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WEAKLY DELAYED SYSTEMS OF LINEAR DISCRETE EQUATIONS IN $\mathbb{R}^3$

SLABĚ ZPOŽDĚNÉ SYSTÉMY LINEÁRNÍCH DISKRÉTNÍCH ROVNIC V $\mathbb{R}^3$

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1 INTRODUCTION

Discrete equations arise naturally in discretizing ordinary or delayed differential equations. As such, they may serve as a powerful mathematical tool for modelling various phenomena in physics, biology, engineering, economics, and many other scientific fields.

Of special importance are applications of discrete equations in electrical engineering, particularly discrete-time signal processing where they are employed to digitize analogous signals. Thus, investigating the properties of systems of discrete equations is part of the main stream of research.

The thesis is devoted to the problems of representing the solutions of what is called discrete systems with weak delay.

The fundamentals of the theory of difference equations are well described, for example, in books by S. Elaydi [33], by I. Győri, G. Ladas [35], by V. L. Kocić, G. Ladas [42] and by R. P. Agarwal, M. Bohner, S. R. Grace, D. O'Regan [1]. Applications to electrical engineering are described, e. g., in books by R. Vích, Z. Smékal [60] , by J. G. Proakis, D. G. Manolakis [48] and by A. V. Oppenheim, R. W. Schafer, J. R. Buck [47].

1.1 CURRENT STATE

The theory of ordinary difference equations is developed intensively in various directions. Many of them copy topics similar to those of ordinary differential equations, but there are topics not analogous to problems considered for them.

The areas of in-depth resarch include, e.g., the theory of representations of solutions of linear discrete systems with delay. We mention at least papers [17], [18], [21]– [29], [40], [45] and monograph [20]. The problem of the existence of positive solutions of discrete equations is studied, e.g., in [2]– [5], [44]. Various problems related to the stability of solutions of discrete equations and systems are analyzed in [19], [37], [41], [43]. Oscillation properties of solutions of discrete equations are studied, e.g., [38], [46], [49]. The asymptotic behavior of the solutions and various qualitative properties of solutions are studied in [6], [7], [15], [16], [30], [31], [34], [50].

The theory of weakly delayed systems is considered in the papers [9]– [11], [21], [39]. As a co-author, the author of the thesis, has recently achieved new results on this topic, e.g., in [12]– [14], [32], [51]– [59].

1.2 AIMS OF THE THESIS

We use the following notation in the sequel: For integers s, q, $s \leq q$, we define a set $\mathbb{Z}_s^q := \{s, s+1, \dots, q-1, q\}$. Similarly, we define a set $\mathbb{Z}_s^\infty := \{s, s+1, \dots\}$.
The aim of this thesis is to analyse weakly delayed linear discrete systems with constant coefficients and delays of the form

$$x(k+1) = Ax(k) + Bx(k-m) \tag{1.1}$$

where $m > 0$ is a positive integer, $k \in \mathbb{Z}_0^\infty$, $A = (a_{ij})$ and $B = (b_{ij})$ are constant 3×3 matrices, and $x\colon \mathbb{Z}_{-m}^\infty \to \mathbb{R}^3$. Throughout the thesis we assume $\det A \neq 0$ (the presence of a zero eigenvalue of A can cause problems in constructing of systems of generalized eigenvectors of some auxiliaty matrices).

Methodically, we will follow the papers [39] and [10, 11] as well. In [39] a planar linear discrete system with a weak delay is considered

$$x(k+1) = Ax(k) + Bx(k-m), \tag{1.2}$$

having a fixed integer $m > 0$, $k \in \mathbb{Z}_0^\infty$, $A = (a_{ij})$ and $B = (b_{ij})$ are constant 2×2 matrices, and $x\colon \mathbb{Z}_{-m}^\infty \to \mathbb{R}^2$.
The system with weak delays is defined in [39] as a system (1.2) for which the equality

$$\det\left(A + \lambda^{-m}B - \lambda E\right) = \det\left(A - \lambda E\right),$$

where E is a 3 by 3 unit matrix, holds for every $\lambda \in \mathbb{C} \setminus \{0\}$.
The relevant general solution of (1.2) is constructed in [39] with the results on the dimensionality of the space of solutions deduced.

In [10, 11] generalizations were investigated of system (1.2) having the form

$$x(k+1) = Ax(k) + Bx(k-m) + Cx(k-n),$$

where $m > n > 0$ are fixed integers, $k \in \mathbb{Z}_0^\infty$, $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ are constant 2×2 matrices, and $x\colon \mathbb{Z}_{-m}^\infty \to \mathbb{R}^2$ and

$$x(k+1) = Ax(k) + \sum_{l=1}^{n} B^l x_l(k - m_l)$$

where $m_1, m_2, \ldots, m_n$ are constant integer delays, $0 < m_1 < m_2 < \cdots < m_n$, $k \in \mathbb{Z}_0^\infty$, $A, B^1, \ldots, B^n$ are constant 2×2 matrices, $A = (a_{ij})$, $B^l = (b_{ij}^l)$, $i, j = 1, 2$, $l = 1, 2, \ldots, n$ and $x\colon \mathbb{Z}_{-m_n}^\infty \to \mathbb{R}^2$.

The dissertation aims to select weakly delayed systems from a general discrete system (1.1).

We will give criteria for (1.1) to be weakly delayed in terms of the coefficients of matrices A and B. The next problem is to find analytical formulas describing the solutions of system (1.1). It is known, that in contrast to non-weakly delayed systems, the solutions of weakly delayed ones depend only on part of the initial data. Therefore, we will investigate the problem of reducing the initial data as well.

1.3 PRELIMINARY NOTIONS AND PROPERTIES

Consider discrete systems

$$x(k+1) = Ax(k) + Bx(k-m) \tag{1.3}$$

where $m > 0$ is a fixed integer, $k \in \mathbb{Z}_0^\infty$, $A = (a_{ij})$ and $B = (b_{ij})$, are constant $l \times l$ matrices, and $x\colon \mathbb{Z}_{-m}^\infty \to \mathbb{R}^l$, $l \geq 2$.

In [21], linear weakly delayed systems were defined for planar systems. This definition can be applied to l-dimensional systems as follows.

Definition 1.3.1. System (1.3) is called weakly delayed if the characteristic equations for (1.3) and for the system without delay

$$x(k+1) = Ax(k) \tag{1.4}$$

have identical roots, that is, if, for every $\lambda \in \mathbb{C} \setminus \{0\}$,

$$\det\left(A + \lambda^{-m} B - \lambda E\right) = \det\left(A - \lambda E\right).$$

In the thesis we use various regular transformations of weakly delayed systems (1.3). The property of a system to be weakly delayed is invariant under such transformations. Therefore, the following result forms the basis for our following constructions and the relevant proofs can be found, for example, in [21] and [10], [11], [36]. We formulate this property and, for the reader's convenience, we give its proof as well.

We consider a linear transformation

$$x(k) = \mathcal{S}y(k) \tag{1.5}$$

with a nonsingular 3×3 matrix $\mathcal{S}$. Then, the discrete system for y is

$$y(k+1) = A_{\mathcal{S}}y(k) + B_{\mathcal{S}}y(k-m) \tag{1.6}$$

with $A_{\mathcal{S}} = \mathcal{S}^{-1}A\mathcal{S}$, $B_{\mathcal{S}} = \mathcal{S}^{-1}B\mathcal{S}$. We show that the property of a system being weakly delayed is preserved by every nonsingular linear transformation.

Lemma 1.3.2. *If the system* (1.1) *is weakly delayed, then its arbitrary linear nonsingular transformation* (1.5) *again leads to a weakly delayed system* (1.6)*.*

1.4 CRITERIA OF WEAKLY DELAYED SYSTEMS

Theorem 1.4.1 ([9])**.** *Let* $l = 3$ *in* (1.3)*. Then,* (1.3) *is a weakly delayed system if and only if conditions* (1.7) – (1.12) *below hold:*

$$b_{11} + b_{22} + b_{33} = 0, \tag{1.7}$$

$$\begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} = 0, \tag{1.8}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} + \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ a_{21} & a_{22} & a_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} + \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0, \tag{1.9}$$

$$\begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ 0 & 1 & 0 \\ b_{31} & b_{32} & b_{33} \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} = 0, \tag{1.10}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0, \tag{1.11}$$

$$\begin{aligned} &\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 1 & 0 \\ b_{31} & b_{32} & b_{33} \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} \\ +&\begin{vmatrix} b_{11} & b_{12} & b_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ b_{21} & b_{22} & b_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0. \end{aligned} \tag{1.12}$$

Theorem 1.4.2. *Let* $l = 3$ *and* λ_{Bi}*,* $i = 1, 2, 3$ *be eigenvalues of matrix* B*. If* (1.7), (1.8) *and* (1.10) *hold, then*

$$\lambda_{Bi} = 0, \ \ i = 1, 2, 3.$$

Theorem 1.4.3. *If* (1.3) *is weakly delayed, then* B *is a nilpotent matrix.*

2 CRITERIA FOR WEAKLY DELAYED SYSTEMS

2.1 JORDAN CANONICAL FORMS OF A AND CRITERIA FOR WEAKLY DELAYED SYSTEMS

Throughout the remaining part of the thesis we will assume that $l = 3$ in (1.3).
It is known that, for every matrix A, there exists a nonsingular matrix $\mathcal{S}$ transforming it to the corresponding Jordan matrix form A_J. This means that

$$A_J = \mathcal{S}^{-1} A \mathcal{S}$$

where A_J has the following seven possible forms (denoted below by $\Lambda_1, \ldots, \Lambda_7$), depending on the roots of the characteristic equation

$$\det(A - \lambda E) = 0. \tag{2.1}$$

If (2.1) has three real distinct roots λ_1, λ_2, λ_3, then

$$\Lambda_1 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}. \tag{2.2}$$

If (2.1) has real single root λ_1 and two-fold real root $\lambda_2 = \lambda_3$, then

$$\Lambda_2 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \tag{2.3}$$

or

$$\Lambda_3 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{pmatrix}. \tag{2.4}$$

In the case of one triple real root $\lambda = \lambda_{1,2,3}$, the following forms are possible

$$\Lambda_4 = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \tag{2.5}$$

$$\Lambda_5 = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \tag{2.6}$$

$$\Lambda_6 = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}. \tag{2.7}$$

Finally, if one root is real and two roots are complex conjugate, i.e. $\lambda_{2,3} = p \pm iq$, with $q \neq 0$, then

$$\Lambda_7 = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & p & q \\ 0 & -q & p \end{pmatrix}. \tag{2.8}$$

In this part, we will simplify the general conditions (1.7)–(1.12) for each of the Jordan forms (2.2)–(2.8).

Since the property of a system to be weakly delayed is preserved (by Lemma 1.3.2, page 8) under arbitrary linear nonsingular transformation (1.5), in the following, we assume, without loss of generality, that matrix A is given in the Jordan form (2.2)–(2.8) and, without defining addditional notations, the matrix B is used again (instead of a transformed matrix B_S, see (1.6)).

2.1.1 Criterion for Weakly Delayed Systems in the Case (2.2)

Consider system (1.3) with the matrix $A = \Lambda_1$, i.e.,

$$x(k+1) = \Lambda_1 x(k) + Bx(k-m). \tag{2.9}$$

In [8] the following result is formulated.

Theorem 2.1.1. *System* (2.9) *is a weakly delayed system if and only if*

$$b_{11} = b_{22} = b_{33} = 0, \tag{2.10}$$

$$b_{12}b_{23}b_{31} + b_{13}b_{21}b_{32} = 0, \tag{2.11}$$

$$b_{12}b_{21} + b_{13}b_{31} + b_{23}b_{32} = 0, \tag{2.12}$$

$$\lambda_3 b_{12}b_{21} + \lambda_2 b_{13}b_{31} + \lambda_1 b_{23}b_{32} = 0. \tag{2.13}$$

2.1.2 Criterion for Weakly Delayed Systems in the Case (2.3)

Consider system (1.3) with the matrix $A = \Lambda_2$, i.e.,

$$x(k+1) = \Lambda_2 x(k) + Bx(k-m). \tag{2.14}$$

Theorem 2.1.2. *System* (2.14) *is a weakly delayed system if and only if*

$$b_{11} = 0, \tag{2.15}$$

$$b_{22} + b_{33} = 0, \tag{2.16}$$

$$b_{12}b_{21} + b_{13}b_{31} = 0, \tag{2.17}$$

$$b_{22}b_{33} - b_{23}b_{32} = 0, \tag{2.18}$$

$$b_{12}b_{23}b_{31} + b_{13}b_{21}b_{32} - b_{13}b_{22}b_{31} - b_{12}b_{21}b_{33} = 0. \tag{2.19}$$

2.1.3 Criterion for Weakly Delayed Systems in the Case (2.4)

Consider system (1.3) with the matrix $A = \Lambda_3$, i.e.,

$$x(k+1) = \Lambda_3 x(k) + Bx(k-m). \tag{2.20}$$

Theorem 2.1.3. *System* (2.20) *is a weakly delayed system if and only if*

$$b_{11} = 0, \tag{2.21}$$

$$b_{22} + b_{33} = 0, \tag{2.22}$$

$$b_{32} = 0, \tag{2.23}$$

$$b_{22}b_{33} - b_{12}b_{21} - b_{13}b_{31} = 0, \tag{2.24}$$

$$(\lambda_1 - \lambda_2)b_{22}b_{33} + b_{12}b_{31} = 0, \tag{2.25}$$

$$b_{12}b_{23}b_{31} - b_{13}b_{22}b_{31} - b_{12}b_{21}b_{33} = 0. \tag{2.26}$$

2.1.4 Criterion for Weakly Delayed Systems in the Case (2.5)

Consider system (1.3) with the matrix $A = \Lambda_4$, i.e.,

$$x(k+1) = \Lambda_4 x(k) + Bx(k-m). \tag{2.27}$$

Theorem 2.1.4. *System* (2.27) *is a weakly delayed system if and only if*

$$b_{11} + b_{22} + b_{33} = 0, \tag{2.28}$$

$$b_{11}b_{22} + b_{11}b_{33} + b_{22}b_{33} - b_{12}b_{21} - b_{13}b_{31} - b_{23}b_{32} = 0, \tag{2.29}$$

$$b_{11}b_{22}b_{33} + b_{12}b_{23}b_{31} + b_{13}b_{21}b_{32} - b_{13}b_{22}b_{31} - b_{12}b_{21}b_{33} - b_{11}b_{23}b_{32} = 0. \tag{2.30}$$

2.1.5 Criterion for Weakly Delayed Systems in the Case (2.6)

Consider system (1.3) with the matrix $A = \Lambda_5$, i.e.,

$$x(k+1) = \Lambda_5 x(k) + Bx(k-m). \tag{2.31}$$

Theorem 2.1.5. *System* (2.31) *is a weakly delayed system if and only if*

$$b_{11} + b_{22} + b_{33} = 0, \tag{2.32}$$

$$b_{21} = 0, \tag{2.33}$$

$$b_{23}b_{31} = 0, \tag{2.34}$$

$$b_{11}b_{22} + b_{11}b_{33} + b_{22}b_{33} - b_{13}b_{31} - b_{23}b_{32} = 0, \tag{2.35}$$

$$b_{11}b_{22}b_{33} - b_{13}b_{22}b_{31} - b_{11}b_{23}b_{32} = 0. \tag{2.36}$$

2.1.6 Criterion for Weakly Delayed Systems in the Case (2.7)

Consider system (1.3) with the matrix $A = \Lambda_6$, i.e.,

$$x(k+1) = \Lambda_6 x(k) + Bx(k-m). \tag{2.37}$$

Theorem 2.1.6. *System* (2.37) *is a weakly delayed system if and only if*

$$b_{11} + b_{22} + b_{33} = 0, \tag{2.38}$$

$$b_{21} + b_{32} = 0, \tag{2.39}$$

$$b_{31} = 0, \tag{2.40}$$

$$b_{21}b_{33} + b_{11}b_{32} = 0, \tag{2.41}$$

$$b_{11}b_{22} + b_{11}b_{33} + b_{22}b_{33} - b_{12}b_{21} - b_{23}b_{32} = 0, \tag{2.42}$$

$$b_{11}b_{22}b_{33} + b_{13}b_{21}b_{32} - b_{12}b_{21}b_{33} - b_{11}b_{23}b_{32} = 0. \tag{2.43}$$

2.1.7 Criterion for Weakly Delayed Systems in the Case (2.8)

Consider system (1.3) with the matrix $A = \Lambda_7$, i.e.,

$$x(k+1) = \Lambda_7 x(k) + Bx(k-m). \tag{2.44}$$

Theorem 2.1.7. *System* (2.44) *is a weakly delayed system if and only if*

$$b_{11} = 0, \tag{2.45}$$
$$b_{22} + b_{33} = 0, \tag{2.46}$$
$$b_{23} - b_{32} = 0, \tag{2.47}$$
$$b_{22}b_{33} - b_{12}b_{21} - b_{13}b_{31} - b_{23}b_{32} = 0, \tag{2.48}$$
$$(\lambda - p)(b_{12}b_{21} + b_{13}b_{31}) + q(b_{12}b_{31} - b_{13}b_{21}) = 0, \tag{2.49}$$
$$b_{12}b_{23}b_{31} + b_{13}b_{21}b_{32} - b_{13}b_{22}b_{31} - b_{12}b_{21}b_{33} = 0. \tag{2.50}$$

3 SOLUTION OF WEAKLY DELAYED DIFFERENCE SYSTEMS IN $\mathbb{R}^3$

3.1 CONSTRUCTION OF GENERAL SOLUTION OF SYSTEM (1.1)

Below we investigate system (1.1) with the initial data

$$x(0) = x_0 = \begin{pmatrix} x_{0,1} \\ x_{0,2} \\ x_{0,3} \end{pmatrix}, \dots, x(-m) = x_{-m} = \begin{pmatrix} x_{m,1} \\ x_{m,2} \\ x_{m,3} \end{pmatrix} \tag{3.1}$$

where $x_{i,j}$, $i = 0, \dots, m$, $j = 1, 2, 3$, are real constants. To avoid complicated notation, we will write just m in the right-hand side of (3.1) rather than $-m$ as one would expect. This will cause no problems in future computations.

Define new dependent 3-dimensional vector functions $y_i(k)$, $i = 1, \dots, 3(m+1)$ by the formulas

$$\begin{aligned} y_s(k) &= x_s(k), \\ y_{s+3}(k) &= x_s(k-1), \\ y_{s+6}(k) &= x_s(k-2), \\ &\vdots \\ y_{s+3m}(k) &= x_s(k-m), \end{aligned} \tag{3.2}$$

where $s = 1, 2, 3$. It is easy to see that

$$\begin{array}{llllll} y_s(k+1) & = Ay_s(k) & & & & +By_{s+3m}(k), \\ y_{s+3}(k+1) & = Ey_s(k), & & & & \\ y_{s+6}(k+1) & = & Ey_{s+3}(k), & & & \\ \vdots & & & \ddots & & \\ y_{s+3m}(k+1) & = & & & Ey_{s+3(m-1)}(k) & \end{array}$$

where $s = 1, 2, 3$. The new system can be written in the matrix form

$$y(k+1) = \mathcal{A}y(k),\ k \geq 0 \tag{3.3}$$

where $y(k) = (y_1(k), \dots, y_{3(m+1)}(k))^T$ and

$$\mathcal{A} = \begin{pmatrix} A & \Theta & \dots & \Theta & B \\ E & \Theta & \dots & \Theta & \Theta \\ \Theta & E & \dots & \Theta & \Theta \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Theta & \Theta & \dots & E & \Theta \end{pmatrix} \tag{3.4}$$

is a $3(m+1) \times 3(m+1)$ matrix, Θ is a 3×3 zero matrix.
The initial data for the system (3.3), in terms of (3.1), are

$$y(0) = y_0 = (x_0, \dots, x_m)^T. \tag{3.5}$$

We will transform system (3.3) using the transformation

$$y(k) = \mathcal{T} w(k) \tag{3.6}$$

where $\mathcal{T}$ is a regular transition $3(m+1) \times 3(m+1)$ matrix and $w(k)$ is a new dependent $3(m+1)$-dimensional vector into a system with a matrix of the Jordan form. We get

$$\mathcal{T} w(k+1) = \mathcal{A}\mathcal{T} w(k)$$

or

$$w(k+1) = \mathcal{G} w(k) \tag{3.7}$$

where

$$\mathcal{G} = \mathcal{T}^{-1}\mathcal{A}\mathcal{T} \tag{3.8}$$

with the initial data for (3.7) being, as it follows from (3.6),

$$w(0) = \mathcal{T}^{-1} y(0). \tag{3.9}$$

Then, the solution of the system (3.7) is

$$w(k) = \mathcal{G}^k w(0), \ \ k = 1, 2, 3, \dots.$$

If the matrix A in (3.4) is in its Jordan form, that is, $A = \Lambda_i$, $i \in \{1, \dots, 7\}$, we will denote the matrix $\mathcal{A}$ as $\mathcal{A}_i$, $i \in \{1, \dots, 7\}$, that is,

$$\mathcal{A}_i = \begin{pmatrix} \Lambda_i & \Theta & \dots & \Theta & B \\ E & \Theta & \dots & \Theta & \Theta \\ \Theta & E & \dots & \Theta & \Theta \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Theta & \Theta & \dots & E & \Theta \end{pmatrix}. \tag{3.10}$$

Considering system (3.3), suppose that an eigenvalue λ of the matrix $\mathcal{A}$ has a geometric multiplicity of $m_g(\lambda)$. Then, the eigenvalue λ of the matrix $\mathcal{G}$ in (3.7) has the same geometric multiplicity $m_g(\lambda)$. We prove this property.

Theorem 3.1.1. *Geometric multiplicities of identical eigenvalues of matrices $\mathcal{A}$ and $\mathcal{G}$ are identical.*

3.2 RELATIONSHIP BETWEEN THE EIGENVALUES OF Λ_I, B, AND $\mathcal{A}_I$

The main purpose of this part is to show that the set of all eigenvalues of matrices $\mathcal{A}_i$, $i = 1, \dots, 7$ can be written as the union of the sets of all eigenvalues of matrices A_i, $i = 1, \dots, 7$, and the relevant matrix B.
In other words, we prove the following theorem.

Theorem 3.2.1. *Let system* (1.1) *be weakly delayed and let the matrix A, having a Jordan form Λ_i, $i \in \{1, \dots, 7\}$, be fixed. Then, the set of all the eigenvalues $\mu^i_j, j = 1, \dots, 3(m+1)$ of the matrix $\mathcal{A}_i$ equals the union of the sets of all the eigenvalues $\lambda^i_j, j = 1, 2, 3$ of the matrix A_i, $\lambda_{Bs}, s = 1, 2, 3$ of the matrix B and the remaining eigenvalues equal zero, i.e.,*

$$\mu^i_j = \lambda^i_j, j = 1, 2, 3$$
$$\mu^i_j = 0, j = 4, \dots, 3(m+1).$$

3.3 SOLUTION OF THE PROBLEM (1.1), (3.1)

The solution of the problem (3.3) on page 12, and (3.5) on page 13, that is, the problem

$$y(k+1) = \mathcal{A}y(k),\ k \geq 0$$

$$y(0) = y_0 = (x_0, \ldots, x_m)^T.$$

is

$$y(k) = \mathcal{A}^k y(0) = (\mathcal{T}\mathcal{G}\mathcal{T}^{-1})^k y(0) = \mathcal{T}\mathcal{G}^k\mathcal{T}^{-1}y(0) = \mathcal{T}\mathcal{G}^k w(0),\ \ k = 1, 2, 3, \ldots$$

where $w(0)$ is given by (3.9).

In (3.3), the matrix $\mathcal{A}$ is defined by (3.4) and takes seven different forms (3.10) depending on the Jordan forms Λ_i, $i = 1, \ldots, 7$, of A. The matrix $\mathcal{G}$ defined by (3.8) takes different forms depending on the Jordan form of $\mathcal{A}$ and on geometric multiplicity of B. Therefore if $\mathcal{A} = \mathcal{A}_i, i = 1, \ldots, 7$ then $\mathcal{G} = \mathcal{G}_{ij}, i = 1, \ldots, 7$ where $j = 1$ if geometric multiplicity of the zero root of B equals 1 and $j = 2$ if geometric multiplicity of the zero root of B equals 2, and $\mathcal{T} = \mathcal{T}_{ij}$, $i = 1, \ldots, 7$, $j = 1, 2$ where $\mathcal{T}_{ij}$ is a transition matrix transforming $\mathcal{A}_i$ to $\mathcal{G}_{ij}$.

Using an auxiliary matrix

$$Q = (E, \underbrace{\Theta, \ldots, \Theta}_{m}),$$

we can write the solution of the initial problem (1.1), page 6, (3.1), page 12, that is, the problem

$$x(k+1) = Ax(k) + Bx(k-m),$$

$$x(0) = x_0 = \begin{pmatrix} x_{0,1} \\ x_{0,2} \\ x_{0,3} \end{pmatrix}, \ldots, x(m) = x_m = \begin{pmatrix} x_{m,1} \\ x_{m,2} \\ x_{m,3} \end{pmatrix},$$

in terms of transformation (3.2), as

$$x(k) = Q\mathcal{T}\mathcal{G}^k w(0),\ \ k = 1, 2, 3, \ldots,$$

where, by (3.9)

$$w(0) = \mathcal{T}^{-1}y(0).$$

Therefore, the following theorems hold.

Theorem 3.3.1. *Let the matrix A have the form* (2.2) *with three real distinct roots λ_1, λ_2, λ_3, let the elements of the matrix B satisfy* (2.10)–(2.13). *Then, the solution of the initial problem* (1.1), (3.1) *is given by the formula*

$$x(k) = Q\mathcal{T}_{1j}\mathcal{G}_{1j}^k w(0),\ \ k = 1, 2, 3, \ldots, \tag{3.11}$$

where $j = 1$ if the geometric multiplicity of the zero eigenvalue of B equals 1 *and $j = 2$ if the geometric multiplicity of the zero eigenvalue of B equals* 2 *and $w(0)$ is given by* (3.9).

If $j = 1$ and $k \geq 3m$ then, by (3.11),

$$x(k) = Q\mathcal{T}_{11}\mathcal{G}_{11}^k w(0) = Q\mathcal{T}_{11}\begin{pmatrix} \lambda_1^k w_1(0) \\ \lambda_2^k w_2(0) \\ \lambda_3^k w_3(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3.12}$$

If $j = 2$ and $k \geq 3m - 1$ then, by (3.11),

$$x(k) = Q\mathcal{T}_{12}\mathcal{G}_{12}^k w(0) = Q\mathcal{T}_{12}\begin{pmatrix} \lambda_1^k w_1(0) \\ \lambda_2^k w_2(0) \\ \lambda_3^k w_3(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3.13}$$

Theorem 3.3.2. *Let the matrix A have the form* (2.3) *with a single real root λ_1 and a double real root λ_2, let the elements of the matrix B satisfy* (2.15)–(2.19)*. Then, the solution of the initial problem* (1.1), (3.1) *is given by the formula*

$$x(k) = Q\mathcal{T}_{2j}\mathcal{G}_{2j}^k w(0), \quad k = 1, 2, 3, \ldots, \tag{3.14}$$

where $j = 1$ if the geometric multiplicity of the zero eigenvalue of B equals 1 *and $j = 2$ if the geometric multiplicity of the zero eigenvalue of B equals* 2 *and $w(0)$ is given by* (3.9)*.*
If $j = 1$ and $k \geq 3m$ then, by (3.14),

$$x(k) = Q\mathcal{T}_{21}\mathcal{G}_{21}^k w(0) = Q\mathcal{T}_{21}\begin{pmatrix} \lambda_1^k w_1(0) \\ \lambda_2^k w_2(0) + k\lambda_2^{k-1} w_3(0) \\ \lambda_2^k w_3(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3.15}$$

If $j = 2$ and $k \geq 3m - 1$ then, by (3.14),

$$x(k) = Q\mathcal{T}_{22}\mathcal{G}_{22}^k w(0) = Q\mathcal{T}_{22}\begin{pmatrix} \lambda_1^k w_1(0) \\ \lambda_2^k w_2(0) + k\lambda_2^{k-1} w_3(0) \\ \lambda_2^k w_3(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3.16}$$

Theorem 3.3.3. *Let the matrix A have the form* (2.4) *with a single real root λ_1 and a double real root λ_2, let the elements of the matrix B satisfy* (2.21)–(2.26)*. Then, the solution of the initial problem* (1.1), (3.1) *is given by the formula*

$$x(k) = Q\mathcal{T}_{3j}\mathcal{G}_{3j}^k w(0), \quad k = 1, 2, 3, \ldots, \tag{3.17}$$

where $j = 1$ if the geometric multiplicity of the zero eigenvalue of B equals 1 and $j = 2$ if the geometric multiplicity of the zero eigenvalue of B equals 2 and $w(0)$ is given by (3.9).
If $j = 1$ and $k \geq 3m$ then, by (3.17),

$$x(k) = Q\mathcal{T}_{31}\mathcal{G}_{31}^k w(0) = Q\mathcal{T}_{31} \begin{pmatrix} \lambda_1^k w_1(0) \\ \lambda_2^k w_2(0) + k\lambda_2^{k-1} w_3(0) \\ \lambda_2^k w_3(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3.18}$$

If $j = 2$ and $k \geq 3m - 1$ then, by (3.17),

$$x(k) = Q\mathcal{T}_{32}\mathcal{G}_{32}^k w(0) = Q\mathcal{T}_{32} \begin{pmatrix} \lambda_1^k w_1(0) \\ \lambda_2^k w_2(0) + k\lambda_2^{k-1} w_3(0) \\ \lambda_2^k w_3(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3.19}$$

Theorem 3.3.4. *Let the matrix A have the form (2.5) with one triple real root $\lambda = \lambda_{1,2,3}$, let the elements of the matrix B satisfy (2.28)–(2.30). Then, the solution of the initial problem (1.1), (3.1) is given by the formula*

$$x(k) = Q\mathcal{T}_{4j}\mathcal{G}_{4j}^k w(0), \quad k = 1, 2, 3, \ldots, \tag{3.20}$$

where $j = 1$ if the geometric multiplicity of the zero eigenvalue of B equals 1 and $j = 2$ if the geometric multiplicity of the zero eigenvalue of B equals 2 and $w(0)$ is given by (3.9).
If $j = 1$ and $k \geq 3m$ then, by (3.20),

$$x(k) = Q\mathcal{T}_{41}\mathcal{G}_{41}^k w(0) = Q\mathcal{T}_{41} \begin{pmatrix} \lambda^k w_1(0) + k\lambda^{k-1} w_2(0) + \frac{k(k-1)}{2}\lambda^{k-2} w_3(0) \\ \lambda^k w_2(0) + k\lambda^{k-1} w_3(0) \\ \lambda^k w_3(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3.21}$$

If $j = 2$ and $k \geq 3m - 1$ then, by (3.20),

$$x(k) = Q\mathcal{T}_{42}\mathcal{G}_{42}^k w(0) = Q\mathcal{T}_{42} \begin{pmatrix} \lambda^k w_1(0) \\ \lambda^k w_2(0) + k\lambda^{k-1} w_3(0) \\ \lambda^k w_3(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3.22}$$

Theorem 3.3.5. *Let the matrix A have the form (2.6) with one triple real root $\lambda = \lambda_{1,2,3}$, let the elements of the matrix B satisfy (2.32)–(2.36). Then, the solution of the initial problem (1.1), (3.1) is given by the formula*

$$x(k) = Q\mathcal{T}_{5j}\mathcal{G}_{5j}^k w(0), \quad k = 1, 2, 3, \ldots, \tag{3.23}$$

where $j = 1$ if the geometric multiplicity of the zero eigenvalue of B equals 1 and $j = 2$ if the geometric multiplicity of the zero eigenvalue of B equals 2 and $w(0)$ is given by (3.9).
If $j = 1$ and $k \geq 3m$ then, by (3.23),

$$x(k) = Q\mathcal{T}_{51}\mathcal{G}_{51}^k w(0) = Q\mathcal{T}_{51}\begin{pmatrix} \lambda^k w_1(0) + k\lambda^{k-1}w_2(0) + \frac{k(k-1)}{2}\lambda^{k-2}w_3(0) \\ \lambda^k w_2(0) + k\lambda^{k-1}w_3(0) \\ \lambda^k w_3(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3.24}$$

If $j = 2$ and $k \geq 3m - 1$ then, by (3.23),

$$x(k) = Q\mathcal{T}_{52}\mathcal{G}_{52}^k w(0) = Q\mathcal{T}_{52}\begin{pmatrix} \lambda^k w_1(0) + k\lambda^{k-1}w_2(0) + \frac{k(k-1)}{2}\lambda^{k-2}w_3(0) \\ \lambda^k w_2(0) + k\lambda^{k-1}w_3(0) \\ \lambda^k w_3(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3.25}$$

Theorem 3.3.6. *Let the matrix A have the form (2.7) with one triple real root $\lambda = \lambda_{1,2,3}$, let the elements of the matrix B satisfy (2.38)–(2.43). Then, the solution of the initial problem (1.1), (3.1) is given by the formula*

$$x(k) = Q\mathcal{T}_{6j}\mathcal{G}_{6j}^k w(0), \quad k = 1, 2, 3, \ldots, \tag{3.26}$$

where $j = 1$ if the geometric multiplicity of the zero eigenvalue of B equals 1 and $j = 2$ if the geometric multiplicity of the zero eigenvalue of B equals 2 and $w(0)$ is given by (3.9).
If $j = 1$ and $k \geq 3m$ then, by (3.26),

$$x(k) = Q\mathcal{T}_{61}\mathcal{G}_{61}^k w(0) = Q\mathcal{T}_{61}\begin{pmatrix} \lambda^k w_1(0) + k\lambda^{k-1}w_2(0) + \frac{k(k-1)}{2}\lambda^{k-2}w_3(0) \\ \lambda^k w_2(0) + k\lambda^{k-1}w_3(0) \\ \lambda^k w_3(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3.27}$$

If $j = 2$ and $k \geq 3m - 1$ then, by (3.26),

$$x(k) = Q\mathcal{T}_{62}\mathcal{G}_{62}^k w(0) = Q\mathcal{T}_{62}\begin{pmatrix} \lambda^k w_1(0) + k\lambda^{k-1}w_2(0) + \frac{k(k-1)}{2}\lambda^{k-2}w_3(0) \\ \lambda^k w_2(0) + k\lambda^{k-1}w_3(0) \\ \lambda^k w_3(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3.28}$$

Theorem 3.3.7. *Let the matrix A have the form (2.8) with one real eigenvalue $\lambda_1 = \lambda$ and two complex conjugate eigenvalues $\lambda_{2,3} = p \pm iq$, let the elements of the matrix B satisfy (2.45)–(2.50). Then, the solution of the initial problem (1.1), (3.1) is given by the formula*

$$x(k) = Q\mathcal{T}_{7j}\mathcal{G}_{7j}^k w(0), \quad k = 1, 2, 3, \ldots, \tag{3.29}$$

where $j = 1$ if the geometric multiplicity of the zero eigenvalue of B equals 1 and $j = 2$ if the geometric multiplicity of the zero eigenvalue of B equals 2 and $w(0)$ is given by (3.9).
If $j = 1$ and $k \geq 3m$ then, by (3.29),

$$x(k) = Q\mathcal{T}_{71}\mathcal{G}_{71}^k w(0) = Q\mathcal{T}_{71}\begin{pmatrix} \lambda^k w_1(0) \\ (r^k\cos k\varphi)w_2(0) + (r^k\sin k\varphi)w_3(0) \\ (-r^k\cos k\varphi)w_2(0) + (r^k\sin k\varphi)w_3(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3.30}$$

If $j = 2$ and $k \geq 3m - 1$ then, by (3.29),

$$x(k) = Q\mathcal{T}_{72}\mathcal{G}_{72}^k w(0) = Q\mathcal{T}_{72}\begin{pmatrix} \lambda^k w_1(0) \\ (r^k\cos k\varphi)w_2(0) + (r^k\sin k\varphi)w_3(0) \\ (-r^k\cos k\varphi)w_2(0) + (r^k\sin k\varphi)w_3(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3.31}$$

3.4 INDEPENDENT INITIAL VALUES. EXAMPLES

From Theorems 3.3.1–3.3.7 we deduce the following

Theorem 3.4.1. *If $k \geq 3m$ and the geometric multiplicity of the zero root of the matrix B is 1, then, as it follows from formulas (3.12), (3.15), (3.18), (3.21), (3.24), (3.27), (3.30), solution $x(k)$ depends only on three independent initial values, that is, among the $3(m + 1)$ initial values, there are only 3 independent.*
If $k \geq 3m - 1$ and the geometric multipicity of the zero root of the matrix B is 2, then, as it follows from formulas (3.13), (3.16), (3.19), (3.22), (3.25), (3.28), (3.31), solution $x(k)$ depends only on three independent initial values, that is, among the $3(m + 1)$ initial values, there are only 3 independent.

4 CONCLUSIONS

To our best knowledge, weakly delayed systems were first defined in [39] for systems of linear delayed differential systems with constant coefficients and, in [21], for planar linear discrete systems with a single delay (in these papers such systems are called systems with a weak delay). Then, for the planar systems with multiple delays, weakly delayed systems, were considered in [10] and [11]. The weakly delayed systems are simplified and their solutions can be found in explicit analytical forms. Analytical forms of solutions can be used to solve several problems for weakly delayed system, for example problems of the asymptotical behavior of their solutions or conditional stability problems (we refer to [14]).

In the thesis, linear weakly delayed systems in $\mathbb{R}^3$ are considered. Rather than a direct approach to solving such a system used in [10], [11], [21], the method of transformation of a given system into a system of $3(m + 1)$ equations without delay has been used. The use of a method adapted from [10], [11], [21] is impossible since it is not possible to predict the “ad-hoc” analytical form of the solutions as in the above paper paper. A new method leads to general results and analytical formulas for solutions have been derived for every possible Jordan form of matrix A in system (1.1).

Moreover, an analysis of the dependence of solutions on the initial values yielded a general result as well – after several steps the solutions only depend on 3 initial values that are suitable linear combinations of the $3(m + 1)$ values that formulate the original problem. That is, after several steps, the behavior of the general solution of system (1.1) is the same as that of general solutions of (1.1) without delayed terms, that is the system (1.4).

This remark, applied to systems considered in [10], [11], [21] will make it possible to reconsider and improve the results given there related to statements on the number of independent initial values. Of course, the general solutions derived in the thesis are well-suited to solving some problems mentioned above, as, for example, the problem about the asymptotic behavior of solutions or conditional stability problems.

As a topic for future research, an investigation of the case of A having a zero eigenvalue, can be suggested. The new approach probably will be applicable to general weakly delayed systems in $\mathbb{R}^n$.

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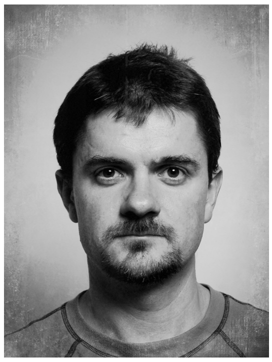
CURRICULUM VITÆ



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ABSTRACT

The present thesis deals with construction of general solution of weakly delayed systems of linear discrete equations in $\mathbb{R}^3$ of the form

$$x(k+1) = Ax(k) + Bx(k-m)$$

where $m > 0$ is a positive integer, $x\colon \mathbb{Z}_{-m}^{\infty} \to \mathbb{R}^3$, $\mathbb{Z}_{-m}^{\infty} := \{-m, -m+1, \dots, \infty\}$, $k \in \mathbb{Z}_0^{\infty}$, $A = (a_{ij})$ and $B = (b_{ij})$ are constant 3×3 matrices. The characteristic equations of weakly delayed systems are identical with those of the same systems but without delayed terms. Criteria ensuring that given system is weakly delayed are developed and then are specified for every possible case of Jordan form od matrix A. System is solved by a method transforming it to a higher-dimensional system but without delays

$$y(k+1) = \mathcal{A}y(k),$$

where $\dim y = 3(m+1)$. Using methods of linear algebra it is possible to found Jordan forms of $\mathcal{A}$ depending on eigenvalues of matrices A and B. Therefore, general solution of new system can be found and, consequently, general solution of initial system is deduced.

ABSTRAKT

Dizertační práce se zabývá konstrukcí obecného řešení slabě zpožděných systémů lineárních diskrétních rovnic v $\mathbb{R}^3$ tvaru

$$x(k+1) = Ax(k) + Bx(k-m),$$

kde $m > 0$ je kladné celé číslo, $x\colon \mathbb{Z}_{-m}^{\infty} \to \mathbb{R}^3$, $\mathbb{Z}_{-m}^{\infty} := \{-m, -m+1, \dots, \infty\}$, $k \in \mathbb{Z}_0^{\infty}$, $A = (a_{ij})$ a $B = (b_{ij})$ jsou konstantní 3×3 matice. Charakteristické rovnice těchto systémů jsou identické s charakteristickými rovnicemi systému, který neobsahuje zpožděné členy. Jsou získána kriteria garantující, že daný systém je slabě zpožděný a následně jsou tato kritéria specifikována pro všechny možné případy Jordanova tvaru matice A. Systém je vyřešen pomocí metody, která ho transformuje na systém vyšší dimenze, ale bez zpoždění

$$y(k+1) = \mathcal{A}y(k),$$

kde $\dim y = 3(m+1)$. Pomocí metod lineární algebry je možné najít Jordanovy formy matice $\mathcal{A}$ v závislosti na vlastních číslech matic A and B. Tudíž lze nalézt obecné řešení nového systému a v důsledku toho pak odvodit obecné řešení počátečního systému.