Invariance of the Null Distribution of the Multiple Coherence

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Abstract. *In this paper we investigate the invariance of the null distribution of the multiple coherence (MC) to the statistics of the examined signals. We show that when the MC is computed between a group of signals $x_i[n]$, $i = 1,\dots,K$ and a signal $y[n]$, the null distribution of the MC is independent of the distribution of $x_i[n]$ and $y[n]$ if at a given frequency the joint distribution of the spectra of the segments of $x_i[n]$ and $y[n]$ is rotationally symmetric with respect to the rotation of the spectra of the segments of $x_i[n]$ or $y[n]$.*

The significance of this result lies in the improvement of the multiple coherence analysis. Hitherto, the null distribution of the MC was known only for signals with the multivariate Gaussian distribution; therefore, an MC estimate could be evaluated for its statistical significance only in this limited case. With the results presented in this paper, it will be possible to evaluate the statistical significance of MC estimates for much wider class of signals.

Keywords

Multiple coherence, null distribution, invariance.

1. Introduction

The multiple coherence (MC) is a measure that indicates a relationship between a single signal $y[n]$ and a group of signals $x_i[n]$ $(i = 1 \dots K)$. It has found numerous applications for example in signal detection (e.g. [19], [20]), blood flow analysis (e.g. [13], [14]), brain signal processing (e.g. [3], [8], [9], [11], [17], [18]), data acquisition (e.g. [15], [16]), geology (e.g. [10]) and many other fields.

A crucial part of the multiple coherence analysis is the evaluation of the MC estimate for its statistical significance. A statistical test is employed to detect if the value of the MC estimate is high enough to indicate that the coupling between the examined signals is statistically significant and not just a random occurrence.

To perform such statistical testing the null distribution of the MC estimate needs to be known. In the past, the null distribution was derived under the condition that the examined signals are multivariate Gaussian [2], [5], [6], [7], [19], [20]. However, the real world signals are often non-Gaussian; therefore, it would be desirable to know the null distribution of the multiple coherence for a more general class of signals.

A partial achievement was already attained in the special case, where $K = 1$. In this case, the MC reflects the connection between two signals, and becomes equivalent to the magnitude squared coherence (MSC). The null distribution of an MSC estimate was shown to have a distribution invariant to the distribution of the second signal if the signals are statistically independent and the first signal is stationary Gaussian [12]. Later, the assumption of Gaussianity was relaxed, requiring only the spectra of the segments of one of the signals to have a distribution that is spherically symmetric at a given frequency [4].

To our knowledge it was never analytically examined, if such generalization applies to MC estimates for $K > 1$. However, such generalization would widen the applicability of the known statistical tests, and be beneficial in cases, where the examined signals are non-Gaussian, or their statistical distribution is unknown.

In this paper we therefore analyze the null distribution of the MC. We show that this distribution is in fact invariant to the distribution of the signals $x_i[n]$ and $y[n]$ if at a given frequency the joint distribution of the spectra of the segments of $x_i[n]$ and $y[n]$ is rotationally symmetric with respect to the rotation of the spectra of segments of $x_i[n]$ *or* $y[n]$.

2. Methods

2.1 MC Definition

If $x_i[n]$ and $y[n]$ are zero mean, wide sense stationary, we can define their power and cross-spectral densities as

$$\mathbf{S}_{XX}(\Omega) = \mathcal{F}\{\mathbf{R}_{xx}[k]\}, \tag{1}$$

$$\mathbf{S}_{XY}(\Omega) = \mathcal{F}\{\mathbf{R}_{xy}[k]\}, \tag{2}$$

$$S_{YY}(\Omega) = \mathcal{F}\{R_{yy}[k]\} \tag{3}$$

where $\mathcal{F}$ denotes the discrete time Fourier transform, and $\mathbf{R}_{xx}[k]$, $\mathbf{R}_{xy}[k]$, $R_{yy}[k]$ are the covariance matrices

$$\mathbf{R}_{xx}[k] = E[\mathbf{x}^*[n]\mathbf{x}^T[n+k]], \tag{4}$$
$$\mathbf{R}_{xy}[k] = E[\mathbf{x}^*[n]y[n+k]], \tag{5}$$
$$R_{yy}[k] = E[y^*[n]y[n+k]] \tag{6}$$

where $\mathbf{x}[n] = [x_1[n], \ldots, x_K[n]]^T$, and * denotes the complex conjugate.

The multiple coherence between $x_i[n]$ and $y[n]$ is given as [2]

$$|\gamma_{xy}(\Omega)|^2 = \frac{\mathbf{S}_{XY}^H(\Omega)\mathbf{S}_{XX}^{-1}(\Omega)\mathbf{S}_{XY}(\Omega)}{S_{YY}(\Omega)} \tag{7}$$

where ^{H} denotes the conjugate transpose. In the following text we shall assume that $|\gamma_{xy}(\Omega)|^2$ exists for at least some Ω.

2.2 MC Estimation

If $x_i[n]$ and $y[n]$ are finite length records of a wide sense stationary process, the MC estimate can be computed in the following way [2].

First, the signals are segmented

$$x_{li}[n] = x_i[(l-1)M+n], \tag{8}$$
$$y_l[n] = y[(l-1)M+n] \tag{9}$$

where M denotes the segment length, and $l = 1, \ldots, L$, where L is the number of segments. Next, let $X_{li}(\Omega)$ and $Y_l(\Omega)$ be the discrete Fourier transforms of $x_{li}[n]$ and $y_l[n]$, respectively (each Fourier spectrum has M frequency points). Also, let us define

$$\mathbf{X}(\Omega) = [X_{li}(\Omega)]_{\substack{l=1\ldots L\\ i=1\ldots K}}, \tag{10}$$
$$\mathbf{Y}(\Omega) = [Y_1(\Omega), \ldots, Y_L(\Omega)]^T. \tag{11}$$

Now, we can compute the estimates of the cross and power spectral densities

$$\widehat{\mathbf{S}}_{XX}(\Omega) = \frac{1}{L}\mathbf{X}^H(\Omega)\mathbf{X}(\Omega), \tag{12}$$
$$\widehat{\mathbf{S}}_{XY}(\Omega) = \frac{1}{L}\mathbf{X}^H(\Omega)\mathbf{Y}(\Omega), \tag{13}$$
$$\widehat{S}_{YY}(\Omega) = \frac{1}{L}\mathbf{Y}^H(\Omega)\mathbf{Y}(\Omega), \tag{14}$$

with which the MC estimate $|\widehat{\gamma}_{xy}(\Omega)|^2$ will be given as

$$|\widehat{\gamma}_{xy}(\Omega)|^2 = \frac{\widehat{\mathbf{S}}_{XY}^H(\Omega)\widehat{\mathbf{S}}_{XX}^{-1}(\Omega)\widehat{\mathbf{S}}_{XY}(\Omega)}{\widehat{S}_{YY}(\Omega)} \tag{15}$$
$$= \frac{\mathbf{Y}^H(\Omega)\mathbf{X}(\Omega)(\mathbf{X}(\Omega)^H\mathbf{X}(\Omega))^{-1}\mathbf{X}(\Omega)^H\mathbf{Y}(\Omega)}{\mathbf{Y}^H(\Omega)\mathbf{Y}(\Omega)}. \tag{16}$$

2.3 Null Distribution of MC Estimate

The null distribution of the MC estimate is defined as

$$G_0(g) = P\left[|\widehat{\gamma}_{xy}(\Omega)|^2 < g \,\middle|\, |\gamma_{xy}(\Omega)|^2 = 0\right] \tag{17}$$

where $G_0(g)$ is the cumulative distribution function (CDF) of the null distribution of the MC estimate, and $P[.|.]$ denotes the conditional probability operator.

In this section we will analyze the invariance of (17). For this purpose we will assume $x_i[n]$ and $y[n]$ to be finite length records of a strict sense stationary random process. Further, to simplify the derivation we will drop the argument (Ω), and assume it implicitly.

Also, we will use the following notation. Let $f_{\mathbf{XY}}(\mathbf{X},\mathbf{Y})$ be the joint probability density function of $\mathbf{X}$ and $\mathbf{Y}$, let $f_{\mathbf{X}}(\mathbf{X})$ be the probability density function of $\mathbf{X}$, and let $f_{\mathbf{Y}}(\mathbf{Y})$ be the probability density function of $\mathbf{Y}$. In the following subsections we will analyze (17) assuming rotational symmetry of $f_{\mathbf{XY}}(\mathbf{X},\mathbf{Y})$ with respect to the rotation of $\mathbf{X}$ or $\mathbf{Y}$.

2.3.1 Rotational Symmetry of $f_{\mathbf{XY}}(\mathbf{X},\mathbf{Y})$ with Respect to Rotation of Y

In this section we will assume that $f_{\mathbf{XY}}(\mathbf{X},\mathbf{Y})$ is invariant to the rotation of the argument $\mathbf{Y}$ – that is we will assume that $f_{\mathbf{XY}}(\mathbf{X},\mathbf{Y}) = f_{\mathbf{XY}}(\mathbf{X},\mathbf{BY})$ if $\mathbf{B}$ is a rotation matrix (i.e. $\det(\mathbf{B}) = 1$ and $\mathbf{B}^H = \mathbf{B}^{-1}$). This means that $f_{\mathbf{XY}}(\mathbf{X},\mathbf{Y})$ and $f_{\mathbf{Y}}(\mathbf{Y})$ can also be expressed as $f_{\mathbf{XY}}(\mathbf{X},\mathbf{Y}) = f'_{\mathbf{XY}}(\mathbf{X},|\mathbf{Y}|)$ and $f_{\mathbf{Y}}(\mathbf{Y}) = f'_{\mathbf{Y}}(|\mathbf{Y}|)$, where $f'_{\mathbf{XY}}$ and $f'_{\mathbf{Y}}$ are new functions, and $|.|$ denotes the L_2 norm.

With these assumptions the MC estimate CDF (17) can be expressed as

$$\int_{\mathcal{C}} f'_{\mathbf{XY}}(\mathbf{X},|\mathbf{Y}|)d\mathbf{X}d\mathbf{Y} \tag{18}$$

where $\mathcal{C}$ is a set of $\mathbf{X}$ and $\mathbf{Y}$ for which the following holds

$$\mathbf{X} \in \mathbb{C}^{LK}, \tag{19}$$
$$\mathbf{Y} \in \mathbb{C}^{L}, \tag{20}$$
$$|\widehat{\gamma}_{xy}|^2 = \frac{\mathbf{Y}^H\mathbf{X}(\mathbf{X}^H\mathbf{X})^{-1}\mathbf{X}^H\mathbf{Y}}{\mathbf{Y}^H\mathbf{Y}} < g \tag{21}$$

where $\mathbb{C}$ denotes the set of the complex numbers. Integral (18) can be rearranged

$$\int_{\mathbb{C}^{LK}} \left(\int_{\mathcal{C}_1} f'_{\mathbf{XY}}(\mathbf{X},|\mathbf{Y}|)d\mathbf{Y} \right) d\mathbf{X} \tag{22}$$

where $\mathcal{C}_1$ denotes a set of $\mathbf{Y}$ for which conditions (20) and (21) hold.

Now, let $\mathbf{X} = \mathbf{R_X}\mathbf{S_X}\boldsymbol{T}_{\mathbf{X}}^H$ be the singular value decomposition of $\mathbf{X}$, where $\mathbf{R_X}$ is a LxL orthonormal matrix, $\mathbf{S_X}$ is a LxK diagonal matrix, and $\boldsymbol{T}_{\mathbf{X}}$ is a KxK orthonormal matrix. We can make a substitution $\mathbf{Y} = \mathbf{R_X}\mathbf{a}$ (where $\mathbf{a} =$

$[a_1,\dots,a_L]^T$). The numerator in (21) will then become

$$\mathbf{Y}^H\mathbf{X}(\mathbf{X}^H\mathbf{X})^{-1}\mathbf{X}^H\mathbf{Y} \tag{23}$$
$$= \mathbf{a}^H\mathbf{R}_\mathbf{X}^H\mathbf{R}_\mathbf{X}\mathbf{S}_\mathbf{X}\boldsymbol{T}_\mathbf{X}^H(\boldsymbol{T}_\mathbf{X}\mathbf{S}_\mathbf{X}^H\mathbf{R}_\mathbf{X}^H\mathbf{R}_\mathbf{X}\mathbf{S}_\mathbf{X}\boldsymbol{T}_\mathbf{X}^H)^{-1}\boldsymbol{T}_\mathbf{X}\mathbf{S}_\mathbf{X}^H\mathbf{R}_\mathbf{X}^H\mathbf{R}_\mathbf{X}\mathbf{a}$$
$$= \mathbf{a}^H\mathbf{S}_\mathbf{X}\boldsymbol{T}_\mathbf{X}^H\boldsymbol{T}_\mathbf{X}^{-H}(\mathbf{S}_\mathbf{X}^H\mathbf{S}_\mathbf{X})^{-1}\boldsymbol{T}_\mathbf{X}^{-1}\boldsymbol{T}_\mathbf{X}\mathbf{S}_\mathbf{X}^H\mathbf{a} \tag{24}$$
$$= \mathbf{a}^H\mathbf{S}_\mathbf{X}(\mathbf{S}_\mathbf{X}^H\mathbf{S}_\mathbf{X})^{-1}\mathbf{S}_\mathbf{X}^H\mathbf{a} = \mathbf{a}^H\boldsymbol{\Gamma}\mathbf{a} = \sum_{l=1}^{K}|a_l|^2 \tag{25}$$

where the expression $\mathbf{S}_\mathbf{X}(\mathbf{S}_\mathbf{X}^H\mathbf{S}_\mathbf{X})^{-1}\mathbf{S}_\mathbf{X}^H$ was denoted as $\boldsymbol{\Gamma}$, and for $\mathbf{X}$ full rank it is a diagonal matrix

$$\boldsymbol{\Gamma} = \mathrm{diag}\{[\underbrace{1,\dots,1}_{K},0,\dots,0]\}\,. \tag{26}$$

If $\mathbf{X}$ is rank deficient, $\boldsymbol{\Gamma}$ is different (some of the trailing ones get replaced by zeros), but the set of the rank deficient matrices $\mathbf{X}$ has zero measure in $\mathbb{C}^{KL}$, does not affect the integral (22), and so needs not be considered.

The denominator in (21) will be

$$\mathbf{Y}^H\mathbf{Y} = \mathbf{a}^H\mathbf{R}_\mathbf{X}^H\mathbf{R}_\mathbf{X}\mathbf{a} = \mathbf{a}^H\mathbf{a} = \sum_{l=1}^{L}|a_l|^2\,. \tag{27}$$

With this notation the entire condition (21) will be transformed into

$$\frac{\sum_{l=1}^{K}|a_l|^2}{\sum_{l=1}^{L}|a_l|^2} < g\,. \tag{28}$$

Consequently, (22) will become

$$\int_{\mathbb{C}^{LK}}\left(\int_{\mathcal{D}_1} f'_{\mathbf{XY}}(\mathbf{X},|\mathbf{a}|)d\mathbf{a}\right)d\mathbf{X} \tag{29}$$

where $\mathcal{D}_1$ is a set of $\mathbf{a}$ for which (28) holds. Because $\mathcal{D}_1$ is independent of $\mathbf{X}$, we can now change the order of the integration, and we will get

$$\int_{\mathcal{D}_1}\left(\int_{\mathbb{C}^{LK}} f'_{\mathbf{XY}}(\mathbf{X},|\mathbf{a}|)d\mathbf{X}\right)d\mathbf{a} = \int_{\mathcal{D}_1} f'_{\mathbf{Y}}(|\mathbf{a}|)d\mathbf{a}\,. \tag{30}$$

So at this point we can see that the MC estimate CDF does not depend on the distribution of $\mathbf{X}$, which means that it does not depend on the distribution of $x_i[n]$.

Now, we will show that (30) does not depend even on $f'_{\mathbf{Y}}(.)$. For this purpose we will denote

$$a_l = z_{2l-1} + jz_{2l}, \quad z_l \in \mathbb{R}, \tag{31}$$
$$\mathbf{a} = [z_1 + jz_2, z_3 + jz_4, \dots, z_{2L-1} + jz_{2L}]^T\,, \tag{32}$$
$$\mathbf{z} = [z_1,\dots,z_{2L}]^T \tag{33}$$

where $\mathbb{R}$ denotes the set of real numbers, and j is the imaginary unit. Next, we will substitute $\mathbf{z}$ with the spherical coordinates [1]

$$z_l = r\left(\prod_{k=1}^{l-1}\sin\alpha_k\right)\cos\alpha_l,\; l = 1,\dots,2L-1\,, \tag{34}$$

$$z_{2L} = r\prod_{k=1}^{2L-1}\sin\alpha_k\,. \tag{35}$$

This substitution can be expressed in a simpler form as

$$z_l = rc_l(\boldsymbol{\alpha}),\ \text{where}\ \boldsymbol{\alpha} = [\alpha_1,\dots,\alpha_{2L-1}]^T \tag{36}$$

where $c_l(.)$ is just a shorthand notation for the products of the trigonometric functions in (34) and (35). Also, note that we have $|\mathbf{a}| = |\mathbf{z}| = r$. Assuming that r is greater than zero, the inequality in (28) will become

$$\frac{\sum_{l=1}^{K}|a_l|^2}{\sum_{l=1}^{L}|a_l|^2} = \frac{\sum_{l=1}^{2K}z_l^2}{|\mathbf{a}|^2} = \frac{\sum_{l=1}^{2K}r^2c_l^2(\boldsymbol{\alpha})}{r^2} = \sum_{l=1}^{2K}c_l^2(\boldsymbol{\alpha}) < g\,, \tag{37}$$

which does not depend on r, and therefore in our integration r will not be bound by any condition (it will range from 0 to ∞).

The Jacobian of substitution (34), (35) is [1]

$$J = r^{2L-1}\prod_{k=1}^{2L-2}\sin^{2K-1-k}\alpha_k\,, \tag{38}$$

which we abbreviate by

$$J = h(\boldsymbol{\alpha})r^{2L-1}\,. \tag{39}$$

Integral (30) thus becomes

$$\int_{\mathcal{G}}\left(\int_0^\infty f'_{\mathbf{Y}}(r)h(\boldsymbol{\alpha})r^{2L-1}dr\right)d\boldsymbol{\alpha} \tag{40}$$

where $\mathcal{G}$ is a properly constructed set of values of $\boldsymbol{\alpha}$, which we do not need to express here. This expression can be rearranged into

$$\int_{\mathcal{G}} h(\boldsymbol{\alpha})d\boldsymbol{\alpha}\int_0^\infty f'_{\mathbf{Y}}(r)r^{2L-1}dr\,. \tag{41}$$

This formula expresses $G_0(g)$ in a form that allows to show its invariance on $f'_{\mathbf{Y}}(.)$. Note that the second integral does not depend on g, and will evaluate into a scaling constant (which for any $f'_{\mathbf{Y}}(.)$ must be such that $G_0(g) = 1$ for $g > 1$, because MC estimate lies within interval $\langle 0, 1\rangle$, and $G_0(g)$ is a CDF). The shape of $G_0(g)$ is therefore given solely by the first integral, which does not depend on $f'_{\mathbf{Y}}(.)$. Consequently, (41) is invariant to $f'_{\mathbf{Y}}(.)$, which means that $G_0(g)$ does not depend on the distribution of $y[n]$.

2.3.2 Rotational Symmetry of $f_{\mathbf{XY}}(\mathbf{X},\mathbf{Y})$ with Respect to Rotation of X

In this subsection we will assume that $f_{\mathbf{XY}}(\mathbf{X},\mathbf{Y}) = f_{\mathbf{XY}}(\mathbf{BX},\mathbf{Y})$ if $\mathbf{B}$ is a rotation. This means $f_{\mathbf{XY}}(\mathbf{X},\mathbf{Y})$ and $f_{\mathbf{X}}(\mathbf{X})$ can also be expressed as $f_{\mathbf{XY}}(\mathbf{X},\mathbf{Y}) = f''_{\mathbf{XY}}(|\mathbf{X}_1|,\dots,|\mathbf{X}_K|,\mathbf{Y})$ and $f_{\mathbf{X}}(\mathbf{X}) = f'_{\mathbf{X}}(|\mathbf{X}_1|,\dots,|\mathbf{X}_K|)$, where $f''_{\mathbf{XY}}$ and $f'_{\mathbf{X}}$ are new functions, and $\mathbf{X}_i$ denotes the i-th column of $\mathbf{X}$.

With these assumptions MC estimate CDF (17) can be expressed as

$$G_0(g) = \int_{\mathcal{C}} f''_{\mathbf{XY}}(|\mathbf{X}_1|,\ldots,|\mathbf{X}_K|,\mathbf{Y})d\mathbf{X}d\mathbf{Y}. \tag{42}$$

This integral can be rearranged into

$$\int_{\mathbb{C}^L}\left(\int_{\mathcal{C}_2} f''_{\mathbf{XY}}(|\mathbf{X}_1|,\ldots,|\mathbf{X}_K|,\mathbf{Y})d\mathbf{X}\right)d\mathbf{Y} \tag{43}$$

where $\mathcal{C}_2$ denotes the set of $\mathbf{X}$, for which conditions (19) and (21) hold.

Now, we will use steps somewhat similar to those used in the previous section.

Let $\mathbf{Y} = \mathbf{Q_Y R_Y}$ be the QR decomposition of $\mathbf{Y}$, where $\mathbf{Q_Y}$ is a LxL orthonormal matrix, and $\mathbf{R_Y} = [\sigma,0,\ldots,0]^T$. We will make a substitution $\mathbf{X} = \mathbf{Q_Y A}$ (where $\mathbf{A} = [a_{li}]_{li}$ is a LxK matrix). The numerator in (21) will thus become

$$\begin{aligned}
&\mathbf{Y}^H\mathbf{X}(\mathbf{X}^H\mathbf{X})^{-1}\mathbf{X}^H\mathbf{Y} && (44)\\
&= \mathbf{R}_\mathbf{Y}^H\mathbf{Q}_\mathbf{Y}^H\mathbf{Q_Y A}(\mathbf{A}^H\mathbf{Q}_\mathbf{Y}^H\mathbf{Q_Y A})^{-1}\mathbf{A}^H\mathbf{Q}_\mathbf{Y}^H\mathbf{Q_Y R_Y} && (45)\\
&= \mathbf{R}_\mathbf{Y}^H\mathbf{A}(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H\mathbf{R_Y} && (46)\\
&= {}_1\mathbf{A}(\mathbf{A}^H\mathbf{A})^{-1}{}_1\mathbf{A}^H|\sigma|^2 && (47)
\end{aligned}$$

where ${}_1\mathbf{A}$ denotes the first row of $\mathbf{A}$. The denominator in (21) will be

$$\mathbf{Y}^H\mathbf{Y} = \mathbf{R}_\mathbf{Y}^H\mathbf{Q}_\mathbf{Y}^H\mathbf{Q_Y R_Y} = \mathbf{R}_\mathbf{Y}^H\mathbf{R_Y} = |\sigma|^2. \tag{48}$$

Thus, the entire condition (21) will transform into

$${}_1\mathbf{A}(\mathbf{A}^H\mathbf{A})^{-1}{}_1\mathbf{A}^H < g. \tag{49}$$

Consequently, (43) will become

$$\int_{\mathbb{C}^L}\left(\int_{\mathcal{D}_2} f''_{\mathbf{XY}}(|\mathbf{A}_1|,\ldots,|\mathbf{A}_K|,\mathbf{Y})d\mathbf{A}\right)d\mathbf{Y} \tag{50}$$

where $\mathbf{A}_i$ denotes the i-th column of $\mathbf{A}$ (note that $|\mathbf{X}_i| = |\mathbf{A}_i|$ due to the orthonormality of $\mathbf{Q_Y}$), and $\mathcal{D}_2$ is a set of $\mathbf{A}$ for which (49) holds.

Because $\mathcal{D}_2$ is independent of $\mathbf{Y}$, we can now change the order of the integration, and we will get

$$\int_{\mathcal{D}_2}\left(\int_{\mathbb{C}^L} f''_{\mathbf{XY}}(|\mathbf{A}_1|,\ldots,|\mathbf{A}_K|,\mathbf{Y})d\mathbf{Y}\right)d\mathbf{A} = \int_{\mathcal{D}_2} f'_{\mathbf{X}}(|\mathbf{A}_1|,\ldots,|\mathbf{A}_K|)d\mathbf{A}. \tag{51}$$

So at this point we can see that the MC estimate CDF does not depend on the distribution of $\mathbf{Y}$, which means it does not depend on the distribution of $y[n]$.

Now, we will show that (51) does not depend even on $f_\mathbf{X}(.)$. For this purpose we will denote

$$\begin{aligned}
a_{li} &= z_{2l-1,i} + jz_{2l,i}, \quad z_{li} \in \mathbb{R}, && (52)\\
\mathbf{A} &= [z_{2l-1,i} + jz_{2l,i}]_{\substack{l=1\ldots L\\ i=1\ldots K}}, && (53)\\
\mathbf{Z} &= [z_{li}]_{\substack{l=1\ldots 2L\\ i=1\ldots K}}. && (54)
\end{aligned}$$

Next, we will substitute $\mathbf{Z}$ with the spherical coordinates

$$z_{li} = r_i\left(\prod_{k=1}^{l-1}\sin\alpha_{ki}\right)\cos\alpha_{li}, \quad \begin{matrix} l = 1,\ldots,2L-1\\ i = 1,\ldots,K\end{matrix}, \tag{55}$$

$$z_{2L,i} = r_i\prod_{k=1}^{2L-1}\sin\alpha_{ki}, \qquad i = 1,\ldots,K. \tag{56}$$

This substitution can be expressed in a simpler form as

$$z_{li} = r_i c'_l(\boldsymbol{\alpha}_i), \text{ where } \boldsymbol{\alpha}_i = [\alpha_{1,i},\ldots,\alpha_{2L-1,i}]^T \tag{57}$$

where $c'_l(.)$ is just a shorthand notation for the product of the trigonometric functions in (55) and (56). (57) can also be written as

$$\begin{aligned}
&\mathbf{Z} = \mathbf{CR}, \text{ where} && (58)\\
&\mathbf{C} = [c'_l(\boldsymbol{\alpha}_i)]_{\substack{l=1\ldots 2L\\ i=1\ldots K}}, \quad \mathbf{R} = \mathrm{diag}\{[r_1,\ldots,r_K]\}. && (59)
\end{aligned}$$

Further, note that using LxL identity matrices $\mathbf{I}_L$, the matrix $\mathbf{A}$ can be expressed as

$$\mathbf{A} = [\mathbf{I}_L, j\mathbf{I}_L]\mathbf{Z} = [\mathbf{I}_L, j\mathbf{I}_L]\mathbf{CR} = \mathbf{C}_c\mathbf{R}, \tag{60}$$

where we denoted $\mathbf{C}_c = [\mathbf{I}_L, j\mathbf{I}_L]\mathbf{C}$. Thus, the condition in (49) will become

$$\begin{aligned}
&{}_1\mathbf{A}(\mathbf{A}^H\mathbf{A})^{-1}{}_1\mathbf{A}^H = {}_1\mathbf{C}_c\mathbf{R}(\mathbf{R}^H\mathbf{C}_c^H\mathbf{C}_c\mathbf{R})^{-1}\mathbf{R}^H{}_1\mathbf{C}_c^H && (61)\\
&= {}_1\mathbf{C}_c\mathbf{R}\mathbf{R}^{-1}(\mathbf{C}_c^H\mathbf{C}_c)^{-1}\mathbf{R}^{-H}\mathbf{R}^H{}_1\mathbf{C}_c^H && (62)\\
&= {}_1\mathbf{C}_c(\mathbf{C}_c^H\mathbf{C}_c)^{-1}{}_1\mathbf{C}_c^H < g && (63)
\end{aligned}$$

where ${}_1\mathbf{C}_c$ denotes the first row of $\mathbf{C}_c$, and r_i were assumed to be greater than zero. Note that (63) is independent of r_i, and therefore in our integration r_i will not be bound by any condition (it will range from 0 to ∞). Also, note that we have $|\mathbf{A}_i| = |\mathbf{Z}_i| = r_i$, for $i = 1,\ldots,K$ (where $\mathbf{Z}_i$ denotes the i-th column of $\mathbf{Z}$).

The Jacobian of substitution (55), (56) is [1]

$$\begin{aligned}
J &= \prod_{i=1}^{K}\left(r_i^{2L-1}\prod_{k=1}^{2L-2}\sin^{2L-1-k}\alpha_{ki}\right) && (64)\\
&= \left(\prod_{i=1}^{K} r_i^{2L-1}\right)\left(\prod_{i=1}^{K}\prod_{k=1}^{2L-2}\sin^{2L-1-k}\alpha_{ki}\right), && (65)
\end{aligned}$$

which we abbreviate by

$$J = h'(\boldsymbol{\alpha})\prod_{i=1}^{K} r_i^{2L-1} \tag{66}$$

where $h'(\boldsymbol{\alpha})$ is just a shorthand notation for the expression in the second bracket in (65). Integral (51) thus becomes

$$\int_{\mathcal{E}}\left(\int_0^{\infty}\cdots\int_0^{\infty} f'_{\mathbf{X}}(r_1,\ldots,r_K)h'(\boldsymbol{\alpha})\prod_{i=1}^{K} r_i^{2L-1}dr_1\ldots dr_K\right)d\boldsymbol{\alpha} \tag{67}$$

where $\mathcal{E}$ is a properly constructed set of values of $\boldsymbol{\alpha}$, which we do not need to express here. This integral can be rearranged into

$$\int_{\mathcal{E}} h'(\boldsymbol{\alpha})d\boldsymbol{\alpha}\int_0^{\infty}\cdots\int_0^{\infty} f'_{\mathbf{X}}(r_1,\ldots,r_K)\prod_{i=1}^{K} r_i^{2L-1}dr_1\ldots dr_K\,. \tag{68}$$

This formula expresses $G_0(g)$ in a form that allows to show its invariance on $f'_{\mathbf{X}}(.)$. Note that all the integrals except the first one do not depend on g, and will evaluate into a scaling constant, which for any $f'_{\mathbf{X}}(.)$ must be such that $G_0(g)=1$ for $g>1$ (because $G_0(g)$ is a CDF). The shape of $G_0(g)$ is therefore given solely by the first integral in (68), which does not depend on $f'_{\mathbf{X}}(.)$. Consequently, (68) is invariant to $f'_{\mathbf{X}}(.)$, which means that $G_0(g)$ does not depend on the distribution of $x_i[n]$.

2.3.3 Alternative Rotational Symmetry in Real and Imaginary Parts of X or Y

If we denote

$$X_{l,i}=U_{2l-1,i}+jU_{2l,i},\quad Y_l=V_{2l-1}+jV_{2l} \tag{69}$$

$$\mathbf{U}=[U_{li}]_{\substack{l=1\ldots 2L\\ i=1\ldots K}},\quad \mathbf{V}=[V_1,\ldots,V_{2L}]^T \tag{70}$$

we have

$$f_{\mathbf{XY}}(\mathbf{X},\mathbf{Y})=f_{\mathbf{UV}}(\mathbf{U},\mathbf{V}) \tag{71}$$

where $f_{\mathbf{UV}}(\mathbf{U},\mathbf{V})$ is the joint PDF of $\mathbf{U}$ and $\mathbf{V}$. We also have $|\mathbf{Y}|=|\mathbf{V}|$ and $|\mathbf{X}_l|=|\mathbf{U}_l|$ (where $\mathbf{U}_l$ denotes the l-th column of $\mathbf{U}$).

Now, if $f_{\mathbf{UV}}(\mathbf{U},\mathbf{V})=f_{\mathbf{UV}}(\mathbf{U},\mathbf{B}_r\mathbf{V})$, where $\mathbf{B}_r$ is a real rotation matrix, we can write

$$f_{\mathbf{XY}}(\mathbf{X},\mathbf{Y})=f_{\mathbf{UV}}(\mathbf{U},\mathbf{V})=f'_{\mathbf{UV}}(\mathbf{U},|\mathbf{V}|)=f'_{\mathbf{UV}}(\mathbf{U},|\mathbf{Y}|)\,. \tag{72}$$

Consequently, (keeping in mind that the L_2 norm is invariant to a rotation) if $f_{\mathbf{UV}}(\mathbf{U},\mathbf{V})$ is rotationally symmetric with respect to the rotation of $\mathbf{V}$, then $f_{\mathbf{XY}}(\mathbf{X},\mathbf{Y})$ is rotationally symmetric with respect to the rotation of $\mathbf{Y}$.

A similar argument applies to the rotational symmetry in $\mathbf{U}$. If $f_{\mathbf{UV}}(\mathbf{U},\mathbf{V})=f_{\mathbf{UV}}(\mathbf{B}_r\mathbf{U},\mathbf{V})$, we can write

$$\begin{aligned}f_{\mathbf{XY}}(\mathbf{X},\mathbf{Y})&=f_{\mathbf{UV}}(\mathbf{U},\mathbf{V})=f'_{\mathbf{UV}}(|\mathbf{U}_1|,\ldots,|\mathbf{U}_K|,\mathbf{V})\\&=f'_{\mathbf{UV}}(|\mathbf{X}_1|,\ldots,|\mathbf{X}_K|,\mathbf{Y}),\end{aligned} \tag{73}$$

which shows that $f_{\mathbf{XY}}(\mathbf{X},\mathbf{Y})$ is rotationally symmetric with respect to the rotation of $\mathbf{X}$.

Consequently, instead of the rotational symmetry of $f_{\mathbf{XY}}(\mathbf{X},\mathbf{Y})$ with respect to the rotation of $\mathbf{X}$ or $\mathbf{Y}$, we could require the rotational symmetry of $f_{\mathbf{UV}}(\mathbf{U},\mathbf{V})$ with respect to the rotation of $\mathbf{U}$ or $\mathbf{V}$.

2.3.4 Explicit Expression for the Null Distribution

The abovementioned rotational symmetries will be achieved at a given Ω if the real and imaginary parts of $\mathbf{X}(\Omega)$ and $\mathbf{Y}(\Omega)$ are zero mean independent Gaussian with equal variance.

Consequently, the explicit expression for the null distribution is equal to the one derived for this kind of Gaussian random variables (e.g. in [7])

$$G_0(g)=Beta(g,K,L) \tag{74}$$

where $Beta(.,K,L)$ denotes the CDF of the Beta distribution with parameters K and L.

2.3.5 Necessity of Null Assumption

The assumption that $|\gamma(\Omega)|^2=0$ did not actually enter the derivation in Sec. 2.3.2 and Sec. 2.3.1. However, our results are still limited to the null distribution of the MC estimate.

Note that if $x_i[n]$ and $y[n]$ are $M\cdot L$ samples long records of wide sense stationary signals, the number of signals K is fixed and the number of segments L tends to infinity, then CDF (74) limits to the Heaviside step function. This means that the MC estimate limits to zero. This can happen only if the cross spectral density estimates in (13) limits to zero. Because (13) is an unbiased estimator of the cross spectral density, this can happen only if the true cross spectral densities of $x_i[n]$ and $y[n]$ are zero, in which case the true MC is also zero. Therefore, the MC estimate can have distribution (74) only if the true MC of $x_i[n]$ and $y[n]$ is zero. Since we have showed that our assumption of the rotational symmetries provides the MC estimate with distribution (74), our assumption also implies that the true MC is zero. Therefore, our results are applicable only for the null distribution of the MC estimate.

3. Conclusions

3.1 Results and Corollaries

When the multiple coherence estimate is computed between a group of signals $x_i[n]$, $i=1,\ldots,K$ and a signal $y[n]$, these signals are segmented into L segments $x_{li}[n]$ and $y_l[n]$, and $X_{li}(\Omega)$, $Y_l(\Omega)$ are the discrete Fourier transforms of the individual segments, then the null distribution of the multiple coherence estimate at a given frequency Ω is invariant to the distribution of $x_i[n]$ and $y[n]$ if the joint distribution the spectra of the segments has either one of the following rotational symmetries:

i) $f_{\mathbf{XY}}(\mathbf{X},\mathbf{Y})=f_{\mathbf{XY}}(\mathbf{BX},\mathbf{Y})$
ii) $f_{\mathbf{XY}}(\mathbf{X},\mathbf{Y})=f_{\mathbf{XY}}(\mathbf{X},\mathbf{BY})$

where $\mathbf{B}$ is a rotation matrix ($\det(\mathbf{B})=1$ and $\mathbf{B}^H=\mathbf{B}^{-1}$), $\mathbf{X}=[X_{li}(\Omega)]_{\substack{l=1\ldots L\\ i=1\ldots K}}$, $\mathbf{Y}=[Y_1(\Omega),\ldots,Y_L(\Omega)]^T$, and

$f_{\mathbf{XY}}(\mathbf{X},\mathbf{Y})$ is the joint PDF of $\mathbf{X}$ and $\mathbf{Y}$.

A less general corollary of this finding is that the null distribution of the multiple coherence estimate will be independent of the distribution of $x_i[n]$ and $y[n]$ if $x_i[n]$ and $y[n]$ are independent for each $i = 1,\ldots,K$,[1] and if at least one of the following conditions hold

i) the distribution of $[X_{1i}(\Omega),\ldots,X_{Li}(\Omega)]^T$ is rotationally symmetric for each i and a given Ω,

ii) the distribution of $[Y_1(\Omega),\ldots,Y_L(\Omega)]^T$ is rotationally symmetric for a given Ω.

A simpler (but even less general) corollary is that the null distribution of the multiple coherence estimate will be independent of the distribution of $x_i[n]$ if $y[n]$ is Gaussian and independent of $x_i[n]$, *or* the null distribution of MC will be independent of the distribution of $y[n]$ if $x_i[n]$ are multivariate Gaussian and independent of $y[n]$.

3.2 Hypothesis Testing

Note that even though we have provided our derivation for a fairly general assumption of the rotational symmetries of $f_{\mathbf{XY}}(\mathbf{X},\mathbf{Y})$, we do not imply that these rotational symmetries should be used directly as a null hypothesis in the hypothesis testing. Assuming these rotational symmetries automatically implies the fact that the true MC is equal to zero (Sec. 2.3.5). For the hypothesis testing, we recommend to use one of the corollaries stated above. For example, the hypothesis testing can be performed in the following manner:

Prior knowledge: the distribution of $[X_{1i}(\Omega),\ldots,X_{Li}(\Omega)]^T$ is rotationally symmetric for each i, *or* the distribution of $[Y_1(\Omega),\ldots,Y_L(\Omega)]^T$ is rotationally symmetric.

Null hypothesis: $x_i[n]$ and $y[n]$ are independent for each $i = 1,\ldots,K$,

If the null hypothesis holds, than the MC estimate should have distribution (74). Now, if the MC estimated from the data exceeds a chosen quantile of this distribution the null hypothesis appears unlikely, and we can choose to reject it, and conclude that the signals are most likely dependent.

3.3 Final Remarks

The significance of this result lies in the improvement of the multiple coherence analysis. Hitherto, the applicability of the formula for the null distribution (74) was guaranteed only if all the examined signals were multivariate Gaussian. With the findings presented in this paper, it will be possible to apply these formulas to a much wider class of signals. This, we believe, will be fairly beneficial in numerous fields that use the multiple coherence for signal detection or examination of relationship between signals.

[1] In this case the MC estimate has the null distribution at all frequencies.

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References

[1] BLUMENSON, L. E. A derivation of *n*-dimensional spherical coordinates. *The Americal Mathematical Montly*, 1960, vol. 67, no. 1, p. 63 - 66.

[2] BRILLINGER, D. R. *Time Series: Data Analysis and Theory*. Society for Industrial and Applied Mathematics, 2001.

[3] FELIX, L. B., MIRANDA DE SÁ, A. M. F. L., INFANTOSI, A. F. C., YEHIA, H. C. Multivariate objective response detectors (MORD): statistical tools for multichanel EEG analysis during rhytmic stimulation. *Annals of Biomedical Engineering*, 2007, vol. 35, no. 3, p. 443 - 452.

[4] GISH, H., COCHRAN, D. Invariance of the magnitude-squared cohrence estimate with respect to second-channel statistics. *IEEE Transactions on Acoustics, Speech, and Signal Processing*, 1987, vol. 35, no. 12, p. 1774 - 1776.

[5] GOODMAN, N. R. Statistical analysis based on a certain multivariate complex Gaussian distribution (an introduction). *Annals of Mathematical Statistics*, 1963, vol. 34, no. 1, p. 152 - 177.

[6] JAMES, A. T. Distributions of matrix variates and latent roots derived from normal samples. *Annals of Mathematical Statistics*, 1964, vol. 35, no. 2, p. 475 - 501.

[7] KHARTI, C. G. Classical statistical analysis based on a certain multivariate complex Gaussian distribution. *Annals of Mathematical Statistics*, 1965, vol. 36, no. 1, p. 98 - 114.

[8] MIRANDA DE SÁ, A. M. F. L., FELIX, L. B., INFANTOSI, A. F. C. A matrix-based algorithm for estimating multiple coherence of a periodic signal and its application to the multichannel EEG during sensory stimulation. *IEEE Transactions on Biomedical Engineering*, 2004, vol. 51, no. 7, p. 1140 - 1146.

[9] MIRANDA DE SÁ, A. M. F. L., INFANTOSI, A. F. C., MELGES, D. B. A multiple coherence-based detector for evoked responses in the EEG during sensory stimulation. In *30th Annual International IEEE EMBS Conference*. Vancouver (British Columbia, Canada), 2008, p. 3516 - 3519.

[10] MOSKALSKI, S., TORRES, R. Influences of tides, weather, and discharge on suspended sediment concentration. *Continental Shelf Research*, 2012, vol. 37, p. 36 - 45.

[11] NEDUNGADI, A. G., DING, M., RANGARAJAN, G. Block coherence: a method for measuring the interdependence between two block of neurobiological time series. *Biological Cybernetics*, 2011, vol. 104, no. 3, p. 197 - 207.

[12] NUTTALL, A. H. Invariance of distribution of coherence estimate to second-channel statistics. *IEEE Transactions on Acoustics, Speech, and Signal Processing*, 1981, vol. 29, no. 1, p. 120 - 122.

[13] PANERAI, R. B., EAMES, P. J., POTTER, J. F. Multiple coherence of cerebral blood flow velocity in humans. *American Journal of Physiology - Heart and Circulatory Physiology*, 2006, vol. 291, no. 1, p. H251 - H259.

[14] PENG, T., ROWLEY, A. B., AINSLIE, P. N., POULIN, M. J., PAYNE, S. J. Multivariate system identification for cerebral autoregulation. *Annals of Biomedical Engineering*, 2008, vol. 36, no. 2, p. 308 - 320.

[15] RICHARDS, T. C. Dynamic testing of A/D converters using the multiple coherence function. *IEEE Transactions on Instrumentation and Measurement*, 2008, vol. 57, no. 11, p. 2596 - 2607.

[16] RICHARDS, T. C. Dynamic testing of data acquisition channels using the multiple coherence function. KARAKEHAYOV, Z. (Ed.) *Data Acquisition Applications*. Intech, 2012, p. 51 - 78.

[17] SALVADOR, R. A simple view of the brain through a frequency-specific functional connectivity measure. *NeuroImage*, 2008, vol. 39, no. 1, p. 279 - 289.

[18] SALVADOR, R., ANGUERA, M., GOMAR, J. J., BULLMORE, E. T., POMAROL-CLOTET, E. Conditional mutual information maps as descriptors of net connectivity levels in the brain. *Frontiers in Neuroinformatics*, 2010, vol. 4, no. 115.

[19] TRUEBLOOD, R. D., ALSPACH, D. L. Multiple coherence. In *11^{th} Asilomar Conference on Circuits, Systems and Computers*. Pacific Grove (CA, USA), 1977, p. 327 - 333.

[20] TRUEBLOOD, R. D., ALSPACH, D. L. Multiple coherence as a detection statistic. *NOSC Technical Report 265*. San Diego: Naval Ocean Systems Center, 1978.