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WEAKLY DELAYED LINEAR PLANAR SYSTEMS OF DISCRETE EQUATIONS

SLABĚ ZPOŽDĚNÉ LINEÁRNÍ ROVINNÉ SYSTÉMY DISKRÉTNÍCH ROVNIC

Short version of Ph.D. thesis

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Klíčová slova

Diskrétní rovnice, lineární systémy diferenčních rovnic, slabě zpožděný systém, rovinný systém, dimenze prostoru řešení, podmíněná stabilita.

Contents

1 Introduction

As most of the time measurements involving variables are discrete, the observed evolution phenomena can be expressed naturally in terms of difference equations and, thus, such equations are important mathematical models in their own right. Difference equations have an important role in the study of discretization methods for differential equations, too. The theory of difference equations is much richer than the corresponding theory of differential equations. For example, a simple difference equation obtained from a first-order differential equation may involve phenomena that can only occur for higher-order differential equations. Thus, the theory of difference equations is interesting by itself and, therefore, likely to take on greater importance in the near future.

The application of the theory of difference equations is rapidly increasing in various fields such as numerical analysis, control theory, finite mathematics, and computer science.

The fundamentals of the theory of difference equations are well described for example in books by S. Elaydi [17], by I. Győri, G. Ladas [19], by V. L. Kocić, G. Ladas [25] and by R. P. Agarwal, M. Bohner, S. R. Grace, D. O'Regan [1].

1.1 Current State

The difference equations have recently been an object of intensive research. Monographs summarizing some outcomes were mentioned above.

Every month, numerous new papers are published on the qualitative theory of difference equations. Some interesting results have been published on the representations of solutions of linear discrete systems with delay, e.g., [11,12,23], on the existence of positive solutions of discrete equations, e.g., [2–5, 27], on the oscillation of solutions of discrete equations, e.g., [21, 28, 29], on the stability of solutions of discrete systems, e.g., [6, 14, 20, 24, 26] and on the asymptotic properties of solutions of the discrete equations and systems such as [7–10, 15, 16, 18, 37].

1.2 Aims of the thesis

The thesis is concened with planar systems of weakly delayed equations

$$x(k+1) = Ax(k) + \sum_{l=1}^{n} B^l x_l(k - m_l) \tag{1}$$

where $m_1, m_2, \ldots, m_n$ are constant integer delays, $0 < m_1 < m_2 < \cdots < m_n$, $k \in \mathbb{Z}_0^\infty$, $A, B^1, ..., B^n$ are constant 2×2 matrices, $A = (a_{ij})$, $B^l = (b^l_{ij})$, $i, j = 1, 2$, $l = 1, 2, \ldots, n$ and $x\colon \mathbb{Z}_{-m_n}^\infty \to \mathbb{R}^2$. In the thesis, we construct general solutions of such systems. Further, we show that, after several steps, the dimension of the space of all solutions is reduced to a less-dimensional space. Moreover, we discuss the stability of the system.

Methodically, we follow the paper [13] where a planar weakly delayed linear discrete system

$$x(k+1) = Ax(k) + Bx(k - m), \tag{2}$$

is considered with $m \geq 0$ being a fixed integer, $k \in \mathbb{Z}_0^\infty$, $A = (a_{ij})$, $B = (b_{ij})$ constant 2×2 matrices, and $x\colon \mathbb{Z}_{-m}^\infty \to \mathbb{R}^2$. A general solution of (2) is constructed and results on the dimensionality of the space of solutions are derived.

Some of the results obtained are published in [38, 39]. In [38], a system (1) with $n = 2$ is considered. The results published in [39] concern system (1) and generalize the results published in [13].

1.3 Preliminary notions and properties

We use the following notation: for integers s, q, $s \leq q$, we define $\mathbb{Z}_s^q := \{s, s+1, \ldots, q\}$ where $s = -\infty$ or $q = \infty$ are admitted, too. Throughout this dissertation, using notation $\mathbb{Z}_s^q$, we always assume $s \leq q$. In the thesis, we deal with the discrete planar systems (1)

$$x(k+1) = Ax(k) + \sum_{l=1}^{n} B^l x_l(k - m_l)$$

where $m_1, m_2, \dots, m_n$ are constant integer delays, $0 < m_1 < m_2 < \dots < m_n$, $k \in \mathbb{Z}_0^\infty$, $A, B^1, \dots, B^n$ are constant 2×2 matrices, $A = (a_{ij})$, $B^l = (b_{ij}^l)$, $i, j = 1, 2$, $l = 1, 2, \dots, n$ and $x \colon \mathbb{Z}_{-m_n}^\infty \to \mathbb{R}^2$. Throughout the dissertation, we assume that

$$B^l \neq \Theta \tag{3}$$

where $l = 1, 2, \dots, n$ and Θ is 2×2 zero matrix. Together with equation (1), we consider an initial (Cauchy) problem

$$x(k) = \varphi(k) \tag{4}$$

where $k = -m_n, -m_n + 1, \dots, 0$ with $\varphi \colon \mathbb{Z}_{-m_n}^0 \to \mathbb{R}^2$. The *existence* and *uniqueness* of the solution of the initial problem (1), (4) on $\mathbb{Z}_{-m_n}^\infty$ is obvious. We recall that the *solution* $x \colon \mathbb{Z}_{-m_n}^\infty \to \mathbb{R}^2$ of (1), (4) is defined as an *infinite sequence*

$$\{x(-m_n) = \varphi(-m_n), x(-m_n + 1) = \varphi(-m_n + 1), \dots,$$

$$x(0) = \varphi(0), x(1), x(2), \dots, x(k), \dots\}$$

such that, for any $k \in \mathbb{Z}_0^\infty$, equality (1) holds.

The space of all initial data (4) with $\varphi \colon \mathbb{Z}_{-m_n}^0 \to \mathbb{R}^2$ is obviously $2(m_n+1)$-dimensional. Below, we describe the fact that, among the systems (1), there are such systems that their space of solutions, being initially $2(m_n+1)$-dimensional, on a reduced interval turns into a space having a dimension less than $2(m_n+1)$. The problem under consideration (pasting property of solutions) is exactly formulated in Part 1.4.

1.4 Weakly delayed systems

We consider the system (1) and look for a solution having the form $x(k) = \xi \lambda^k$ where $k \in \mathbb{Z}_{-m_n}^\infty$, $\lambda = \text{const}$ with $\lambda \neq 0$ and $\xi = (\xi_1, \xi_2)^T$ is a nonzero constant vector. The usual procedure leads to a characteristic equation

$$D := \det \left(A + \sum_{l=1}^{n} \lambda^{-m_l} B^l - \lambda I \right) = 0 \tag{5}$$

where I is the unit 2×2 matrix. Together with (1), we consider a system with the terms containing delays omitted

$$x(k+1) = Ax(k) \tag{6}$$

and its characteristic equation

$$\det (A - \lambda I) = 0. \tag{7}$$

Definition 1.1. *The system* (1) *is called a weakly delayed system if the characteristic equations* (5), (7) *corresponding to systems* (1) *and* (6) *are equal, i.e. if, for every* $\lambda \in \mathbb{C} \setminus \{0\}$,

$$D = \det \left(A + \sum_{l=1}^{n} \lambda^{-m_l} B^l - \lambda I \right) = \det (A - \lambda I). \tag{8}$$

We consider a linear transformation

$$x(k) = \mathcal{S}y(k) \tag{9}$$

with a nonsingular 2×2 matrix $\mathcal{S}$. Then, the discrete system for y is

$$y(k+1) = A_{\mathcal{S}}y(k) + \sum_{l=1}^{n} B_{\mathcal{S}}^{l} y(k - m_l) \tag{10}$$

with $A_{\mathcal{S}} = \mathcal{S}^{-1}A\mathcal{S}$, $B_{\mathcal{S}}^{l} = \mathcal{S}^{-1}B^{l}\mathcal{S}$ where $l = 1, 2, \dots, n$. We show that a system's property of being one weakly delayed is preserved by every nonsingular linear transformation.

Lemma 1.2. *If the system* (1) *is a weakly delayed system, then its arbitrary linear nonsingular transformation* (9) *again leads to a weakly delayed system* (10)*.*

1.5 Necessary and sufficient conditions determining weakly delayed systems

In the below theorem, we give conditions, in terms of determinants, indicating whether a system is weakly delayed.

Theorem 1.3. *System* (1) *is a weakly delayed system if and only if the following* $3n + n(n-1)/2$ *conditions hold simultaneously:*

$$b_{11}^{l} + b_{22}^{l} = 0, \tag{11}$$

$$\begin{vmatrix} b_{11}^{l} & b_{12}^{l} \\ b_{21}^{l} & b_{22}^{l} \end{vmatrix} = 0, \tag{12}$$

$$\begin{vmatrix} a_{11} & a_{12} \\ b_{21}^{l} & b_{22}^{l} \end{vmatrix} + \begin{vmatrix} b_{11}^{l} & b_{12}^{l} \\ a_{21} & a_{22} \end{vmatrix} = 0, \tag{13}$$

$$\begin{vmatrix} b_{11}^{l} & b_{12}^{l} \\ b_{21}^{v} & b_{22}^{v} \end{vmatrix} + \begin{vmatrix} b_{11}^{v} & b_{12}^{v} \\ b_{21}^{l} & b_{22}^{l} \end{vmatrix} = 0 \tag{14}$$

where $l, v = 1, 2, \dots, n$ *and* $v > l$*.*

Lemma 1.4. *Conditions* (11)–(14) *are equivalent to*

$$\operatorname{tr} B^{l} = \det B^{l} = 0,$$

$$\det(A + B^{l}) = \det A,$$

$$\det(B^{l} + B^{v}) = 0,$$

where $l, v = 1, 2, \dots, n$ *and* $v > l$*.*

1.6 Problem under consideration

The aim of this thesis is to give explicit formulas for the solutions of weakly delayed systems. This is done in Chapter 2. Moreover, we show (in Chapter 3) that, after several steps, the dimension of the space of all solutions, being initially equal to the dimension $2(m_n+1)$ of the space of initial data (4) generated by discrete functions φ, is reduced to a dimension less than the initial one on an interval of the form $\mathbb{Z}_s^{\infty}$ with $s>0$. In other words, we will show that the $2(m_n+1)$-dimensional space of all solutions of (1) is pasted to a less-dimensional space of solutions on $\mathbb{Z}_s^{\infty}$. This problem is solved directly by explicitly computing the corresponding solutions of the Cauchy problems with each of the cases arising being considered. The underlying idea for such investigation is simple. If (1) is a weakly delayed system, then the corresponding characteristic equation has only two eigenvalues instead of $2(m_n+1)$ eigenvalues in the case of systems with non-weak delays. This explains why the dimension of the space of solutions becomes less than the initial one. The final results (Theorems 3.1 – 3.4 below) provide the dimension of the space of solutions. Our results (published in [38] and [39]) generalize the results in [13]. Paper [38] considers the system (1) with $n=2$ and, in [39], a general case (1) is treated. From explicit formulas we deduce (in Chapter 4) stability and so-called conditional stability of the system (1).

1.7 Auxiliary formula

Recall one explicit formula (see e.g. [17]) for the solutions of linear scalar discrete non-delayed equations used in this thesis. We consider initial - value problem for the first order linear discrete nonhomogeneous equation

$$w(k+1)=aw(k)+g(k),\quad w(k_0)=w_0,\quad k\in\mathbb{Z}_{k_0}^{\infty}$$

with $a\in\mathbb{C}$ and $g\colon\mathbb{Z}_{k_0}^{\infty}\to\mathbb{C}$. Then, it is easy to verify that unique solution of this problem is

$$w(k)=a^{k-k_0}\,w_0+\sum_{r=k_0}^{k-1}a^{k-1-r}g(r),\quad k\in\mathbb{Z}_{k_0+1}^{\infty}. \tag{15}$$

Throughout this thesis, we adopt the customary notation for the sum: $\sum_{i=\ell+t}^{\ell}\mathcal{F}(i)=0$ where ℓ is an integer, t is a positive integer and "$\mathcal{F}$" denotes the function considered independently of whether it is defined for indicated arguments or not.

Note that the formula (15) is many times used in recent literature to analyze asymptotic properties of solutions of various classes of difference equations, including nonlinear equations. We refer, e.g., to [30]– [36] and to relevant references therein.

2 General solution

In this chapter we derive general solution of weakly delayed system (1). If (8) holds, then equations (5) and (7) have only two (and the same) roots simultaneously. In order to prove the properties of the family of solutions of (1) formulated in Introduction, we will discuss each combination of roots, i.e., the cases of two real and distinct roots, a pair of complex conjugate roots, and, finally, a double real root.

Although computations in Parts 1.4 and 1.5 were performed under assumption $\lambda \neq 0$, results of this part remain valid also if one or both roots of characteristic equation (7) are zero.

2.1 Jordan forms of the matrix A and corresponding solutions of the problem (1), (4)

It is known that, for every matrix A, there exists a nonsingular matrix S transforming it to the corresponding Jordan matrix form Λ. This means that

$$\Lambda = S^{-1}AS,$$

where Λ has the following four possible forms (denoted below as $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$), depending on the roots of the characteristic equation (7), i.e. on the roots of

$$\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0. \tag{16}$$

If (16) has two real distinct roots λ_1, λ_2, then

$$\Lambda_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{17}$$

if the roots are complex conjugate, i.e. $\lambda_{1,2} = p \pm iq$ with $q \neq 0$, then

$$\Lambda_2 = \begin{pmatrix} p & q \\ -q & p \end{pmatrix} \tag{18}$$

and, finally, in the case of one double real root $\lambda_{1,2} = \lambda$, we have either

$$\Lambda_3 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \tag{19}$$

or

$$\Lambda_4 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}. \tag{20}$$

The transformation $y(k) = S^{-1}x(k)$ transforms (1) into a system

$$y(k+1) = \Lambda y(k) + \sum_{l=1}^{n} B^{*l} y(k - m_l),\ \ k \in \mathbb{Z}_0^{\infty} \tag{21}$$

with $B^{*l} = S^{-1}B^lS$, $B^{*l} = (b_{ij}^{*l})$, $l = 1, \dots, n$ and $i, j = 1, 2$. Together with (21), we consider an initial problem

$$y(k) = \varphi^*(k), \tag{22}$$

$k \in \mathbb{Z}_{-m_n}^0$ with $\varphi^* \colon \mathbb{Z}_{-m_n}^0 \to \mathbb{R}^2$ where $\varphi^*(k) = S^{-1}\varphi(k)$ is the initial function corresponding to the initial function φ in (4).

Below, we consider all four possible cases (17)–(20) separately.

We define

$$\Phi_1(k) := (0, \varphi_1^*(k))^T, \quad \Phi_2(k) := (\varphi_2^*(k), 0)^T, \quad k \in \mathbb{Z}_{-m_n}^0.$$

Assuming that (1) is a weakly delayed system, by Lemma 1.2, the system (21) is weakly delayed system again.

2.1.1 The case (17) of two real distinct roots

In this case, we have $\Lambda = \Lambda_1$ and $\Lambda_1^k = \operatorname{diag}(\lambda_1^k, \lambda_2^k)$. The necessary and sufficient conditions (11)–(14) for (21) turn into

$$b_{11}^{*l} + b_{22}^{*l} = 0, \tag{23}$$

$$\begin{vmatrix} b_{11}^{*l} & b_{12}^{*l} \\ b_{21}^{*l} & b_{22}^{*l} \end{vmatrix} = b_{11}^{*l} b_{22}^{*l} - b_{12}^{*l} b_{21}^{*l} = 0, \tag{24}$$

$$\begin{vmatrix} \lambda_1 & 0 \\ b_{21}^{*l} & b_{22}^{*l} \end{vmatrix} + \begin{vmatrix} b_{11}^{*l} & b_{12}^{*l} \\ 0 & \lambda_2 \end{vmatrix} = \lambda_1 b_{22}^{*l} + \lambda_2 b_{11}^{*l} = 0, \tag{25}$$

$$\begin{vmatrix} b_{11}^{*l} & b_{12}^{*l} \\ b_{21}^{*v} & b_{22}^{*v} \end{vmatrix} + \begin{vmatrix} b_{11}^{*v} & b_{12}^{*v} \\ b_{21}^{*l} & b_{22}^{*l} \end{vmatrix} = 0. \tag{26}$$

Since $\lambda_1 \neq \lambda_2$, equations (23), (25) yield $b_{11}^{*l} = b_{22}^{*l} = 0$. Then, from (24), we get $b_{12}^{*l} b_{21}^{*l} = 0$, so that either $b_{21}^{*l} = 0$ or $b_{12}^{*l} = 0$. In view of assumptions $B^l \neq \Theta$, $l = 1, 2, \dots, n$ we conclude that only the following cases I, II are possible

- **I)** $b_{11}^{*l} = b_{22}^{*l} = b_{21}^{*l} = 0$, $b_{12}^{*l} \neq 0$, $l = 1, 2, \dots, n$,
- **II)** $b_{11}^{*l} = b_{22}^{*l} = b_{12}^{*l} = 0$, $b_{21}^{*l} \neq 0$, $l = 1, 2, \dots, n$.

In Theorem 1.5 below are both cases I, II analyzed.

Theorem 2.1. *Let* (1) *be a weakly delayed system and equation* (16) *has two real distinct roots* λ_1, λ_2. *If the case I) hold, then the solution of the initial problem* (1), (4) *is* $x(k) = Sy(k)$, $k \in \mathbb{Z}_{-m_n}^{\infty}$ *where* $y(k)$ *has the form*

$$y(k) = \begin{cases} \varphi^*(k) \quad \text{if} \quad k \in \mathbb{Z}_{-m_n}^{0}, \\[4pt] \Lambda_1^k \varphi^*(0) + \sum\limits_{r=0}^{k-1} \lambda_1^{k-1-r} \left[\sum\limits_{l=1}^{n} b_{12}^{*l} \Phi_2(r - m_l) \right] \quad \text{if} \quad k \in \mathbb{Z}_1^{m_1+1}, \\[4pt] \dots \\[4pt] \Lambda_1^k \varphi^*(0) + \sum\limits_{r=0}^{k-1} \lambda_1^{k-1-r} \left[\sum\limits_{l=s+1}^{n} b_{12}^{*l} \Phi_2(r - m_l) \right] \\ \qquad + \sum\limits_{l=1}^{s} b_{12}^{*l} \left[\sum\limits_{r=0}^{m_l} \lambda_1^{k-1-r} \Phi_2(r - m_l) \right. \\ \qquad\qquad \left. + \Phi_2(0) \sum\limits_{r=m_l+1}^{k-1} \lambda_1^{k-1-r} \lambda_2^{r-m_l} \right] \\ \qquad \text{if} \quad k \in \mathbb{Z}_{m_s+2}^{m_{s+1}+1}, \ s = 1, 2, \dots, n-1, \\[4pt] \dots \\[4pt] \Lambda_1^k \varphi^*(0) + \sum\limits_{l=1}^{n} b_{12}^{*l} \left[\sum\limits_{r=0}^{m_l} \lambda_1^{k-1-r} \Phi_2(r - m_l) \right. \\ \qquad\qquad \left. + \Phi_2(0) \sum\limits_{r=m_l+1}^{k-1} \lambda_1^{k-1-r} \lambda_2^{r-m_l} \right] \quad \text{if} \quad k \in \mathbb{Z}_{m_n+2}^{\infty}. \end{cases}$$

If the case II) is true, then the solution of initial problem (1), (4) *is* $x(k) = Sy(k)$, $k \in \mathbb{Z}_{-m_n}^{\infty}$ *where* $y(k)$ *has the form*

$$
y(k) = \begin{cases}
\varphi^*(k) \quad \text{if} \quad k \in \mathbb{Z}_{-m_n}^{0}, \\
\Lambda_1^k \varphi^*(0) + \sum\limits_{r=0}^{k-1} \lambda_2^{k-1-r} \left[\sum\limits_{l=1}^{n} b_{21}^{*l} \Phi_1(r-m_l) \right] \quad \text{if} \quad k \in \mathbb{Z}_1^{m_1+1}, \\
\dots \\
\Lambda_1^k \varphi^*(0) + \sum\limits_{r=0}^{k-1} \lambda_2^{k-1-r} \left[\sum\limits_{l=s+1}^{n} b_{21}^{*l} \Phi_1(r-m_l) \right] \\
\qquad + \sum\limits_{l=1}^{s} b_{21}^{*l} \left[\sum\limits_{r=0}^{m_l} \lambda_2^{k-1-r} \Phi_1(r-m_l) \right. \\
\qquad\qquad \left. + \Phi_1(0) \sum\limits_{r=m_l+1}^{k-1} \lambda_1^{r-m_l} \lambda_2^{k-1-r} \right] \\
\qquad \text{if} \quad k \in \mathbb{Z}_{m_s+2}^{m_{s+1}+1}, \; s = 1, 2, \dots, n-1, \\
\dots \\
\Lambda_1^k \varphi^*(0) + \sum\limits_{l=1}^{n} b_{21}^{*l} \left[\sum\limits_{r=0}^{m_l} \lambda_2^{k-1-r} \Phi_1(r-m_l) \right. \\
\qquad\qquad \left. + \Phi_1(0) \sum\limits_{r=m_l+1}^{k-1} \lambda_1^{r-m_l} \lambda_2^{k-1-r} \right] \quad \text{if} \quad k \in \mathbb{Z}_{m_n+2}^{\infty}.
\end{cases}
$$

2.1.2 The case (18) of two complex conjugate roots

The necessary and sufficient conditions (11)–(14) take the forms (23), (24), (26) and

$$
\begin{vmatrix} p & q \\ b_{21}^{*l} & b_{22}^{*l} \end{vmatrix} + \begin{vmatrix} b_{11}^{*l} & b_{12}^{*l} \\ -q & p \end{vmatrix} = p(b_{11}^{*l} + b_{22}^{*l}) + q(b_{12}^{*l} - b_{21}^{*l}) = 0 \tag{27}
$$

where $l, v = 1, 2, \dots, n$ and $v > l$.

The system of conditions (23), (24) and (27) gives $b_{12}^{*l} = b_{21}^{*l}$, $(b_{11}^{*l})^2 = -(b_{12}^{*l})^2$ and admits only one possibility, namely,

$$
b_{11}^{*l} = b_{22}^{*l} = b_{12}^{*l} = b_{21}^{*l} = 0.
$$

Consequently, $B^{*l} = \Theta$, $B^l = \Theta$.

The initial problem (1), (4) reduces to a problem without delay

$$
\begin{cases}
x(k+1) = Ax(k), \\
x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-m_n}^{0}
\end{cases}
$$

and, obviously,

$$
x(k) = \begin{cases}
\varphi(k) & \text{if} \quad k \in \mathbb{Z}_{-m_n}^{0}, \\
A^k \varphi(0) & \text{if} \quad k \in \mathbb{Z}_1^{\infty}.
\end{cases}
$$

From this discussion, the next theorem follows.

Theorem 2.2. *There exists no weakly delayed system* (1) *if* Λ *has the form* (18).

Finally, we note that the assumptions (3) alone exclude this case.

2.1.3 The case (19) of double real root

In this case we have $\Lambda = \Lambda_3$ and $\Lambda_3^k = \mathrm{diag}(\lambda^k, \lambda^k)$. For (21), the necessary and sufficient conditions (11)–(14) are reduced to (23), (24), (26) and

$$\begin{vmatrix} \lambda & 0 \\ b_{21}^{*l} & b_{22}^{*l} \end{vmatrix} + \begin{vmatrix} b_{11}^{*l} & b_{12}^{*l} \\ 0 & \lambda \end{vmatrix} = \lambda(b_{11}^{*l} + b_{22}^{*l}) = 0 \tag{28}$$

where $l = 1, 2, \ldots, n$.

From (23), (24) and (28), we get $b_{12}^{*l} b_{21}^{*l} = -\left(b_{11}^{*l}\right)^2$. From the condition (26) we get

$$b_{11}^{*l} b_{22}^{*v} - b_{12}^{*l} b_{21}^{*v} + b_{22}^{*l} b_{11}^{*v} - b_{21}^{*l} b_{12}^{*v} = 0 \tag{29}$$

where $l, v = 1, 2, \ldots, n$ and $v > l$. Multiplying (29) by $b_{12}^{*l} b_{12}^{*v}$, we have

$$b_{11}^{*l} b_{22}^{*v} b_{12}^{*l} b_{12}^{*v} - (b_{12}^{*l})^2 b_{21}^{*v} b_{12}^{*v} + b_{22}^{*l} b_{11}^{*v} b_{12}^{*l} b_{12}^{*v} - b_{21}^{*l} b_{12}^{*l} (b_{12}^{*v})^2 = 0. \tag{30}$$

Substituting $b_{12}^{*l} b_{21}^{*l} = -\left(b_{11}^{*l}\right)^2$, $b_{12}^{*v} b_{21}^{*v} = -\left(b_{11}^{*v}\right)^2$ into (30) and using (23) we obtain

$$-b_{11}^{*l} b_{11}^{*v} b_{12}^{*l} b_{12}^{*v} + (b_{12}^{*l})^2 (b_{11}^{*v})^2 - b_{11}^{*l} b_{11}^{*v} b_{12}^{*l} b_{12}^{*v} + (b_{11}^{*l})^2 (b_{12}^{*v})^2 = 0. \tag{31}$$

The equation (31) can be written as

$$(b_{12}^{*l} b_{11}^{*v} - b_{12}^{*v} b_{11}^{*l})^2 = 0$$

and

$$b_{12}^{*l} b_{11}^{*v} = b_{12}^{*v} b_{11}^{*l}. \tag{32}$$

We analyse the two possible cases: $b_{12}^{*l} b_{21}^{*l} = 0$ and $b_{12}^{*l} b_{21}^{*l} \neq 0$.

For the case $b_{12}^{*l} b_{21}^{*l} = 0$, we have from (23), (24) that $b_{11}^{*l} = b_{22}^{*l} = 0$ and $b_{12}^{*l} = 0$ or $b_{21}^{*l} = 0$. For $b_{12}^{*l} = 0$ and $b_{21}^{*l} \neq 0$, condition (26) gives $b_{12}^{*v} = 0$, where $l, v = 1, 2, \ldots, n$ and $v > l$. Then, from (23), (24) for $l = v$, we get $b_{11}^{*v} = b_{22}^{*v} = 0$ and $b_{21}^{*v} \neq 0$.
For $b_{21}^{*l} = 0$ and $b_{12}^{*l} \neq 0$, condition (26) gives $b_{21}^{*v} = 0$, where $l, v = 1, 2, \ldots, n$ and $v > l$. Then, from (23), (24) for $l = v$, we get $b_{11}^{*v} = b_{22}^{*v} = 0$ and $b_{12}^{*v} \neq 0$.

Now we discuss the case $b_{12}^{*l} b_{21}^{*l} \neq 0$. From conditions (23), (24), we have $b_{12}^{*l} b_{21}^{*l} = -\left(b_{11}^{*l}\right)^2$ and $b_{11}^{*l} b_{22}^{*l} \neq 0$. This yields $b_{11}^{*l} \neq 0$, $b_{22}^{*l} \neq 0$ and, from (32), we have $b_{11}^{*v} \neq 0$, $b_{12}^{*v} \neq 0$. By conditions (23), (24) for $v = l$, we get $b_{22}^{*v} \neq 0$, $b_{21}^{*v} \neq 0$.

From the assumptions $B^l \neq \Theta$, we conclude that only the following cases I, II, III are possible

I) $b_{11}^{*l} = b_{22}^{*l} = b_{21}^{*l} = 0$, $b_{12}^{*l} \neq 0$,

II) $b_{11}^{*l} = b_{22}^{*l} = b_{12}^{*l} = 0$, $b_{21}^{*l} \neq 0$,

III) $b_{12}^{*l} b_{21}^{*l} \neq 0$,

where $l = 1, 2, \ldots, n$.

The case $b_{12}^{*l} b_{21}^{*l} = 0$.

Theorem 2.3. *Let* (1) *be a weakly delayed system, equation* (16) *has a two-fold root* $\lambda_{1,2} = \lambda$, $b_{12}^{*l} b_{21}^{*l} = 0$ *and the matrix* Λ *has the form* (19). *Then the solution of the initial problem* (1), (4) *is* $x(k) = Sy(k)$, $k \in \mathbb{Z}_{-m_n}^{\infty}$ *where in the case* $b_{21}^{*l} = 0$, $y(k)$ *has the form*

$$y(k) = \begin{cases} \varphi^*(k) \quad \text{if} \quad k \in \mathbb{Z}_{-m_n}^{0}, \\ \Lambda_3^k \varphi^*(0) + \sum\limits_{r=0}^{k-1} \lambda^{k-1-r} \left[\sum\limits_{l=1}^{n} b_{12}^{*l} \Phi_2(r - m_l) \right] \quad \text{if} \quad k \in \mathbb{Z}_1^{m_1+1}, \\ \dots \\ \Lambda_3^k \varphi^*(0) + \sum\limits_{r=0}^{k-1} \lambda^{k-1-r} \left[\sum\limits_{l=s+1}^{n} b_{12}^{*l} \Phi_2(r - m_l) \right] \\ \qquad + \sum\limits_{l=1}^{s} b_{12}^{*l} \left[\sum\limits_{r=0}^{m_l} \lambda^{k-1-r} \Phi_2(r - m_l) \right. \\ \qquad\quad \left. + (k - 1 - m_l) \lambda^{k-1-m_l} \Phi_2(0) \right] \\ \qquad \text{if} \quad k \in \mathbb{Z}_{m_s+2}^{m_{s+1}+1}, \ s = 1, 2, \dots, n-1, \\ \dots \\ \Lambda_3^k \varphi^*(0) + \sum\limits_{l=1}^{n} b_{12}^{*l} \left[\sum\limits_{r=0}^{m_l} \lambda^{k-1-r} \Phi_2(r - m_l) \right. \\ \qquad\quad \left. + (k - 1 - m_l) \lambda^{k-1-m_l} \Phi_2(0) \right] \quad \text{if} \quad k \in \mathbb{Z}_{m_n+2}^{\infty}. \end{cases}$$

If $b_{12}^{*l} = 0$ *is true then the solution of initial problem* (1), (4) *is* $x(k) = Sy(k)$, $k \in \mathbb{Z}_{-m_n}^{\infty}$ *where* $y(k)$ *has the form*

$$y(k) = \begin{cases} \varphi^*(k) \quad \text{if} \quad k \in \mathbb{Z}_{-m_n}^{0}, \\ \Lambda_3^k \varphi^*(0) + \sum\limits_{r=0}^{k-1} \lambda^{k-1-r} \left[\sum\limits_{l=1}^{n} b_{21}^{*l} \Phi_1(r - m_l) \right] \quad \text{if} \quad k \in \mathbb{Z}_1^{m_1+1}, \\ \dots \\ \Lambda_3^k \varphi^*(0) + \sum\limits_{r=0}^{k-1} \lambda^{k-1-r} \left[\sum\limits_{l=s+1}^{n} b_{21}^{*l} \Phi_1(r - m_l) \right] \\ \qquad + \sum\limits_{l=1}^{s} b_{21}^{*l} \left[\sum\limits_{r=0}^{m_l} \lambda^{k-1-r} \Phi_1(r - m_l) \right. \\ \qquad\quad \left. + (k - 1 - m_l) \lambda^{k-1-m_l} \Phi_1(0) \right] \\ \qquad \text{if} \quad k \in \mathbb{Z}_{m_s+2}^{m_{s+1}+1}, \ s = 1, 2, \dots, n-1 \\ \dots \\ \Lambda_3^k \varphi^*(0) + \sum\limits_{l=1}^{n} b_{21}^{*l} \left[\sum\limits_{r=0}^{m_l} \lambda^{k-1-r} \Phi_1(r - m_l) \right. \\ \qquad\quad \left. + (k - 1 - m_l) \lambda^{k-1-m_l} \Phi_1(0) \right] \quad \text{if} \quad k \in \mathbb{Z}_{m_n+2}^{\infty}. \end{cases}$$

The case $b^{*l}_{12}b^{*l}_{21} \neq 0$.

For $k \in \mathbb{Z}^0_{-m_n}$, we define

$$\Phi^*_l(k) := \left(b^{*l}_{11}\left[\varphi^*_1(k) + \frac{b^{*1}_{12}}{b^{*1}_{11}}\,\varphi^*_2(k)\right], -\frac{(b^{*l}_{11})^2}{b^{*l}_{12}}\left[\varphi^*_1(k) + \frac{b^{*1}_{12}}{b^{*1}_{11}}\,\varphi^*_2(k)\right]\right)^T.$$

Theorem 2.4. *Let the system* (1) *be a weakly delayed system, equation* (16) *admits two repeated roots* $\lambda_{1,2} = \lambda$, $b^{*l}_{12}b^{*l}_{21} \neq 0$ *and the matrix* Λ_3 *has the form* (19). *Then the solution of the initial problem* (1), (4) *is given by* $x(k) = Sy(k)$, $k \in \mathbb{Z}^\infty_{-m_n}$ *where* $y(k)$ *has the form*

$$y(k) = \begin{cases} \varphi^*(k) \quad \text{if} \quad k \in \mathbb{Z}^0_{-m_n}, \\ \Lambda^k_3\varphi^*(0) + \sum\limits_{r=0}^{k-1} \lambda^{k-1-r}\left[\sum\limits_{l=1}^{n} \Phi^*_l(r-m_l)\right] \quad \text{if} \quad k \in \mathbb{Z}^{m_1+1}_1, \\ \dots \\ \Lambda^k_3\varphi^*(0) + \sum\limits_{r=0}^{k-1} \lambda^{k-1-r}\left[\sum\limits_{l=s+1}^{n} \Phi^*_l(r-m_l)\right] \\ \qquad + \sum\limits_{l=1}^{s}\left[\sum\limits_{r=0}^{m_l} \lambda^{k-1-r}\Phi^*_l(r-m_l) \right. \\ \qquad\qquad \left. +(k-1-m_l)\lambda^{k-1-m_l}\Phi^*_l(0)\right] \\ \qquad \text{if} \quad k \in \mathbb{Z}^{m_{s+1}+1}_{m_s+2},\ s = 1, 2, \dots, n-1, \\ \dots \\ \Lambda^k_3\varphi^*(0) + \sum\limits_{l=1}^{n}\left[\sum\limits_{r=0}^{m_l} \lambda^{k-1-r}\Phi^*_l(r-m_l) \right. \\ \qquad\qquad \left. +(k-1-m_l)\lambda^{k-1-m_l}\Phi^*_l(0)\right] \quad \text{if} \quad k \in \mathbb{Z}^\infty_{m_n+2}. \end{cases}$$

2.1.4 The case (20) of a double real root

If the matrix Λ has the form (20), the necessary and sufficient conditions (11)–(14), for (21) are reduced to (23), (24), (26) and

$$\begin{vmatrix} \lambda & 1 \\ b^{*l}_{21} & b^{*l}_{22} \end{vmatrix} + \begin{vmatrix} b^{*l}_{11} & b^{*l}_{12} \\ 0 & \lambda \end{vmatrix} = \lambda(b^{*l}_{11} + b^{*l}_{22}) - b^{*l}_{21} = 0. \tag{33}$$

Then (23), (24), and (33) give $b^{*l}_{11} = b^{*l}_{22} = b^{*l}_{21} = 0$.

Theorem 2.5. *Let* (1) *be a weakly delayed system, equation* (16) *has a double root* $\lambda_{1,2} = \lambda$ *and the matrix* Λ *has the form* (20). *Then* $b^{*l}_{11} = b^{*l}_{22} = b^{*l}_{21} = 0$ *and the solution of the* *initial problem* (1), (4) *is* $x(k)=\mathcal{S}y(k)$, $y(k)=(y_1(k),y_2(k))^T$ *and*

$$y_1(k)=\begin{cases}\varphi_1^*(k) \quad \textit{if} \quad k\in\mathbb{Z}_{-m_n}^0,\\[4pt] \lambda^k\varphi_1^*(0)+k\lambda^{k-1}\varphi_2^*(0)+\sum\limits_{r=0}^{k-1}\lambda^{k-1-r}\left[\sum\limits_{l=1}^{n}b_{12}^{*l}\varphi_2^*(r-m_l)\right]\\ \qquad \textit{if} \quad k\in\mathbb{Z}_1^{m_1+1},\\ \dots\\ \lambda^k\varphi_1^*(0)+k\lambda^{k-1}\varphi_2^*(0)+\sum\limits_{r=0}^{k-1}\lambda^{k-1-r}\left[\sum\limits_{l=s+1}^{n}b_{12}^{*l}\varphi_2^*(r-m_l)\right]\\ \qquad +\sum\limits_{l=1}^{s}b_{12}^{*l}\left[\sum\limits_{r=0}^{m_l}\lambda^{k-1-r}\varphi_2^*(r-m_l)\right.\\ \qquad\quad \left.+(k-1-m_l)\lambda^{k-1-m_l}\varphi_2^*(0)\right]\\ \qquad \textit{if} \quad k\in\mathbb{Z}_{m_s+2}^{m_{s+1}+1},\ s=1,2,\dots,n-1,\\ \dots\\ \lambda^k\varphi_1^*(0)+k\lambda^{k-1}\varphi_2^*(0)+\sum\limits_{l=1}^{n}b_{12}^{*l}\left[\sum\limits_{r=0}^{m_l}\lambda^{k-1-r}\varphi_2^*(r-m_l)\right.\\ \qquad\quad \left.+(k-1-m_l)\lambda^{k-1-m_l}\varphi_2^*(0)\right]\\ \qquad \textit{if} \quad k\in\mathbb{Z}_{m_n+2}^{\infty},\end{cases}$$

$$y_2(k)=\begin{cases}\varphi_2^*(k) \quad \textit{if} \quad k\in\mathbb{Z}_{-m_n}^0,\\ \lambda^k\varphi_2^*(0) \quad \textit{if} \quad k\in\mathbb{Z}_1^{\infty}.\end{cases}$$

3 Dimension of the set of solutions

Since all the possible cases of the planar system (1) with weak delay have been analysed, we are ready to formulate results concerning the dimension of the space of solutions of (1) assuming that initial conditions (4) are variable. Although the case $b_{11}^{*l}=b_{22}^{*l}=b_{12}^{*l}=b_{21}^{*l}=0$ does not lead to a weakly delayed system and is excluded by (3), for completeness of analysis we incorporate such possibility in our analysis as well (such a case can be considered as a degenerated weakly delayed system). Before formulation we remark that if an assumption in the following theorem is assumed to be valid for a fixed index $l\in\{1,2,\dots,n\}$, it is easy to see that it must be valid for all indices $l=1,2,\dots,n$.

Theorem 3.1. *Let* (1) *be a weakly delayed system and* (16) *have both roots different from zero and* $l\in\{1,2,\dots,n\}$ *be fixed. Then the space of solutions, being initially* $2(m_n+1)$*-dimensional, becomes on* $\mathbb{Z}_{m_n+2}^{\infty}$ *only*

1) (m_n+2)*-dimensional if equation* (16) *has*

- **a)** *two real distinct roots and* $(b_{12}^{*l})^2+(b_{21}^{*l})^2>0$.
- **b)** *a double real root,* $b_{12}^{*l}b_{21}^{*l}=0$ *and* $(b_{12}^{*l})^2+(b_{21}^{*l})^2>0$.

c) *a double real root and* $b^{*l}_{12}b^{*l}_{21} \neq 0$.

2) 2-*dimensional if equation* (16) *has*

a) *two real distinct roots and* $b^{*l}_{12} = b^{*l}_{21} = 0$.

b) *a pair of complex conjugate roots.*

c) *a double real root and* $b^{*l}_{12} = b^{*l}_{21} = 0$.

Theorem 3.1 can be formulated simply as

Theorem 3.2. *Let* (1) *be a weakly delayed system and let* (16) *have both roots different from zero. Then the space of solutions, being initially* $2(m_n+1)$-*dimensional, is on* $\mathbb{Z}^{\infty}_{m_n+2}$ *only*

1) (m_n+2)-*dimensional if* $(b^{*l}_{12})^2 + (b^{*l}_{21})^2 > 0$.

2) 2-*dimensional if* $b^{*l}_{12} = b^{*l}_{21} = 0$.

Theorem 3.3. *Let* (1) *be a weakly delayed system and let* (16) *have a simple root* $\lambda = 0$. *Then the space of solutions, being initially* $2(m_n+1)$-*dimensional, is either* (m_n+1)-*dimensional or* 1-*dimensional on* $\mathbb{Z}^{\infty}_{m_n+2}$.

Theorem 3.4. *Let* (1) *be a weakly delayed system and let* (16) *have a double root* $\lambda = 0$. *Then the space of solutions, being initially* $2(m_n+1)$-*dimensional, turns into a* 0-*dimensional space on* $\mathbb{Z}^{\infty}_{m_n+2}$, *namely, into the zero solution.*

4 Discussion of stability

In this chapter we use the explicit formulas derived in Chapter 3 for stability analysis of linear system (1).

Define a norm of a 2×2 matrix $A = \{a_{ij}\}^2_{i,j=1}$ as

$$\|A\| = \max\{|a_{11}| + |a_{12}|, |a_{21}| + |a_{22}|\}$$

and, for 2×1 vectors $x = (x_1, x_2)^T$, an induced vector norm

$$\|x\| = \max\{|x_1|, |x_2|\}.$$

For a discrete vector $\psi\colon \mathbb{Z}^0_{-m_n} \to \mathbb{R}^2$ we define

$$\|\psi\|_{m_n} := \max\{\|\psi(-m_n)\|, \|\psi(-m_n+1)\|, \dots, \|\psi(0)\|\}.$$

Now we define the stability of the zero solution of linear system (1)

$$x(k+1) = Ax(k) + \sum_{l=1}^{n} B^l x_l(k-m_l)$$

where $k \in \mathbb{Z}^{\infty}_{-m_n}$, B^l are 2×2 constant matrices.

Definition 4.1. *The zero solution $x(k) = 0$, $k \in \mathbb{Z}_{-m_n}^{\infty}$ of* (1) *is said to be*

a) *Stable if, given $\varepsilon > 0$ and $k_0 \geq 0$, there exists $\delta = \delta(\varepsilon, k_0)$ such that $\varphi(k)$, $k \in \mathbb{Z}_{k_0-m_n}^{k_0}$, $\|\varphi\|_{m_n} < \delta$ implies $\|x(k, k_0, \varphi)\| < \varepsilon$ for all $k \geq k_0$, uniformly stable if δ may be chosen independently of k_0, unstable if it is not stable;*

b) *Asymptotically stable if it is stable and $\lim_{k\to\infty} \|x(k)\| = 0$;*

c) *Conditionally stable (conditionally asymptotically stable) if it is stable (asymptotically stable) under the condition that a subspace P of the space all initial data with $\dim P$ satisfying*

$$1 < \dim P < 2(m_n + 1)$$

is fixed.

It is easy to see that

$$\max\{\|\varphi^*(0)\|, \|\Phi_2(0)\|, \|\Phi_2(-1)\|, \dots, \|\Phi_2(-m_l)\|\} \leq \|\varphi^*\|_{m_n}.$$

Assume

$$q = \max\{|\lambda_1|, |\lambda_2|\} < 1.$$

In the following Theorems 4.2–4.16, we present the stability, asymptotic stability, conditional asymptotic stability, and conditional stability of system (21), (22). Tracing the statements of these theorems we conclude that it is sufficient to perform stability investigation only on the interval $k \in \mathbb{Z}_{m_n+2}^{\infty}$. Therefore we omit the technical details connected with investigating the stability on $k \in \mathbb{Z}_0^{m_n+1}$.

Theorem 4.2. *If Theorem 2.1 holds and $q < 1$, then the zero solution of* (1) *is stable.*

Theorem 4.3. *If Theorem 2.1 holds and $q < 1$, then the zero solution of* (1) *is asymptotically stable.*

Theorem 4.4. *If Theorem 2.3 holds and $q < 1$, then the zero solution of* (1) *is asymptotically stable.*

Theorem 4.5. *If Theorem 2.4 holds and $q < 1$, then the zero solution of* (1) *is asymptotically stable.*

Theorem 4.6. *If Theorem 2.5 holds and $q < 1$, then the zero solution of* (1) *is asymptotically stable.*

Theorem 4.7. *If Theorem 2.1 holds and $|\lambda_1| = 1, |\lambda_2| \leq q < 1$, then the zero solution of* (1) *is stable.*

Theorem 4.8. *If Theorem 2.1 holds and $|\lambda_1| \leq q < 1, |\lambda_2| = 1$, then the zero solution of* (1) *is stable.*

Theorem 4.9. *If Theorem 2.1 holds, the case I) occurs, $|\lambda_1| \leq q < 1, |\lambda_2| \geq 1$ and $\varphi_2^*(0) = 0$, then the zero solution of* (1) *is conditionally asymptotically stable.*

Theorem 4.10. *If Theorem 2.1 holds, the case II) occurs,* $|\lambda_2| \le q < 1, |\lambda_1| \ge 1$ *and* $\varphi_1^*(0) = 0$*, then the zero solution of* (1) *is conditionally asymptotically stable.*

Theorem 4.11. *If Theorem 2.1 holds, the case I) occurs,* $|\lambda_1| = 1, |\lambda_2| > 1$ *and* $\varphi_2^*(0) = 0$*, then the zero solution of* (1) *is conditionally stable.*

Theorem 4.12. *If Theorem 2.1 holds, the case II) occurs,* $|\lambda_2| = 1, |\lambda_1| > 1$ *and* $\varphi_1^*(0) = 0$*, then the zero solution of* (1) *is conditionally stable.*

Theorem 4.13. *If Theorem 2.3 holds, the case* $b_{21}^{*l} = 0$ *occurs,* $|\lambda| = 1$ *and* $\varphi_2^*(0) = 0$*, then the zero solution of* (1) *is conditionally stable.*

Theorem 4.14. *If Theorem 2.3 holds, the case* $b_{12}^{*l} = 0$ *occurs,* $|\lambda| = 1$ *and* $\varphi_1^*(0) = 0$*, then the zero solution of* (1) *is conditionally stable.*

Theorem 4.15. *If Theorem 2.4 holds and* $|\lambda| = 1$ *and* $\varphi_1^*(0) = \varphi_2^*(0) = 0$*, then the zero solution of* (1) *is conditionally stable.*

Theorem 4.16. *If Theorem 2.5 holds and* $|\lambda| = 1$ *and* $\varphi_2^*(0) = 0$*, then the zero solution of* (1) *is conditionally stable.*

5 Conclusions

To our best knowledge, weakly delayed systems were first defined in [22] for systems of linear delayed differential systems with constant coefficients and, in [13], for planar linear discrete systems with a single delay (in these papers such systems are called systems with a weak delay). The weakly delayed systems analyzed in this paper can be simplified and their solutions can be found in explicit analytical forms (results obtained and published in [39] generalize those in [13] and [38]). Consequently, analytical forms of solutions can be used directly to solve several problems for weakly delayed systems, e.g., problems of asymptotical behavior of their solutions, stability problems, boundary-value problems, and some problems of control theory.

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Abstract

The present thesis deals with planar weakly delayed linear discrete systems

$$x(k+1) = Ax(k) + \sum_{l=1}^{n} B^l x_l(k - m_l)$$

where $k \in \mathbb{Z}_0^\infty := \{0, 1, \ldots, \infty\}$, $m_1, m_2, \ldots, m_n$ are constant integer delays, $0 < m_1 < m_2 < \cdots < m_n$, $A, B^1, \ldots, B^n$ are constant 2×2 matrices and $x\colon \mathbb{Z}_{-m_n}^\infty \to \mathbb{R}^2$. The characteristic equations of weakly delayed systems are identical with those of the same systems but without delayed terms. In this case, after several steps, the space of solutions with a given starting dimension $2(m_n + 1)$ is pasted into a space with a dimension less than the starting one. In a sense, this situation is analogous to one known in the theory of linear differential systems with constant coefficients and special delays when the initially infinite dimensional space of solutions on the initial interval turns (after several steps) into a finite dimensional set of solutions. For every possible case, explicit general solutions are constructed and, finally, results on the dimensionality of the space of solutions are obtained. The stability of solutions is investigated as well.

Abstrakt

Dizertační práce se zabývá slabě zpožděnými lineárními rovinnými systémemy s konstantními koeficienty tvaru

$$x(k+1) = Ax(k) + \sum_{l=1}^{n} B^l x_l(k - m_l)$$

kde $k \in \mathbb{Z}_0^\infty := \{0, 1, \ldots, \infty\}$, $m_1, m_2, \ldots, m_n$ jsou konstatní celá čísla, $0 < m_1 < m_2 < \cdots < m_n$, $A, B^1, \ldots, B^n$ jsou konstantní 2×2 matice a $x\colon \mathbb{Z}_{-m_n}^\infty \to \mathbb{R}^2$ je hledané řešení. Charakteristická rovnice těchto systémů je identická s charakteristickou rovnicí systému, který neobsahuje zpožděné členy. V takovém případě se počáteční dimenze prostoru řešení $2(m_n + 1)$ mění po několika krocích na menší. V jistém smyslu je tato situace analogická podobnému jevu v teorii lineárních diferenciálních systémů s konstantními koeficienty a speciálním zpožděním, kdy původně nekonečně rozměrný prostor řešení (na počátečním intervalu) přejde po několika krocích do konečného prostoru řešení. V práci je pro každý možný případ kombinace kořenů charakteristické rovnice konstruováno obecné řešení daného systému a jsou formulovány výsledky o dimenzi prostoru řešení. Také je zkoumána stabilita řešení.