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Fredholm's third theorem for second-order singular Dirichlet problem

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Abstract

Consider the singular Dirichlet problem

$$\begin{aligned}u'' &= p(t)u + q(t); \\ u(a) &= 0, \quad u(b) = 0,\end{aligned}$$

where $p, q :]a, b[\rightarrow \mathbb{R}$ are locally Lebesgue integrable functions. It is proved that if

$$\int_a^b (s-a)(b-s)[p(s)]_- ds < +\infty \quad \text{and} \quad \int_a^b (s-a)(b-s)|q(s)| ds < +\infty,$$

then Fredholm's third theorem remains true.

MSC: 34B05

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1 Introduction

Consider the boundary value problem

$$u'' = p(t)u + q(t), \tag{1}$$

$$u(a) = 0, \quad u(b) = 0, \tag{2}$$

where $p, q \in L_{\text{loc}}(]a, b[)$. We are mainly interested in the case, when the functions p and q are not integrable on $[a, b]$. In this case, the problem (1), (2) is said to be singular. It is proved in [1] that if

$$\int_a^b (s-a)(b-s)[p(s)]_- ds < +\infty \tag{3}$$

and

$$\int_a^b (s-a)(b-s)|q(s)| ds < +\infty, \tag{4}$$

then, for the singular problem (1), (2), the Fredholm alternative holds. More precisely, the following theorem is true.

Theorem 1.1 ([1, Theorem 0.1]) *Let (3) hold. Then the problem (1), (2) is uniquely solvable for any q satisfying (4) iff the corresponding homogeneous equation*

$$u'' = p(t)u \tag{1a}$$

has no nontrivial solution satisfying (2).

The aim of this paper is to show that, under the assumption (3), the Fredholm’s third theorem remains true. Before formulation of the main results, we introduce the following notation.

\mathbb{R} is the set of real numbers.

For $x \in \mathbb{R}$, we put $[x]_- = \frac{1}{2}(|x| - x)$.

$C(I)$, where $I \subset \mathbb{R}$, is a set of continuous functions $u : I \rightarrow \mathbb{R}$.

For $u \in C([\alpha, \beta])$, we put $\|u\|_{[\alpha, \beta]} = \max\{|u(t)| : t \in [\alpha, \beta]\}$.

$AC'_{loc}]\alpha, \beta[$ is the set of functions $u :]\alpha, \beta[\rightarrow \mathbb{R}$, which are absolutely continuous together with their first derivative on every closed subinterval of $] \alpha, \beta [$.

$L_{loc}]\alpha, \beta[$ is the set of functions $p :]\alpha, \beta[\rightarrow \mathbb{R}$, which are Lebesgue integrable on every closed subinterval of $] \alpha, \beta [$.

By $f(a)$ (resp., $f(b)$) we denote the right (resp., left) limit of the function $f :]a, b[\rightarrow \mathbb{R}$ at the point a (resp., b).

By a solution of equation (1) we understand a function $u \in AC'_{loc}]a, b[$, which satisfies it almost everywhere in $]a, b[$. A solution of equation (1) satisfying (2) is said to be a solution of the problem (1), (2).

We will say that a certain property holds in $] \alpha, \beta [$ if it takes place on every closed subinterval of $] \alpha, \beta [$.

Recall that we consider the problem (1), (2), where $p, q \in L_{loc}]a, b[$.

Theorem 1.2 *Let (3) hold. Then the homogeneous problem (1a), (2) has no more than one, up to a constant multiple, nontrivial solution.*

Remark 1.1 Below we will show (see Proposition 2.1) that if (3) holds and u_0 is a nontrivial solution of (1a), (2), then there exists $r_0 > 0$ such that

$$|u_0(t)| \leq r_0(t - a)(b - t) \quad \text{for } t \in [a, b].$$

Theorem 1.3 *Let (3) hold and the homogeneous problem (1a), (2) have a nontrivial solution u_0 . Then the problem (1), (2), where the function q satisfies (4), is solvable iff the condition*

$$\int_a^b q(s)u_0(s) ds = 0 \tag{5}$$

is fulfilled.

Remark 1.2 In view of Remark 1.1 and condition (4), the function qu_0 is integrable on $[a, b]$ and, therefore, condition (5) is meaningful.

2 Auxiliary statements

First of all, for convenience of references, we recall two lemmas from [1].

Lemma 2.1 ([1, Lemma 2.1]) *Let (3) and (4) hold. Then, for any $\alpha \in]a, b[$ and $\beta \in]\alpha, b[$, every solution u of equation (1) satisfying*

$$u(\alpha) = 0, \quad u(\beta) = 0$$

admits the estimate

$$(t - a)(b - t)|u'(t)| \leq \|u\|_{[\alpha, \beta]} \left(b - a + \int_a^b (s - a)(b - s)[p(s)]_- ds \right) + \int_a^b (s - a)(b - s)|q(s)| ds \quad \text{for } t \in]\alpha, \beta[.$$

Lemma 2.2 ([1, Lemma 2.2]) *Let (3) hold. Then there exist $a_0 \in]a, b[$, $b_0 \in]a_0, b[$, and $\varrho > 0$ such that, for any $\alpha \in]a, a_0[$, $\beta \in]b_0, b[$, and q satisfying (4), every solution u of equation (1) satisfying*

$$u(\alpha) = 0$$

admits the estimate

$$|u(t)| \leq \varrho \left((t - a)\|u\|_{[\alpha, a_0]} + \int_a^t (s - a)|q(s)| ds + (t - a) \int_t^{a_0} |q(s)| ds \right) \quad \text{for } t \in]\alpha, a_0[,$$

while every solution u of equation (1) satisfying

$$u(\beta) = 0$$

admits the estimate

$$|u(t)| \leq \varrho \left((b - t)\|u\|_{[b_0, \beta]} + \int_t^b (b - s)|q(s)| ds + (b - t) \int_{b_0}^t |q(s)| ds \right) \quad \text{for } t \in [b_0, \beta[.$$

Next proposition immediately follows from Lemma 2.2.

Proposition 2.1 *Let (3) hold and u_0 be a nontrivial solution of the homogeneous problem (1a), (2). Then there exists $r_0 > 0$ such that*

$$|u_0(t)| \leq r_0(t - a)(b - t) \quad \text{for } t \in [a, b].$$

Proposition 2.2 *Let (3) hold and u_0 be a nontrivial solution of (1a) satisfying $u_0(a) = 0$ (respectively, $u_0(b) = 0$). Then there exists $a_1 \in]a, b[$ (respectively, $b_1 \in]a, b[$) such that*

$$u_0(t) \neq 0 \quad \text{for } t \in]a, a_1[\quad (\text{respectively, } u_0(t) \neq 0 \quad \text{for } t \in [b_1, b[). \quad (6)$$

Proof In view of (3) there exists $a_0 \in]a, b[$ (respectively, $b_0 \in]a, b[$) such that

$$\int_a^{a_0} (s - a)[p(s)]_- ds < 1 \quad \left(\text{respectively, } \int_{b_0}^b (b - s)[p(s)]_- ds < 1 \right).$$

Hence, the inequality

$$\int_a^{a_0} (s-a)(a_0-s)[p(s)]_- ds < a_0 - a$$

$$\left(\text{respectively, } \int_{b_0}^b (s-b_0)(b-s)[p(s)]_- ds < b - b_0 \right)$$

holds, as well. The latter inequality, by virtue of [2, Lemma 4.1], implies that for any $a < t_1 < t_2 < a_0$ (respectively, $b_0 < t_1 < t_2 < b$), the problem

$$u'' = p(t)u; \quad u(t_1) = 0, \quad u(t_2) = 0$$

has no nontrivial solution.

Now suppose that u_0 is a nontrivial solution of (1a) satisfying $u_0(a) = 0$ (respectively, $u_0(b) = 0$). Then it follows from the above that either

$$u_0(t) \neq 0 \quad \text{for } t \in]a, a_0] \quad \left(\text{respectively, } u_0(t) \neq 0 \quad \text{for } t \in [b_0, b[\right), \tag{7}$$

or there is a $t_0 \in]a, a_0]$ (respectively, $t_0 \in [b_0, b[$) such that

$$u_0(t) \neq 0 \quad \text{for } t \in]a, t_0[, \quad u_0(t_0) = 0$$

$$\left(\text{respectively, } u_0(t) \neq 0 \quad \text{for } t \in]t_0, b[, \quad u_0(t_0) = 0 \right). \tag{8}$$

It is now clear that (6) holds with $a_1 = a_0$ (respectively, $b_1 = b_0$) if (7) holds, and with $a_1 = \frac{a+t_0}{2}$ (respectively, $b_1 = \frac{t_0+b}{2}$) if (8) is satisfied. \square

Lemma 2.3 *Let (3) and (4) hold. Let, moreover, u be a solution of the problem (1), (2) and u_0 be a solution of the problem (1a), (2). Then*

$$\lim_{t \rightarrow a^+} (u'(t)u_0(t) - u(t)u'_0(t)) = 0, \quad \lim_{t \rightarrow b^-} (u'(t)u_0(t) - u(t)u'_0(t)) = 0. \tag{9}$$

Proof It is clear that

$$(u'(t)u_0(t) - u(t)u'_0(t))' = q(t)u_0(t) \quad \text{for } t \in]a, b[.$$

Hence,

$$u'(t)u_0(t) - u(t)u'_0(t) = \delta - \int_t^c q(s)u_0(s) ds \quad \text{for } t \in]a, b[, \tag{10}$$

where

$$c = \frac{a+b}{2} \quad \text{and} \quad \delta = u'(c)u_0(c) - u(c)u'_0(c).$$

By virtue of Proposition 2.1 and condition (4), the function qu_0 is integrable on $[a, b]$. Thus, it follows from (10) that there exists a finite limit

$$\lim_{t \rightarrow a^+} |u'(t)u_0(t) - u(t)u'_0(t)| = \varepsilon_0. \tag{11}$$

Now we will show that $\varepsilon_0 = 0$. Suppose the contrary, let

$$\varepsilon_0 > 0. \tag{12}$$

Then there is $\alpha \in]a, b[$ such that

$$|u'(t)u_0(t) - u(t)u'_0(t)| > \frac{\varepsilon_0}{2} \quad \text{for } t \in]a, \alpha]. \tag{13}$$

On account of Proposition 2.2, we can assume without loss of generality that

$$u_0(t) \neq 0 \quad \text{for } t \in]a, \alpha]. \tag{14}$$

Then it follows from (13) that

$$\left| \left(\frac{u(t)}{u_0(t)} \right)' \right| > \frac{\varepsilon_0}{2u_0^2(t)} \quad \text{for } t \in]a, \alpha].$$

Hence

$$|\mu u_0(t) - u(t)| > \frac{\varepsilon_0}{2} |u_0(t)| \int_t^\alpha \frac{ds}{u_0^2(s)} \quad \text{for } t \in]a, \alpha], \tag{15}$$

where $\mu = \frac{u(\alpha)}{u_0(\alpha)}$.

Taking now into account Proposition 2.1, we get from (15) that

$$|\mu u_0(t) - u(t)| > \varepsilon_1 |u_0(t)| \left(\frac{1}{t-a} - \frac{1}{\alpha-a} \right) \quad \text{for } t \in]a, \alpha],$$

where $\varepsilon_1 = \frac{\varepsilon_0}{2r_0^2(b-a)^2}$. The latter inequality, in view of the conditions $u_0(a) = 0$ and $u(a) = 0$, implies that

$$\lim_{t \rightarrow a^+} \frac{|u_0(t)|}{t-a} = 0. \tag{16}$$

On the other hand, by virtue of Lemma 2.1, there is $M > 0$ such that

$$(t-a)|u'(t)| \leq M \quad \text{for } t \in]a, \alpha]. \tag{17}$$

In view of (16) and (17), we get

$$\lim_{t \rightarrow a^+} |u'(t)u_0(t)| = \lim_{t \rightarrow a^+} (t-a)|u'(t)| \frac{|u_0(t)|}{t-a} = 0,$$

and therefore, on account of (11), we obtain

$$\lim_{t \rightarrow a^+} |u(t)u'_0(t)| = \varepsilon_0.$$

Now, let $\alpha_0 \in]a, \alpha[$ be such that

$$|u(t)u'_0(t)| > \frac{\varepsilon_0}{2} \quad \text{for } t \in]a, \alpha_0].$$

Then it is clear that

$$\|u\|_{[a,b]} |u'_0(t)| > \frac{\varepsilon_0}{2} \quad \text{for } t \in]a, \alpha_0]$$

and consequently

$$\|u\|_{[a,b]} |u_0(t)| > \frac{\varepsilon_0}{2}(t - a) \quad \text{for } t \in]a, \alpha_0].$$

However, the latter inequality and (16) yield that $\varepsilon_0 \leq 0$, which contradicts (12). The contradiction obtained proves the first equality in (9). By the same arguments one can prove the second equality in (9). \square

We will need the next lemma in the proof of the sufficiency part of Theorem 1.3 and thus, we will suppose that Theorem 1.2 and the necessity part of Theorem 1.3 are true.

Lemma 2.4 *Let (3) hold and the homogeneous problem (1a), (2) have a nontrivial solution u_0 . Then there exist $n_0 \in \mathbb{N}$ and $r > 0$ such that, for any q satisfying (4) and (5) and every $n > n_0$, the solution u of the problem*

$$u'' = \left(p(t) + \frac{1}{n} [p(t)]_- \right) u + q(t); \quad u(a) = 0, \quad u(b) = 0$$

admits the estimate

$$|u(t)| \leq r \int_a^b (s - a)(b - s) |q(s)| ds \quad \text{for } t \in [a, b].$$

Proof Suppose the contrary, let the assertion of the lemma be violated. Then, for any $n \in \mathbb{N}$, there exist $k_n \geq n$, $q_n \in L_{\text{loc}}(]a, b[)$, and $u_n \in AC'_{\text{loc}}(]a, b[)$ such that

$$\int_a^b (s - a)(b - s) |q_n(s)| ds < +\infty, \quad \int_a^b q_n(s) u_0(s) ds = 0,$$

$$u''_n(t) = \left(p(t) + \frac{1}{k_n} [p(t)]_- \right) u_n(t) + q_n(t) \quad \text{for } t \in]a, b[,$$

$$u_n(a) = 0, \quad u_n(b) = 0$$

and

$$\|u_n\|_{[a,b]} > n \int_a^b (s - a)(b - s) |q_n(s)| ds.$$

Introduce the notation

$$\tilde{u}_n(t) = \frac{1}{\|u_n\|_{[a,b]}} u_n(t), \quad \tilde{q}_n(t) = \frac{1}{\|u_n\|_{[a,b]}} q_n(t) \quad \text{for } t \in]a, b[.$$

Then it is clear that

$$\tilde{u}''_n(t) = \left(p(t) + \frac{1}{k_n} [p(t)]_- \right) \tilde{u}_n(t) + \tilde{q}_n(t) \quad \text{for } t \in]a, b[, \tag{18}$$

$$\begin{aligned} \tilde{u}_n(a) = 0, \quad \tilde{u}_n(b) = 0, \\ \|\tilde{u}_n\|_{[a,b]} = 1, \end{aligned} \tag{19}$$

$$\int_a^b (s-a)(b-s)|\tilde{q}_n(s)| ds < \frac{1}{n} \tag{20}$$

and

$$\int_a^b \tilde{q}_n(s)u_0(s) ds = 0. \tag{21}$$

By virtue of Lemma 2.1 (with $q(t) = \frac{1}{k_n} [p(t)]_- \tilde{u}_n(t) + \tilde{q}_n(t)$) and (19), we have

$$\begin{aligned} (t-a)(b-t)|\tilde{u}'_n(t)| \leq b-a + \frac{n+1}{n} \int_a^b (s-a)(b-s)[p(s)]_- ds \\ + \int_a^b (s-a)(b-s)|\tilde{q}_n(s)| ds \quad \text{for } t \in]a, b[, \end{aligned} \tag{22}$$

while, by virtue of Lemma 2.2 (with $q(t) = \frac{1}{k_n} [p(t)]_- \tilde{u}_n(t) + \tilde{q}_n(t)$), there exist $a_0 \in]a, b[$, $b_0 \in]a_0, b[$, and $\varrho > 0$ such that

$$\begin{aligned} |\tilde{u}_n(t)| \leq \varrho \left[t-a + \int_a^{a_0} (s-a) \left| \frac{1}{k_n} [p(s)]_- \tilde{u}_n(s) + \tilde{q}_n(s) \right| ds \right] \quad \text{for } t \in]a, a_0[, \\ |\tilde{u}_n(t)| \leq \varrho \left[b-t + \int_t^{b_0} (b-s) \left| \frac{1}{k_n} [p(s)]_- \tilde{u}_n(s) + \tilde{q}_n(s) \right| ds \right] \quad \text{for } t \in [b_0, b]. \end{aligned} \tag{23}$$

On account of (19) and (22), the sequence $\{u_n\}_{n=1}^{+\infty}$ is uniformly bounded and equicontinuous in $]a, b[$. Thus, by virtue of the Arzelà-Ascoli lemma, we can assume without loss of generality that

$$\lim_{n \rightarrow +\infty} \tilde{u}_n(t) = v_0(t) \quad \text{uniformly in }]a, b[, \tag{24}$$

where $v_0 \in C([a, b])$ and, moreover,

$$\lim_{n \rightarrow +\infty} \tilde{u}'_n\left(\frac{a+b}{2}\right) = c_0. \tag{25}$$

In view of (18) it is clear that

$$\begin{aligned} \tilde{u}_n(t) = \tilde{u}_n\left(\frac{a+b}{2}\right) + \left(t - \frac{a+b}{2}\right) \tilde{u}'_n\left(\frac{a+b}{2}\right) \\ + \int_{\frac{a+b}{2}}^t \left(\int_{\frac{a+b}{2}}^s \left[\left(p(\xi) + \frac{1}{k_n} [p(\xi)]_- \right) \tilde{u}_n(\xi) + \tilde{q}_n(\xi) \right] d\xi \right) ds \quad \text{for } t \in]a, b[. \end{aligned}$$

Hence, on account of (19), (20), (24), and (25), we get

$$v_0(t) = v_0\left(\frac{a+b}{2}\right) + c_0 \left(t - \frac{a+b}{2}\right) + \int_{\frac{a+b}{2}}^t \left(\int_{\frac{a+b}{2}}^s p(\xi) v_0(\xi) d\xi \right) ds \quad \text{for } t \in]a, b[.$$

Therefore, $v_0 \in AC'_{loc}(]a, b[)$ and v_0 is a solution of equation (1a). On the other hand, it follows from (23), in view of (19), (20), and (24), that

$$|v_0(t)| \leq \varrho(t - a) \quad \text{for } t \in]a, a_0] \quad \text{and} \quad |v_0(t)| \leq \varrho(b - t) \quad \text{for } t \in [b_0, b[,$$

and thus v_0 is a solution of the problem (1a), (2).

By virtue of (20) and (23), it is clear that there are $n_1 \in \mathbb{N}$, $a_1 \in]a, a_0]$, and $b_1 \in [b_0, b[$ such that

$$|\tilde{u}_n(t)| < 1 \quad \text{for } t \in [a, a_1] \cup [b_1, b], n > n_1.$$

Therefore, $\|\tilde{u}_n\|_{[a_1, b_1]} = 1$ for $n > n_1$. Taking now into account (24), we get that $\|v_0\|_{[a_1, b_1]} = 1$ and, therefore, v_0 is a nontrivial solution of the problem (1a), (2).

By virtue of Theorem 1.2, there is $\lambda \neq 0$ such that

$$v_0(t) = \lambda u_0(t) \quad \text{for } t \in [a, b]. \tag{26}$$

Moreover, in view of the necessity part of Theorem 1.3 (with $q(t) = \frac{1}{k_n}[p(t)]_- \tilde{u}_n(t) + \tilde{q}_n(t)$), (18), (19), (21), and (26), we get

$$\int_a^b [p(s)]_- \tilde{u}_n(s) v_0(s) ds = 0. \tag{27}$$

Let now $\alpha \in]a, b[$ and $\beta \in]\alpha, b[$ be arbitrary. Then, in view of (24), we have

$$\lim_{n \rightarrow +\infty} \int_\alpha^\beta [p(s)]_- \tilde{u}_n(s) v_0(s) ds = \int_\alpha^\beta [p(s)]_- v_0^2(s) ds. \tag{28}$$

On account of (3), (26), and Proposition 2.1, the function $[p]_- v_0$ is integrable on $[a, b]$. Taking into account (19), we get

$$\left| \int_a^\alpha [p(s)]_- \tilde{u}_n(s) v_0(s) ds \right| \leq \int_a^\alpha [p(s)]_- |v_0(s)| ds$$

and

$$\left| \int_\beta^b [p(s)]_- \tilde{u}_n(s) v_0(s) ds \right| \leq \int_\beta^b [p(s)]_- |v_0(s)| ds.$$

Hence, (27) implies the inequality

$$\int_\alpha^\beta [p(s)]_- \tilde{u}_n(s) v_0(s) ds \leq \int_a^\alpha [p(s)]_- |v_0(s)| ds + \int_\beta^b [p(s)]_- |v_0(s)| ds,$$

which, together with (28), results in

$$\int_\alpha^\beta [p(s)]_- v_0^2(s) ds \leq \int_a^\alpha [p(s)]_- |v_0(s)| ds + \int_\beta^b [p(s)]_- |v_0(s)| ds.$$

Since α and β were arbitrary, we get from the latter inequality that

$$\int_a^b [p(s)]_- v_0^2(s) ds = 0.$$

Taking now into account that $v_0 \not\equiv 0$, we get $[p]_- \equiv 0$, i.e., $p(t) \geq 0$ for $t \in]a, b[$. However, in this case the problem (1a), (2) has no nontrivial solution, which contradicts the assumption of the lemma. \square

3 Proofs

Proof of Theorem 1.2 Let u_0 and v_0 be any nontrivial solutions of (1a). By virtue of Lemma 2.3 (with $u \equiv v_0$ and $q \equiv 0$), we get

$$\lim_{t \rightarrow a^+} (u_0'(t)v_0(t) - u_0(t)v_0'(t)) = 0.$$

On the other hand, clearly

$$(u_0'(t)v_0(t) - u_0(t)v_0'(t))' = 0 \quad \text{for } t \in]a, b[,$$

and, therefore,

$$u_0'(t)v_0(t) - u_0(t)v_0'(t) = 0 \quad \text{for } t \in [a, b]. \tag{29}$$

Choose $t_0 \in]a, b[$ such that

$$u_0'(t_0) = 0.$$

It is clear that $u_0(t_0) \neq 0$ since otherwise $u_0 \equiv 0$. Then it follows from (29) that

$$v_0'(t_0) = 0$$

and as above $v_0(t_0) \neq 0$. Put $\lambda = \frac{u_0(t_0)}{v_0(t_0)}$ and

$$w(t) = u_0(t) - \lambda v_0(t) \quad \text{for } t \in [a, b].$$

Evidently, w is a solution of equation (1a) and $w(t_0) = 0$. However, it follows from (29) that $w'(t_0) = 0$. Consequently, $w \equiv 0$ and thus $u_0 \equiv \lambda v_0$. \square

Proof of Theorem 1.3 Let u_0 be a nontrivial solution of (1a), (2) while u be a solution of (1), (2). Put

$$f(t) = u'(t)u_0(t) - u(t)u_0'(t) \quad \text{for } t \in]a, b[.$$

It is clear that

$$f'(t) = q(t)u_0(t) \quad \text{for } t \in]a, b[.$$

Hence,

$$f\left(\frac{a+b}{2}\right) - f(t) = \int_t^{\frac{a+b}{2}} q(s)u_0(s) ds \quad \text{for } t \in]a, b[. \tag{30}$$

By virtue of Lemma 2.3, Proposition 2.1, and condition (4), we get from (30) that

$$f\left(\frac{a+b}{2}\right) = \int_a^{\frac{a+b}{2}} q(s)u_0(s) ds \quad \text{and} \quad f\left(\frac{a+b}{2}\right) = - \int_{\frac{a+b}{2}}^b q(s)u_0(s) ds,$$

and therefore (5) is fulfilled.

Let now u_0 be a nontrivial solution of (1a), (2), $q \in L_{loc}(]a, b[)$ satisfy (4), and (5) be fulfilled. Let, moreover, $n_0 \in \mathbb{N}$ and $r > 0$ be from the assertion of Lemma 2.4. By virtue of Lemma 2.4, for any $n > n_0$, the problem

$$u'' = \left(p(t) + \frac{1}{n} [p(t)]_- \right) u; \quad u(a) = 0, \quad u(b) = 0$$

has no nontrivial solution. Since

$$\left[p(t) + \frac{1}{n} [p(t)]_- \right]_- = \frac{n-1}{n} [p(t)]_- \tag{31}$$

and (3) holds, it follows from Theorem 1.1 that, for any $n > n_0$, the problem

$$u'' = \left(p(t) + \frac{1}{n} [p(t)]_- \right) u + q(t); \quad u(a) = 0, \quad u(b) = 0 \tag{32}$$

has a unique solution u_n .

In view of Lemma 2.4, the inequalities

$$|u_n(t)| \leq M \quad \text{for } t \in]a, b[, n > n_0 \tag{33}$$

are fulfilled, where

$$M = r \int_a^b (s-a)(b-s) |q(s)| ds.$$

On the other hand, on account of Lemma 2.1, (31), and (33), we get

$$(t-a)(b-t) |u'_n(t)| \leq M_1 \quad \text{for } t \in]a, b[, n > n_0, \tag{34}$$

where

$$M_1 = M \left(b-a + 2 \int_a^b (s-a)(b-s) [p(s)]_- ds \right) + \int_a^b (s-a)(b-s) |q(s)| ds.$$

It follows from (33) and (34) that the sequence $\{u_n\}_{n=n_0}^{+\infty}$ is uniformly bounded and equicontinuous in $]a, b[$. Hence, by virtue of the Arzelà-Ascoli lemma, we can assume without loss

of generality that

$$\lim_{n \rightarrow +\infty} u_n = u(t) \quad \text{uniformly in }]a, b[, \tag{35}$$

where $u \in C(]a, b[)$ and, moreover,

$$\lim_{n \rightarrow +\infty} u'_n \left(\frac{a+b}{2} \right) = c_0. \tag{36}$$

In view of (32), it is clear that

$$\begin{aligned} u_n(t) &= u_n \left(\frac{a+b}{2} \right) + \left(t - \frac{a+b}{2} \right) u'_n \left(\frac{a+b}{2} \right) \\ &\quad + \int_{\frac{a+b}{2}}^t \left(\int_{\frac{a+b}{2}}^s \left[\left(p(\xi) + \frac{1}{n} [p(\xi)]_- \right) u_n(\xi) + q(\xi) \right] d\xi \right) ds \quad \text{for } t \in]a, b[. \end{aligned}$$

Hence, on account of (35) and (36), we get that

$$\begin{aligned} u(t) &= u \left(\frac{a+b}{2} \right) + c_0 \left(t - \frac{a+b}{2} \right) \\ &\quad + \int_{\frac{a+b}{2}}^t \left(\int_{\frac{a+b}{2}}^s [p(\xi)u(\xi) + q(\xi)] d\xi \right) ds \quad \text{for } t \in]a, b[. \end{aligned}$$

Therefore, $u \in AC'_{\text{loc}}(]a, b[)$ and u is a solution of equation (1).

On the other hand, by virtue of Lemma 2.2 and (33), there are $a_0 \in]a, b[$, $b_0 \in]a_0, b[$, and $\varrho > 0$ such that, for any $n > n_0$, the inequalities

$$\begin{aligned} |u_n(t)| &\leq \varrho \left[M(t-a) + \int_a^t (s-a) \left(\frac{M}{n} [p(s)]_- + |q(s)| \right) ds \right. \\ &\quad \left. + (t-a) \int_t^{a_0} \left(\frac{M}{n} [p(s)]_- + |q(s)| \right) ds \right] \quad \text{for } t \in]a, a_0] \end{aligned}$$

and

$$\begin{aligned} |u_n(t)| &\leq \varrho \left[M(b-t) + \int_t^b (b-s) \left(\frac{M}{n} [p(s)]_- + |q(s)| \right) ds \right. \\ &\quad \left. + (b-t) \int_{b_0}^t \left(\frac{M}{n} [p(s)]_- + |q(s)| \right) ds \right] \quad \text{for } t \in [b_0, b[\end{aligned}$$

are fulfilled. Hence, in view of (35), we get

$$|u(t)| \leq \varrho \left[M(t-a) + \int_a^t (s-a) |q(s)| ds + (t-a) \int_t^{a_0} |q(s)| ds \right] \quad \text{for } t \in]a, a_0]$$

and

$$|u(t)| \leq \varrho \left[M(b-t) + \int_t^b (b-s) |q(s)| ds + (b-t) \int_{b_0}^t |q(s)| ds \right] \quad \text{for } t \in [b_0, b[.$$

Consequently, u satisfies (2) and thus u is a solution of the problem (1), (2). □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors read and approved the final manuscript.

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