by solving problem (54) we get two auxiliary functions $\hat{\chi}_1$ (on Figure 9) and $\hat{\chi}_2$.

  In the second step the problem (55) is solved to obtain the homogenized solution. A comparison of functions u_ε and homogenized solution u_0 is on Figure 11. Graph of function $u_{1/4}$ is on Figure 10.

In the following two cases we only present numerical results without any theoretical result.

- For $r(\varepsilon) = \varepsilon^2$ (so called small holes), the results are on Figure 12. The sequence of domains is on the middle line on Figure 6.

- For $r(\varepsilon) = \varepsilon(2-\varepsilon)$, the results are on Figure 13. The sequence of domains is on the lower line on Figure 6.

The numerical results are obtained by finite element method implemented in MATLAB. Some aspects of implementation are described in Appendix A.

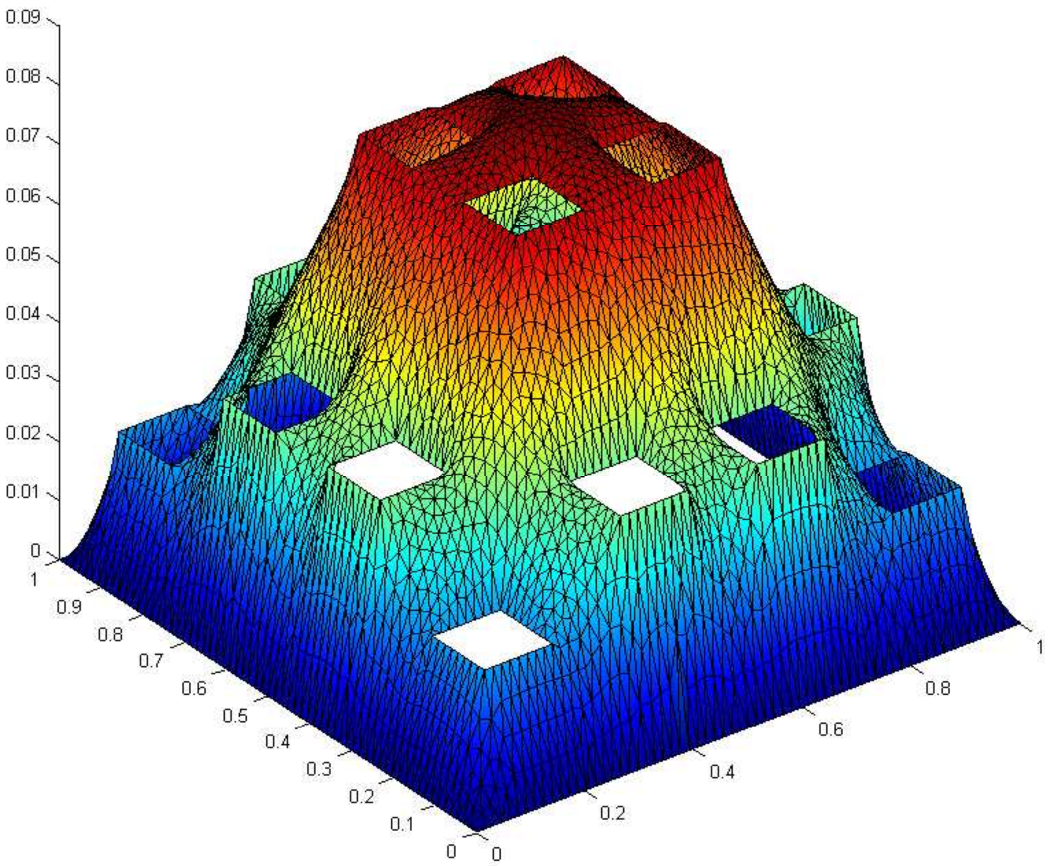

Figure 10: Graph of function $u_{1/4}$.

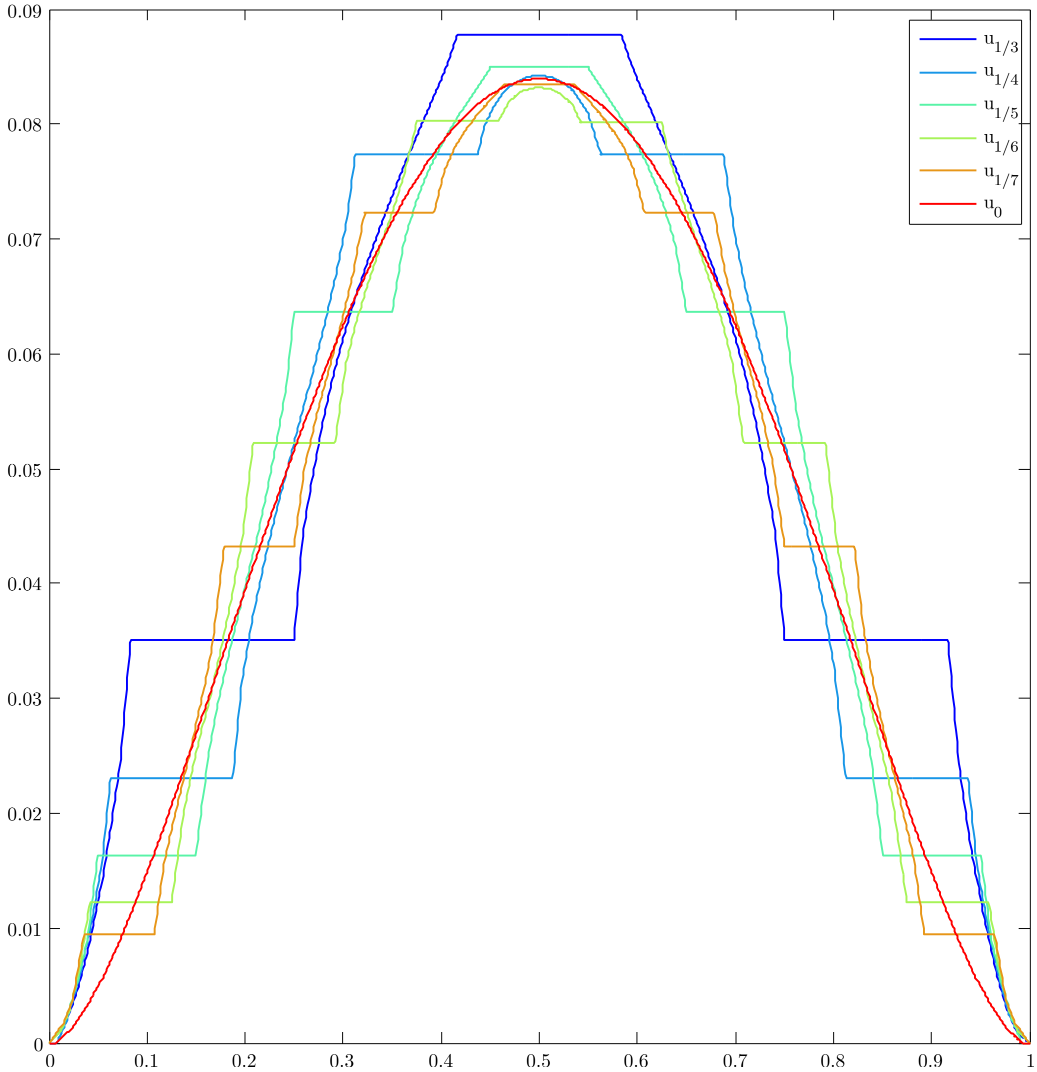

Figure 11: Diagonal cuts of functions u_ε, for $\varepsilon = {}^1\!/_3, {}^1\!/_4, {}^1\!/_5, {}^1\!/_6, {}^1\!/_7$, and of homogenized solution u_0, the behavior of holes is described by $r(\varepsilon) = \varepsilon$.

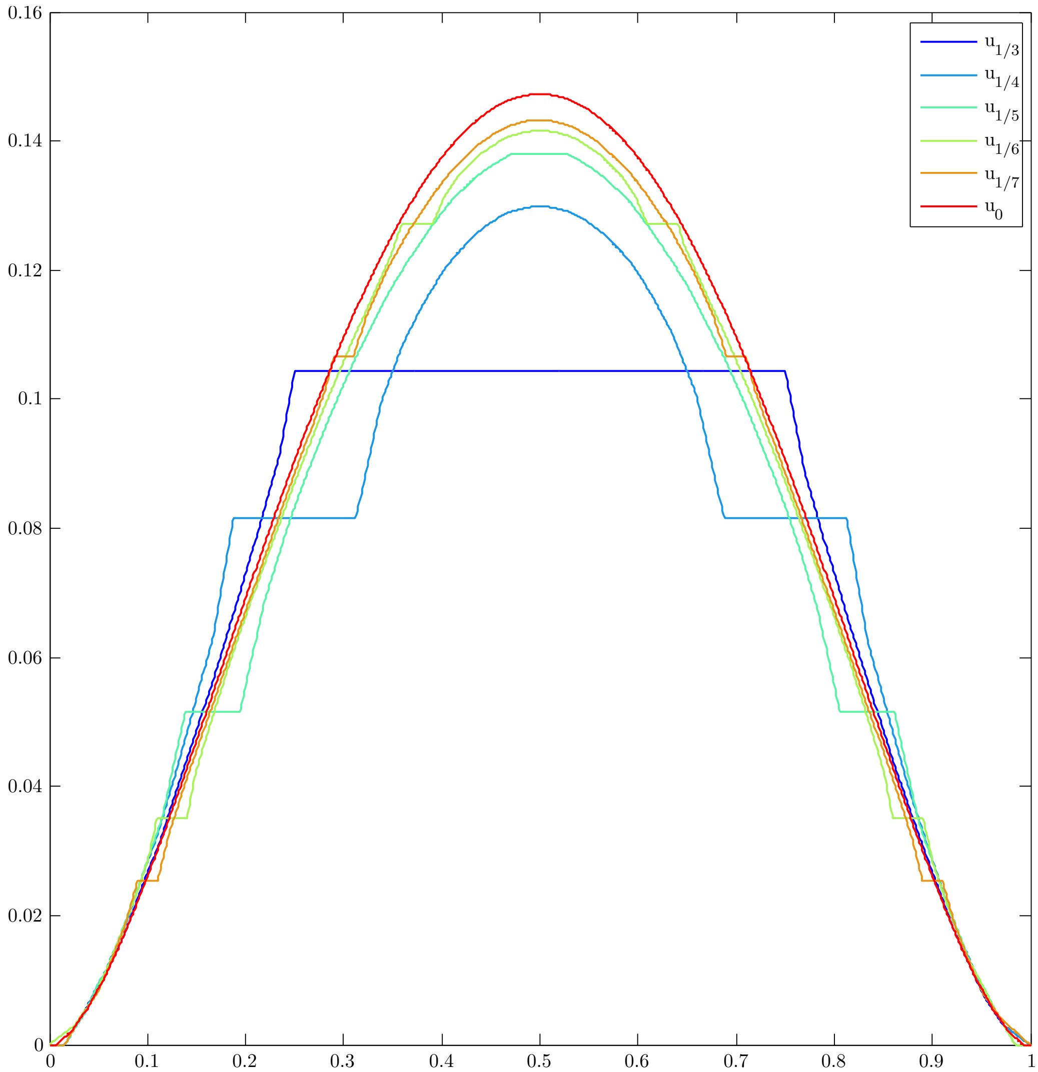

Figure 12: Diagonal cuts of functions u_ε, for $\varepsilon = {}^1\!/_3, {}^1\!/_4, {}^1\!/_5, {}^1\!/_6, {}^1\!/_7$, and solution u_0 of torsion problem on domain without holes (simply connected domain), the behavior of holes is described by $r(\varepsilon) = \varepsilon^2$.

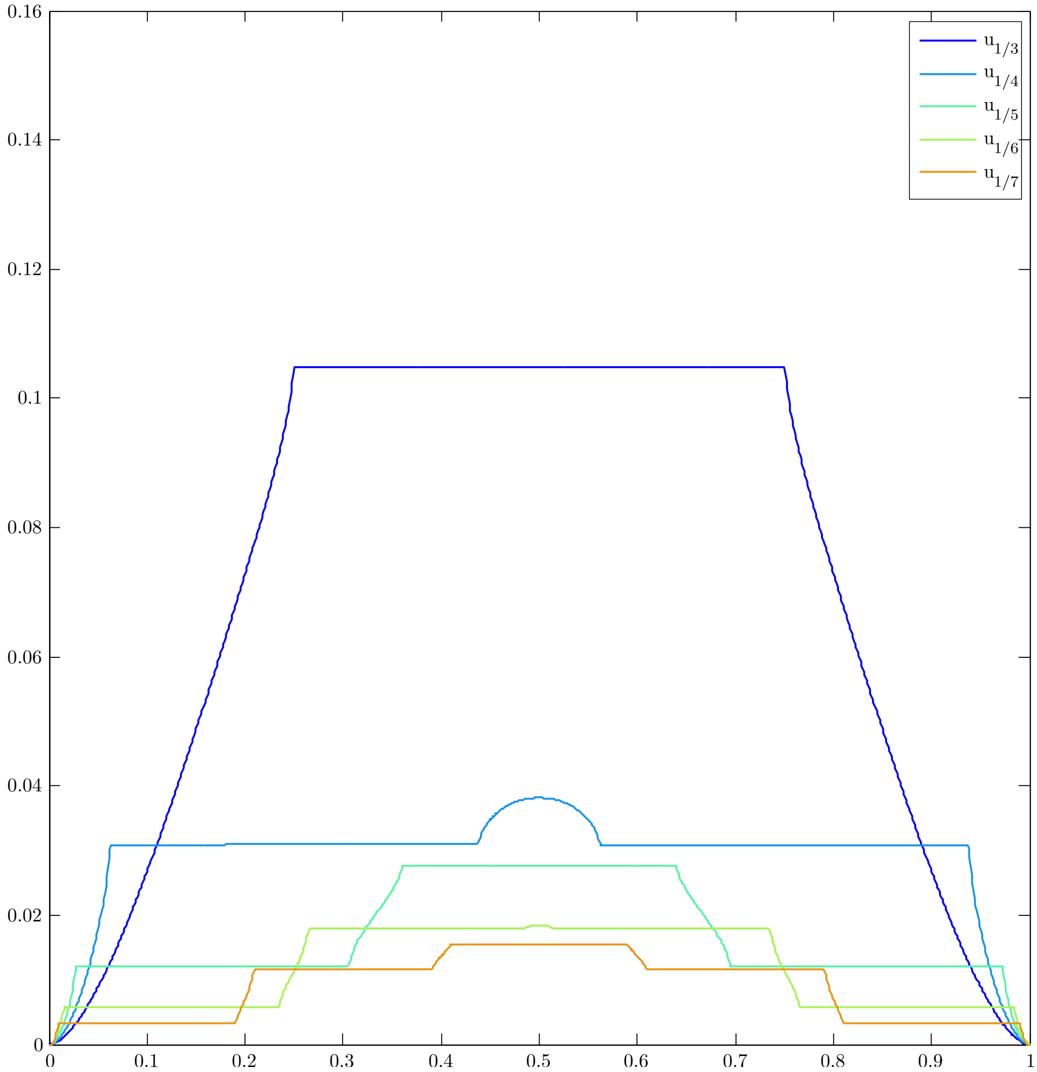

Figure 13: Diagonal cuts of functions u_ε, for $\varepsilon = {}^1/_3, {}^1/_4, {}^1/_5, {}^1/_6, {}^1/_7$, the behavior of holes is described by $r(\varepsilon) = \varepsilon(2-\varepsilon)$.

7. Conclusion

In problems which are set on perforated domains Ω_ε^*, where the shape and distribution of holes depends on the parameter ε, it may be difficult to define convergence for the sequence of solutions $\{u_\varepsilon\}$. There exist some approaches to solve this difficulty but their usage is usually limited. Limiting factors are usually the shape of the perforations or boundary conditions on inner boundaries.

The two-scale convergence, the approach presented in this thesis, is based on periodic unfolding operator for perforated domains $\mathcal{T}_\varepsilon^*$. This method is suitable for periodically distributed holes. The unfolded sequence $\{\mathcal{T}_\varepsilon^*(u_\varepsilon)\}$ is defined on fixed domains which removes difficulties with the convergence.

This technique was applied to the problem describing torsion of the bar (and its more general version). We derived a homogenized equation defined on a simply connected domain (without holes). We also presented numerical aspect of solving such a homogenized problem and in the last section there are some numerical examples.

Moreover, we proved some interesting properties which make it suitable for more general situations than that presented here. Unfolding operator $\mathcal{T}_\varepsilon^*$, used in this thesis, is slightly different than the one used in e.g. [CD00], [CDG02]. This change in definition allowed us to prove some properties in a more elegant way.

Appendices

A. Implementation of numerical experiments

Algorithms producing the numerical examples presented in Chapter 6 were implemented in MATLAB. Here, we shortly describe some aspects of implementation.

A.1. Homogenized problem

Solving of problem (53) consists of three steps: firstly, two auxiliary cell problem (54) are solved in order to find $\hat{\chi}_1, \hat{\chi}_2$. Secondly, these functions are used in (56) to evaluate the elements of matrix $\mathcal{A}$. Finally, using $\mathcal{A}$, the homogenized solution is found by solving problem (55).

Auxiliary cell problem and homogenized problem are solved by finite element method implemented in MATLAB. The domain is decomposed into conforming unstructured triangular mesh. Basis and test functions are piecewise linear. Hence, all integrals resulting from finite element formulation can be precomputed analytically.

Periodic boundary conditions prescribed in the formulation (54) can be replaced, in our case, by Dirichlet and homogenous Neumann condition. Let us remind that in our model problem the reference cell $Y = \langle 0, l_1) \times \langle 0, l_2)$ and reference hole $T = \left(\frac{l_1-a}{2}, \frac{l_1+a}{2}\right) \times \left(\frac{l_2-b}{2}, \frac{l_2+b}{2}\right)$, where $0 < a < l_1$ and $0 < b < l_2$. So perforated reference cell Y^* is symmetric with respect to axes $y_2 = \frac{l_2}{2}$ and $y_1 = \frac{l_1}{2}$.

If a function $\hat{\chi}_1(y_1, y_2)$ is a solution of the problem (54), then also the function $-\hat{\chi}_1(l_1 - y_1, y_2)$ is a solution of the same problem. Indeed, if a function $w = w(y_1, y_2)$ belongs to $L^2(\Omega, H^1_{\text{per}}(Y))$ then also the function $-w(l_1 - y_1, y_2)$ belongs to the same space. Let us choose it in (54) as a test function Φ. If a function $\hat{\chi}_1$ solves the problem (54), then we can derive:

$$\int_0^{l_2}\int_0^{l_1} \nabla(\hat{\chi}_1(y_1,y_2) - y_1) \cdot \nabla[-w(l_1 - y_1, y_2)]\ 1_{Y^*}\,\mathrm{d}y_1\,\mathrm{d}y_2 \left| \begin{array}{l} l_1 - y_1 = t \\ -\mathrm{d}y_1 = \mathrm{d}t \\ y_1 = 0 \Rightarrow t = l_1 \\ y_1 = l_1 \Rightarrow t = 0 \end{array} \right| =$$

$$= \int_0^{l_2}\int_{l_1}^{0} \nabla(\hat{\chi}_1(l_1 - t, y_2) - (l_1 - t)) \cdot \nabla w(t, y_2)\ 1_{Y^*}\,\mathrm{d}t\,\mathrm{d}y_2 =$$

$$= -\int_0^{l_2}\int_0^{l_1} \nabla(\hat{\chi}_1(l_1 - t, y_2) + t) \cdot \nabla w(t, y_2)\,\mathrm{d}t\ 1_{Y^*}\,\mathrm{d}y_2 =$$

$$= \int_0^{l_2}\int_0^{l_1} \nabla(-\hat{\chi}_1(l_1 - t, y_2) - t) \cdot \nabla w(t, y_2)\,\mathrm{d}t\ 1_{Y^*}\,\mathrm{d}y_2 = 0.$$

Thus, the function $-\hat{\chi}_1(l_1 - y_1, y_2)$ also solves the problem (54). From the uniqueness of the solution of the problem (54) we get $\hat{\chi}_1(y_1, y_2) = -\hat{\chi}_1(l_1 - y_1, y_2)$, which means that $\hat{\chi}_1$ "odd in variable y_1 with respect to point $y_1 = \frac{l1}{2}$". Finally, since this symmetry holds and the function $\hat{\chi}_1 \in H^1_{\mathrm{per}}(Y)$ we can prescribe to the boundary $y_1 = 0$ and $y_1 = l_1$ Dirichlet conditions.

Similar reasoning leads to Neumann conditions on the boundary $y_2 = 0$ and $y_2 = l_2$. Indeed, let us choose in (54) as a test function $\Phi = w(y_1, l_2 - y_2)$, which belongs to $L^2(\Omega, H^1_{\mathrm{per}}(Y))$. If a function $\hat{\chi}_1$ solves the problem (54), then we can derive:

$$\int_0^{l_1}\int_0^{l_2} \nabla(\hat{\chi}_1(y_1, y_2) - y_1) \cdot \nabla w(y_1, l_2 - y_2)\ 1_{Y^*}\, \mathrm{d}y_2\, \mathrm{d}y_1 \left| \begin{array}{l} l_2 - y_2 = t \\ -\,\mathrm{d}y_2 = \mathrm{d}t \\ y_2 = 0 \Rightarrow t = l_2 \\ y_2 = l_2 \Rightarrow t = 0 \end{array} \right| =$$

$$= -\int_0^{l_1}\int_{l_2}^{0} \nabla(\hat{\chi}_1(y_1, l_2 - t) - y_1) \cdot \nabla w(y_1, t)\ 1_{Y^*}\, \mathrm{d}t\, \mathrm{d}y_2 =$$

$$= \int_0^{l_1}\int_0^{l_2} \nabla(\hat{\chi}_1(y_1, l_2 - t) - y_1) \cdot \nabla w(y_1, t)\ 1_{Y^*}\, \mathrm{d}t\, \mathrm{d}y_2$$

Thus, the function $\hat{\chi}_1(y_1, l_2 - y_2)$ also solves the problem (54). Since the solution of (54) is unique it means that $\hat{\chi}_1(y_1, y_2) = \hat{\chi}_1(y_1, l_2 - y_2)$, i.e. the function $\hat{\chi}_1$ is "even in variable y_2 with respect to axis $y_2 = \frac{l2}{2}$". Finally, because this symmetry holds and the function $\hat{\chi}_1 \in H^1_{\mathrm{per}}(Y)$ we can prescribe to the boundary $y_2 = 0$ and $y_2 = l_2$ homogenous Neumann conditions.

So, the boundary conditions for function $\hat{\chi}$ are:

$$\hat{\chi}_1(y_1, 0) = c, \qquad \hat{\chi}_1(y_1, l_2) = c,$$

$$\frac{\partial \hat{\chi}_1}{\partial y_2}(0, y_2) = 0, \qquad \frac{\partial \hat{\chi}_1}{\partial y_2}(l_2, y_2) = 0,$$

where constant c is determined from the condition $\mathcal{M}_Y(\hat{\chi}_i) = 0$. For the function $\hat{\chi}_2$ the reasoning is analogical.

Due to the symmetry it would be possible to solve the auxiliary problems on one quarter of period.

Matrix $\mathcal{A}$: Integrals in the formula (56) for matrix $\mathcal{A}$ are computed numerically element by element by using 2D quadrature rule of order 1, which is sufficient because $\hat{\chi}_1, \hat{\chi}_2$ are approximated only by piecewise linear functions. Since the formula for matrix $\mathcal{A}$ contains partial derivatives of functions $\hat{\chi}_1, \hat{\chi}_2$ to achieve better accuracy it would be useful to use higher polynomial basis in finite element formulation at least for the cell problem.

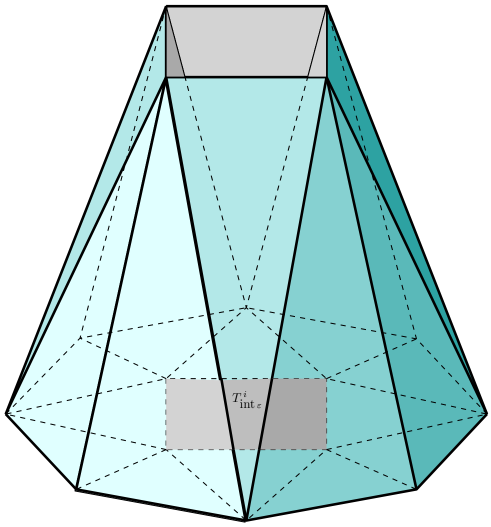

Figure 14: Basis function associated with vertices on the boundary of an inner hole $T^i_{\mathrm{int},\,\varepsilon}$.

A.2. Problem on perforated domain

Problem (37) formulated in Chapter 5 is also solved by finite element method on triangular mesh.

Basis and test functions are piecewise linear and belong to the space $\mathcal{S}_\varepsilon(\Omega)$ (defined by (36)). To fulfill conditions required by definition of this space (especially requirement that functions in $\mathcal{S}_\varepsilon$ equal constant on each inner hole) we chose slightly different basis functions than is usual. The basis function associate with a vertex which is not on the boundary (neither outer nor inner) is a classical hat function, which equals one at its associated vertex and zero at all other vertices. For all vertices belonging to the boundary of a inner hole $T^i_{\mathrm{int},\,\varepsilon}$ there is the only one basis function, which equals one in all these vertices and zero at all others, see Figure 14.

A.3. Meshes

Decomposition of domains for all problems mentioned above were generated by using toolbox *MESH2D - Automatic Mesh Generation* by Darren Engwirda. The code is covered by the BSD Licence.

MESH2D is a toolbox of 2D meshing routines that allows for the automatic generation of unstructured triangular meshes for general 2D geometry. The resulting mesh achieves high quality.

Mesh2D is suitable for domain with holes and also for domain with multiple connected faces. In addition to the fully automatic settings, MESH2D allows the user to specify sizing information, allowing for varying levels of mesh resolution within the domain.

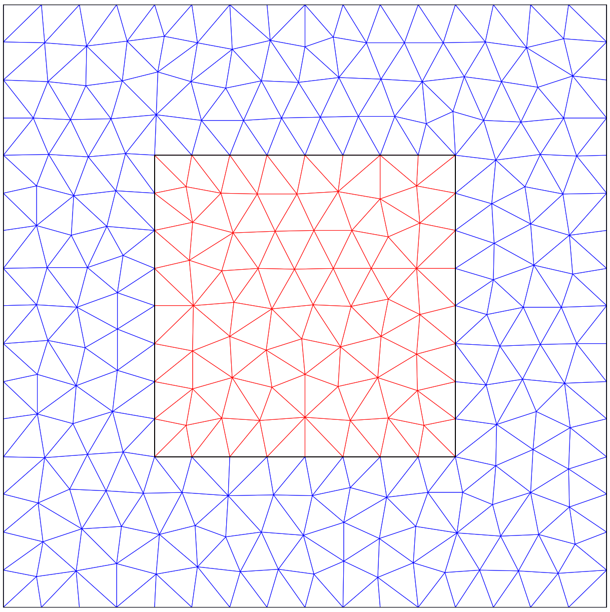

Figure 15: Mesh used to solve auxiliary cell problem (54).

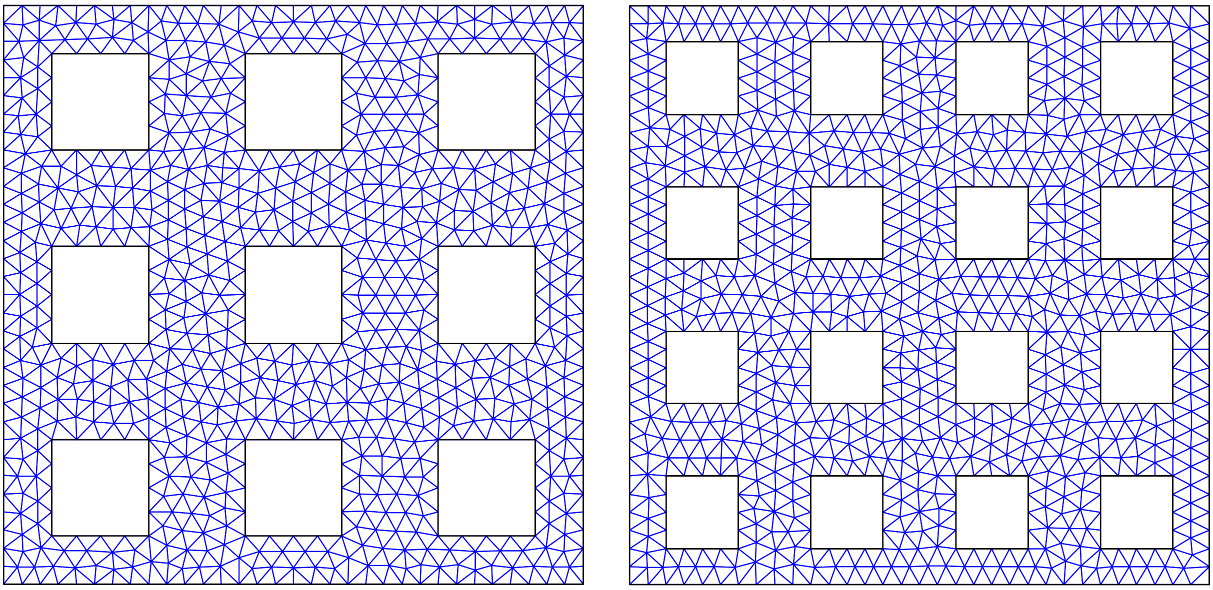

Figure 16: Meshes used to solve problem on perforated domain (37), for $\varepsilon = {}^1/_3, {}^1/_4$.

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