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ÚSTAV MATEMATIKY

STOCHASTIC CALCULUS AND ITS APPLICATIONS IN BIOMEDICAL PRACTICE
STOCHASTICKÝ KALKULUS A JEHO APLIKACE V BIOMEDICÍNSKÉ PRAXI

DOCTORAL THESIS
DIZERTAČNÍ PRÁCE

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ABSTRACT

In the presented dissertation is defined the stochastic differential equation and its basic properties are listed. Stochastic differential equations are used to describe physical phenomena, which are also influenced by random effects. Solution of the stochastic model is a random process. Objective of the analysis of random processes is the construction of an appropriate model, which allows understanding the mechanisms. On their basis observed data are generated. Knowledge of the model also allows forecasting the future and it is possible to control and optimize the activity of the applicable system. In this dissertation is at first defined probability space and Wiener process. On this basis is defined the stochastic differential equation and the basic properties are indicated. The final part contains biology model illustrating the use of the stochastic differential equations in practice.

KEYWORDS

Stochastic process, stochastic differential equation, Brownian motion, Wiener process, matrix equations

ABSTRAKT

V předložené práci je definována stochastická diferenciální rovnice a jsou uvedeny její základní vlastnosti. Stochastické diferenciální rovnice se používají k popisu fyzikálních jevů, které jsou ovlivněny i náhodnými vlivy. Řešením stochastického modelu je náhodný proces. Cílem analýzy náhodných procesů je konstrukce vhodného modelu, který umožní porozumět mechanismům, na jejichž základech jsou generována sledovaná data. Znalost modelu také umožňuje předvídání budoucnosti a je tak možné kontrolovat a optimalizovat činnost daného systému. V práci je nejdříve definován pravděpodobnostní prostor a Wienerův proces. Na tomto základě je definována stochastická diferenciální rovnice a jsou uvedeny její základní vlastnosti. Závěrečná část práce obsahuje biologický model ilustrující použití stochastických diferenciálních rovnic v praxi.

KLÍČOVÁ SLOVA

Náhodný proces, stochastická diferenciální rovnice, Brownův pohyb, Wienerův proces, maticové rovnice

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DECLARATION

I declare that I have written the Doctoral Thesis titled "Stochastic calculus and its applications in biomedical practice" independently, under the guidance of the advisor and using exclusively the technical references and other sources of information cited in the thesis and listed in the comprehensive bibliography at the end of the thesis.

As the author I furthermore declare that, with respect to the creation of this Doctoral Thesis, I have not infringed any copyright or violated anyone's personal and/or ownership rights. In this context, I am fully aware of the consequences of breaking Regulation § 11 of the Copyright Act No. 121/2000 Coll. of the Czech Republic, as amended, and of any breach of rights related to intellectual property or introduced within amendments to relevant Acts such as the Intellectual Property Act or the Criminal Code, Act No. 40/2009 Coll., Section 2, Head VI, Part 4.

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1 INTRODUCTION

Theory of stochastic differential equations is used to describe the physical and technical phenomena. We have chosen this topic because randomness is actual issue for the last years. The solution of the stochastic model is a random process. The aim of the study of random phenomena is the construction of a suitable model to understand its working. Knowledge of the model provides to predict the future behavior of the system and making it possible to control and optimize the operation of the corresponding system.

Stochastic processes can be applied to solve problems in various fields of science. In practical situations we meet with random events which take place in time. Stochasticity is very important in physics, technology, economy and biology.

In biological science, introducing stochastic noise has been found to help improve the signal strength of the internal feedback loops for balance and other communication. It has been found to help diabetic and stroke patients with balance control. Many biochemical events also lend themselves to stochastic analysis. Gene expression, for example, has a stochastic component through the molecular collisions as during binding and unbinding of RNA polymerase to a gene promoter via the solution's Brownian motion. In biology, it also can be a monitoring of various parameters of air pollution, EEG, EKG records in medicine, multiplication processes (bacteria), etc.

In medicine science, stochastic effect is one classification of radiation effects that refers to the random, statistical nature of the damage. In contrast to the deterministic effect, severity is independent of dose. Only the probability of an effect increases with dose.

In epidemic science, Markov processes are used to model of epidemic diseases in small populations, among many other phenomena. There exist algorithms like the SSA that simulates a single trajectory with the exact distribution of the process, but it can be time-consuming when many reactions take place during a short time interval.

In social science, there it can be processes of mortality and disability of the population, changes in population.

In physical and technical science, it can be solved the seismic record in geophysics, series of daily maximum temperatures in meteorology, course of the output signal of the electric devices, changes in the number calls on a phone line.

In mathematical finance science, we can suppose a person has an asset or resource (e.g. a house, stocks, oil...) that she is planning to sell. At what time should she decide to sell? Or we can suppose that the person is offered to buy one unit of the risky asset at a specified price. How much should the person be willing to pay for such an option? These and other problems were solved using stochastic analysis.

The submitted thesis deals with finding solutions of stochastic differential equations and systems of stochastic differential equations, and determining their stability. The main part of the thesis is based on the theory Stochastic Differential Equations - an introduction with applications by B. Ôksendal [114]. B. Maslowski discusses stochastic equations and stochastic methods in partial differential equations [106]. The book Stochastic bio mathematical models: with applications to neuronal modeling by S. Ditlevsen at al. [38] concerns with noise in living systems. Fundamental knowledge of probability and mathematical statistics is in textbook by M. Navara [111]. The book by X. Mao [102] describes the basic principles and applications of various types of stochastic systems. The book by R. Z. Khasminskii [75] deals with the stochastic stability of differential equations, exact formulas for the Lyapunov exponent, the criteria for the moment and almost sure stability, and for the existence of solutions of stochastic differential equations have been widely used. There are derived conditions for the stability of the mean zero solution stochastic equations with Brownian motion. There is used the Lyapunov method to determine the stability of the solution of the stochastic system. This method for analyzing the behavior of stochastic differential equations provides useful information for the study of stability and its properties for special types of stochastic dynamical systems, allows to specify conditions for the existence of stationary solutions of stochastic differential equations and related problems.

Main results determined the solution and the stability of solutions of differential systems of order 3 and 4. This basic is extended by the stochastic process and there is looked for the solution and the stability of stochastic differential equations and stochastic differential systems (matrix equations). Theoretical results are illustrated on the model of medical practice.

2 ORDINARY SYSTEM THEORY

The thesis assumes knowledge of differential equations. Kalas and Ráb [71] and many other authors present ordinary differential equation theory.

The behavior of systems is described by the system's own solution, which can generally have infinitely many solutions corresponding to the different choices of its initial state. For most systems, we require their behavior is in some sense close to one of the advance given behavior of the system (staying at rest, periodic motion, etc.). Diblík at al. [37] describe the ordinary differential theory and the stability of solutions of ordinary differential equations.

The simplicity of the fundamental principles of the theory of linear differential equations helped towards a development in the theory of linear oscillations, see Asymptotic Methods in the Theory of Non-Linear Oscillations [20] by N. N. Bogoliubov at al.

Differential-difference equations were studied by R. E. Bellman and K. L. Cooke in [17].

Introduction to functional differential equations is presented by J. Hale at al. in [67]-[69].

Authors M. A. Aizeman and F. R. Gantmacher studied absolute stability of regulator systems in [2].

Nonlinear systems and their stability is studied in [58] -[59] by A. H. Gelig at al.

There are many publications and studies related the behavior of ordinary differential equations and systems. However, this issue is not an objective of this thesis hence within this chapter we suggest the basic knowledge of ODE systems, and we confine to the differential system in the form

$$\mathrm{d}X_t = AX_t\mathrm{d}t,$$

which we will extend about the stochastic argument and will study in detail in the following chapters.

2.1 Ordinary Differential Systems

We have a system of ordinary differential equations in the form

$$\mathrm{d}X_t = AX_t\mathrm{d}t, \tag{2.1}$$

where $X_t = \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix}$ is a vector of unknown functions, $\mathrm{d}X_t = \begin{pmatrix} \mathrm{d}X_1(t) \\ \mathrm{d}X_2(t) \end{pmatrix}$, $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ is a matrix, where $a_1, a_2, a_3, a_4 \in \mathbb{R}$.

We determine a determinant of the matrix A

$$|A| = \begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix} = a_1a_4 - a_2a_3,$$

and distinguish two basic cases according to whether the matrix A is singular or regular.

2.1.1 Stability of the Singular Matrix Solution

The matrix A is singular, when $|A| = 0, a_1a_4 = a_2a_3$, i.e. the matrix A is in the form

$$A_s = \begin{pmatrix} a_1 & a_2 \\ ka_1 & ka_2 \end{pmatrix},$$

$k \in \mathbb{R}$. We look for a solution of the equation in the form $\mathrm{d}X_t = A_sX_t\mathrm{d}t$, i.e.

$$\begin{pmatrix} \mathrm{d}X_1(t) \\ \mathrm{d}X_2(t) \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ ka_1 & ka_2 \end{pmatrix} \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} \mathrm{d}t.$$

The system is itemized into a system of equations and look for solutions $X_1(t)$ a $X_2(t)$

$$\begin{aligned} \mathrm{d}X_1(t) &= a_1X_1(t)\mathrm{d}t + a_2X_2(t)\mathrm{d}t \\ \mathrm{d}X_2(t) &= ka_1X_1(t)\mathrm{d}t + ka_2X_2(t)\mathrm{d}t = k\mathrm{d}X_1(t). \end{aligned}$$

Using the eigenvalues finding solutions. We solve $|A - \lambda I| = 0$.

$$\begin{vmatrix} a_1 - \lambda & a_2 \\ ka_1 & ka_2 - \lambda \end{vmatrix} = \lambda^2 - \lambda(a_1 + a_2k) = 0,$$

$$\lambda_1 = 0, \lambda_2 = a_1 + a_2k.$$

We determine the eigenvectors v_1 a v_2 corresponding to their eigenvalues λ_1 a λ_2

$$\begin{pmatrix} a_1 - \lambda_{1,2} & a_2 \\ ka_1 & ka_2 - \lambda_{1,2} \end{pmatrix} v_{1,2} = \Theta,$$

where Θ is a zero vector.
The vector $v_1 = (v_{11}, v_{12})^T$ for $\lambda_1 = 0$:

$$\begin{pmatrix} a_1 & a_2 \\ ka_1 & ka_2 \end{pmatrix} v_1 = \Theta \Rightarrow a_1v_{11} + a_2v_{12} = 0,$$

$$v_{11} = -\frac{a_2}{a_1} v_{12} \Rightarrow v_1 = \begin{pmatrix} -a_2 \\ a_1 \end{pmatrix}.$$

The vector $v_2 = (v_{21}, v_{22})^T$ for $\lambda_2 = a_1 + ka_2$:

$$\begin{pmatrix} -ka_2 & a_2 \\ ka_1 & -a_1 \end{pmatrix} v_2 = \Theta \Rightarrow$$

$$\begin{aligned} -ka_2 v_{21} + a_2 v_{22} &= 0 \\ ka_1 v_{21} - a_1 v_{22} &= 0, \end{aligned}$$

$$kv_{21} = v_{22} \Rightarrow v_2 = \begin{pmatrix} 1 \\ k \end{pmatrix}.$$

$$X_t = C_1 v_1 e^{0t} + C_2 v_2 e^{(a_1+ka_2)t} = C_1 \begin{pmatrix} -a_2 \\ a_1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ k \end{pmatrix} e^{(a_1+ka_2)t},$$

$$\begin{aligned} X_1(t) &= -C_1 a_2 + C_2 e^{(a_1+ka_2)t} \\ X_2(t) &= C_1 a_1 + C_2 k e^{(a_1+ka_2)t}. \end{aligned}$$

Theorem 2.1.1. *The solution of the equation* (2.1) *is stable if* $\lambda_2 < 0$, *i.e.:*

(i) for $k > 0$ *must be* $a_1 < 0 \wedge a_2 < 0$ *or* $a_1 > 0 \wedge a_2 < 0 \wedge (a_1 + ka_2) < 0$,

(ii) for $k < 0$ *must be* $a_1 < 0 \wedge a_2 > 0$ *or* $a_1 < 0 \wedge a_2 < 0 \wedge (a_1 + ka_2) < 0$,

(iii) for $k = 0$ *must be* $a_1 < 0$.

Proof. Follows from [37].

2.1.2 Stability of the Regular Matrix Solution

The matrix A is regular, when $|A| \neq 0$, i.e. $a_1 a_4 \neq a_2 a_3$,

$$\begin{aligned} \mathrm{d}X_1(t) &= a_1 X_1(t)\mathrm{d}t + a_2 X_2(t)\mathrm{d}t, \\ \mathrm{d}X_2(t) &= a_3 X_1(t)\mathrm{d}t + a_4 X_2(t)\mathrm{d}t. \end{aligned}$$

From the first equation follows

$$X_2(t) = \frac{1}{a_2} X_1(t)' - \frac{a_1}{a_2} X_1(t)$$

and after derivation we get

$$X_2(t)' = \frac{1}{a_2} X_1(t)'' - \frac{a_1}{a_2} X_1(t)'.$$

Substituting this equation into the second equation we have

$$\frac{1}{a_2}X_1(t)'' - \frac{a_1}{a_2}X_1(t)' = a_3X_1(t) + a_4\left(\frac{X_1(t)' - a_1X_1(t)}{a_2}\right)$$

and after adjusting we get

$$X_1(t)'' - (a_1 + a_4)X_1(t)' + (a_1a_4 - a_2a_3)X_1(t) = 0.$$

Its characteristic equation is

$$\lambda^2 - (a_1 + a_4)\lambda + (a_1a_4 - a_2a_3) = 0.$$

It is a quadratic equation with roots

$$\begin{aligned} \lambda_1 &= \frac{a_1 + a_4 + \sqrt{(a_1 - a_4)^2 + 4a_2a_3}}{2}, \\ \lambda_2 &= \frac{a_1 + a_4 - \sqrt{(a_1 - a_4)^2 + 4a_2a_3}}{2}. \end{aligned}$$

Example 2.1.2. We choose the matrix A is symmetrical, i.e. $A_r = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 \end{pmatrix}$, $|a_1| \neq |a_2|$.
Eigenvalues

$$\lambda_{1,2} = a_1 \pm |a_2|.$$

For

$$\begin{array}{cc} a_2 > 0 & a_2 \leq 0 \\ \lambda_1 = a_1 + a_2 & \lambda_1 = a_1 - a_2 \\ \lambda_2 = a_1 - a_2 & \lambda_2 = a_1 + a_2 \end{array}$$

we get

$$\begin{aligned} X_1(t) &= C_1e^{(a_1+a_2)t} + C_2e^{(a_1-a_2)t} \\ X_2(t) &= C_1e^{(a_1+a_2)t} - C_2e^{(a_1-a_2)t}. \end{aligned}$$

Theorem 2.1.3. *For the solution of the system* (2.1), *the solution:*

(i) for $a_1 < 0 \wedge a_2 < 0$ *is stable, if*

$$\begin{aligned} \lambda_1 < 0 &\Rightarrow (a_1 + a_2) < 0 \Leftrightarrow |a_1| > |a_2| \\ \lambda_2 < 0 &\Rightarrow (a_1 - a_2) < 0 \Leftrightarrow |a_1| > |a_2|, \end{aligned}$$

(ii) for $a_1 < 0 \wedge a_2 > 0$ *is stable, if*

$$\begin{aligned} \lambda_1 < 0 &\Rightarrow (a_1 + a_2) < 0 \Leftrightarrow |a_1| > a_2 \\ \lambda_2 < 0 &\Rightarrow (a_1 - a_2) < 0 \textit{ is always valid,} \end{aligned}$$

(iii) for $a_1 > 0 \wedge a_2 < 0$

$$\begin{aligned}
\lambda_1 < 0 &\quad\Rightarrow\quad (a_1 + a_2) < 0 \Leftrightarrow |a_1| < |a_2| \\
\lambda_2 < 0 &\quad\Rightarrow\quad (a_1 - a_2) < 0 \Leftrightarrow \textit{never},
\end{aligned}$$

is always unstable,

(iv) pro $a_1 > 0 \wedge a_2 > 0$

$$\begin{aligned}
\lambda_1 < 0 &\quad\Rightarrow\quad (a_1 + a_2) < 0 \Leftrightarrow \textit{never} \\
\lambda_2 < 0 &\quad\Rightarrow\quad (a_1 - a_2) < 0 \Leftrightarrow |a_1| < |a_2|\,,
\end{aligned}$$

is always unstable.

Proof. Follows from [37].

3 STOCHASTIC SYSTEM THEORY

The first English-language text to offer detailed coverage of boundless, stability, and asymptotic behavior of linear and nonlinear differential equations was issued in the 50s of 20th century by R. Bellman [17].

The basic probability theory is introduced in the work Probability through problems [21] of authors M. Capinski and T. J. Zastawniak and in [39] by R. Durrett. Theory of matrices, their applications is described in the following literature [3], [48], [56]-[57], [60]-[61], [72]-[74], [91], [93], [99], [107]-[108], [121]-[122], [128]-[133], [135]-[137], etc.

In the paper [5] authors J. Baštinec and I. Dzahalladova investigate sufficient conditions for stability of solutions of systems of nonlinear differential equations with right-hand side depending on Markov's process and the basic role in proof have Lyapunov functions.

Stochastic differential equations and applications is presented in [55] by A. Friedman, in [62] by J. I. Gikhman and A. V. Skorokhod, in [66] by D. V. Gusak, in [79]-[80] by E. Kolářová and L. Brančík, in [124] by E. Renshaw and in [126] by M. Růžičková at al.

B. P. Demidovich discuss mathematical theory of stability in [26].

Stability of motion is stated in [23] by N. G. Chetaev.

Stability and time-optimal control of hereditary systems is stated in [24] by E. N. Chukwu.

J. A. Daletskii and M. G. Crane investigate the stability of differential equations solutions in Banach space [25].

J. Carkovs at al. present stochastic stability of Markov dynamical systems in [22].

I. Dzhalladova at al. deal with stability for solutions of stochastic systems and stochastic systems with delay in papers [41]-[44]. I. Dzhalladova analyzes optimization of stochastic systems in [40].

J. Diblík at al. investigate in papers [27]-[36] stability and estimation of solutions of differential systems and systems with delay.

Method of Lyapunov in stability theory is investigated by A. O. Ignatyev at al. in [70], by D. Y. Khusainov in [76]-[78], by N. N. Krasovskii in [87]-[90], by V. M. Kuntsevich at al. in [92], by J. P. La Salle at al. in [94], by A. M. Lyapunov in [100], by H. Rush at al. in [125], by K. G. Valeev at al. in [134], and by V. I. Zubov in [140].

Stability of functional differential equations is described in [81]-[86], [95], [101], [127], [138]-[139], etc.

3.1 Probability Spaces, Random Variables

Definition 3.1.1. [114] If Ω is a given set, then a σ-algebra $\mathcal{F}$ on Ω is a family $\mathcal{F}$ of subsets of Ω with the following properties:

(i) $\emptyset \in \mathcal{F}$

(ii) $F \in \mathcal{F} \Rightarrow F^C \in \mathcal{F}$, where $F^C = \Omega \setminus \mathcal{F}$ is the complement of $\mathcal{F}$ in Ω

(iii) $A_1, A_2, \cdots \in \mathcal{F} \Rightarrow A := \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

The pair $(\Omega, \mathcal{F})$ is called a measurable space.

Definition 3.1.2. [114] A probability measure P on a measurable space $(\Omega, \mathcal{F})$ is a function $P : \mathcal{F} \longrightarrow [0, 1]$ such that

(i) $P(\emptyset) = 0, P(\Omega) = 1$.

(ii) if $A_1, A_2, \cdots \in \mathcal{F}$ and $\{A_i\}_{i=1}^{\infty}$ is disjoint (i.e. $A_i \cap A_j = \emptyset$ if $i \neq j$) then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

The triple $(\Omega, \mathcal{F}, P)$ is called a probability space. It is called a complete probability space if $\mathcal{F}$ contains all subsets G of Ω with P-outer measure zero, i.e. with

$$P^*(G) := \inf\{P(F); F \in \mathcal{F}, G \subset F\} = 0.$$

3.2 Brownian Motion

One of the simplest continuous-time stochastic processes is Brownian motion. This was first observed by botanist Robert Brown. He observed that pollen grains suspended in liquid performed an irregular motion. The motion was later explained by the random collisions with the molecules of the liquid. The motion was describe mathematically by Norbert Wiener who used the concept of a stochastic process $W_t(\omega)$, interpreted as the position at time t of the pollen grain ω. Thus, this process is also called Wiener process.

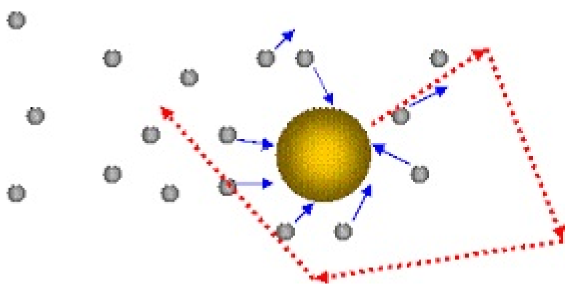

Fig. 3.1: Brownian motion [19]

3.2.1 Basic Properties of Brownian Motion

Definition 3.2.1. The stochastic process B_t is called Brownian motion (or Wiener process) if the process has some basic properties:

(i) $B_0 = 0$

(ii) $B_t - B_s$ has the distribution $N(0, t-s)$ for $t \geq s \geq 0$

(iii) B_t has independent increments, i.e.

$$B_{t_1},\ B_{t_2} - B_{t_1},\ \ldots,\ B_{t_k} - B_{t_{k-1}}$$

are independent for all $0 \leq t_1 < t_2 \cdots < t_k$.

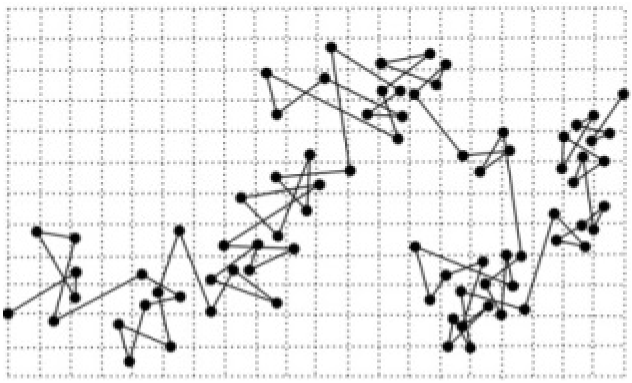

Fig. 3.2: A sample path of Brownian motion [47]

Remark. The unconditional probability density function at a fixed time t

$$f_{B_t}(x) = \frac{1}{\sqrt{2\pi t}} \mathrm{e}^{-\frac{x^2}{2t}}.$$

It holds that

(i) $E[B_t] = 0$ for $t > 0$.

(ii) $E[B_t^2] = t$.

Theorem 3.2.1. *Let B_t be Brownian motion. Then*

$$E[B_t B_s] = min\{t, s\}\ \ for\ t \geq 0,\ s \geq 0.$$

Proof. [114], pp. 14.

Definition 3.2.2. Let $B_i(t), t = 1, 2, \ldots, m$, be a stochastic process. Then $B(t) = (B_1(t), ..., B_m(t))$ denote m-dimensional Brownian motion.

3.3 Itô Integrals

Suppose $0 \leq S < T$ and $f(t,\omega)$ is given. We want to define

$$\int_S^T f(t,\omega) dB_t(\omega), \tag{3.1}$$

where $B_t(\omega)$ is 1-dimensional Brownian motion, ω is a sample point on $\omega \in \Omega$. The variations of the paths of B_t are too big to enable us to define the integral (3.1) in the Riemann-Stieltjes sense (the variation of the path is infinite). The solution is to approximate a given function $f(t,\omega)$ by

$$\sum_j f(t_j^*,\omega) \cdot \chi_{[t_j,t_{j+1}]}(t),$$

where the points t_j^* belong to the intervals $[t_j, t_{j+1}]$, and then define (3.1) as $\lim_{n\to\infty} \sum_j f(t_j^*,\omega)\Big[B_{t_{j+1}} - B_{t_j}\Big](\omega)$, where n is a number of subinterval on $[S,T]$. It does make a difference here what points t_j^* we choose: for $t_j^* = t_j$ we obtain **Itô integral**

$$\int_S^T f(t,\omega) dB_t(\omega).$$

Definition 3.3.1. [114] Let $B_t(\omega)$ be n-dimensional Brownian motion. Then we define $\mathcal{F}_t = \mathcal{F}_t^n$ to be the σ-algebra generated by the random variables $B_s(\cdot)$; $s \leq t$. In other words, $\mathcal{F}_t$ is the smallest σ-algebra containing all sets of the form

$$\{\omega, B_{t_1}(\omega) \in F_1, \cdots, B_{t_k}(\omega) \in F_k\},$$

where $t_j \leq t$ and $F_j \subset \mathbb{R}^n$ are Borel sets, $j \leq k = 1, 2, \ldots$ (We assume that all sets of measure zero are included in $\mathcal{F}_t$).

Definition 3.3.2. [114] Let $\{\mathcal{N}_t\}_{t\geq 0}$ be an increasing family of σ-algebras of subsets of Ω. A process $g(t,\omega) : [0;\infty) \times \Omega \to \mathbb{R}^n$ is called $\mathcal{N}_t$-adapted if for each $t \geq 0$ the function

$$\omega \to g(t,\omega)$$

is $\mathcal{N}_t$-measurable.

Definition 3.3.3. [114] Let $\mathcal{V} = \mathcal{V}(S,T)$ be the class of functions

$$f(t,\omega) : [0;\infty) \times \Omega \to \mathbb{R}$$

such that

(i) $(t,\omega) \to f(t,\omega)$ is $\mathcal{B} \times \mathcal{F}$-measurable, where $\mathcal{B}$ denotes the Borel σ-algebra on $[0,\infty)$.

(ii) $f(t,\omega)$ is $\mathcal{F}_t$-adapted.

(iii) $E\left[\int_S^T f(t,\omega)^2 dt\right] < \infty$.

A function $\phi \in \mathcal{V}$ is called elementary if it has the form

$$\phi(t,\omega) = \sum_j e_j(\omega) \cdot \chi_{[t_j, t_{j+1}]}(t).$$

Note that since $\phi \in \mathcal{V}$ each function e_j must be $\mathcal{F}_{t_j}$-measurable. For elementary functions $\phi(t,\omega)$ we define the integral

$$\int_S^T \phi(t,\omega) dB_t(\omega) = \sum_{j \geq 0} e_j(\omega) \left[B_{t_{j+1}} - B_{t_j}\right](\omega).$$

Definition 3.3.4. [114] Let $f \in \mathcal{V}(S,T)$. Then the Itô integral of f (from S to T) is defined by

$$\int_S^T f(t,\omega) dB_t(\omega) = \lim_{n\to\infty} \int_S^T \phi_n(t,\omega) dB_t(\omega),$$

(limit in $L^2(P)$), where $\{\phi_n\}$ is a sequence of elementary functions such that

$$E\left[\int_S^T (f(t,\omega) - \phi_n(t,\omega))^2 dt\right] \to 0 \text{ as } n \to \infty.$$

3.3.1 Itô Isometry

Corollary 3.3.1. [114] If $\phi(t,\omega)$ is bounded and elementary function then

$$E\left[\left(\int_S^T \phi(t,\omega) dB_t(\omega)\right)^2\right] = E\left[\int_S^T \phi(t,\omega)^2 dt\right].$$

Proof. [114], pp. 26.

3.3.2 Some Properties of the Itô Integral

Theorem 3.3.2. *Let $f, g \in \mathcal{V}(0,T)$ and let $0 \leq S < U < T$. Then*

(i) $\int_S^T f dB_t = \int_S^U f dB_t + \int_U^T f dB_t$

(ii) $\int_S^T (cf + g) dB_t = c\int_S^T f dB_t + \int_S^T g dB_t$ *for* $c \in \mathbb{R}$

(iii) $E\left[\int_S^T f dB_t\right] = 0$

(iv) $\int_S^T f dB_t$ *is* $\mathcal{F}_T$*-measurable.*

Proof. This holds for all elementary functions, so by taking limits we obtain this for all $f, g \in \mathcal{V}(0,T)$.

3.3.3 Martingale Representation Theorem

An important property of the Itô integral is that it is a martingale.

Definition 3.3.5. [114] A filtration on $(\Omega, \mathcal{F})$ is a family $\mathcal{M} = \{\mathcal{M}_t\}_{t\geq 0}$ of σ-algebras $\mathcal{M}_t \subset \mathcal{F}$ such that $0 \leq s < t \Rightarrow \mathcal{M}_s \subset \mathcal{M}_t$. An n-dimensional stochastic process $\{\mathcal{M}_t\}_{t\geq 0}$ on $(\Omega, \mathcal{F}, P)$ is called a martingale with respect to a filtration $\{\mathcal{M}_t\}_{t\geq 0}$ (and with respect to P) if
(i) M_t is $\mathcal{M}_t$-measurable for all t,
(ii) $E\left[|M_t|\right] < \infty$ for all t,
(iii) $E\left[M_s|\mathcal{M}_t\right] = M_t$ for all $s \geq t$.

Theorem 3.3.3. ***(Doob's martingale inequality)***
If M_t is a martingale such that $t \to M_t(\omega)$ is continuous a.s., then for all $p \geq 1$, $T \geq 0$ and all $\lambda > 0$

$$P\left[\sup_{0\leq t\leq T} |M_t| \geq \lambda\right] \leq \frac{1}{\lambda^p} \cdot E\left[|M_t|^p\right].$$

Theorem 3.3.4. *Let $f \in \mathcal{V}(0, T)$. Then there exists a t-continuous version of*

$$\int_0^t f(s, \omega) dB_s(\omega);\ 0 \leq t \leq T,$$

i.e. there exists a t-continuous stochastic process J_t on $(\Omega, \mathcal{F}, P)$ such that

$$P\left[J_t = \int_0^t f dB\right] = 1,\ 0 \leq t \leq T.$$

Proof. [114], pp. 32. Any $\mathcal{F}_t^{(n)}$ -martingale can be represented as an Itô integral. This is called the martingale representation theorem.

3.3.4 Itô Formula

Theorem 3.3.5. *Let X_t be an Itô process given by*

$$dX_t = udt + vdB_t.$$

Let $g(t, x) \in C^2([0, \infty)\times\mathbb{R})$ (i.e. g is twice continuously differentiable on $[0, \infty)\times\mathbb{R}$). Then

$$Y_t = g(t, X_t)$$

is again an Itô process, and

$$dY_t = \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial g}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial^2 x^2}(t, X_t)(dX_t)^2,$$

where $(dX_t)^2 = (dXt) \cdot (dXt)$ *is computed according to the rules*

$$dt \cdot dt = dt \cdot dBt = dBt \cdot dt = 0, \tag{3.2}$$

$$dBt \cdot dBt = dt. \tag{3.3}$$

Proof.[114], pp. 46.

Theorem 3.3.6. ***(The Multi-dimensional Itô formula)***
Let

$$dX_t = udt + vdB_t$$

be an n-dimensional Itô process. Let $g(t, x) = (g_1(t, x), \dots, g_p(t, x))$ *be a* C^2 *map from* $[0, \infty) \times \mathbb{R}^n$ *into* $\mathbb{R}^p$. *Then the process*

$$Y_t = g(t, X_t)$$

is again an Itô process, whose component number k, Y_k *is given by*

$$dY_k = \frac{\partial g_k}{\partial t}(t, X)dt + \sum_i \frac{\partial g_k}{\partial x_i}(t, X)dX_i + \frac{1}{2}\sum_{i,j} \frac{\partial^2 g_k}{\partial x_i \partial x_j}(t, X)dX_i dX_j,$$

where

$$dB_i dB_j = \delta_{i,j} dt, \tag{3.4}$$

$$dB_i dt = dtdB_i = 0, \tag{3.5}$$

where $\delta_{i,j}$ *is the Kronecker delta,*

$$\delta_{i,j} = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

3.4 Stochastic Differential Equations

Definition 3.4.1. Let $W_t = (W_1(t), ..., W_m(t))$ be m-dimensional Wiener process and $b : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$ be measurable functions. Then the process $X_t = (X_1(t), ..., X_m(t))$, $t \in [0, T]$ is the solution of the stochastic differential equation

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = b(t, X_t) + \sigma(t, X_t)W_t, \tag{3.6}$$

$b(t, X_t) \in \mathbb{R}$, $\sigma(t, X_t)W_t \in \mathbb{R}$, where W_t is 1-dimensional white noise. Equation (3.6) can be written as the differential form

$$dX_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}B_t. \tag{3.7}$$

We formally replace the white noise W_t by $\frac{\mathrm{d}B_t}{\mathrm{d}t}$ and multiply by $\mathrm{d}t$. After the integration of equation (3.7) we give the stochastic integral equation

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s. \tag{3.8}$$

This implies that X_t is the solution of the following modified Itô equation:

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \frac{1}{2}\int_0^t \sigma'(s, X_s)\sigma(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s,$$

where σ' denotes the derivative of $\sigma(t, x)$ with respect to x.

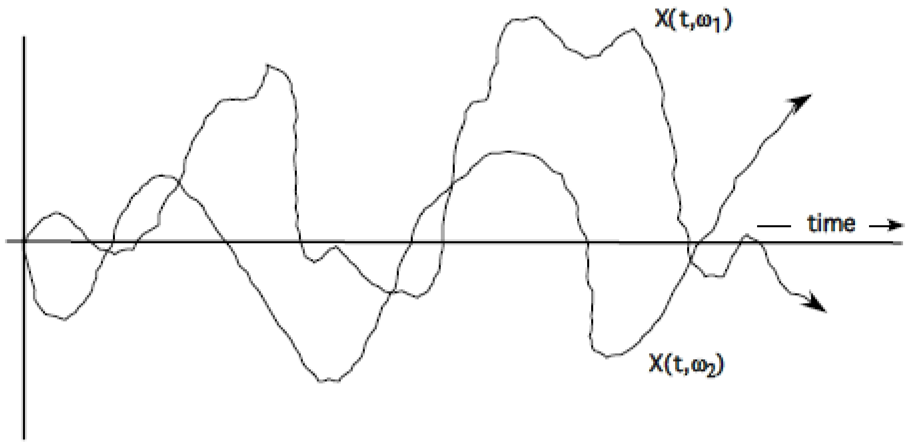

Fig. 3.3: Two sample paths of a stochastic process [47]

3.4.1 Existence and Uniqueness of Solution

Definition 3.4.2. Let $T > 0$ and $b : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ be measurable functions satisfying next conditions:

(i) Exist some constant C such that

$$\mid b(t, x) \mid + \mid \sigma(t, x) \mid \leq C(1+ \mid x \mid)$$

for $x \in \mathbb{R}^n, t \in [0, T]$.

(ii) Exist some constant D such that

$$\mid b(t, x) - b(t, y) \mid + \mid \sigma(t, x) - \sigma(t, y) \mid \leq D \mid x - y \mid)$$

for $x, y \in \mathbb{R}^n, t \in [0, T]$.

(iii) Let Z be a random variable which is independent of the σ-algebra $\mathcal{F}_\infty^m$ and $E\left[\mid Z^2 \mid\right] < \infty$.

Then the stochastic differential equation (3.8) has a unique continuous solution X_t such that

$$E\left[\int_0^T \mid X_t \mid^2 \mathrm{d}t\right] < \infty$$

for $t \in [0, T]$.

Proof. [114], pp. 65.

3.5 Stability of Stochastic Differential Equations

In the year 1892, A.M. Lyapunov introduced the concept of stability of a dynamic system. The stability means insensitivity of the state of the system to small changes in the initial state or the parameters of the system. For a stable system, the trajectories which are close to each other at a specific instant should therefore remain close to each other at all subsequent instants.

Lyapunov developed a method for determining stability without solving the equation, and this method is now known as the Lyapunov direct or second method. To explain the method, let us introduce a few necessary notations. Let K denote the family of all continuous nondecreasing functions $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\mu(0) = 0$ and $\mu(r) > 0$ if $r > 0$. For $h > 0$, let $S_h = \{x \in \mathbb{R}^n : |x| < h\}$. A continuous function $V(x, t)$ defined on $S_h \times [t_0, \infty)$ is said to be **positive-definite** (in the sense of Lyapunov) if $V(0, t) \equiv 0$ and, for some $\mu \in K$,

$$V(x, t) \geq \mu(|x|)$$

for all $(x, t) \in S_h \times [t_0, \infty)$. A function $V(x, t)$ is said to be **negative-definite** if $(-V)$ is positive-definite. A continuous non-negative function $V(x, t)$ is said to be **decrescent** (i.e. to have an arbitrarily small upper bound) if for some $\mu \in K$,

$$V(x, t) \leq \mu(|x|)$$

for all $(x, t) \in S_h \times [t_0, \infty)$. A function $V(x, t)$ defined on $\mathbb{R}^n \times [t_0, \infty)$ is said to be **radially unbounded** if

$$\lim_{|x| \to \infty} \inf_{t \geq t_0} V(x, t) = \infty.$$

Let $C^{1,1}(S_h \times [t_0, \infty), \mathbb{R}_+)$ denote the family of all continuous functions $V(x, t)$ from $S_h \times [t_0, \infty)$ to $\mathbb{R}_+$ with continuous first partial derivatives with respect to every component of x and to t. Then $v(t) = V(t, X_t)$ represents a function of t with the derivative

$$\dot{v}(t) = V_t(t, X_t) + V_x(t, X_t)b(t, X_t) = \frac{\partial V}{\partial t}(t, X_t) + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(t, X_t)b_i(t, X_t).$$

If $\dot{v}(t) \leq 0$, then $v(t)$ will not increase so the distance of X_t from the equilibrium point measured by $V(t, X_t)$ does not increase. If $\dot{v}(t) < 0$, then $v(t)$ will decrease to zero so the distance will decrease to zero, that is $X_t \to 0$. We determine the stability of the zero solution.

Theorem 3.5.1. ***(Lyapunov theorem)*** *If there exists a positive-definite function $V(x,t) \in C^{1,1}(S_h \times [t_0, \infty), \mathbb{R}_+)$ such that*

$$\dot{V}(x,t) := V_t(t, X_t) + V_x(t, X_t)b(t, X_t) \leq 0$$

for all $(x,t) \in S_h \times [t_0, \infty)$, then the trivial solution of equation (3.7) *is stable. If there exists a positive-definite decrescent function $V(x,t) \in C^{1,1}(S_h \times [t_0, \infty), \mathbb{R}_+)$ such that $\dot{V}(x,t)$ is negative-definite, then the trivial solution is asymptotically stable.*

A function $V(x,t)$ that satisfies the stability conditions of Theorem (3.5.1) is called a Lyapunov function corresponding to the ordinary differential equation. The next text carries over the principles of the Lyapunov stability theory for deterministic systems to stochastic ones.

In the next subsections it will be investigated various types of stability for the n-dimensional stochastic differential equation (3.7).

3.5.1 Stability in Probability

Definition 3.5.1. The trivial solution of equation (3.7) is said to be

(i) stochastically **stable** or **stable in probability** if for every pair of $\epsilon \in (0,1)$ and $r > 0$, there exists $\delta = \delta(\epsilon, r, t_0) > 0$ such that

$$P\left\{|x(t, t_0, x_0)| < r\right\} \geq 1 - \epsilon$$

for all $t \geq t_0$, whenever $|x_0| < \delta$. Otherwise, it is said to be stochastically **unstable**.

(ii) stochastically **asymptotically stable** if it is stochastically stable and, moreover, for every $\epsilon \in (0,1)$, there exists $\delta_0 = \delta_0(\epsilon, t_0) > 0$ such that

$$P\left\{\lim_{t\to\infty} x(t, t_0, x_0) = 0\right\} \geq 1 - \epsilon$$

whenever $|x_0| < \delta_0$.

(iii) stochastically **asymptotically stable in the large** if it is stochastically stable and, moreover, for all $x_0 \in \mathbb{R}^n$

$$P\left\{\lim_{t\to\infty} x(t, t_0, x_0) = 0\right\} = 1.$$

Suppose one would like to let the initial value be a random variable. It should also be pointed out that when $\sigma^{(x,t)} = 0$, these definitions reduce to the corresponding deterministic ones. We now extend the Lyapunov Theorem (3.5.1) to the stochastic case. Let $0 < h \leq \infty$. Denote by $C^{2,1}(S_h \times \mathbb{R}_+, \mathbb{R}_+)$ the family of all nonnegative functions $V(x,t)$ defined on $S_h \times \mathbb{R}_+$ such that they are continuously twice differentiable in x and once in t. Define the differential operator L associated with equation (3.7) by

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(t, X_t) b_i(x,t) + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} \left[\sigma(x,t)\sigma^T(x,t)\right]_{ij}.$$

The inequality $\dot{V}(x,t) \leq 0$ will be replaced by $LV(x,t) \leq 0$ in order to get the stochastic stability assertions.

Theorem 3.5.2. *If there exists a positive-definite*

(i) function $V(x,t) \in C^{2,1}(S_h \times [t_0,\infty), \mathbb{R}_+)$ such that $LV(x,t) \leq 0$ for all $(x,t) \in S_h \times [t_0,\infty)$, then the trivial solution of equation (3.7) *is stochastically* ***stable****.*

(ii) decrescent function $V(x,t) \in C^{2,1}(S_h \times [t_0,\infty), \mathbb{R}_+)$ such that $LV(x,t)$ is negative-definite, then the trivial solution of equation (3.7) *is stochastically* ***asymptotically stable****.*

(iii) decrescent radially unbounded function $V(x,t) \in C^{2,1}(\mathbb{R}^n \times [t_0,\infty), \mathbb{R}_+)$ such that $LV(x,t)$ is negative-definite, then the trivial solution of equation (3.7) *is stochastically* ***asymptotically stable in the large.***

Proof.[102], pp. 111.

The functions $V(X_t)$ used in Theorem (3.5.2) are called stochastic Lyapunov functions, and the use of these theorems depends on the construction of the functions.

3.5.2 Almost Sure Exponential Stability

Definition 3.5.2. The trivial solution of equation (3.7) is said to be almost surely exponentially stable if

$$\lim_{t\to\infty} \sup \frac{1}{t} \log |x(t, t_0, x_0)| < 0 \tag{3.9}$$

almost surely for all $x_0 \in \mathbb{R}^n$. The left-hand side of (3.9) is called the sample Lyapunov exponents of the solution. The trivial solution is almost surely exponentially stable if and only if the sample Lyapunov exponents are negative. The almost sure exponential stability means that almost all sample paths of the solution will tend to the equilibrium position $x = 0$ exponentially fast.

Theorem 3.5.3. *For all $x_0 \neq 0$ in $\mathbb{R}^n$*

$$P\left\{x(t, t_0, x_0) \neq 0 \quad on \quad t \geq t_0\right\} = 1.$$

That is, almost all the sample path of any solution starting from a non-zero state will never reach the origin.

Theorem 3.5.4. *Assume that there exists a function $V \in C^{2,1}(\mathbb{R}^n \times [t_0, \infty), \mathbb{R}_+)$ and constants $p > 0, c_1 > 0, c_2 \in \mathbb{R}, c_3 \geq 0$, such that for all $x_0 \neq 0$ and $t \geq t_0$,*

(i) $c_1 |x|^p \leq V(x,t)$,

(ii) $LV(x,t) \leq c_2 V(x,t)$

(iii) $|V_x(x,t)\sigma(x,t)|^2 \geq c_3 V^2(x,t)$

Then

$$\limsup_{t\to\infty} \frac{1}{t} \log |x(t,t_0,x_0)| \leq -\frac{c_3 - 2c_2}{2p}$$

almost surely for all $x_0 \in \mathbb{R}^n$. In particular, if $c_3 > 2c_2$, the trivial solution of equation (3.7) *is almost surely exponentially stable.*

Proof. [102], pp. 121.

Theorem 3.5.5. *Assume that there exists a function $V \in C^{2,1}(\mathbb{R}^n \times [t_0, \infty), \mathbb{R}_+)$ and constants $p > 0, c_1 > 0, c_2 \in \mathbb{R}, c_3 \geq 0$, such that for all $x_0 \neq 0$ and $t \geq t_0$,*

(i) $c_1 |x|^p \geq V(x,t) > 0$,

(ii) $LV(x,t) \geq c_2 V(x,t)$

(iii) $|V_x(x,t)\sigma(x,t)|^2 \leq c_3 V^2(x,t)$

Then

$$\liminf_{t\to\infty} \frac{1}{t} \log |x(t,t_0,x_0)| \geq -\frac{2c_2 - c_3}{2p}$$

almost surely for all $x_0 \in \mathbb{R}^n$. In particular, if $2c_2 > c_3$, then almost all the sample paths of $|x(t,t_0,x_0)|$ will tend to infinity, and we say in this case that the trivial solution of equation (3.7) *is almost surely exponentially unstable.*

3.5.3 Moment Exponential Stability

Definition 3.5.3. The trivial solution of equation (3.7) is said to be $p-$th moment exponentially stable if there is a pair of positive constants λ and C such that

$$E\,|x(t,t_0,x_0)|^p \leq C\,|x_0|^p\, \mathrm{e}^{-\lambda(t-t_0)} \quad \text{on} \quad t \geq t_0$$

for all $x_0 \in \mathbb{R}^n$. When $p = 2$, it is usually said to be exponentially stable in mean square. It also follows that

$$\limsup_{t\to\infty} \frac{1}{t} \log \left(E\,|x(t,t_0,x_0)|^p\right) < 0. \tag{3.10}$$

The $p-$th moment exponential stability means that the $p-$th moment of the solution will tend to 0 exponentially fast. The left-hand side of (3.10) is called the $p-$th moment Lyapunov exponent of the solution.

Theorem 3.5.6. *Assume that there is a positive constant K such that*

$$x^T b(x,t) \vee |\sigma(x,t)|^2 \leq K |x|^2 \quad for \;\; all \quad (x,t) \in \mathbb{R}^n \times [t_0, \infty).$$

Then the pth moment exponential stability of the trivial solution of equation (3.7) *implies the almost surely exponential stability.*

Proof. [102], pp. 128.

Theorem 3.5.7. *Assume that there is a function $V(x,t) \in C^{2,1}(\mathbb{R}^n \times [t_0,\infty), \mathbb{R}_+)$ and positive constants c_1, c_2, c_3, such that*

$$c_1 |x|^p \leq V(x,t) \leq c_2 |x|^p \quad and \quad LV(x,t) \leq -c_3 V(x,t)$$

for all $(x,t) \in \mathbb{R}^n \times [t_0,\infty)$. Then

$$E\,|x(t,t_0,x_0)|^p \leq \frac{c_2}{c_1} |x_0|^p \,\mathrm{e}^{-c_3(t-t_0)} \quad on \;\; t \geq t_0$$

for all $x_0 \in \mathbb{R}^n$. In other words, the trivial solution of equation (3.7) *is $p-th$ moment exponentially stable and the $p-th$ moment Lyapunov exponent should not be greater than $-c_3$.*

Proof. [102], pp. 130.

Theorem 3.5.8. *Let $q > 0$. Assume that there is a function $V(x,t) \in C^{2,1}(\mathbb{R}^n \times [t_0,\infty), \mathbb{R}_+)$ and positive constants c_1, c_2, c_3, such that*

$$c_1 |x|^q \leq V(x,t) \leq c_2 |x|^q \quad and \quad LV(x,t) \geq c_3 V(x,t)$$

for all $(x,t) \in \mathbb{R}^n \times [t_0,\infty)$. Then

$$E\,|x(t,t_0,x_0)|^q \geq \frac{c_1}{c_2} |x_0|^q \,\mathrm{e}^{c_3(t-t_0)} \quad on \;\; t \geq t_0$$

for all $x_0 \in \mathbb{R}^n$, and we say in this case that the trivial solution of equation (3.7) *is $q-th$ moment exponentially unstable.*

Proof. [102], pp. 131.

3.5.4 Stochastic Stability and Nonstability

It is not surprising that noise can destabilize a stable system. And the noise can stabilized the unstable system. In this section we shall establish a general theory of stochastic stabilization and destabilization for a given nonlinear system. Suppose that the given system is described by a nonlinear ordinary differential equation

$$\dot{y}(t) = f(y(t)) \;\; \text{on} \;\; t \geq t_0, y(t_0) = X_0 \in \mathbb{R}^d.$$

Here $f : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^d$ is a locally Lipschitz continuous function and particularly, for some $K > 0$,

$$|f(X_t, t)| \leq K |X_t| \quad \text{for all} \quad (X_t, t) \in \mathbb{R}^d \times \mathbb{R}_+. \tag{3.11}$$

We now use the m-dimensional Brownian motion $B(t) = (B_1(t), \ldots, B_m(t))^T$ as the source of noise to perturb the given system. For simplicity, suppose the stochastic perturbation is of a linear form, that is the stochastically perturbed system is described by the semilinear Itô equation

$$\mathrm{d}X_t = f(X_t, t)\mathrm{d}t + \sum_{i=1}^{m} G_i X_t \mathrm{d}B_i(t) \quad \text{on} \quad t \geq t_0, X(t_0) = X_0 \in \mathbb{R}^d, \tag{3.12}$$

where all $G_i, 1 \leq i \leq m$ are $d \times d$ matrices. Clearly, equation (3.12) has a unique solution denoted by $X(t; t_0, X_0)$ again and, moreover, it admits a trivial solution $X_t \equiv 0$.

Theorem 3.5.9. *Let* (3.11) *hold. Assume that there are two constants* $\lambda > 0$ *and* $\rho \geq 0$ *such that*

$$\sum_{i=1}^{m} \left| G_i X_t^2 \right| \leq \lambda |X_t|^2 \quad \textit{and} \quad \sum_{i=1}^{m} \left| X_t^T G_i X_t^2 \right| \geq \rho |X_t|^4$$

for all $X_t \in \mathbb{R}^d$. *Then*

$$\lim_{t \to \infty} sup \frac{1}{t} \log |X(t; t_0, X_0)| \leq -\left(\rho - K - \frac{\lambda}{2}\right)$$

almost surely for all $X_0 \in \mathbb{R}^d$. *In particular, if* $\rho > K + \frac{1}{2}\lambda$, *then the trivial solution of equation* (3.12) *is almost surely exponentially stable.*

Proof. [102], pp. 137.

4 STOCHASTIC SYSTEM RESEARCH

4.1 One-Dimensional Brownian Motion

4.1.1 Solution of Stochastic Differential Equations

We derived sufficient conditions for finding solution of the stochastic differential equation using Itô formula.

Theorem 4.1.1. [*114*] *Let X_t be an Itô process given by*

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}B_t. \tag{4.1}$$

Let $g(t,x) \in C^2([0,\infty) \times \mathbb{R})$(i.e. f is twice continuosly differentiable on $[0,\infty) \times \mathbb{R}$). Then

$$Y_t = g(t, X_t)$$

is again an Itô process and

$$\mathrm{d}Y_t = \frac{\partial g}{\partial t}(t, X_t)\mathrm{d}t + \frac{\partial g}{\partial x}(t, X_t)\mathrm{d}X_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t)(\mathrm{d}X_t)^2. \tag{4.2}$$

Theorem 4.1.2. *Let X_t be an Itô process given by*

$$\mathrm{d}X_t = AX_t\mathrm{d}t + GX_t\mathrm{d}B_t. \tag{4.3}$$

Let $f(t,x) \in C^2([0,\infty) \times \mathbb{R})$(i.e. f is twice continuosly differentiable on $[0,\infty) \times \mathbb{R}$). Then

$$Y_t = f(t, X_t)$$

is again an Itô process and

$$\mathrm{d}Y_t = \frac{\partial f}{\partial t}(t, X_t)\mathrm{d}t + \frac{\partial f}{\partial x}(t, X_t)(AX_t\mathrm{d}t + GX_t\mathrm{d}B_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X_t)(G^2X_t^2\mathrm{d}t), \tag{4.4}$$

where $(\mathrm{d}X_t)^2 = (\mathrm{d}X_t)\cdot(\mathrm{d}X_t) = \mathrm{d}t$ is computed according to the rule (3.3)*.*

Proof. First observe that if we substitute

$$\mathrm{d}X_t = AX_t\mathrm{d}t + GX_t\mathrm{d}B_t$$

in equation (4.4) and use rules (3.2) and (3.3), we get the equivalent expression

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \int_0^t \left(\frac{\partial f}{\partial s}(s, X_s) + AX_s\frac{\partial f}{\partial x}(s, X_s) + G^2X_s^2\frac{\partial^2 f}{2\partial x^2}(s, X_s)\right)\mathrm{d}s \\ &+ \int_0^t GX_s\frac{\partial f}{\partial x}(s, X_s)\mathrm{d}B_s. \end{aligned}$$

Assume that AX_t and GX_t are elementary functions. Using Taylor's theorem we get

$$\begin{aligned} f(t,X_t) &= f(0,X_0)+\sum_j \triangle f(t_j,X_j) = f(0,X_0)+\sum_j \frac{\partial f}{\partial t}\triangle t_j + \sum_j \frac{\partial f}{\partial x}\triangle X_j \\ &+ \frac{1}{2}\sum_j \frac{\partial^2 f}{\partial t^2}(\triangle t_j)^2 + \sum_j \frac{\partial^2 f}{\partial t \partial x}(\triangle t_j)(\triangle X_j) \\ &+ \frac{1}{2}\sum_j \frac{\partial^2 f}{\partial x^2}(\triangle X_j)^2 + \sum_j R_j, \end{aligned}$$

where $\frac{\partial f}{\partial t}, \frac{\partial f}{\partial x}$, etc. are evaluated at the points $(t_j, X_j), \triangle t_j = t_{j+1} - t_j$, $\triangle X_j = X_{t_{j+1}} - X_{t_j}, \triangle f(t_j, X_j) = f(t_{j+1}, X_{t_{j+1}}) + f(t_j, X_j)$ and $R_j = o(|\triangle t_j|^2 + |\triangle X_j|^2)$ for all j. If $\triangle t_j \to 0$ then

$$\begin{aligned} \sum_j \frac{\partial f}{\partial t}\triangle t_j &= \sum_j \frac{\partial f}{\partial t}(t_j,X_j)\triangle t_j \to \int_0^t \frac{\partial f}{\partial s}(s,X_s)\mathrm{d}s, \\ \sum_j \frac{\partial f}{\partial x}\triangle X_j &= \sum_j \frac{\partial f}{\partial x}(t_j,X_j)\triangle X_j \to \int_0^t \frac{\partial f}{\partial x}(s,X_s)\mathrm{d}X_s. \end{aligned}$$

Moreover, since AX_t and GX_t are elementary we get

$$\begin{aligned} \sum_j \frac{\partial^2 f}{\partial x^2}(\triangle X_j)^2 &= \sum_j \frac{\partial^2 f}{\partial x^2}(A_j X_{t_j})^2(\triangle t_j)^2 + 2\sum_j \frac{\partial^2 f}{\partial x^2} A_j X_{t_j} G_j X_{t_j}(\triangle t_j)(\triangle B_j) \\ &+ \sum_j \frac{\partial^2 f}{\partial x^2}(G_j X_{t_j})^2(\triangle B_j)^2. \end{aligned}$$

The first two terms here tend to 0 as $\triangle t_j \to 0$. For example,

$$\begin{aligned} &\mathbb{E}\left[\left(\sum_j \frac{\partial^2 f}{\partial x^2} A_j X_{t_j} G_j X_{t_j}(\triangle t_j)(\triangle B_j)\right)^2\right] \\ &= \sum_j \mathbb{E}\left[\left(\frac{\partial^2 f}{\partial x^2} A_j X_{t_j} G_j X_{t_j}\right)^2\right](\triangle t_j)^3 \to 0. \end{aligned}$$

We claim that the last term tends to

$$\int_0^t G^2 X_s^2 \frac{\partial^2 f}{\partial x^2}(s,X_s)\mathrm{d}s \to 0.$$

□

That completes the proof of the formula (4.4).

Corollary 4.1.3. Let's consider the following stochastic differential equation (4.3) with constant coefficients A and $G, X_t \neq 0, X_0 = \eta, B_0 = 0$. We can rewrite this equation as follows

$$\frac{\mathrm{d}X_t}{X_t} = A\mathrm{d}t + G\mathrm{d}B_t$$

and integrate both sides over $[0,t]$ and get the expression

$$\int_0^t \frac{\mathrm{d}X_s}{X_s} = A\int_0^t \mathrm{d}s + G\int_0^t \mathrm{d}B_s.$$

First of all we evaluate the part $I = \int_0^t \mathrm{d}B_s$ using Itô's formula (4.4). Choose $X_t = B_t$ and $f(t,X_t) = X_t$. Then

$$Y_t = f(t,B_t) = B_t.$$

Then

$$\begin{aligned} \mathrm{d}Y_t &= 0 + 1\cdot \mathrm{d}B_t + 0\cdot \mathrm{d}t \\ \mathrm{d}B_t &= \mathrm{d}B_t, \end{aligned}$$

hence

$$\int_0^t \mathrm{d}B_s = B_t.$$

We get the expression

$$\int_0^t \frac{\mathrm{d}X_s}{X_s} = At + GB_t.$$

Now we solve the left-hand side. Replacing $AX_t\mathrm{d}t + GX_t\mathrm{d}B_t$ for $\mathrm{d}X_t$ in the equation (4.4) we give

$$\begin{aligned} \mathrm{d}Y_t &= \frac{\partial f}{\partial x}(t,X_t)(AX_t\mathrm{d}t + GX_t\mathrm{d}B_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X_t)(AX_t\mathrm{d}t + GX_t\mathrm{d}B_t)^2 \\ &+ \frac{\partial f}{\partial t}(t,X_t)\mathrm{d}t = \frac{\partial f}{\partial t}(t,X_t)\mathrm{d}t + \frac{\partial f}{\partial x}(t,X_t)(AX_t\mathrm{d}t) + \frac{\partial f}{\partial x}(t,X_t)(GX_t\mathrm{d}B_t) \\ &+ \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X_t)(G^2X_t^2\mathrm{d}t). \end{aligned}$$

We use the Itô's formula of the function $f(t,X_t) = \ln X_t$. Then

$$Y_t = f(t,X_t) = \ln X_t.$$

$$\begin{aligned} \mathrm{d}Y_t &= \left(0 + \frac{1}{X_t}AX_t + \frac{1}{2}\cdot\frac{-1}{X_t^2}G^2X_t^2\right)\mathrm{d}t + \frac{1}{X_t}GX_t\mathrm{d}B_t \\ &= \frac{-1}{2}G^2\mathrm{d}t + A\mathrm{d}t + G\mathrm{d}B_t. \end{aligned}$$

We get

$$\mathrm{d}(\ln X_t) = \frac{-1}{2}G^2\mathrm{d}t + \frac{\mathrm{d}X_t}{X_t} \Rightarrow \frac{\mathrm{d}X_t}{X_t} = \mathrm{d}(\ln X_t) + \frac{1}{2}G^2\mathrm{d}t,$$

hence

$$\begin{aligned} \int_0^t \mathrm{d}(\ln X_s) + \int_0^t \frac{1}{2}G^2\mathrm{d}s &= At + GB_t, \\ \ln X_t - \ln X_0 &= At + GB_t - \frac{1}{2}G^2 t, \\ X_t &= X_0\mathrm{e}^{At+GB_t-\frac{1}{2}G^2t}. \end{aligned}$$

For $X_0 = \eta$ we obtain $X_t = \eta\mathrm{e}^{At+GB_t-\frac{1}{2}G^2t}$.

Corollary 4.1.4. Let X_t be an Itô process given by

$$\mathrm{d}X_t = AX_t\mathrm{d}t + G\mathrm{d}B_t$$

with constant coefficients A and $G, X_t \neq 0, X_0 = \eta, B_0 = 0$. We can rewrite the stochastic equation as follows

$$\frac{\mathrm{d}X_t}{X_t} = A\mathrm{d}t + \frac{G}{X_t}\mathrm{d}B_t$$

and integrate both sides over $[0,t]$

$$\int_0^t \frac{\mathrm{d}X_s}{X_s} = A\int_0^t \mathrm{d}s + G\int_0^t \frac{1}{X_s}\mathrm{d}B_s$$

and get the expression

$$\int_0^t \frac{\mathrm{d}X_s}{X_s} = At + G\int_0^t \frac{1}{X_t}\mathrm{d}B_s.$$

Let's solve the left-hand side. Replacing $AX_t\mathrm{d}t + G\mathrm{d}B_t$ for $\mathrm{d}X_t$ in the Itô's formula according to the theorem (4.1.1) we give

$$\begin{aligned}
\mathrm{d}Y_t &= \frac{\partial f}{\partial t}(t,X_t)\mathrm{d}t + \frac{\partial f}{\partial x}(t,X_t)(AX_t\mathrm{d}t + G\mathrm{d}B_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X_t)(AX_t\mathrm{d}t + G\mathrm{d}B_t)^2 \\
&= \frac{\partial f}{\partial t}(t,X_t)\mathrm{d}t + \frac{\partial f}{\partial x}(t,X_t)(AX_t\mathrm{d}t) + \frac{\partial f}{\partial x}(t,X_t)(G\mathrm{d}B_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X_t)(G^2\mathrm{d}t).
\end{aligned}$$

We use the Itô's formula of the function $f(t,X_t) = \ln X_t$. Then

$$Y_t = f(t,X_t) = \ln X_t,$$

$$\mathrm{d}Y_t = 0 + \frac{1}{X_t}AX_t\mathrm{d}t + \frac{1}{2}\cdot\frac{-1}{X_t^2}G^2\mathrm{d}t + \frac{1}{X_t}G\mathrm{d}B_t.$$

We get

$$\mathrm{d}(\ln X_t) = -\frac{G^2}{2X_t^2}\mathrm{d}t + \frac{\mathrm{d}X_t}{X_t} \Rightarrow \frac{\mathrm{d}X_t}{X_t} = \mathrm{d}(\ln X_t) + \frac{G^2}{2X_t^2}\mathrm{d}t,$$

hence

$$\begin{aligned}
\int_0^t \mathrm{d}(\ln X_s) + \int_0^t \frac{G^2}{2X_s^2}\mathrm{d}s &= At + G\int_0^t \frac{1}{X_s}\mathrm{d}B_s, \\
\ln X_t - \ln X_0 &= At + G\int_0^t \frac{1}{X_s}\mathrm{d}B_s - \frac{G^2}{2}\int_0^t \frac{1}{X_s^2}\mathrm{d}s.
\end{aligned}$$

Now we solve the part $\int_0^t \frac{1}{X_s}\mathrm{d}B_s$ replacing $X_t \equiv B_t$

$$\int_0^t \frac{1}{X_s}\mathrm{d}X_s = \ln X_t - \ln X_0.$$

We get the expression

$$\begin{aligned}\ln X_t - \ln X_0 &= At + G(\ln X_t - \ln X_0) - \frac{G^2}{2}\int_0^t \frac{1}{X_s^2}\mathrm{d}s,\\ (\ln X_t - \ln X_0)(1-G) &= At - \frac{G^2}{2}\int_0^t \frac{1}{X_s^2}\mathrm{d}s.\end{aligned}$$

Finally result is given

$$\begin{aligned}\ln X_t - \ln X_0 &= \frac{At}{1-G} - \frac{G^2}{2(1-G)}\int_0^t \frac{1}{X_s^2}\mathrm{d}s,\\ X_t &= X_0 \mathrm{e}^{\frac{At}{1-G} - \frac{G^2}{2(1-G)}\int_0^t \frac{1}{X_s^2}\mathrm{d}s}.\end{aligned}$$

For $X_0 = \eta$ we obtain $X_t = \eta \mathrm{e}^{\frac{At}{1-G} - \frac{G^2}{2(1-G)}\int_0^t \frac{1}{X_s^2}\mathrm{d}s}$.

4.1.2 Stability of Solution Using Lyapunov Method

The stability of the solution can be determined without the knowledge of the solution.

Definition 4.1.1. [102] It is given Lyapunov quadratic function V

$$V(X_t) = X_t^T Q\, X_t, \tag{4.5}$$

where Q is a symmetric positive definite matrix.

Theorem 4.1.5. *Let the function LV*

$$LV(X_t) = 2X_t^T Q\, b(t,X_t) + \sigma(t,X_t)^T Q\, \sigma(t,X_t), \tag{4.6}$$

be negative definite around of the point $X_t = 0$ pro $t \geq t_0$, then the trivial solution of the equation (4.1) *is stochastically asymptotically stable.*

According to the Theorem (3.5.2) we first calculate the derivation of Lyapunov function along the solution of the equation (4.1)

$$\begin{aligned}\mathrm{d}V(X_t) &= V(X_t + \mathrm{d}X_t) - V(X_t) = (X_t^T + \mathrm{d}X_t^T)Q\,(X_t + \mathrm{d}X_t) - X_t^T Q\, X_t\\ &= (X_t^T + b(t,X_t)^T\mathrm{d}t + \sigma(t,X_t)^T\mathrm{d}B_t)Q\,(X_t + b(t,X_t)\mathrm{d}t + \sigma(t,X_t)\mathrm{d}B_t)\\ &- X_t^T Q\, X_t = X_t^T Q\, X_t + X_t^T Q\, b(t,X_t)\mathrm{d}t + X_t^T Q\, \sigma(t,X_t)\mathrm{d}B_t\\ &+ b(t,X_t)^T dtQ\, X_t + b(t,X_t)^T\mathrm{d}tQ\, b(t,X_t)\mathrm{d}t + b(t,X_t)^T\mathrm{d}tQ\, \sigma(t,X_t)\mathrm{d}B_t\\ &+ \sigma(t,X_t)^T\mathrm{d}B_tQ\, X_t + \sigma(t,X_t)^T\mathrm{d}B_tQ\, b(t,X_t)\mathrm{d}t\\ &+ \sigma(t,X_t)^T\mathrm{d}B_tQ\, \sigma(t,X_t)\mathrm{d}B_t - X_t^T Q\, X_t.\end{aligned}$$

We use rules (3.2) a (3.3) and get

$$\begin{aligned} \mathrm{d}V(X_t) &= X_t^T Q\, b(t,X_t)\mathrm{d}t + X_t^T Q\, \sigma(t,X_t)\mathrm{d}B_t + b(t,X_t)^T \mathrm{d}t Q\, X_t + \\ &+ \sigma(t,X_t)^T \mathrm{d}B_t Q\, X_t + \sigma(t,X_t)^T Q\, \sigma(t,X_t)\mathrm{d}t. \end{aligned}$$

We determine the mean value $\mathbb{E}\left\{dV(X_t)\right\}$

$$\mathbb{E}\left\{\mathrm{d}V(X_t)\right\} = X_t^T Q\, b(t,X_t)\mathrm{d}t + b(t,X_t)^T Q\, X_t \mathrm{d}t + \sigma(t,X_t)^T Q\, \sigma(t,X_t)\mathrm{d}t = LV(X_t)\mathrm{d}t,$$

$$-LV(X_t) \geq kV(X_t), \quad \mathrm{k} = \mathrm{const.}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left\{V(X_t)\right\} \leq -k\mathbb{E}\left\{V(X_t)\right\},$$

$$\mathbb{E}\left\{V(X_t)\right\} \leq \mathrm{e}^{-kt}.$$

For

$$\lim_{t\to\infty} \mathbb{E}^2\left\{X_t\right\} = \lim_{t\to\infty} \mathbb{E}\left\{X_t X_t^T\right\} = \Theta,$$

the solution is almost asymptotically stable. If $LV(X_t)$ is a positive definite function around the point $X_t = 0$ of the equation (4.1), then the trivial solution of the equation (4.1) is unstable (according to the Theorem (3.5.2)).

4.2 Two-Dimensional Brownian Motion

4.2.1 Solution of Stochastic Differential Equations

In this part we derive general solution of SDEs for multidimensional Brownian motion. Using of two-dimensional Brownian motion we demonstrate on the example.

For many dimensions, there is a very useful analogue of Itô formula (4.2). Let $B_t = (B_1(t), \ldots, B_m(t))$ denote m-dimensional Brownian motion.

Theorem 4.2.1. *Let*

$$\mathrm{d}X_t = u\mathrm{d}t + v\mathrm{d}B_t$$

be an n-dimensional Itô process, where

$X_t = \begin{pmatrix} X_1(t) \\ \vdots \\ X_n(t) \end{pmatrix}, u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, v = \begin{pmatrix} v_{11} & \cdots & v_{1m} \\ \vdots & & \vdots \\ v_{n1} & \cdots & v_{nm} \end{pmatrix}, v_{ik} \in \mathbb{R}, i = 1, \ldots, n,$

$k = 1, \ldots, m, B_t = \begin{pmatrix} B_1(t) \\ \vdots \\ B_m(t) \end{pmatrix}.$

Let $g(t,x) = (g_1(t,x), ..., g_p(t,x))$ be a twice continuously differentiable function from $\mathbb{R}^n$ into $\mathbb{R}^p$. Then the process

$$Y(t) = g(t, X(t))$$

is again an Itô process, whose component number k, Y_k, is given by

$$\mathrm{d}Y_k = \frac{\partial g_k}{\partial t}(t, X_t)\mathrm{d}t + \sum_i \frac{\partial g_k}{\partial x_i}(t, X_t)\mathrm{d}X_{i,t} + \frac{1}{2}\sum_{i,j} \frac{\partial^2 g_k}{\partial x_i \partial x_j}(t, X_t)\mathrm{d}X_{i,t}\mathrm{d}X_{j,t},$$

where $\mathrm{d}X_{i,t}\mathrm{d}X_{j,t}$ is computed according to rules (3.4) *and* (3.5).

Theorem 4.2.2. *Let*

$$\mathrm{d}X_t = AX_t\mathrm{d}t + G\mathrm{d}B_t \tag{4.7}$$

be an n-dimensional Itô process, where

$$X_t = \begin{pmatrix} X_1(t) \\ \vdots \\ X_n(t) \end{pmatrix}, A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}, G = \begin{pmatrix} g_{11} & \cdots & g_{1m} \\ \vdots & & \vdots \\ g_{n1} & \cdots & g_{nm} \end{pmatrix}, B_t = \begin{pmatrix} B_1(t) \\ \vdots \\ B_m(t) \end{pmatrix}.$$

Let $f(t,x) = (f_1(t,x), ..., f_p(t,x))$ be a twice continuously differentiable function from $\mathbb{R}^n$ into $\mathbb{R}^p$. Then the process

$$Y(t) = f(t, X(t))$$

is again an Itô process, whose component number k, Y_k, is given by

$$\mathrm{d}Y_k = \frac{\partial f_k}{\partial t}(t, X_t)\mathrm{d}t + \sum_i \frac{\partial f_k}{\partial x_i}(t, X_t)(AX_{i,t}\mathrm{d}t + G\mathrm{d}B_{i,t}) \tag{4.8}$$

$$+\frac{1}{2}\sum_{i,j} \frac{\partial^2 f_k}{\partial x_i \partial x_j}(t, X_t)(G^2\mathrm{d}B_{i,t}^2),$$

where $\mathrm{d}X_{i,t}\mathrm{d}X_{j,t}$ is computed according to rules (3.4) *and* (3.5).

Proof. The proof will be performed according to the proof for 1-dimensional Brownian motion and adjusted for the 2-dimensional version. First observe that if we substitute

$$\mathrm{d}X_t = AX_t\mathrm{d}t + G\mathrm{d}B_t$$

in equation (4.8) and use rules (3.4) and (3.5), we get the equivalent expression

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \int_0^t \left(\frac{\partial f}{\partial s}(s, X_s) + AX_s\frac{\partial f}{\partial x}(s, X_s) + G^2\frac{\partial^2 f}{\partial x^2}(s, X_s) \right) \mathrm{d}s \\ &+ \int_0^t G\frac{\partial f}{\partial x}(s, X_s)\mathrm{d}B_s \end{aligned}$$

Assume that AX_t and G are elementary functions. Using Taylor's theorem we get

$$\begin{aligned} f(t,X_t) &= f(0,X_0) + \sum_j \triangle f(t_j,X_j) = f(0,X_0) + \sum_j \frac{\partial f}{\partial t} \triangle t_j + \sum_j \frac{\partial f}{\partial x} \triangle X_j \\ &+ \frac{1}{2}\sum_j \frac{\partial^2 f}{\partial t^2}(\triangle t_j)^2 + \sum_j \frac{\partial^2 f}{\partial t \partial x}(\triangle t_j)(\triangle X_j) \\ &+ \frac{1}{2}\sum_j \frac{\partial^2 f}{\partial x^2}(\triangle X_j)^2 + \sum_j R_j, \end{aligned}$$

where $\dfrac{\partial f}{\partial t}, \dfrac{\partial f}{\partial x}$, etc. are evaluated at the points $(t_j, X_j), \triangle t_j = t_{j+1} - t_j$,
$\triangle X_j = X_{t_{j+1}} - X_{t_j}, \triangle f(t_j, X_j) = f(t_{j+1}, X_{t_{j+1}}) + f(t_j, X_j)$ and
$R_j = o(|\triangle t_j|^2 + |\triangle X_j|^2)$ for all j. If $\triangle t_j \to 0$ then

$$\begin{aligned} \sum_j \frac{\partial f}{\partial t} \triangle t_j &= \sum_j \frac{\partial f}{\partial t}(t_j,X_j) \triangle t_j \to \int_0^t \frac{\partial f}{\partial s}(s,X_s)\mathrm{d}s, \\ \sum_j \frac{\partial f}{\partial x} \triangle X_j &= \sum_j \frac{\partial f}{\partial x}(t_j,X_j) \triangle X_j \to \int_0^t \frac{\partial f}{\partial x}(s,X_s)\mathrm{d}X_s. \end{aligned}$$

Moreover, since AX_t and G are elementary we get

$$\begin{aligned} \sum_j \frac{\partial^2 f}{\partial x^2}(\triangle X_j)^2 &= \sum_j \frac{\partial^2 f}{\partial x^2}(A_j X_{t_j})^2(\triangle t_j)^2 + 2\sum_j \frac{\partial^2 f}{\partial x^2} A_j X_{t_j} G_j (\triangle t_j)(\triangle B_j) \\ &+ \sum_j \frac{\partial^2 f}{\partial x^2}(G_j)^2(\triangle B_j)^2. \end{aligned}$$

The first two terms here tend to 0 as $\triangle t_j \to 0$. For example,

$$\mathbb{E}\left[\left(\sum_j \frac{\partial^2 f}{\partial x^2} A_j X_{t_j} G_j (\triangle t_j)(\triangle B_j)\right)^2\right] = \sum_j \mathbb{E}\left[\left(\frac{\partial^2 f}{\partial x^2} A_j X_{t_j} G_j\right)^2\right](\triangle t_j)^3 \to 0.$$

We claim that the last term tends to

$$\int_0^t G^2 \frac{\partial^2 f}{\partial x^2}(s,X_s)\mathrm{d}s \to 0.$$

That completes the proof of the formula (4.8). $\square$

Corollary 4.2.3. Suppose the stochastic system (4.7) with $X_t \neq 0, X_0 = \eta, \eta$ is a constant vector, $B_0 = 0$,

$$\mathrm{d}X_t = AX_t\mathrm{d}t + G\mathrm{d}B_t.$$

First we compute the deterministic part

$$\begin{aligned} \mathrm{d}X_t &= AX_t\mathrm{d}t, \\ \frac{\mathrm{d}X_t}{X_t} &= A\mathrm{d}t, \\ \int_0^t \frac{1}{X_s}\mathrm{d}X_s &= A\int_0^t \mathrm{d}s, \\ \ln X_t - \ln X_0 &= At, \\ X_t &= \mathrm{e}^{At}\eta. \end{aligned}$$

Suppose that $\eta = \phi(t), \phi(t)$ is a function

$$\begin{aligned} X_t &= \mathrm{e}^{At}\phi(t) \\ \mathrm{d}X_t &= \mathrm{e}^{At}\mathrm{d}\phi(t) + \mathrm{e}^{At}A\phi(t) \\ \mathrm{e}^{At}\mathrm{d}\phi(t) + \mathrm{e}^{At}A\phi(t) &= A\mathrm{e}^{At}\phi(t) + G\mathrm{d}B_t \\ \mathrm{e}^{At}\mathrm{d}\phi(t) &= G\mathrm{d}B_t \\ G^{-1}\mathrm{e}^{At}\mathrm{d}\phi(t) &= \mathrm{d}B_t \end{aligned}$$

At this moment let's solve the right-hand side using Itô formula (4.8). Choose $X_t \equiv B_t$ and $f(t, X_t) = X_t$. Then

$$Y_t = f(t, B_t) = B_t.$$

Then by Itô's formula,

$$\begin{aligned} \mathrm{d}Y_t &= 0 + 1 \cdot \mathrm{d}B_t + 0 \cdot \mathrm{d}t \\ \mathrm{d}B_t &= \mathrm{d}B_t = Bt \end{aligned}$$

hence

$$\begin{aligned} G^{-1}\mathrm{e}^{At}\mathrm{d}\phi(t) &= \mathrm{d}B_t, \\ G^{-1}\mathrm{e}^{At}\int \mathrm{d}\phi(t) &= \int \mathrm{d}B_t, \\ G^{-1}\mathrm{e}^{At}\phi(t) &= B_t, \\ \phi(t) &= G\mathrm{e}^{-At}B_t. \end{aligned}$$

Solution of the stochastic system (4.7) is

$$X_t = \mathrm{e}^{At}G\mathrm{e}^{-At}B_t.$$

Example 4.2.4. It is given the stochastic differential equation

$$\mathrm{d}X_t = AX_t\mathrm{d}t + G\mathrm{d}B_t,$$

where A is a drift coefficient, G is a diffuse coefficient, $X_t = \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix}$,
$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$, $G = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix}$, $\mathrm{d}B_t = \begin{pmatrix} \mathrm{d}B_1(t) \\ \mathrm{d}B_2(t) \end{pmatrix}$.
There are several possibilities:

- **Singular matrix A and singular matrix G**

$$|A| = \begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix} = a_1a_4 - a_2a_3 = 0 \qquad |G| = \begin{vmatrix} g_1 & g_2 \\ g_3 & g_4 \end{vmatrix} = g_1g_4 - g_2g_3 = 0.$$

Matrices A and G we can write as

$$A_s = \begin{pmatrix} a_1 & a_2 \\ ka_1 & ka_2 \end{pmatrix} \qquad G_s = \begin{pmatrix} g_1 & g_2 \\ mg_1 & mg_2 \end{pmatrix},$$

$$\mathrm{d}X_t = A_s X_t \mathrm{d}t + G_s \mathrm{d}Bt,$$

$$\begin{pmatrix} \mathrm{d}X_1(t) \\ \mathrm{d}X_2(t) \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ ka_1 & ka_2 \end{pmatrix} \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} \mathrm{d}t + \begin{pmatrix} g_1 & g_2 \\ mg_1 & mg_2 \end{pmatrix} \begin{pmatrix} \mathrm{d}B_1(t) \\ \mathrm{d}B_2(t) \end{pmatrix}.$$

We look for a solution of the stochastic equation by substitution
$\mathrm{d}(X_t) = \mathrm{d}(t + X_t)$

$$\begin{aligned} \mathrm{d}(t + X_1) &= \mathrm{d}t + \mathrm{d}X_1 = \mathrm{d}t + a_1 X_1 \mathrm{d}t + a_2 X_2 \mathrm{d}t + g_1 \mathrm{d}B_1 + g_2 \mathrm{d}B_2, \\ t + X_1(t) - X_1(0) &= t + a_1 \int_0^t X_1(s)\mathrm{d}s + a_2 \int_0^t X_2(s)\mathrm{d}s + g_1 \int_0^t \mathrm{d}B_1(s) \\ &+ g_2 \int_0^t \mathrm{d}B_2(s), \\ X_1(t) &= X_1(0) + a_1 \int_0^t X_1(s)\mathrm{d}s + a_2 \int_0^t X_2(s)\mathrm{d}s \\ &+ g_1 \left(B_1(t) - B_1(0)\right) + g_2 \left(B_2(t) - B_2(0)\right). \end{aligned}$$

and

$$\begin{aligned} \mathrm{d}(t + X_2) &= \mathrm{d}t + \mathrm{d}X_2 = \mathrm{d}t + k\left(a_1 X_1 \mathrm{d}t + a_2 X_2 \mathrm{d}t\right) + m\left(g_1 \mathrm{d}B_1 + g_2 \mathrm{d}B_2\right), \\ X_2(t) &= X_2(0) + k\left(a_1 \int_0^t X_1(s)\mathrm{d}s + a_2 \int_0^t X_2(s)\mathrm{d}s\right) \\ &+ m\left[g_1 \left(B_1(t) - B_1(0)\right) + g_2 \left(B_2(t) - g_2 B_2(0)\right)\right]. \end{aligned}$$

- **Singular matrix A and regular matrix G**

$$A_s = \begin{pmatrix} a_1 & a_2 \\ ka_1 & ka_2 \end{pmatrix}. \qquad G_r = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix}, g_1 g_4 - g_2 g_3 \neq 0.$$

$$\begin{pmatrix} \mathrm{d}X_1 \\ \mathrm{d}X_2 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ ka_1 & ka_2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \mathrm{d}t + \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} \begin{pmatrix} \mathrm{d}B_1 \\ \mathrm{d}B_2 \end{pmatrix}.$$

The system solution is

$$\begin{aligned} X_1(t) &= X_1(0) + a_1 \int_0^t X_1(s)\mathrm{d}s + a_2 \int_0^t X_2(s)\mathrm{d}s + g_1 \left(B_1(t) - B_1(0)\right) \\ &+ g_2 \left(B_2(t) - B_2(0)\right), \\ X_2(t) &= X_2(0) + k\left(a_1 \int_0^t X_1(s)\mathrm{d}s + a_2 \int_0^t X_2(s)\mathrm{d}s\right) + g_3 \left(B_1(t) - B_1(0)\right) \\ &+ g_4 \left(B_2(t) - B_2(0)\right). \end{aligned}$$

- **Regular matrix A and singular matrix G**

$$\begin{pmatrix} \mathrm{d}X_1 \\ \mathrm{d}X_2 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \mathrm{d}t + \begin{pmatrix} g_1 & g_2 \\ mg_1 & mg_2 \end{pmatrix} \begin{pmatrix} \mathrm{d}B_1 \\ \mathrm{d}B_2 \end{pmatrix}.$$

The system solution is

$$\begin{aligned} X_1(t) &= X_1(0) + a_1 \int_0^t X_1(s)\mathrm{d}s + a_2 \int_0^t X_2(s)\mathrm{d}s + g_1 \left(B_1(t) - B_1(0)\right) \\ &+ g_2 \left(B_2(t) - B_2(0)\right), \\ X_2(t) &= X_2(0) + a_3 \int_0^t X_1(s)\mathrm{d}s + a_4 \int_0^t X_2(s)\mathrm{d}s + m \left[g_1 \left(B_1(t) - B_1(0)\right)\right. \\ &+ \left. g_2 \left(B_2(t) - B_2(0)\right)\right]. \end{aligned}$$

- **Regular matrix A and regular matrix G**

$$\begin{pmatrix} \mathrm{d}X_1 \\ \mathrm{d}X_2 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \mathrm{d}t + \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} \begin{pmatrix} \mathrm{d}B_1 \\ \mathrm{d}B_2 \end{pmatrix}.$$

The system solution is

$$\begin{aligned} X_1(t) &= X_1(0) + a_1 \int_0^t X_1(s)\mathrm{d}s + a_2 \int_0^t X_2(s)\mathrm{d}s + g_1 \left(B_1(t) - B_1(0)\right) \\ &+ g_2 \left(B_2(t) - B_2(0)\right) \\ X_2(t) &= X_2(0) + a_3 \int_0^t X_1(s)\mathrm{d}s + a_4 \int_0^t X_2(s)\mathrm{d}s + g_3 \left(B_1(t) - B_1(0)\right) \\ &+ g_4 \left(B_2(t) - B_2(0)\right). \end{aligned}$$

4.2.2 Stability of the Solution Using the Lyapunov Method

We have a homogenous linear stochastic differential equation

$$\mathrm{d}X_t = AX_t\mathrm{d}t + G\mathrm{d}B_t, \tag{4.9}$$

where $X_t = \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix}$, $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$, $G = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix}$, $B_t = \begin{pmatrix} B_1(t) \\ B_2(t) \end{pmatrix}$, $a_1, a_2, a_3, a_4, g_1, g_2, g_3, g_4$ are constants.

Definition 4.2.1. Lyapunov quadratic function V is given

$$V(X_t) = X_t^T Q\, X_t,$$

where $Q = \begin{pmatrix} q_1 & q_2 \\ q_2 & q_1 \end{pmatrix}$ is a symmetric positive-definite matrix, i.e. $q_1 > 0$, $q_1^2 - q_2^2 > 0$.

Theorem 4.2.5. *Zero solution of equation* (4.9) *is stochastically stable if holds* $LV < 0$, *where*

$$LV = 2\left[a_1X_1^2(t) + a_4X_2^2(t) + (a_2+a_3)X_1(t)X_2(t) + g_1^2 + g_2^2 + g_3^2 + g_4^2\right].$$

Proof. We compute derivation of Lyapunov function of equation (4.9)

$$\begin{aligned}
\mathrm{d}V(X_t) &= V(X_t + \mathrm{d}X_t) - V(X_t) \\
&= (X_t^T + (AX_t)^T\mathrm{d}t + (G\mathrm{d}B_t)^T)Q(X_t + AX_t\mathrm{d}t + G\mathrm{d}B_t) - X_t^TQX_t \\
&= X_t^TQX_t + X_t^TQAX_t\mathrm{d}t + X_t^TQG\mathrm{d}B_t + (AX_t)^T\mathrm{d}tQX_t \\
&+ (AX_t)^T\mathrm{d}tQAX_t\mathrm{d}t + (AX_t)^T\mathrm{d}tQG\mathrm{d}B_t + (G\mathrm{d}B_t)^TQX_t \\
&+ (G\mathrm{d}B_t)^TQAX_t\mathrm{d}t + (G\mathrm{d}B_t)^TQG\mathrm{d}B_t - X_t^TQX_t \\
&= X_t^TQAX_t\mathrm{d}t + X_t^TQG\mathrm{d}B_t + X_t^TA^T\mathrm{d}tQX_t + X_t^TA^T\mathrm{d}tQAX_t\mathrm{d}t \\
&+ X_t^TA^T\mathrm{d}tQG\mathrm{d}B_t + \mathrm{d}B_t^TG^TQX_t + \mathrm{d}B_t^TG^TQAX_t\mathrm{d}t + \mathrm{d}B_t^TG^TQG\mathrm{d}B_t.
\end{aligned}$$

We use the rules [114], pp. 44,

$$\mathrm{d}t\cdot\mathrm{d}t = \mathrm{d}t\cdot\mathrm{d}B_1(t) = \mathrm{d}t\cdot\mathrm{d}B_2(t) = \mathrm{d}B_1(t)\cdot\mathrm{d}B_2(t) = 0.$$

After that we get

$$\mathrm{d}V(X_t) = X_t^TQAX_t\mathrm{d}t + X_t^TQG\mathrm{d}B_t + X_t^TA^T\mathrm{d}tQX_t + \mathrm{d}B_t^TG^TQX_t + \mathrm{d}B_t^TG^TQG\mathrm{d}B_t.$$

In matrix form

$$\begin{aligned}
\mathrm{d}V\begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} &= \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix}^T \begin{pmatrix} q_1 & q_2 \\ q_2 & q_1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} \mathrm{d}t \\
&+ \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix}^T \begin{pmatrix} q_1 & q_2 \\ q_2 & q_1 \end{pmatrix} \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} \begin{pmatrix} \mathrm{d}B_1(t) \\ \mathrm{d}B_2(t) \end{pmatrix} \\
&+ \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix}^T \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}^T \begin{pmatrix} q_1 & q_2 \\ q_2 & q_1 \end{pmatrix} \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} \mathrm{d}t \\
&+ \begin{pmatrix} \mathrm{d}B_1(t) \\ \mathrm{d}B_2(t) \end{pmatrix}^T \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix}^T \begin{pmatrix} q_1 & q_2 \\ q_2 & q_1 \end{pmatrix} \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} \\
&+ \begin{pmatrix} \mathrm{d}B_1(t) \\ \mathrm{d}B_2(t) \end{pmatrix}^T \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix}^T \begin{pmatrix} q_1 & q_2 \\ q_2 & q_1 \end{pmatrix} \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} \begin{pmatrix} \mathrm{d}B_1(t) \\ \mathrm{d}B_2(t) \end{pmatrix}.
\end{aligned}$$

We denote the last addend

$$\begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix}^T \begin{pmatrix} q_1 & q_2 \\ q_2 & q_1 \end{pmatrix} \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} = M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}.$$

Then we have

$$
\begin{aligned}
& \begin{pmatrix} \mathrm{d}B_1(t) \\ \mathrm{d}B_2(t) \end{pmatrix}^T \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix}^T \begin{pmatrix} q_1 & q_2 \\ q_2 & q_1 \end{pmatrix} \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} \begin{pmatrix} \mathrm{d}B_1(t) \\ \mathrm{d}B_2(t) \end{pmatrix} \\
= & \begin{pmatrix} \mathrm{d}B_1(t) \\ \mathrm{d}B_2(t) \end{pmatrix}^T \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \begin{pmatrix} \mathrm{d}B_1(t) \\ \mathrm{d}B_2(t) \end{pmatrix} \\
= & \begin{pmatrix} m_1 \mathrm{d}B_1(t) + m_3 \mathrm{d}B_2(t), & m_2 \mathrm{d}B_1(t) + m_4 \mathrm{d}B_2(t) \end{pmatrix} \begin{pmatrix} \mathrm{d}B_1(t) \\ \mathrm{d}B_2(t) \end{pmatrix} \\
= & \; m_1 \mathrm{d}B_1(t)\mathrm{d}B_1(t) + m_3 \mathrm{d}B_2(t)\mathrm{d}B_1(t) + m_2 \mathrm{d}B_1(t)\mathrm{d}B_2(t) + m_4 \mathrm{d}B_2(t)\mathrm{d}B_2(t) \\
= & \; m_1 \mathrm{d}t + m_4 \mathrm{d}t = tr(M)\mathrm{d}t,
\end{aligned}
$$

where $tr(M)$ is trace of square matrix M. We used the rules [114], pp. 44,

$$\mathrm{d}B_1(t) \cdot \mathrm{d}B_1(t) = \mathrm{d}B_2(t) \cdot \mathrm{d}B_2(t) = \mathrm{d}t.$$

We get

$$
\begin{aligned}
\mathrm{d}V(X_t) \;=\; & 2\Big[(a_1 q_1 + a_3 q_2) X_1^2(t) + ((a_2 + a_3) q_1 + (a_1 + a_4) q_2) X_1(t) X_2(t) \\
+ \; & (a_4 q_1 + a_2 q_2) X_2^2(t) + (2 q_2 (g_4 g_2 + g_1 g_3) + q_1 (g_1^2 + g_2^2 + g_3^2 + g_4^2))\Big] \mathrm{d}t \\
+ \; & 2\left[(g_1 q_1 + g_3 q_2) X_1(t) + (g_3 q_1 + g_1 q_2) X_2(t)\right] \mathrm{d}B_1(t) \\
+ \; & 2\left[(g_2 q_1 + g_4 q_2) X_1(t) + (g_4 q_1 + g_2 q_2) X_2(t)\right] \mathrm{d}B_2(t).
\end{aligned}
$$

We apply expectation $\mathbb{E}\left\{\mathrm{d}V(X_t)\right\}$

$$
\begin{aligned}
\mathbb{E}\left\{\mathrm{d}V(X_t)\right\} \;=\; & 2\Big[(a_1 q_1 + a_3 q_2) X_1^2(t) + (a_4 q_1 + a_2 q_2) X_2^2(t) + 2 q_2 (g_4 g_2 + g_1 g_3) \\
+ \; & ((a_2 + a_3) q_1 + (a_1 + a_4) q_2) X_1(t) X_2(t) \\
+ \; & q_1 (g_1^2 + g_2^2 + g_3^2 + g_4^2)\Big] \mathrm{d}t = LV \mathrm{d}t.
\end{aligned}
$$

For $Q = I$ we get

$$LV = 2\left[a_1 X_1^2(t) + a_4 X_2^2(t) + (a_2 + a_3) X_1(t) X_2(t) + g_1^2 + g_2^2 + g_3^2 + g_4^2\right].$$

□

Now we can do a discussion under which conditions the system will be stable. The Euclidean matrix norm on the space $\mathbb{R}^n$ can be define as

$$\|A\|_E := \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij}^2},$$

where a_{ij} is a matrix element of the i-th line and of the j-th column of the matrix, n is number of matrix raws, m is number of matrix columns.
We denote $g_1^2 + g_2^2 + g_3^2 + g_4^2 = \|G\|^2$ and give

$$LV = 2\left[a_1 X_1^2(t) + a_4 X_2^2(t) + (a_2 + a_3) X_1(t) X_2(t) + \|G\|^2\right]. \tag{4.10}$$

The Lyapunov function LV will be negative definite if and only if when

$$a_1X_1^2(t) + a_4X_2^2(t) + (a_2 + a_3)X_1(t)X_2(t) + \|G\|^2 \leq 0,$$

because $\|G\|^2 \geq 0$, therefore the matrix A must be sufficiently negative, to obtain a negative definite function. We use the **Sylvester's criterion** which is a necessary and sufficient criterion to determine whether a matrix is positive-definite. [63]

Theorem 4.2.6. ***(Sylvester's criterion)***
Let A be a real symetric matrix of the n-th order. For $k = 1, \ldots, n$ we denote the main subdeterminants D_k of the matrix A

$$D_k = det \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{pmatrix}.$$

Then the matrix A is positive definite if and only when $D_k > 0$ pro $k = 1, \ldots, n$. And analogously the matrix A is negative definite if and only when $(-1)^k D_k > 0$ for $k = 1, \ldots, n$.

Corollary 4.2.7. First, we consider a diagonal matrix A and G of equation (4.9) in the form

$$A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, G = \begin{pmatrix} \frac{a}{10} & 0 \\ 0 & \frac{a}{10} \end{pmatrix}.$$

The matrix A will be negative definite under following conditions:

$$\begin{aligned} D_1 &= a < 0, \\ D_2 &= a^2 > 0 \end{aligned}$$

if holds D_1 then the condition D_2 is obvious. Then from (4.10) follows

$$\begin{aligned} aX_1^2(t) + aX_2^2(t) &\leq -\|G\|^2, \\ a\|X_t\|^2 &\leq -\|G\|^2. \end{aligned}$$

If the variable a is negative and also inequality $a\|X_t\|^2 \leq -\|G\|^2$ is valid, then the system is stochastically stable.
We find a solution of the stochastic system based on eigenvalues. If $a_{12} = a_{21} = 0$, then $\lambda_1 = a_{11}, \lambda_2 = a_{22} \Rightarrow \lambda_{1,2} = a$. Because a is negative we make substitution $a = -\alpha, \alpha > 0$. We give a solution of the system

$$\begin{aligned} X_1(t) &= C_1 e^{-\alpha t}, \\ X_2(t) &= C_2 t e^{-\alpha t}, \end{aligned}$$

when C_1, C_2 are constants. Zero solution of equation (4.9) with a matrix A is stochastically stable if holds the inequality $a\left\|X_t\right\|^2 \leq -\left\|G\right\|^2$. We determine stability of solution for $Q = I$

$$\begin{aligned} \mathrm{d}V(X_t) &= 2\left[aX_1^2(t) + aX_2^2(t) + \frac{a^2}{50}\right]\mathrm{d}t + \frac{aX_1(t)}{5}\mathrm{d}B_1(t) + \frac{aX_2(t)}{5}\mathrm{d}B_2(t), \\ \mathbb{E}\left\{\mathrm{d}V(X_t)\right\} &= 2\left[aX_1^2(t) + aX_2^2(t) + \frac{a^2}{50}\right]\mathrm{d}t = LV\mathrm{d}t. \end{aligned}$$

There has to hold the inequality $a\left\|X_t\right\|^2 \leq -\left\|G\right\|^2$, so

$$a^2 + 50a\left\|X_t\right\|^2 < 0 \Leftrightarrow a < 0 \vee a < -50\left\|X_t\right\|^2.$$

For $X_1(t) = C_1\mathrm{e}^{-\alpha t}, X_2(t) = C_2 t\mathrm{e}^{-\alpha t}$ we get

$$a < -50\left(C_1^2\mathrm{e}^{-2\alpha t} + C_2^2 t^2\mathrm{e}^{-2\alpha t}\right).$$

Stochastic differential system is stable for $a < 0$ or $a < -50\left(C_1^2\mathrm{e}^{-2\alpha t} + C_2^2 t^2\mathrm{e}^{-2\alpha t}\right)$.

Corollary 4.2.8. We consider a diagonal matrix A and G of equation (4.9) in the form

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, G = \begin{pmatrix} \frac{a}{10} & 0 \\ 0 & \frac{b}{10} \end{pmatrix}.$$

The matrix A will be negative definite under following conditions:

$$\begin{aligned} D_1 &= a < 0, \\ D_2 &= ab > 0 \Rightarrow b < 0. \end{aligned}$$

Then from (4.10) follows $aX_1^2(t) + bX_2^2(t) \leq -\left\|G\right\|^2$. We find a solution of the stochastic system based on eigenvalues. $\lambda_1 = a, \lambda_2 = b$. We substitute $a = -\alpha$, $\alpha > 0, b = -\beta, \beta > 0$. We give a solution of the system

$$\begin{aligned} X_1(t) &= C_1\mathrm{e}^{-\alpha t}, \\ X_2(t) &= C_2 t\mathrm{e}^{-\beta t}, \end{aligned}$$

C_1, C_2 are constants. Zero solution of equation (4.9) with a matrix A is stochastically stable if holds the inequality $aX_1^2(t) + bX_2^2(t) \leq -\left\|G\right\|^2$. We determine stability of solution for $Q = I$

$$\begin{aligned} \mathrm{d}V(X_t) &= 2\left[aX_1^2(t) + bX_2^2(t) + \frac{a^2 + b^2}{100}\right]\mathrm{d}t + \frac{aX_1(t)}{5}\mathrm{d}B_1(t) + \frac{bX_2(t)}{5}\mathrm{d}B_2(t), \\ \mathbb{E}\left\{\mathrm{d}V(X_t)\right\} &= 2\left[aX_1^2(t) + bX_2^2(t) + \left(\frac{a}{10}\right)^2 + \left(\frac{b}{10}\right)^2\right]\mathrm{d}t = LV\mathrm{d}t. \end{aligned}$$

There has to hold the inequality $aX_1^2(t)+bX_2^2(t)\leq -\|G\|^2$, so if for $X_1(t)=C_1\mathrm{e}^{-\alpha t}, X_2(t)=C_2t\mathrm{e}^{-\beta t}$ holds the inequality

$$aC_1^2\mathrm{e}^{-2\alpha t}+bC_2^2t^2\mathrm{e}^{-2\beta t}\leq -\frac{a^2+b^2}{100},$$

then the system is stable.

Corollary 4.2.9. We consider a symmetric matrix A and G of equation (4.9) in the form

$$A=\begin{pmatrix} a & b \\ b & a \end{pmatrix}, G=\begin{pmatrix} \frac{a}{10} & \frac{b}{10} \\ \frac{b}{10} & \frac{a}{10} \end{pmatrix}.$$

The matrix A will be negative definite under following conditions:

$$\left.\begin{array}{l} D_1=a<0, \\ D_2=a^2-b^2>0\Rightarrow |a|>|b| \end{array}\right\} \text{ i.e. must be valid } |a|>|b|>0.$$

Then from (4.10) follows

$$aX_1^2(t)+aX_2^2(t)+2bX_1(t)X_2(t) \quad\leq\quad -\|G\|^2.$$

The variable a must be sufficiently negative and also inequality

$$a\|X(t)\|^2+2bX_1(t)X_2(t)\leq -\|G\|^2$$

must be valid, then we can say that the system is stochastically stable.
We find eigenvalues of matrix A as the solution of the characteristic equation

$$|A-\lambda I|=0,$$

where I is the unit matrix.

$$\begin{aligned} |A-\lambda I|=(a-\lambda)^2-b^2 &= 0, \\ (a-\lambda)^2 &= b^2, \\ |a-\lambda| &= |b|. \end{aligned}$$

Eigenvalues are

$$\begin{aligned} -a+\lambda_1=|b|\Rightarrow \lambda_1=a+|b|, \\ a-\lambda_2=|b|\Rightarrow \lambda_2=a-|b|. \end{aligned}$$

We substitute $a=-\alpha, \alpha>0, |b|>0, \alpha<|b|$, i.e. $\lambda_1=-\alpha+|b|, \lambda_2=-\alpha-|b|$. For the eigenvalue $\lambda_1=-\alpha+|b|$ we find the eigenvector $v_1=(v_{11},v_{12})$. There is any nonzero vector which fulfills a following relation

$$\begin{aligned} (A-\lambda_1 I)\, v_1 &= 0, \\ \begin{pmatrix} a-(a+|b|) & b \\ b & a-(a+|b|) \end{pmatrix} v_1 &= 0. \end{aligned}$$

For $b > 0$ we choose an arbitrary vector $v_1 = (1,1)^T$, for $b < 0$ we choose $v_1 = (-1,1)^T$. Then

$$\begin{aligned} &\text{for} \quad b > 0 \quad \text{is} \;\; X_1(t) = (1,1)^T \mathrm{e}^{(-\alpha+b)t}, \\ &\text{for} \quad b < 0 \quad \text{is} \;\; X_1(t) = (-1,1)^T \mathrm{e}^{(-\alpha+b)t}. \end{aligned}$$

For the eigenvalue $\lambda_1 = -\alpha - |b|$ we find an eigenvector $v_2 = (v_{21}, v_{22})$

$$(A - \lambda_1 I)\, v_2 = 0,$$

$$\begin{pmatrix} a - (a - |b|) & b \\ b & a - (a - |b|) \end{pmatrix} v_2 = 0.$$

For $b > 0$ we choose an arbitrary vector $v_2 = (1,-1)^T$, for $b < 0$ we choose $v_2 = (1,1)^T$. Then

$$\begin{aligned} &\text{for} \quad b < 0 \quad \text{is} \;\; X_2(t) = (1,1)^T \mathrm{e}^{-(\alpha+b)t}, \\ &\text{for} \quad b > 0 \quad \text{is} \;\; X_2(t) = (1,-1)^T \mathrm{e}^{-(\alpha+b)t}. \end{aligned}$$

The general solution is given by a linear combination $X_t = C_1 X_1(t) + C_2 X_2(t)$, with arbitrary constants C_1, C_2. Zero solution of equation (4.9) with a matrix A is stochastically stable if holds the inequality $a \|X(t)\|^2 + 2bX_1(t)X_2(t) \leq -\|G\|^2$. We determine stability of solution for $Q = I$

$$\begin{aligned} \mathrm{d}V(X_t) &= 2\left[a\left(X_1^2(t) + X_2^2(t)\right) + 2bX_1(t)X_2(t) + \frac{a^2}{50} + \frac{b^2}{50}\right]\mathrm{d}t \\ &+ \frac{aX_1(t) + bX_2(t)}{5}\mathrm{d}B_1(t) + \frac{bX_1(t) + aX_2(t)}{5}\mathrm{d}B_2(t), \\ \mathbb{E}\left\{\mathrm{d}V(X_t)\right\} &= 2\left[a\left(X_1^2(t) + X_2^2(t)\right) + 2bX_1(t)X_2(t) + \frac{a^2 + b^2}{50}\right]\mathrm{d}t = LV\mathrm{d}t. \end{aligned}$$

There has to hold the inequality $a\|X(t)\|^2 + 2bX_1(t)X_2(t) \leq -\|G\|^2$, so if holds the inequality

$$a\|X(t)\|^2 + 2bX_1(t)X_2(t) \leq -\frac{a^2 + b^2}{50},$$

for $b > 0, X_1(t) = (1,1)^T \mathrm{e}^{(-\alpha+b)t}, X_2(t) = (1,-1)^T \mathrm{e}^{-(\alpha+b)t}$; for $b < 0$, $X_1(t) = (-1,1)^T \mathrm{e}^{(-\alpha+b)t}, X_2(t) = (1,1)^T \mathrm{e}^{-(\alpha+b)t}$, then the system is stable.

4.3 Three-Dimensional Brownian Motion

4.3.1 Solution of Stochastic Differential Equations

See subsection Solution of stochastic differential equations (4.2.1) in section Two-dimensional Browninan motion, where is described solution of SDE for multidimensional Brownian motion.

4.3.2 Stability of Solution Using Lyapunov Method

We have a homogenous linear stochastic differential equation

$$\mathrm{d}X_t = AX_t\mathrm{d}t + G\mathrm{d}B_t, \tag{4.11}$$

where $X_t = \begin{pmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{pmatrix}$, $A = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}$, $G = \begin{pmatrix} g_1 & g_2 & g_3 \\ g_4 & g_5 & g_6 \\ g_7 & g_8 & g_9 \end{pmatrix}$,
$B_t = \begin{pmatrix} B_1(t) \\ B_2(t) \\ B_3(t) \end{pmatrix}$, a_i, g_i for $i = 1, \ldots, 9$ are constants.

Definition 4.3.1. Lyapunov quadratic function V is given

$$V(X_t) = X_t^T Q\, X_t,$$

where $Q = \begin{pmatrix} q_1 & q_2 & q_3 \\ q_2 & q_1 & q_2 \\ q_3 & q_2 & q_1 \end{pmatrix}$ is a symmetric positive-definite matrix, i.e.

$$D_1 = q_1 > 0, D_2 = q_1^2 - q_2^2 > 0, D_3 = q_1^3 + 2q_2^2q_3 - q_3^2q_1 - 2q_2^2q_1 > 0.$$

Theorem 4.3.1. *Zero solution of equation* (4.11) *is stochastically stable if holds* $LV < 0$*, where*

$$\begin{aligned} LV &= 2a_1X_1^2(t) + 2a_5X_2^2(t) + 2a_9X_3^2(t) + 2\left(a_4 + a_2\right)X_1(t)X_2(t) \\ &+ 2\left(a_3 + a_7\right)X_1(t)X_3(t) + 2\left(a_6 + a_8\right)X_2(t)X_3(t) + 2\left(g_1 + g_5 + g_9\right) \\ &+ g_1^2 + g_2^2 + g_3^2 + g_4^2 + g_5^2 + g_6^2 + g_7^2 + g_8^2 + g_9^2. \end{aligned}$$

Proof. We compute derivation of Lyapunov function of equation (4.11)

$$\begin{aligned} \mathrm{d}V(X_t) &= V(X_t + \mathrm{d}X_t) - V(X_t) \\ &= (X_t^T + (AX_t)^T\mathrm{d}t + (G\mathrm{d}B_t)^T)Q(X_t + AX_t\mathrm{d}t + G\mathrm{d}B_t) - X_t^TQX_t \\ &= X_t^TQX_t + X_t^TQAX_t\mathrm{d}t + X_t^TQG\mathrm{d}B_t + (AX_t)^T\mathrm{d}tQX_t \\ &+ (AX_t)^T\mathrm{d}tQAX_t\mathrm{d}t + (AX_t)^T\mathrm{d}tQG\mathrm{d}B_t + (G\mathrm{d}B_t)^TQX_t \\ &+ (G\mathrm{d}B_t)^TQAX_t\mathrm{d}t + (G\mathrm{d}B_t)^TQG\mathrm{d}B_t - X_t^TQX_t \\ &= X_t^TQAX_t\mathrm{d}t + X_t^TQG\mathrm{d}B_t + X_t^TA^T\mathrm{d}tQX_t + X_t^TA^T\mathrm{d}tQAX_t\mathrm{d}t \\ &+ X_t^TA^T\mathrm{d}tQG\mathrm{d}B_t + \mathrm{d}B_t^TG^TQX_t + \mathrm{d}B_t^TG^TQAX_t\mathrm{d}t + \mathrm{d}B_t^TG^TQG\mathrm{d}B_t. \end{aligned}$$

We use the rules [114], pp. 44,

$$\mathrm{d}t \cdot \mathrm{d}t = \mathrm{d}t \cdot \mathrm{d}B_1(t) = \mathrm{d}t \cdot \mathrm{d}B_2(t) = \mathrm{d}B_1(t) \cdot \mathrm{d}B_2(t) = 0.$$

After that we get

$$\begin{aligned}\mathrm{d}V(X_t) &= X_t^T QAX_t\mathrm{d}t + X_t^T QG\mathrm{d}B_t + X_t^T A^T \mathrm{d}t QX_t + \mathrm{d}B_t^T G^T QX_t \\ &+ \mathrm{d}B_t^T G^T QG\mathrm{d}B_t.\end{aligned}$$

In matrix form

$$\begin{aligned}\mathrm{d}V\begin{pmatrix} X_1(t) \\ X_2(t) \\ X_3(t)\end{pmatrix} &= \begin{pmatrix} X_1(t) \\ X_2(t) \\ X_3(t)\end{pmatrix}^T \begin{pmatrix} q_1 & q_2 & q_3 \\ q_2 & q_1 & q_2 \\ q_3 & q_2 & q_1\end{pmatrix}\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9\end{pmatrix}\begin{pmatrix} X_1(t) \\ X_2(t) \\ X_3(t)\end{pmatrix}\mathrm{d}t \\ &+ \begin{pmatrix} X_1(t) \\ X_2(t) \\ X_3(t)\end{pmatrix}^T \begin{pmatrix} q_1 & q_2 & q_3 \\ q_2 & q_1 & q_2 \\ q_3 & q_2 & q_1\end{pmatrix}\begin{pmatrix} g_1 & g_2 & g_3 \\ g_4 & g_5 & g_6 \\ g_7 & g_8 & g_9\end{pmatrix}\begin{pmatrix} \mathrm{d}B_1(t) \\ \mathrm{d}B_2(t) \\ \mathrm{d}B_3(t)\end{pmatrix} \\ &+ \begin{pmatrix} X_1(t) \\ X_2(t) \\ X_3(t)\end{pmatrix}^T \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9\end{pmatrix}^T \begin{pmatrix} q_1 & q_2 & q_3 \\ q_2 & q_1 & q_2 \\ q_3 & q_2 & q_1\end{pmatrix}\begin{pmatrix} X_1(t) \\ X_2(t) \\ X_3(t)\end{pmatrix}\mathrm{d}t \\ &+ \begin{pmatrix} \mathrm{d}B_1(t) \\ \mathrm{d}B_2(t) \\ \mathrm{d}B_3(t)\end{pmatrix}^T \begin{pmatrix} g_1 & g_2 & g_3 \\ g_4 & g_5 & g_6 \\ g_7 & g_8 & g_9\end{pmatrix}^T \begin{pmatrix} q_1 & q_2 & q_3 \\ q_2 & q_1 & q_2 \\ q_3 & q_2 & q_1\end{pmatrix}\begin{pmatrix} X_1(t) \\ X_2(t) \\ X_3(t)\end{pmatrix} \\ &+ \begin{pmatrix} \mathrm{d}B_1(t) \\ \mathrm{d}B_2(t) \\ \mathrm{d}B_3(t)\end{pmatrix}^T \begin{pmatrix} g_1 & g_2 & g_3 \\ g_4 & g_5 & g_6 \\ g_7 & g_8 & g_9\end{pmatrix}^T \begin{pmatrix} q_1 & q_2 & q_3 \\ q_2 & q_1 & q_2 \\ q_3 & q_2 & q_1\end{pmatrix}\begin{pmatrix} g_1 & g_2 & g_3 \\ g_4 & g_5 & g_6 \\ g_7 & g_8 & g_9\end{pmatrix} \\ &\times \begin{pmatrix} \mathrm{d}B_1(t) \\ \mathrm{d}B_2(t) \\ \mathrm{d}B_3(t)\end{pmatrix}.\end{aligned}$$

We get

$$\begin{aligned}\mathrm{d}V(X_t) &= 2\left(a_1q_1 + a_4q_2 + a_7q_3\right)X_1^2(t)\mathrm{d}t + 2\left(a_2q_2 + a_5q_1 + a_8q_2\right)X_2^2(t)\mathrm{d}t \\ &+ 2\left(a_3q_3 + a_6q_2 + a_9q_1\right)X_3^2(t)\mathrm{d}t + 2\left(a_1q_2 + a_4q_1 + a_7q_2 + a_2q_1 + a_5q_2\right. \\ &+ \left. a_8q_3\right)X_1(t)X_2(t)\mathrm{d}t + 2\left(a_3q_1 + a_6q_2 + a_9q_3 + a_1q_3 + a_4q_2\right. \\ &+ \left. a_7q_1\right)X_1(t)X_3(t)\mathrm{d}t + 2\left(a_3q_2 + a_6q_1 + a_9q_2 + a_2q_3 + a_5q_2\right. \\ &+ \left. a_8q_1\right)X_2(t)X_3(t)\mathrm{d}t + \left[q_1\left(g_1^2 + g_2^2 + g_3^2 + g_4^2 + g_5^2 + g_6^2 + g_7^2 + g_8^2 + g_9^2\right)\right. \\ &+ 2q_2\left(g_1g_4 + g_2g_5 + g_3g_6 + g_4g_7 + g_5g_8 + g_6g_9\right) + 2q_3\left(g_1g_7 + g_2g_8\right. \\ &+ \left.\left. g_3g_9\right)\right]\mathrm{d}t + 2\left[X_1(t)\left(q_3g_7 + q_2g_4 + q_1g_1\right) + X_2(t)\left(q_2g_7 + q_1g_4 + q_2g_1\right)\right. \\ &+ \left. X_3(t)\left(q_1g_7 + q_2g_4 + q_3g_1\right)\right]\mathrm{d}B_1(t) + 2\left[X_1(t)\left(q_3g_8 + q_2g_5 + q_1g_2\right)\right. \\ &+ \left. X_2(t)\left(q_2g_8 + q_1g_5 + q_2g_2\right) + X_3(t)\left(q_1g_8 + q_2g_5 + q_3g_2\right)\right]\mathrm{d}B_2(t) \\ &+ 2\left[X_1(t)\left(q_3g_9 + q_2g_6 + q_1g_3\right) + X_2(t)\left(q_2g_9 + q_1g_6 + q_2g_3\right)\right. \\ &+ \left. X_3(t)\left(q_1g_9 + q_2g_6 + q_3g_3\right)\right]\mathrm{d}B_3(t)\end{aligned}$$

We apply expectation $\mathbb{E}\{\mathrm{d}V(X_t)\}$

$$\begin{aligned}
\mathbb{E}\{\mathrm{d}V(X_t)\} &= \Big[2\left(a_1q_1+a_4q_2+a_7q_3\right)X_1^2(t)+2\left(a_2q_2+a_5q_1+a_8q_2\right)X_2^2(t)\\
&+ 2\left(a_3q_3+a_6q_2+a_9q_1\right)X_3^2(t)+2\left(a_1q_2+a_4q_1+a_7q_2+a_2q_1\right.\\
&+ \left.a_5q_2+a_8q_3\right)X_1(t)X_2(t)+2\left(a_3q_1+a_6q_2+a_9q_3+a_1q_3+a_4q_2\right.\\
&+ \left.a_7q_1\right)X_1(t)X_3(t)+2\left(a_3q_2+a_6q_1+a_9q_2+a_2q_3+a_5q_2+a_8q_1\right)\\
&\times X_2(t)X_3(t)+\Big[q_1\left(g_1^2+g_2^2+g_3^2+g_4^2+g_5^2+g_6^2+g_7^2+g_8^2+g_9^2\right)\\
&+ 2q_2\left(g_1g_4+g_2g_5+g_3g_6+g_4g_7+g_5g_8+g_6g_9\right)+2q_3\left(g_1g_7+g_2g_8\right.\\
&+ \left.g_3g_9\right)\Big]\Big]\,\mathrm{d}t = LV\mathrm{d}t.
\end{aligned}$$

For $Q=I$ we get

$$\begin{aligned}
LV &= 2\Big[a_1X_1^2(t)+a_5X_2^2(t)+a_9X_3^2(t)+\left(a_4+a_2\right)X_1(t)X_2(t)+\left(a_3+a_7\right)X_1(t)\\
&\times X_3(t)+\left(a_6+a_8\right)X_2(t)X_3(t)\Big]+g_1^2+g_2^2+g_3^2+g_4^2+g_5^2+g_6^2+g_7^2+g_8^2+g_9^2.
\end{aligned}$$

□

Let us find conditions the system will be stable for. We denote $g_1^2+g_2^2+g_3^2+g_4^2+g_5^2+g_6^2+g_7^2+g_8^2+g_9^2=\|G\|^2$ and give

$$\begin{aligned}
LV &= 2a_1X_1^2(t)+2a_5X_2^2(t)+2a_9X_3^2(t)+2\left(a_4+a_2\right)X_1(t)X_2(t)\\
&+ 2\left(a_3+a_7\right)X_1(t)X_3(t)+2\left(a_6+a_8\right)X_2(t)X_3(t)+\|G\|^2.
\end{aligned}$$

The Lyapunov function LV will be negative definite if and only if when

$$\begin{aligned}
&2a_1X_1^2(t)+2a_5X_2^2(t)+2a_9X_3^2(t)+2\left(a_4+a_2\right)X_1(t)X_2(t)\\
+&\ 2\left(a_3+a_7\right)X_1(t)X_3(t)+2\left(a_6+a_8\right)X_2(t)X_3(t)+\|G\|^2\le 0,
\end{aligned}$$

because $\|G\|^2\ge 0$, therefore the matrix A must be sufficiently negative, to obtain a negative definite function.

Corollary 4.3.2. *We consider symmetric matrices A, G of equation* (4.11) *in the form*

$$A=\begin{pmatrix} a & 0 & b\\ 0 & a & 0\\ b & 0 & a\end{pmatrix}, G=\begin{pmatrix} \frac{a}{10} & 0 & \frac{b}{10}\\ 0 & \frac{a}{10} & 0\\ \frac{b}{10} & 0 & \frac{a}{10}\end{pmatrix}.$$

The matrix A will be negative definite for following conditions:

$$\begin{aligned}
&D_1=a<0,\\
&D_2=a^2>0,\ D_2\ \text{ follows from }\ D_1,\\
&D_3=a^3-ab^2<0\Rightarrow a(a^2-b^2)<0\Leftrightarrow a<0\wedge a^2>b^2,
\end{aligned}$$

Based on these conditions it is evident that $|a| > |b|, a < 0$. First of all we find solution of the differential system A. We find eigenvalues of matrix A as the solution of the characteristic equation

$$\begin{vmatrix} a-\lambda & 0 & b \\ 0 & a-\lambda & 0 \\ b & 0 & a-\lambda \end{vmatrix} = 0,$$

$$\lambda_1 = a \Rightarrow X_1(t) = \mathrm{e}^{at}, \lambda_{2,3} = a \pm |b|.$$

We substitute $a = -\alpha, \alpha > 0, |b| > 0, \alpha > |b|$, i.e.

$$\lambda_2 = -\alpha + |b|, \lambda_3 = -\alpha - |b|.$$

For the eigenvalue $\lambda_2 = -\alpha + |b|$ we find the eigenvector $v_2 = (v_{21}, v_{22}, v_{23})$. There is any nonzero vector which fulfills a following relation $(A - \lambda_2 E)\, v_2 = \mathcal{O}$, where $\mathcal{O}$ is a zero vector,

$$\begin{pmatrix} a-(a+|b|) & 0 & b \\ 0 & a-(a+|b|) & 0 \\ b & 0 & a-(a+|b|) \end{pmatrix} v_2 = \mathcal{O}.$$

For $b > 0$ we choose an arbitrary vector $v_2 = (1,0,1)^T$, for $b < 0$ we choose $v_2 = (1,0,-1)^T$. Then

$$\begin{aligned} &\text{for} \quad b > 0 \quad \text{is} \quad X_2(t) = (1,0,1)^T \mathrm{e}^{(-\alpha+b)t}, \\ &\text{for} \quad b < 0 \quad \text{is} \quad X_2(t) = (1,0,-1)^T \mathrm{e}^{(-\alpha+b)t}. \end{aligned}$$

For the eigenvalue $\lambda_3 = -\alpha - |b|$ we find an eigenvector $v_3 = (v_{31}, v_{32}, v_{33})$ in the following relation $(A - \lambda_3 E)\, v_3 = \mathcal{O}$,

$$\begin{pmatrix} a-(a-|b|) & 0 & b \\ 0 & a-(a-|b|) & 0 \\ b & 0 & a-(a-|b|) \end{pmatrix} v_3 = \mathcal{O}.$$

For $b > 0$ we choose an arbitrary vector $v_3 = (1,0,-1)^T$, for $b < 0$ we choose $v_3 = (1,0,1)^T$. Then

$$\begin{aligned} &\text{for} \quad b < 0 \quad \text{is} \quad X_3(t) = (1,0,1)^T \mathrm{e}^{-(\alpha+b)t}, \\ &\text{for} \quad b > 0 \quad \text{is} \quad X_3(t) = (1,0,-1)^T \mathrm{e}^{-(\alpha+b)t}. \end{aligned}$$

The general solution with arbitrary constants C_1, C_2, C_3 is given by a linear combination $X_t = C_1X_1(t) + C_2X_2(t) + C_3X_3(t)$.
It is a solution of differential equation without a stochastic element.

At this moment we find the stability of solution of the stochastic system. We determine stability of solution for $Q = I$

$$\begin{aligned}
\mathrm{d}V(X_t) &= 2\left[aX_1^2(t) + aX_2^2(t) + aX_3^2(t) + 2bX_1(t)X_3(t) + \frac{3}{2}\left(\frac{a}{10}\right)^2\right. \\
&+ \left.\left(\frac{b}{10}\right)^2\right]\mathrm{d}t + \frac{aX_1(t) + bX_3(t)}{5}\mathrm{d}B_1(t) + \frac{aX_2(t)}{5}\mathrm{d}B_2(t) \\
&+ \frac{bX_1(t) + aX_3(t)}{5}\mathrm{d}B_3(t), \\
\mathbb{E}\left\{\mathrm{d}V(X_t)\right\} &= 2\left[a(X_1^2(t) + X_2^2(t) + X_3^2(t)) + 2bX_1(t)X_3(t) + \frac{3a^2 + 2b^2}{200}\right]\mathrm{d}t \\
&= LV\mathrm{d}t.
\end{aligned}$$

If holds the inequality

$$2a\left\|X(t)\right\|^2 + 4bX_1(t)X_3(t) \leq -\frac{3a^2 + 2b^2}{200},$$

for $a > b, b > 0, X_1(t) = \mathrm{e}^{at}, X_2(t) = (1,0,1)^T\mathrm{e}^{(-\alpha+b)t}, X_3(t) = (1,0,-1)^T\mathrm{e}^{-(\alpha+b)t}$;
for $a > b, b < 0, X_1(t) = \mathrm{e}^{at}, X_2(t) = (1,0,-1)^T\mathrm{e}^{(-\alpha+b)t}, X_3(t) = (1,0,1)^T\mathrm{e}^{-(\alpha+b)t}$,
then the system is stable.

4.4 Four-Dimensional Brownian Motion

4.4.1 Solution of Stochastic Differential Equations

See subsection Solution of stochastic differential equations (4.2.1) in section Two-dimensional Brownian motion, where is described solution of SDE for multidimensional Brownian motion.

4.4.2 Stability of Solution Using Lyapunov Method

We have a matrix linear stochastic differential equation

$$\mathrm{d}X_t = AX_t\mathrm{d}t + G\mathrm{d}B_t, \tag{4.12}$$

where $X_t = \begin{pmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \\ X_4(t) \end{pmatrix}$, $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$, $G = \begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{pmatrix}$,

$B_t = \begin{pmatrix} B_1(t) \\ B_2(t) \\ B_3(t) \\ B_4(t) \end{pmatrix}$, a_{ij}, g_{ij} for $i, j = 1, 2, 3, 4$ are constants.

Definition 4.4.1. Lyapunov quadratic function V is given

$$V(X_t) = X_t^T Q\, X_t,$$

where Q is a symmetric positive-definite matrix.

4.4.3 Special Matrix Q Results

Definition 4.4.2. Lyapunov quadratic function V is given

$$V(X_t) = X_t^T Q\, X_t,$$

where $Q = \begin{pmatrix} q_1 & q_2 & q_3 & q_4 \\ q_2 & q_1 & q_2 & q_3 \\ q_3 & q_2 & q_1 & q_2 \\ q_4 & q_3 & q_2 & q_1 \end{pmatrix}$ is a symmetric positive-definite matrix, $q_i \in \mathbb{R}$, $i = 1, 2, 3, 4$. Positive-definite matrix is verified by the Sylvester's criterion. There have to apply these conditions together

$$\begin{aligned} D_1 &= q_1 > 0, \\ D_2 &= q_1^2 - q_2^2 > 0, \\ D_3 &= q_1^3 + 2q_2^2 q_3 - q_1 q_3^2 - 2q_1 q_2^2 > 0, \\ D_4 &= q_1 q_2^3 + q_1 q_2 q_3^2 + q_1^3 q_4 - q_1 q_2^2 q_4 - 2q_1^2 q_2 q_3 - q_1^2 q_2^2 - 2q_2^2 q_3^2 - q_2^3 q_4 + q_2^4 + q_3^4 \\ &+ 2q_1 q_2^2 q_3 + 4q_1 q_2 q_3 q_4 + q_2^2 q_4^2 - 2q_2 q_3^2 q_4 - q_1^2 q_3^2 - q_2^3 q_4 - q_1^2 q_4^2 > 0. \end{aligned}$$

Theorem 4.4.1. *Zero solution of equation* (4.12) *is stochastically stable if holds*

$LV < 0$, *where*

$$\begin{aligned}
LV &= 2\left(a_{11}q_1 + a_{21}q_2 + a_{31}q_3 + a_{41}q_4\right)X_1^2(t) + 2\left(a_{12}q_2 + a_{22}q_1 + a_{32}q_2\right. \\
&+ \left. a_{42}q_3\right)X_2^2(t) + 2\left(a_{13}q_3 + a_{23}q_2 + a_{33}q_1 + a_{43}q_2\right)X_3^2(t) + 2\left(a_{14}q_4\right. \\
&+ \left. a_{24}q_3 + a_{34}q_2 + a_{44}q_1\right)X_4^2(t) + 2\left(a_{12}q_1 + a_{11}q_2 + a_{22}q_2 + a_{21}q_1 + a_{32}q_3\right. \\
&+ \left. a_{31}q_2 + a_{42}q_4 + a_{41}q_3\right)X_1(t)X_2(t) + 2\left(a_{13}q_1 + a_{11}q_3 + a_{23}q_2 + a_{23}q_1\right. \\
&+ \left. a_{21}q_2 + a_{33}q_3 + a_{31}q_1 + a_{43}q_4 + a_{41}q_2\right)X_1(t)X_3(t) + 2\left(a_{14}q_1 + a_{11}q_4\right. \\
&+ \left. a_{24}q_2 + a_{21}q_3 + a_{34}q_3 + a_{31}q_2 + a_{44}q_4 + a_{41}q_1\right)X_1(t)X_4(t) + 2\left(a_{13}q_2\right. \\
&+ \left. a_{12}q_3 + a_{22}q_2 + a_{33}q_2 + a_{32}q_1 + a_{43}q_3 + a_{42}q_2\right)X_2(t)X_3(t) + 2\left(a_{14}q_2\right. \\
&+ \left. a_{24}q_1 + a_{22}q_3 + a_{34}q_2 + a_{32}q_2 + a_{44}q_3 + a_{42}q_1\right)X_2(t)X_4(t) + 2\left(a_{14}q_3\right. \\
&+ \left. a_{24}q_2 + a_{23}q_3 + a_{34}q_1 + a_{33}q_2 + a_{44}q_2 + a_{43}q_1\right)X_3(t)X_4(t) + q_1\left(g_{11}^2\right. \\
&+ g_{12}^2 + g_{13}^2 + g_{14}^2 + g_{21}^2 + g_{22}^2 + g_{23}^2 + g_{24}^2 + g_{31}^2 + g_{32}^2 + g_{33}^2 + g_{34}^2 + g_{41}^2 \\
&+ \left. g_{42}^2 + g_{43}^2 + g_{44}^2\right) + 2q_2\left(g_{11}g_{21} + g_{12}g_{22} + g_{13}g_{23} + g_{14}g_{24} + g_{21}g_{31} + g_{22}g_{32}\right. \\
&+ \left. g_{23}g_{33} + g_{24}g_{34} + g_{31}g_{41} + g_{32}g_{42} + g_{33}g_{43} + g_{34}g_{44}\right) + 2q_3\left(g_{11}g_{31}\right. \\
&+ \left. g_{12}g_{32} + g_{13}g_{33} + g_{14}g_{34} + g_{21}g_{41} + g_{22}g_{42} + g_{23}g_{43} + g_{24}g_{44}\right) + 2q_4 \\
&\times \left(g_{11}g_{41} + g_{12}g_{42} + g_{13}g_{43} + g_{14}g_{44}\right).
\end{aligned}$$

Proof. After derivation of Lyapunov function of equation (4.12) we get

$$\begin{aligned}
\mathrm{d}V(X_t) &= X_t^T QAX_t\mathrm{d}t + X_t^T QG\mathrm{d}B_t + X_t^T A^T \mathrm{d}tQX_t + \mathrm{d}B_t^T G^T QX_t \\
&+ \mathrm{d}B_t^T G^T QG\mathrm{d}B_t.
\end{aligned}$$

In matrix form

$$
\begin{aligned}
&\mathrm{d}V\begin{pmatrix} X_1(t)\\ X_2(t)\\ X_3(t)\\ X_4(t)\end{pmatrix}\\
=\ &\begin{pmatrix} X_1(t)\\ X_2(t)\\ X_3(t)\\ X_4(t)\end{pmatrix}^T\begin{pmatrix} q_1 & q_2 & q_3 & q_4\\ q_2 & q_1 & q_2 & q_3\\ q_3 & q_2 & q_1 & q_2\\ q_4 & q_3 & q_2 & q_1\end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{pmatrix}\begin{pmatrix} X_1(t)\\ X_2(t)\\ X_3(t)\\ X_4(t)\end{pmatrix}\mathrm{d}t\\
+\ &\begin{pmatrix} X_1(t)\\ X_2(t)\\ X_3(t)\\ X_4(t)\end{pmatrix}^T\begin{pmatrix} q_1 & q_2 & q_3 & q_4\\ q_2 & q_1 & q_2 & q_3\\ q_3 & q_2 & q_1 & q_2\\ q_4 & q_3 & q_2 & q_1\end{pmatrix}\begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14}\\ g_{21} & g_{22} & g_{23} & g_{24}\\ g_{31} & g_{32} & g_{33} & g_{34}\\ g_{41} & g_{42} & g_{43} & g_{44}\end{pmatrix}\begin{pmatrix} \mathrm{d}B_1(t)\\ \mathrm{d}B_2(t)\\ \mathrm{d}B_3(t)\\ \mathrm{d}B_4(t)\end{pmatrix}\\
+\ &\begin{pmatrix} X_1(t)\\ X_2(t)\\ X_3(t)\\ X_4(t)\end{pmatrix}^T\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{pmatrix}^T\begin{pmatrix} q_1 & q_2 & q_3 & q_4\\ q_2 & q_1 & q_2 & q_3\\ q_3 & q_2 & q_1 & q_2\\ q_4 & q_3 & q_2 & q_1\end{pmatrix}\begin{pmatrix} X_1(t)\\ X_2(t)\\ X_3(t)\\ X_4(t)\end{pmatrix}\mathrm{d}t\\
+\ &\begin{pmatrix} \mathrm{d}B_1(t)\\ \mathrm{d}B_2(t)\\ \mathrm{d}B_3(t)\\ \mathrm{d}B_4(t)\end{pmatrix}^T\begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14}\\ g_{21} & g_{22} & g_{23} & g_{24}\\ g_{31} & g_{32} & g_{33} & g_{34}\\ g_{41} & g_{42} & g_{43} & g_{44}\end{pmatrix}^T\begin{pmatrix} q_1 & q_2 & q_3 & q_4\\ q_2 & q_1 & q_2 & q_3\\ q_3 & q_2 & q_1 & q_2\\ q_4 & q_3 & q_2 & q_1\end{pmatrix}\begin{pmatrix} X_1(t)\\ X_2(t)\\ X_3(t)\\ X_4(t)\end{pmatrix}\\
+\ &\begin{pmatrix} \mathrm{d}B_1(t)\\ \mathrm{d}B_2(t)\\ \mathrm{d}B_3(t)\end{pmatrix}^T\begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14}\\ g_{21} & g_{22} & g_{23} & g_{24}\\ g_{31} & g_{32} & g_{33} & g_{34}\\ g_{41} & g_{42} & g_{43} & g_{44}\end{pmatrix}^T\begin{pmatrix} q_1 & q_2 & q_3 & q_4\\ q_2 & q_1 & q_2 & q_3\\ q_3 & q_2 & q_1 & q_2\\ q_4 & q_3 & q_2 & q_1\end{pmatrix}\\
\times\ &\begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14}\\ g_{21} & g_{22} & g_{23} & g_{24}\\ g_{31} & g_{32} & g_{33} & g_{34}\\ g_{41} & g_{42} & g_{43} & g_{44}\end{pmatrix}\begin{pmatrix} \mathrm{d}B_1(t)\\ \mathrm{d}B_2(t)\\ \mathrm{d}B_3(t)\\ \mathrm{d}B_4(t)\end{pmatrix}.
\end{aligned}
$$

We get

$$
\begin{aligned}
& \mathrm{d}V(X_t) \\
= \quad & 2\left(a_{11}q_1 + a_{21}q_2 + a_{31}q_3 + a_{41}q_4\right) X_1^2(t)\mathrm{d}t + 2\left(a_{12}q_2 + a_{22}q_1 + a_{32}q_2 + a_{42}q_3\right) \\
\times \quad & X_2^2(t)\mathrm{d}t + 2\left(a_{13}q_3 + a_{23}q_2 + a_{33}q_1 + a_{43}q_2\right) X_3^2(t)\mathrm{d}t + 2\left(a_{14}q_4 + a_{24}q_3\right. \\
+ \quad & \left. a_{34}q_2 + a_{44}q_1\right) X_4^2(t)\mathrm{d}t + 2\left(a_{12}q_1 + a_{11}q_2 + a_{22}q_2 + a_{21}q_1 + a_{32}q_3 + a_{31}q_2\right. \\
+ \quad & \left. a_{42}q_4 + a_{41}q_3\right) X_1(t)X_2(t)\mathrm{d}t + 2\left(a_{13}q_1 + a_{11}q_3 + a_{23}q_2 + a_{23}q_1 + a_{21}q_2 + a_{33}q_3\right. \\
+ \quad & \left. a_{31}q_1 + a_{43}q_4 + a_{41}q_2\right) X_1(t)X_3(t)\mathrm{d}t + 2\left(a_{14}q_1 + a_{11}q_4 + a_{24}q_2 + a_{21}q_3 + a_{34}q_3\right. \\
+ \quad & \left. a_{31}q_2 + a_{44}q_4 + a_{41}q_1\right) X_1(t)X_4(t)\mathrm{d}t + 2\left(a_{13}q_2 + a_{12}q_3 + a_{22}q_2 + a_{33}q_2 + a_{32}q_1\right. \\
+ \quad & \left. a_{43}q_3 + a_{42}q_2\right) X_2(t)X_3(t)\mathrm{d}t + 2\left(a_{14}q_2 + a_{24}q_1 + a_{22}q_3 + a_{34}q_2 + a_{32}q_2 + a_{44}q_3\right. \\
+ \quad & \left. a_{42}q_1\right) X_2(t)X_4(t)\mathrm{d}t + 2\left(a_{14}q_3 + a_{24}q_2 + a_{23}q_3 + a_{34}q_1 + a_{33}q_2 + a_{44}q_2 + a_{43}q_1\right) \\
\times \quad & X_3(t)X_4(t)\mathrm{d}t + q_1\left(g_{11}^2 + g_{12}^2 + g_{13}^2 + g_{14}^2 + g_{21}^2 + g_{22}^2 + g_{23}^2 + g_{24}^2 + g_{31}^2 + g_{32}^2\right. \\
+ \quad & \left. g_{33}^2 + g_{34}^2 + g_{41}^2 + g_{42}^2 + g_{43}^2 + g_{44}^2\right) \mathrm{d}t + 2q_2\left(g_{11}g_{21} + g_{12}g_{22} + g_{13}g_{23} + g_{14}g_{24}\right. \\
+ \quad & \left. g_{21}g_{31} + g_{22}g_{32} + g_{23}g_{33} + g_{24}g_{34} + g_{31}g_{41} + g_{32}g_{42} + g_{33}g_{43} + g_{34}g_{44}\right) \mathrm{d}t \\
+ \quad & 2q_3\left(g_{11}g_{31} + g_{12}g_{32} + g_{13}g_{33} + g_{14}g_{34} + g_{21}g_{41} + g_{22}g_{42} + g_{23}g_{43} + g_{24}g_{44}\right) \mathrm{d}t \\
+ \quad & 2q_4\left(g_{11}g_{41} + g_{12}g_{42} + g_{13}g_{43} + g_{14}g_{44}\right) \mathrm{d}t + 2\left[\left(q_1X_1(t) + q_2X_2(t) + q_3X_3(t)\right.\right. \\
+ \quad & \left. q_4X_4(t)\right)\left(g_{11}\mathrm{d}B_1(t) + g_{12}\mathrm{d}B_2(t) + g_{13}\mathrm{d}B_3(t) + g_{14}\mathrm{d}B_4(t)\right) + \left(q_2X_1(t)\right. \\
+ \quad & \left. q_1X_2(t) + q_2X_3(t) + q_3X_4(t)\right)\left(g_{21}\mathrm{d}B_1(t) + g_{22}\mathrm{d}B_2(t) + g_{23}\mathrm{d}B_3(t)\right. \\
+ \quad & \left. g_{24}\mathrm{d}B_4(t)\right) + \left(q_3X_1(t) + q_2X_2(t) + q_1X_3(t) + q_2X_4(t)\right)\left(g_{31}\mathrm{d}B_1(t) + g_{32}\mathrm{d}B_2(t)\right. \\
+ \quad & \left. g_{33}\mathrm{d}B_3(t) + g_{34}\mathrm{d}B_4(t)\right) + \left(q_4X_1(t) + q_3X_2(t) + q_2X_3(t) + q_1X_4(t)\right)\left(g_{41}\mathrm{d}B_1(t)\right. \\
+ \quad & \left.\left. g_{42}\mathrm{d}B_2(t) + g_{43}\mathrm{d}B_3(t) + g_{44}\mathrm{d}B_4(t)\right)\right].
\end{aligned}
$$

We apply expectation $\mathbb{E}\left\{\mathrm{d}V(X_t)\right\}$

$$\begin{aligned}
\mathbb{E}\left\{\mathrm{d}V(X_t)\right\} &= 2\left(a_{11}q_1 + a_{21}q_2 + a_{31}q_3 + a_{41}q_4\right)X_1^2(t) + 2\left(a_{12}q_2 + a_{22}q_1 + a_{32}q_2\right.\\
&+ \left. a_{42}q_3\right)X_2^2(t) + 2\left(a_{13}q_3 + a_{23}q_2 + a_{33}q_1 + a_{43}q_2\right)X_3^2(t) + 2\left(a_{14}q_4\right.\\
&+ \left. a_{24}q_3 + a_{34}q_2 + a_{44}q_1\right)X_4^2(t) + 2\left(a_{12}q_1 + a_{11}q_2 + a_{22}q_2 + a_{21}q_1\right.\\
&+ \left. a_{32}q_3 + a_{31}q_2 + a_{42}q_4 + a_{41}q_3\right)X_1(t)X_2(t) + 2\left(a_{13}q_1 + a_{11}q_3\right.\\
&+ \left. a_{23}q_2 + a_{23}q_1 + a_{21}q_2 + a_{33}q_3 + a_{31}q_1 + a_{43}q_4 + a_{41}q_2\right)X_1(t)X_3(t)\\
&+ 2\left(a_{14}q_1 + a_{11}q_4 + a_{24}q_2 + a_{21}q_3 + a_{34}q_3 + a_{31}q_2 + a_{44}q_4 + a_{41}q_1\right)\\
&\times X_1(t)X_4(t) + 2\left(a_{13}q_2 + a_{12}q_3 + a_{22}q_2 + a_{33}q_2 + a_{32}q_1 + a_{43}q_3\right.\\
&+ \left. a_{42}q_2\right)X_2(t)X_3(t) + 2\left(a_{14}q_2 + a_{24}q_1 + a_{22}q_3 + a_{34}q_2 + a_{32}q_2\right.\\
&+ \left. a_{44}q_3 + a_{42}q_1\right)X_2(t)X_4(t) + 2\left(a_{14}q_3 + a_{24}q_2 + a_{23}q_3 + a_{34}q_1\right.\\
&+ \left. a_{33}q_2 + a_{44}q_2 + a_{43}q_1\right)X_3(t)X_4(t) + q_1\left(g_{11}^2 + g_{12}^2 + g_{13}^2 + g_{14}^2 + g_{21}^2\right.\\
&+ \left. g_{22}^2 + g_{23}^2 + g_{24}^2 + g_{31}^2 + g_{32}^2 + g_{33}^2 + g_{34}^2 + g_{41}^2 + g_{42}^2 + g_{43}^2 + g_{44}^2\right)\\
&+ 2q_2\left(g_{11}g_{21} + g_{12}g_{22} + g_{13}g_{23} + g_{14}g_{24} + g_{21}g_{31} + g_{22}g_{32} + g_{23}g_{33}\right.\\
&+ \left. g_{24}g_{34} + g_{31}g_{41} + g_{32}g_{42} + g_{33}g_{43} + g_{34}g_{44}\right) + 2q_3\left(g_{11}g_{31} + g_{12}g_{32}\right.\\
&+ \left. g_{13}g_{33} + g_{14}g_{34} + g_{21}g_{41} + g_{22}g_{42} + g_{23}g_{43} + g_{24}g_{44}\right) + 2q_4\left(g_{11}g_{41}\right.\\
&+ \left. g_{12}g_{42} + g_{13}g_{43} + g_{14}g_{44}\right) = LV\mathrm{d}t
\end{aligned}$$

.

For $Q = I$, where I is a unit matrix, we get

$$\begin{aligned}
LV &= 2a_{11}X_1^2(t) + 2a_{22}X_2^2(t) + 2a_{33}X_3^2(t) + 2a_{44}X_4^2(t) + 2\left(a_{12} + a_{21}\right)X_1(t)X_2(t)\\
&+ 2\left(a_{13} + a_{23} + a_{31}\right)X_1(t)X_3(t) + 2\left(a_{14} + a_{41}\right)X_1(t)X_4(t) + 2a_{32}X_2(t)X_3(t)\\
&+ 2\left(a_{24} + a_{42}\right)X_2(t)X_4(t) + 2\left(a_{34} + a_{43}\right)X_3(t)X_4(t) + \left(g_{11}^2 + g_{12}^2 + g_{13}^2 + g_{14}^2\right.\\
&+ \left. g_{21}^2 + g_{22}^2 + g_{23}^2 + g_{24}^2 + g_{31}^2 + g_{32}^2 + g_{33}^2 + g_{34}^2 + g_{41}^2 + g_{42}^2 + g_{43}^2 + g_{44}^2\right).
\end{aligned}$$

□

Now we can find conditions of a stability system. The system will be stable if the Lyapunov function LV is negative definite, so

$$\begin{aligned}
& 2a_{11}X_1^2(t) + 2a_{22}X_2^2(t) + 2a_{33}X_3^2(t) + 2a_{44}X_4^2(t) + 2\left(a_{12} + a_{21}\right)X_1(t)X_2(t)\\
+ & 2\left(a_{13} + a_{23} + a_{31}\right)X_1(t)X_3(t) + 2\left(a_{14} + a_{41}\right)X_1(t)X_4(t) + 2a_{32}X_2(t)X_3(t)\\
+ & 2\left(a_{24} + a_{42}\right)X_2(t)X_4(t) + 2\left(a_{34} + a_{43}\right)X_3(t)X_4(t) + \left\|G\right\|^2 \le 0.
\end{aligned}$$

Remark: Because $\left\|G\right\|^2 \ge 0$, therefore the matrix A must be sufficiently negative, to obtain a negative definite function. We will demonstrate that the matrix A must be more dominant than the matrix G for the stability of the stochastic system,

$$\left\|A\right\| \gg \left\|G\right\|.$$

Corollary 4.4.2. We consider matrices A and G in the form

$$A = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, G = \begin{pmatrix} \frac{a}{10} & 0 & 0 & 0 \\ 0 & \frac{a}{10} & 0 & 0 \\ 0 & 0 & \frac{a}{10} & 0 \\ 0 & 0 & 0 & \frac{a}{10} \end{pmatrix}.$$

The matrix A will be negative definite under following conditions:

$$\begin{aligned}
&D_1 = a < 0, \\
&D_2 = a^2 > 0,\ D_2 \text{ follows from } D_1, \\
&D_3 = a^3 < 0 \Leftrightarrow a < 0 \wedge a^2 > 0, D_3 \text{ follows from } D_1, D_2, \\
&D_4 = a^4 > 0 \Leftrightarrow a^2 > 0,\ D_4 \text{ follows from } D_2.
\end{aligned}$$

Based on these conditions follows $a < 0$ or the first condition D_1. First of all, we will find the solution of the differential system A. We find eigenvalues of matrix A as the solution of the characteristic equation

$$\begin{vmatrix} a-\lambda & 0 & 0 & 0 \\ 0 & a-\lambda & 0 & 0 \\ 0 & 0 & a-\lambda & 0 \\ 0 & 0 & 0 & a-\lambda \end{vmatrix} = 0,$$

$$(a-\lambda)^4 = 0 \Rightarrow \lambda_{1,2,3,4} = a.$$

Then

$$X_1(t) = \mathrm{e}^{at}, X_2(t) = t\mathrm{e}^{at}, X_3(t) = t^2\mathrm{e}^{at}, X_4(t) = t^3\mathrm{e}^{at}.$$

The general solution is given by a linear combination

$$X_t = C_1X_1(t) + C_2X_2(t) + C_3X_3(t) + C_4X_4(t)$$

with arbitrary constants $C_1, C_2, C_3, C_4, t \in \mathbb{R}$, and because $a < 0$, then this solution is stable.

At this moment, we find stability of solution of the stochastic system. We determine stability of solution for $Q = I$

$$\begin{aligned}
\mathrm{d}V(X_t) &= 2\left(aX_1^2(t) + aX_2^2(t) + aX_3^2(t) + aX_4^2(t) + \frac{a^2}{50}\right)\mathrm{d}t + \frac{a}{5}X_1(t)\mathrm{d}B_1(t) \\
&+ \frac{a}{5}X_2(t)\mathrm{d}B_2(t) + \frac{a}{5}X_3(t)\mathrm{d}B_3(t) + \frac{a}{5}X_4(t)\mathrm{d}B_4(t).
\end{aligned}$$

$$\mathbb{E}\left\{\mathrm{d}V(X_t)\right\} = 2\left(aX_1^2(t) + aX_2^2(t) + aX_3^2(t) + aX_4^2(t) + \frac{a^2}{50}\right)\mathrm{d}t = LV\mathrm{d}t.$$

If holds the inequality $LV \leq 0$, thus

$$a\left\|X(t)\right\|^2 \leq -\frac{a^2}{50},$$

for $X_t = C_1\mathrm{e}^{at} + C_2t\mathrm{e}^{at} + C_3t^2\mathrm{e}^{at} + C_4t^3\mathrm{e}^{at}, t \in \mathbb{R}$, then the system is stochastic stable.

Corollary 4.4.3. We consider matrices A and G in the form

$$A = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{pmatrix}, G = \begin{pmatrix} \frac{a_1}{10} & 0 & 0 & 0 \\ 0 & \frac{a_2}{10} & 0 & 0 \\ 0 & 0 & \frac{a_3}{10} & 0 \\ 0 & 0 & 0 & \frac{a_4}{10} \end{pmatrix},$$

where $a_i \neq a_j$ for $i \neq j; i, j = 1, 2, 3, 4$. The matrix A will be negative definite with following conditions:

$$\begin{aligned}
&D_1 = a_1 < 0, \\
&D_2 = a_1a_2 > 0 \Leftrightarrow a_2 < 0,\ D_2 \text{ follows from } D_1, \\
&D_3 = a_1a_2a_3 < 0 \Leftrightarrow a_3 < 0, D_3 \text{ follows from } D_2, \\
&D_4 = a_1a_2a_3a_4 > 0 \Leftrightarrow a_4 < 0,\ D_4 \text{ follows from } D_3.
\end{aligned}$$

Based on these conditions, it follows $a_i < 0, i = 1, 2, 3, 4$. First of all we find solution of the differential system A. We find eigenvalues of matrix A as the solution of the characteristic equation

$$\begin{vmatrix} a_1 - \lambda & 0 & 0 & 0 \\ 0 & a_2 - \lambda & 0 & 0 \\ 0 & 0 & a_3 - \lambda & 0 \\ 0 & 0 & 0 & a_4 - \lambda \end{vmatrix} = 0.$$

Then

$$\lambda_i = a_i \Rightarrow X_i(t) \;\; = \;\; \mathrm{e}^{a_it}.$$

The general solution with arbitrary constants C_1, C_2, C_3, C_4 is given by

$$X_t = \sum_{i=1}^{4} C_i\mathrm{e}^{a_it}, t \in \mathbb{R},$$

and because $a_i < 0$, then this solution is stable.

We find stability of solution of the stochastic system. We determine stability of solution for $Q = I$

$$\begin{aligned}
\mathrm{d}V(X_t) \;\; &= \;\; 2\left(a_1X_1^2(t) + a_2X_2^2(t) + a_3X_3^2(t) + a_4X_4^2(t) + \frac{a_1^2 + a_2^2 + a_3^2 + a_4^2}{200}\right)\mathrm{d}t \\
&+ \;\; \frac{a_1}{5}X_1(t)\mathrm{d}B_1(t) + \frac{a_2}{5}X_2(t)\mathrm{d}B_2(t) + \frac{a_3}{5}X_3(t)\mathrm{d}B_3(t) + \frac{a_4}{5}X_4(t)\mathrm{d}B_4(t).
\end{aligned}$$

$$
\begin{aligned}
\mathbb{E}\left\{\mathrm{d}V(X_t)\right\} &= 2\Big(a_1X_1^2(t)+a_2X_2^2(t)+a_3X_3^2(t)+a_4X_4^2(t)\\
&+ \frac{a_1^2+a_2^2+a_3^2+a_4^2}{200}\Big)\,\mathrm{d}t = LV\mathrm{d}t.
\end{aligned}
$$

If holds the inequality $LV \leq 0$, thus

$$
a_1X_1^2(t)+a_2X_2^2(t)+a_3X_3^2(t)+a_4X_4^2(t) \leq -\frac{a_1^2+a_2^2+a_3^2+a_4^2}{100},
$$

for $X_t = \sum_{i=1}^{4} C_i \mathrm{e}^{a_i t}, t \in \mathbb{R}$, then the system is stochastic stable.

Corollary 4.4.4. We consider matrices A and G in the form

$$
A = \begin{pmatrix} a_1 & 1 & 1 & 1\\ 0 & a_2 & 1 & 1\\ 0 & 0 & a_3 & 1\\ 0 & 0 & 0 & a_4 \end{pmatrix}, G = \begin{pmatrix} \frac{a_1}{10} & 1 & 1 & 1\\ 0 & \frac{a_2}{10} & 1 & 1\\ 0 & 0 & \frac{a_3}{10} & 1\\ 0 & 0 & 0 & \frac{a_4}{10} \end{pmatrix}.
$$

The matrix A will be negative definite with following conditions:

$$
\begin{aligned}
&D_1 = a_1 < 0,\\
&D_2 = a_1a_2 > 0 \Leftrightarrow a_2 < 0,\ D_2 \text{ follows from } D_1,\\
&D_3 = a_1a_2a_3 < 0 \Leftrightarrow a_3 < 0, D_3 \text{ follows from } D_2,\\
&D_4 = a_1a_2a_3a_4 > 0 \Leftrightarrow a_4 < 0,\ D_4 \text{ follows from } D_3.
\end{aligned}
$$

Based on these conditions it is evident that $a_i < 0, i = 1,2,3,4$.
First of all we find solution of the differential system A. We find eigenvalues of matrix A as the solution of the characteristic equation

$$
\begin{vmatrix} a_1-\lambda & 1 & 1 & 1\\ 0 & a_2-\lambda & 1 & 1\\ 0 & 0 & a_3-\lambda & 1\\ 0 & 0 & 0 & a_4-\lambda \end{vmatrix} = 0.
$$

According to previous example the general solution with arbitrary constants C_1, C_2, C_3, C_4 is given by

$$
X_t = C_1\mathrm{e}^{a_1t} + C_2\mathrm{e}^{a_2t} + C_3\mathrm{e}^{a_3t} + C_4\mathrm{e}^{a_4t}, t \in \mathbb{R}.
$$

We can write for a general matrix H

$$
H = \begin{pmatrix} a_1 & \alpha & \beta & \gamma\\ 0 & a_2 & \delta & \epsilon\\ 0 & 0 & a_3 & \kappa\\ 0 & 0 & 0 & a_4 \end{pmatrix},
$$

where $\alpha, \beta, \gamma, \delta, \epsilon, \kappa \in \mathbb{R}$, the general solution is

$$X_t = C_1 \mathrm{e}^{a_1 t} + C_2 \mathrm{e}^{a_2 t} + C_3 \mathrm{e}^{a_3 t} + C_4 \mathrm{e}^{a_4 t}, t \in \mathbb{R},$$

where C_1, C_2, C_3, C_4 are constants. We find stability of solution of the stochastic system. We determine stability of solution for $Q = I$.

$$\begin{aligned}
\mathrm{d}V(X_t) &= 2\Big(3 + a_1 X_1^2(t) + a_2 X_2^2(t) + a_3 X_3^2(t) + a_4 X_4^2(t) + X_1(t)X_2(t) \\
&+ 2X_1(t)X_3(t) + X_1(t)X_4(t) + X_2(t)X_4(t) + X_3(t)X_4(t) \\
&+ \frac{a_1^2 + a_2^2 + a_3^2 + a_4^2}{200}\Big)\mathrm{d}t + 2X_3(t)\left(\frac{a_3}{10}\mathrm{d}B_3(t) + \mathrm{d}B_4(t)\right) \\
&+ 2X_1(t)\left(\frac{a_1}{10}\mathrm{d}B_1(t) + \mathrm{d}B_2(t) + \mathrm{d}B_3(t) + \mathrm{d}B_4(t)\right) + 2X_4(t)\frac{a_4}{10}\mathrm{d}B_4(t) \\
&+ 2X_2(t)\left(\frac{a_2}{10}\mathrm{d}B_2(t) + \mathrm{d}B_3(t) + \mathrm{d}B_4(t)\right).
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left\{\mathrm{d}V(X_t)\right\} &= 2\Big(3 + a_1 X_1^2(t) + a_2 X_2^2(t) + a_3 X_3^2(t) + a_4 X_4^2(t) + X_1(t)X_2(t) \\
&+ 2X_1(t)X_3(t) + X_1(t)X_4(t) + X_2(t)X_4(t) + X_3(t)X_4(t) \\
&+ \frac{a_1^2 + a_2^2 + a_3^2 + a_4^2}{200}\Big)\mathrm{d}t = LV\mathrm{d}t.
\end{aligned}$$

If holds the inequality $LV \leq 0$, thus

$$\begin{aligned}
& a_1 X_1^2(t) + a_2 X_2^2(t) + a_3 X_3^2(t) + a_4 X_4^2(t) + X_1(t)X_2(t) + 2X_1(t)X_3(t) \\
+ \; & X_1(t)X_4(t) + X_2(t)X_4(t) + X_3(t)X_4(t) \leq -\frac{a_1^2 + a_2^2 + a_3^2 + a_4^2}{100} - 6,
\end{aligned}$$

for $X_t = C_1 \mathrm{e}^{a_1 t} + C_2 \mathrm{e}^{a_2 t} + C_3 \mathrm{e}^{a_3 t} + C_4 \mathrm{e}^{a_4 t}, t \in \mathbb{R}$, then the system is stochastic stable.

Corollary 4.4.5. We consider symmetric matrices A and G in the form

$$A = \begin{pmatrix} a_1 & 0 & 0 & a_2 \\ 0 & a_1 & a_2 & 0 \\ 0 & a_2 & a_1 & 0 \\ a_2 & 0 & 0 & a_1 \end{pmatrix}, G = \begin{pmatrix} \frac{a_1}{10} & 0 & 0 & \frac{a_2}{10} \\ 0 & \frac{a_1}{10} & \frac{a_2}{10} & 0 \\ 0 & \frac{a_2}{10} & \frac{a_1}{10} & 0 \\ \frac{a_2}{10} & 0 & 0 & \frac{a_1}{10} \end{pmatrix}.$$

The matrix A will be negative definite with following conditions:

$D_1 = a_1 < 0,$
$D_2 = a_1^2 > 0, \; D_2$ follows from $D_1,$
$D_3 = a_1^3 - a_1 a_2^2 < 0 \Leftrightarrow a_1 < 0 \wedge a_1^2 - a_2^2 > 0 \Rightarrow |a_2| < |a_1|.$
$D_4 = a_1^4 - 2a_1^2 a_2^2 + a_2^4 > 0 \Leftrightarrow (a_1^2 - a_2^2)^2 > 0, \; D_4$ holds for arbitrary $|a_1| \neq |a_2|.$

Based on these conditions it is evident that $a_1 < 0$ and $|a_2| < |a_1|$.
We find solution of the differential system A. We find eigenvalues of matrix A as

the solution of the characteristic equation

$$\begin{vmatrix} a_1-\lambda & 0 & 0 & a_2 \\ 0 & a_1-\lambda & a_2 & 0 \\ 0 & a_2 & a_1-\lambda & 0 \\ a_2 & 0 & 0 & a_1-\lambda \end{vmatrix} = 0,$$

Then according to Example (3.2) in paper [148] we get

$$\begin{aligned}
&\text{for} \quad a_2>0 \quad \text{is} \quad X_{1,2}(t) = (1,1)^T \mathrm{e}^{(-a_1+a_2)t}, \\
&\text{for} \quad a_2<0 \quad \text{is} \quad X_{1,2}(t) = (-1,1)^T \mathrm{e}^{(-a_1+a_2)t}, \\
&\text{for} \quad a_2<0 \quad \text{is} \quad X_{3,4}(t) = (1,1)^T \mathrm{e}^{(-a_1-a_2)t}, \\
&\text{for} \quad a_2>0 \quad \text{is} \quad X_{3,4}(t) = (1,-1)^T \mathrm{e}^{(-a_1-a_2)t}.
\end{aligned}$$

The general solution is given by $X_t = C_1X_1(t)+C_2X_2(t)+C_3X_3(t)+C_4X_4(t)$, with arbitrary constants C_1, C_2, C_3, C_4. We find stability of solution of the stochastic system. We determine stability of solution for $Q = I$.

$$\begin{aligned}
\mathrm{d}V(X_t) &= 2\Big[a_1(X_1^2(t)+X_2^2(t)+X_3^2(t)+X_4^2(t)) + a_2(X_1(t)X_3(t)+2X_1(t)X_4(t) \\
&+ X_2(t)X_3(t)) + \frac{a_1^2}{50} + \frac{a_2^2}{50}\Big]\mathrm{d}t + 2X_1(t)\left(\frac{a_1}{10}\mathrm{d}B_1(t) + \frac{a_2}{10}\mathrm{d}B_4(t)\right) \\
&+ 2X_2(t)\left(\frac{a_1}{10}\mathrm{d}B_2(t) + \frac{a_2}{10}\mathrm{d}B_3(t)\right) + 2X_3(t)\left(\frac{a_2}{10}\mathrm{d}B_2(t) + \frac{a_1}{10}\mathrm{d}B_3(t)\right) \\
&+ 2X_4(t)\left(\frac{a_2}{10}\mathrm{d}B_1(t) + \frac{a_1}{10}\mathrm{d}B_4(t)\right).
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\{\mathrm{d}V(X_t)\} &= 2\Big[a_1(X_1^2(t)+X_2^2(t)+X_3^2(t)+X_4^2(t)) + a_2(X_1(t)X_3(t) \\
&+ 2X_1(t)X_4(t) + X_2(t)X_3(t)) + \frac{a_1^2}{50} + \frac{a_2^2}{50}\Big]\mathrm{d}t = LV\mathrm{d}t.
\end{aligned}$$

If holds the inequality $LV \leq 0$, thus

$$a_1\|X(t)\|^2 + a_2(X_1(t)X_3(t)+2X_1(t)X_4(t)+X_2(t)X_3(t)) \leq -\frac{a_1^2+a_2^2}{50},$$

for $X_t = C_1X_1(t)+C_2X_2(t)+C_3X_3(t)+C_4X_4(t), t \in \mathbb{R}$, then the system is stochastic stable.

Corollary 4.4.6. If we use in the equation (4.12) a general matrix A, then we do not receive any usable results with using this method. There were chosen the types of matrices used in medicine.

5 TIME-DELAY STOCHASTIC SYSTEMS

This chapter deals with delayed stochastic differential equations and systems and the next chapter builds on their application, specifically modeling the immune system's response to infection. The following literature deals with ordinary differential equations with delay and on the basis of this theory the results for stochastic differential equations and systems with delay were derived.

Authors Azbelev, N. V. and Simonov, P. M. present stability theory for differential equations concentrating on functional differential equations with delay and related topics in their book Stability of Differential Equations with After-effect [4].

Authors Baštinec J. and Piddubna G. devote of solutions and stability of solutions of a matrix linear differential system with delay [6]-[9]. Within the next articles, they focus on controllability and solutions on systems of differential equations with delay [11]-[16].

Baštinec J., at al. describe Stability and stabilization of linear systems with aftereffect [10].

Deterministic and stochastic time delay systems are presented in [45] by B. El-Kebir and L. Zi-Kuan and in [46] by L. E. Elsgolts and S. B. Norkin.

Differential equations with delay argument is investigated by K. Gopalsamy in [64], by K. Gu at al. in [65], by M.-G. Liu in [96], by X.-X. Liu at al. in [97]-[98], by U. A. Mitropolskii in [109], by A. D. Myshkis in [110], by S. B. Norkin in [113], by J. H. Park at al. in [115], by G. Piddubna in [117]-[120], and by B. S. Razumihin in [123].

5.1 Time-Delay Systems Theory

A general form of the time-delay differential equation for $X(t) \in \mathbb{R}^n$ is

$$\mathrm{d}X(t) = f(t, X(t), X(t-\tau))\mathrm{d}t,$$

where $X(t-\tau)$ represents the trajectory of the solution in the past, f is functional operator from $\mathbb{R} \times \mathbb{R}^n \times \mathbb{C}^1(\mathbb{R}, \mathbb{R}^n)$ to $\mathbb{R}^n$.

The basic method is the method of steps, the principle of which is the sequence of solving the initial problems of ODE on a sequence of consecutive intervals. The length of these intervals is based on delay that occurs in the equation.

For uniques solution on ODE we need one initial point. For solution of ODE with delay we need initial function. On the interval $I_0 = [t_0 - \tau, t_0]$, we put the initial function $X_0(t) \equiv \phi(t)$. We extend this solution on $t \in I_1 = [t_0, t_0 + \tau]$ by solving the system of ODEs

$$\mathrm{d}X_1(t) = f(t, X_1(t), X_1(t-\tau))\mathrm{d}t = f(t, X_1(t), \phi(t))\mathrm{d}t,$$

for $X_1(t_0) = \phi(t_0)$. In the k-th step, we solve the system

$$\mathrm{d}X_k(t) = f(t, X_k(t), X_k(t-\tau))\mathrm{d}t = f(t, X_k(t), X_{k-1}(t)),$$

for $\phi(t), X_k(t_0+(k-1)\tau) = X_{k-1}(t_0+(k-1)\tau)$. The final solution consists of partial solutions $X_k(t)$

$$X(t) = \begin{cases} \phi(t) & t \in [t_0-\tau, t_0]\,, \\ X_1(t) & t \in [t_0, t_0+\tau]\,, \\ X_2(t) & t \in [t_0+\tau, t_0+2\tau]\,, \\ \vdots & \\ X_k(t) & t \in [t_0+(k-1)\tau, t_0+k\tau]\,. \end{cases}$$

5.2 Time-Delay Stochastic Systems

On the basis of delay ODEs, the solution of delay SDEs is derived. There is used the method of steps which consists in the local transformation of the SDE with a delay to an ordinary SDE.

5.2.1 One-Dimensional Time-Delay Stochastic Equation

Theorem 5.2.1. *Let be a given initial problem*

$$\mathrm{d}X(t) = AX(t-\tau)\mathrm{d}t + G\mathrm{d}B(t) \tag{5.1}$$

for the initial condition $X(t) \equiv \phi(t)$ on the interval $t \in I_0 = [t_0-\tau, t_0]$, where $A, G \in \mathbb{R}, \tau > 0,$.
Then

$$X_1(t) = \phi(t_0) + AF(t-t_0) + GB(t-t_0)$$

is solution of the delayed SDE (5.1) *on the interval $I_1 = [t_0, t_0+\tau]\,,$*

$$\begin{aligned} X_2(t) &= \phi(t_0) + GB(t-t_0) + AF(\tau) + A\int_{t_0+\tau}^{t} \phi(s_0)\mathrm{d}(s-s_0-\tau) \\ &+ A^2\int_{t_0+\tau}^{t} F(s-s_0-\tau)\mathrm{d}(s-s_0-\tau) \\ &+ AG\int_{t_0+\tau}^{t} B(s-s_0-\tau)\mathrm{d}(s-s_0-\tau). \end{aligned}$$

is solution of the delayed SDE (5.1) *on the interval $I_2 = [t_0+\tau, t_0+2\tau]$, etc.*

Proof. First of all, let us put the solution for $t \in I_0$ equals to initial function, e.g. $X(t) \equiv \phi(t)$.
Within the 1^{st} step, the initial problem for $t \in I_1 = [t_0, t_0+\tau]$ is

$$\mathrm{d}X_1(t) = A\phi(t)\mathrm{d}t + G\mathrm{d}B(t),$$

initial condition $X_1(t_0) = \phi(t_0)$. This equation can be written as the stochastic integral equation

$$\int_0^t \mathrm{d}X_1(s) = A\int_0^t \phi(s)\mathrm{d}s + G\int_0^t \mathrm{d}B(s),$$

The left side of the equation will be solved according to the Itô formula (Theorem 3.3.5)

$$\int_0^t \mathrm{d}X_1(s) \Longrightarrow Y_1(t) = g(t, X_1(t)) = X_1(t),$$

where $Y_1(t)$ is an Itô process.

$$\mathrm{d}Y_1(t) = \frac{\partial X_1(t)}{\partial t}\mathrm{d}t + \frac{\partial X_1(t)}{\partial X_1(t)}\mathrm{d}X_1(t) + \frac{1}{2}\frac{\partial^2 X_1(t)}{\partial (X_1(t))^2}(\mathrm{d}X_1(t))^2,$$

$$\mathrm{d}Y_1(t) = \mathrm{d}X_1(t),$$

let us put the the left side to the right side

$$\int_0^t \mathrm{d}X_1(s) = A\int_0^t \phi(s)\mathrm{d}s + G\int_0^t \mathrm{d}B(s).$$

For $\int_0^t \mathrm{d}B(s) = B(t) - B(0)$ see Corrolary 4.1.3. The general solution is

$$X_1(t) = X(0) + A\int_0^t \phi(s)\mathrm{d}s + GB(t) + c_1,$$

where c_1 is integrating constant, $X_1(0) = X(0), B(0) = 0$, substitution

$$\int_0^t \phi(s)\mathrm{d}s = F(t).$$

The constant c_1 will be expressed for the initial condition
$X_1(t_0) = \phi(t_0)$

$$X(0) + AF(t_0) + GB(t_0) + c_1 = \phi(t_0),$$

$$c_1 = \phi(t_0) - X(0) - AF(t_0) - GB(t_0).$$

The particular solution on the interval $I_1 = [t_0, t_0 + \tau]$ is

$$X_1(t) = \phi(t_0) + AF(t - t_0) + GB(t - t_0).$$

For the 2^{nd} step, there will be solved the initial problem for $t \in I_2 = [t_0 + \tau, t_0 + 2\tau]$

$$\mathrm{d}X_2(t) = AX_1(t-\tau)\mathrm{d}t + G\mathrm{d}B(t), X_2(t_0+\tau) = X_1(t_0+\tau).$$

This equation will be rewritten as

$$\int_0^t \mathrm{d}X_2(s) \quad = \quad \int_0^t A\left[\phi(s_0) + AF(s-\tau-s_0) + GB(s-\tau-s_0)\right]\mathrm{d}s + \int_0^t G\mathrm{d}B(s)$$

and the Itô formula will be also applied

$$\int_0^t \mathrm{d}X_2(s) \Longrightarrow Y_2(t) = g(t, X_2(t)) = X_2(t),$$

$$\mathrm{d}Y_2(t) = \frac{\partial X_2(t)}{\partial t}\mathrm{d}t + \frac{\partial X_2(t)}{\partial X_2(t)}\mathrm{d}X_2(t) + \frac{1}{2}\frac{\partial^2 X_2(t)}{\partial (X_2(t))^2}(\mathrm{d}X_2(t))^2,$$
$$\mathrm{d}Y_2(t) = \mathrm{d}X_2(t)$$

and let us solve the following equation

$$\int_0^t \mathrm{d}X_2(s) \;=\; \int_0^t A\left[\phi(s_0) + AF(s-\tau-s_0) + GB(s-\tau-s_0)\right]\mathrm{d}s + \int_0^t G\mathrm{d}B(s).$$

The general solution is

$$\begin{aligned} X_2(t) &= X(0) + A\int_0^t \phi(s_0)\mathrm{d}s + A^2\int_0^t F(s-\tau-s_0)\mathrm{d}s \\ &+ AG\int_0^t B(s-\tau-s_0)\mathrm{d}s + GB(t) + c_2. \end{aligned}$$

For the initial condition $X_2(t_0+\tau) = X_1(t_0+\tau)$, we get the constant c_2

$$\begin{aligned} c_2 &= \phi(t_0) + AF(\tau) + GB(-t_0) - X(0) - A\int_0^{t_0+\tau} \phi(s_0)\mathrm{d}(s_0+\tau) \\ &- A^2\int_0^{t_0+\tau} F(0)\mathrm{d}(s_0+\tau) - AG\int_0^{t_0+\tau} B(0)\mathrm{d}(s_0+\tau). \end{aligned}$$

The particular solution of delayed SDE on the interval $I_2 = [t_0+\tau, t_0+2\tau]$ is

$$\begin{aligned} X_2(t) &= \phi(t_0) + GB(t-t_0) + AF(\tau) + A\int_{t_0+\tau}^t \phi(s_0)\mathrm{d}(s-s_0-\tau) \\ &+ A^2\int_{t_0+\tau}^t F(s-s_0-\tau)\mathrm{d}(s-s_0-\tau) \\ &+ AG\int_{t_0+\tau}^t B(s-s_0-\tau)\mathrm{d}(s-s_0-\tau). \end{aligned}$$

For the $3rd$ step, the solution X_3 on the interval $t \in I_3 = [t_0+2\tau, t_0+3\tau]$ would be found for

$$\mathrm{d}X_3(t) = AX_2(t-\tau)\mathrm{d}t + G\mathrm{d}B(t)$$

and initial condition $X_3(t_0+2\tau) = X_2(t_0+2\tau)$ by the same method as well as the previous steps. The method of steps continues in this proceeding.

□

5.2.2 Multi-Dimensional Time-Delay Stochastic Systems

If we consider that $X(t), B(t)$ and $\phi(t)$ are vectors of function, A and G are constant matrices of dimensions $(n\times n)$, then the Theorem 5.2.1 is valid for multi-dimensional time-delay stochastic systems.

Theorem 5.2.2. *Let be a given initial problem*

$$\mathrm{d}X(t) = AX(t)\mathrm{d}t + HX(t-\tau)\mathrm{d}t + G\mathrm{d}B(t) \tag{5.2}$$

and $X(t) \equiv \phi(t)$ *is the initial condition on the interval* $t \in I_0 = [t_0 - \tau, t_0]$*, where* $X(t) = (X^1(t), \ldots, X^n(t))^T, B(t) = (B^1(t), \ldots, B^n(t))^T$ *and*
$\phi(t) = (\phi^1(t), \ldots, \phi^n(t))^T$ *are vectors of function,* A, H *and* G *are constant matrices of dimensions* $(n \times n)$ *and* $\tau > 0$ *is a constant delay. Then*

$$\begin{aligned} X_1(t) &= \mathrm{e}^{At}X(0) - \mathrm{e}^{At}A^{-1}\mathrm{e}^{-At}H\phi(t) + \mathrm{e}^{At}G\mathrm{e}^{-At}B(t) + \mathrm{e}^{A(t-t_0)}\left[\phi(t_0)\right. \\ &- \left.\mathrm{e}^{At_0}X(0) + \mathrm{e}^{At_0}A^{-1}\mathrm{e}^{-At_0}H\phi(t_0) - \mathrm{e}^{At_0}G\mathrm{e}^{-At_0}B(t_0)\right] \end{aligned}$$

is the solution of the delayed stochastic system (5.2) *on the interval* $I_1 = [t_0, t_0 + \tau]$*, and* e^{At} *is a matrix exponential.*

Proof. Let be $X(t) \equiv \phi(t)$ the initial condition on the interval $t \in I_0 = [t_0 - \tau, t_0]$ of the time-delay stochastic system (5.2).
Within the 1^{st} step, let us solve the following stochastic system on the interval $t \in I_1 = [t_0, t_0 + \tau]$ for the initial condition $X_1(t_0) = \phi(t_0)$

$$\mathrm{d}X_1(t) = AX_1(t)\mathrm{d}t + H\phi(t_0 - \tau)\mathrm{d}t + G\mathrm{d}B(t).$$

The method of the integration factor is used

$$\mathrm{e}^{-At}\mathrm{d}X_1(t) - \mathrm{e}^{-At}AX_1(t)\mathrm{d}t = \mathrm{e}^{-At}H\phi(t_0 - \tau)\mathrm{d}t + \mathrm{e}^{-At}G\mathrm{d}B(t),$$

$$\int_0^t \mathrm{d}(\mathrm{e}^{-As}X_1(s)) = \int_0^t \mathrm{e}^{-As}H\phi(s_0 - \tau)\mathrm{d}s + \int_0^t \mathrm{e}^{-As}G\mathrm{d}B(s).$$

The Multi-dimensional Itô formula (Theorem 3.3.6) is applied

$$\int_0^t \mathrm{d}(\mathrm{e}^{-As}X_1(s)) \Longrightarrow Y_1(t) = g(t, X_1(t)) = \mathrm{e}^{-At}X_1(t),$$

where $Y_1(t)$ is Itô process.

$$\mathrm{d}Y_1(t) = \frac{\partial \mathrm{e}^{-At}X_1(t)}{\partial t}\mathrm{d}t + \frac{\partial \mathrm{e}^{-At}X_1(t)}{\partial X_1(t)}\mathrm{d}X_1(t) + \frac{1}{2}\frac{\partial^2 \mathrm{e}^{-At}X_1(t)}{\partial (X_1(t))^2}(\mathrm{d}X_1(t))^2,$$

$$\mathrm{d}Y_1(t) = -A\mathrm{e}^{-At}X_1(t)\mathrm{d}t + \mathrm{e}^{-At}\mathrm{d}X_1(t).$$

$\mathrm{d}X_1(t)$ will be substituted by $AX_1(t)\mathrm{d}t + H\phi(t_0 - \tau)\mathrm{d}t + G\mathrm{d}B(t)$

$$\mathrm{d}Y_1(t) = -A\mathrm{e}^{-At}X_1(t)\mathrm{d}t + \mathrm{e}^{-At}(AX_1(t)\mathrm{d}t + H\phi(t_0 - \tau)\mathrm{d}t + G\mathrm{d}B(t)),$$

$$\mathrm{d}Y_1(t) = \mathrm{e}^{-At}H\phi(t_0 - \tau)\mathrm{d}t + \mathrm{e}^{-At}G\mathrm{d}B(t)$$

and the system is rewritten into integral form

$$\int_0^t \mathrm{d}(\mathrm{e}^{-As}X_1(s)) = \int_0^t \mathrm{e}^{-As}H\phi(s_0 - \tau)\mathrm{d}s + \int_0^t \mathrm{e}^{-As}G\mathrm{d}B(s).$$

The general solution is

$$\mathrm{e}^{-At}X_1(t) - X_1(0) = -A^{-1}\mathrm{e}^{-At}H\phi(t_0 - \tau) + \mathrm{e}^{-At}GB(t) + c_1, X_1(0) = X(0)$$

or

$$X_1(t) = \mathrm{e}^{At}X(0) - \mathrm{e}^{At}A^{-1}\mathrm{e}^{-At}H\phi(t_0-\tau) + \mathrm{e}^{At}G\mathrm{e}^{-At}B(t) + \mathrm{e}^{At}c_1.$$

For initial condition $X_1(t_0) = \phi(t_0)$

$$\mathrm{e}^{At_0}X(0) - \mathrm{e}^{At_0}A^{-1}\mathrm{e}^{-At_0}H\phi(t_0) + \mathrm{e}^{At_0}G\mathrm{e}^{-At_0}B(t_0) + \mathrm{e}^{At_0}c_1 = \phi(t_0),$$

the integration constant c_1 is determined

$$c_1 = \mathrm{e}^{-At_0}\left[\phi(t_0) - \mathrm{e}^{At_0}X(0) + \mathrm{e}^{At_0}A^{-1}\mathrm{e}^{-At_0}H\phi(t_0-\tau) - \mathrm{e}^{At_0}G\mathrm{e}^{-At_0}B(t_0)\right].$$

The particular solution $X_1(t)$ of stochastic system (5.2) on the interval $I_1 = [t_0, t_0+\tau]$ is

$$\begin{aligned} X_1(t) &= \mathrm{e}^{At}X(0) - \mathrm{e}^{At}A^{-1}\mathrm{e}^{-At}H\phi(t_0-\tau) + \mathrm{e}^{At}G\mathrm{e}^{-At}B(t) + \mathrm{e}^{A(t-t_0)}\left[\phi(t_0)\right. \\ &- \left.\mathrm{e}^{At_0}X(0) + \mathrm{e}^{At_0}A^{-1}\mathrm{e}^{-At_0}H\phi(t_0-\tau) - \mathrm{e}^{At_0}G\mathrm{e}^{-At_0}B(t_0)\right]. \end{aligned}$$

By the 2^{nd} step, the stochastic differential system

$$\mathrm{d}X_2(t) = AX_2(t)\mathrm{d}t + HX_1(t_0+\tau)\mathrm{d}t + G\mathrm{d}B(t)$$

would be solved for the initial condition $X_2(t_0+\tau) = X_1(t_0+\tau)$ on the interval $I_2 = [t_0+\tau, t_0+2\tau]$, etc. □

The concrete example of the time-delay stochastic system application will be demonstrated through the following chapter on the model of the immune system.

6 BIOLOGICAL MODEL

Within this thesis, a stochastic differential system based on a Marchuk model is investigated. The available literature related to the Marchuk mathematical model of infectious disease and immune response is presented in ([1], [49]-[53], [103]-[104]).

Real biological systems will always be exposed to influences that are not fully understood, and therefore there is an increasing need to spread the deterministic models to models that include more difficult differences in the dynamics. A method of demonstrating these elements is by including stochastic influences or noise. A natural extension of an ODE model is a system of SDEs.

All biological dynamical systems evolve under stochastic influence, if we define stochasticity as the parts of the dynamics that we cannot predict or understand. To be realistic, models of biological systems should include random influences, since they are concerned with subsystems of the real world that cannot be sufficiently isolated from outer effects to the model.

The physiological explanation to include erratic behaviors in a model can be found in the many factors that cannot be controlled, like hormonal oscillations, blood pressure variations, respiration, variable neural control of muscle activity, enzymatic processes, energy requirements, the cellular metabolism, sympathetic nerve activity, or individual characteristics like BMI, genes, smoking, stress impacts, etc.

Also, external influences, like small differences in the experimental procedure, temperature, differences in preparation and administration of drugs, if this is included in the experiment or maybe the experiments are conducted by different experimenters that inevitably will exhibit small differences in procedures within the protocols. Different sources of errors will require different modeling of the noise, and these factors should be considered as carefully as the modeling of the deterministic part, in order to make the model predictions and parameter values possible to interpret.

6.1 Immune System Response to Infection

The oldest documented use of immunological methods dates to the 10th century in China, where it was used to inhale dried smallpox scabs to protect against smallpox. This method was improved at the end of the 18th century by English physician Edward Jenner (1749 – 1823), when the cow smallpox virus was used to vaccinate against smallpox and the fundamentals of vaccination were laid.

Because of vaccination, the immune system will learn to recognize the appropriate antigens and the vaccinated person should be protected from infection, or at least from serious progress of the disease, if that person encounters the disease agent.

Antigens and pathogens are dangerous foreign substances or altered own body cells (cancer cells) that the immune system recognizes and reacts to them. Within the Marchuk model we understand antigens as structures on the surface of pathogens or products of their metabolism.

6.1.1 Basic Types of Infectious Diseases

Immune responses to antigen are classified as follows:

- sub clinical form - the disease is hidden, without physiological symptoms, the antigen concentration does not reach the level at which the immune response of the organism becomes observable,
- acute form - the concentration of antigen reaches a level that exceeds the appreciable physiological changes level (the classical form of the disease, which leads to the physiological reactions),
- chronic form - a stable form of the immune process, non-zero concentration of antigens persists in the organism and increased amount of antibodies,
- lethal form - if the antigen concentration exceeds a critical level, the result is a death of organism,
- hyper toxic form of viral disease - this form has an unpredictable ending, is characterized by abundant viral infection of cells, and during the epidemics it will kill many patients.

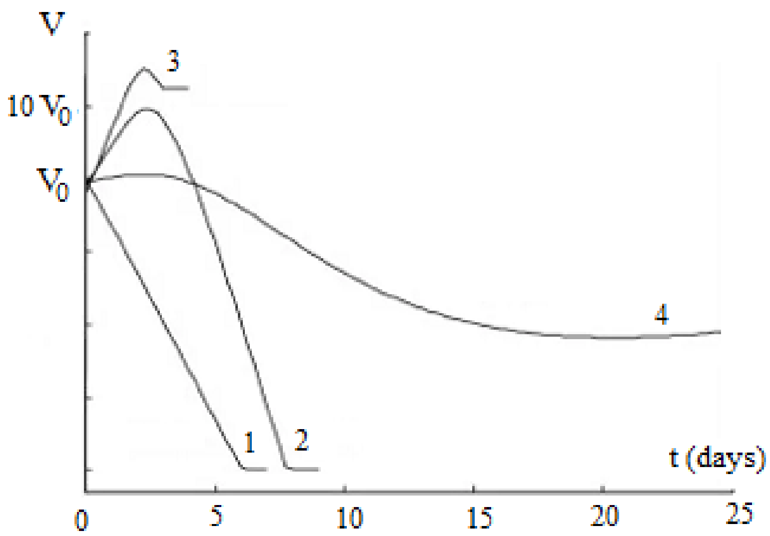

Fig. 6.1: Basic forms of infectious diseases [141]

Fig. 6.1 presents the growth dynamics of the antigen on the axis V and time on the axis t. Curve 1 corresponds to the sub clinical form of the disease, curve 2 to the acute form of the disease, curve 3 to the lethal form, and curve 4 to the chronic form of the disease.

6.1.2 Basic Types of Immune Responses

The main purpose of the immune system is the immune response, i.e. the organism response to antigens that are biologically foreign and potentially dangerous to the organism. There are two types of immune response:

- humoral response - protects against extracellular microorganisms, the basis of the immune response is the reaction of B cells responsible for the formation of antibodies, see Fig. 6.2,

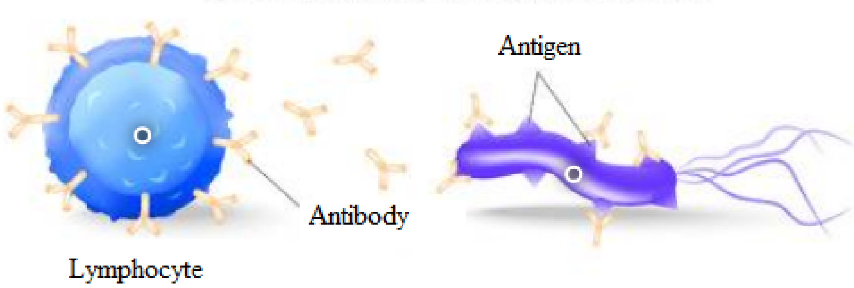

Fig. 6.2: Humoral immune response [116]

- cell response - protects against viruses, tumor cells, intracellular microorganisms and parasites, the basis of the immune response is the reaction of T cells and macrophages, see Fig. 6.3.

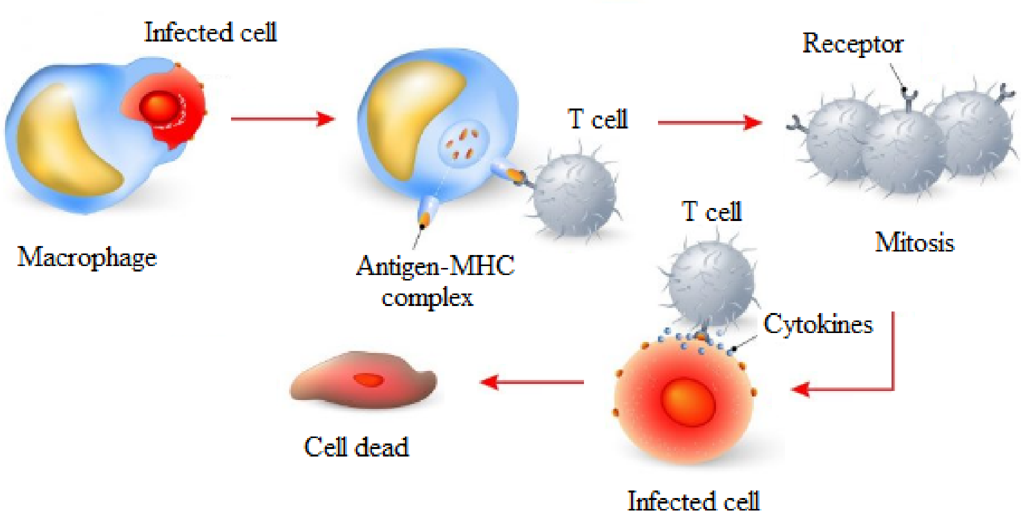

Fig. 6.3: Cell immune response [116]

At the beginning of the immune response, antigen is identified by specific lymphocytes, then these lymphocytes are activated, multiplied and matured into cells that are responsible for elimination the antigen from the organism. After this elimination, memory cells appear in the immune system, which are responsible for an accelerated response of the immune system upon repeated intervention of these same antigens.

6.2 Model of Immune System Response to Infection

The immunological reaction of a human organism attacked by bacteria or viruses will be simulated. The mathematical model takes the form of a system of four ordinary differential equations with a delayed argument.

The first equation describes the process of viral population growth, the second equation describes antibody dynamics, the third equation describes lymphatic cell antibody dynamics, and finally the fourth equation describes how organs are functionally affected. Such a system reflects the basic patterns of the organism's response to the intervention of antigens (viruses or bacteria).

But every person is individuality and dynamics in the real world is necessarily influenced by environmental noise. Due to environmental noise, parameters as immune reactivity, antigen reproduction, antigen-antibody interactions, antibody production, plasma cell production, the birth rate and others involved in the system present more or less random fluctuation.

6.2.1 Antigen Dynamics

The following equation describes the dynamics of antigens and their elimination by reaction with antibodies

$$\mathrm{d}V(t) = \beta V(t)\mathrm{d}t - \gamma F(t)V(t)\mathrm{d}t, \tag{6.1}$$

where $V(t)$ describes the amount of antigens in time t, $F(t)$ describes the amount of antibodies in time t, $\beta > 0$ is the antigen reproduction rate coefficient and $\gamma > 0$ is the suppression coefficient reflecting the probability of antigen neutralization in case antigen-antibody meeting and their interactions (the immune system effectivity rate).

$\mathrm{d}V(t)$ describes the increment of antigen in time t, which is proportional to $V(t)$ and β. However, upon encountering an antigen with an antibody, some antigens are neutralized. This condition is included by $\gamma F(t)V(t)\mathrm{d}t$.

Let us to assume that the white noise influences the antigen reproduction rate coefficient β by $\beta \to \beta + \vartheta\xi(t)$

$$\mathrm{d}V(t) = \beta V(t)\mathrm{d}t - \gamma F(t)V(t)\mathrm{d}t + \vartheta(V(t) - V(0))\xi(t)\mathrm{d}t,$$

where ϑ represents the intensity of the white noise and $\xi(t)$ represents white noise which can be rewritten into the form

$$\mathrm{d}V(t) = (\beta - \gamma F(t))V(t)\mathrm{d}t + \vartheta(V(t) - V(0))\mathrm{d}B(t), \tag{6.2}$$

where $B(t)$ is a Brownian motion.

Denote $V(t) \equiv X(t), \beta - \gamma F(t) \equiv A, V(0) \equiv X(0) = 0$ and $\vartheta \equiv G$. Then we get the equation

$$\mathrm{d}X(t) = AX(t)\mathrm{d}t + GX(t)\mathrm{d}B(t)$$

which has been studied in the chapter 4.

6.2.2 Plasma Cell Dynamics

Plasma cells are a type of white blood cells that form antibodies. Their maturation and activation takes about 5 days, then they are able to form thousands of units (molecules) of antibody per second. The amount of plasma cells is described by the equation

$$\mathrm{d}C(t) = \alpha F(t-\tau)V(t-\tau)\mathrm{d}t - \mu_C(C(t) - C(0))\mathrm{d}t, \tag{6.3}$$

where $C(t)$ is the constant amount of plasma cells in the health organism in time t, τ is the delay with which plasma cells are formed (antibodies and antigens are produced almost immediately), $\alpha > 0$ is the immune system reactivity coefficient (it indicates the number of antigen-antibody reactions in the past that is directly proportional to the formation of plasma cells), $C(0)$ is the level at which the amount of plasma cells in a healthy organism is kept, μ_C is the plasma cell coefficient, it is inversely proportional to plasma cell lifetime, indicates plasma cell lifetime.

The equation (6.3) describes the growth of plasma cells. Let the process of a plasma cell cascading population formation be simplified: B cell stimulated by antigen in the presence of a specific T helper signal activated by antigen on macrophages leads to the initiation of a cascade process of cells formation synthesizing antibodies which neutralize antigens.

The amount of lymphocytes synthesized in this way is proportional to $F(t)V(t)$. There will be a delay τ because of the process of plasma cell formation does not occur immediately, but takes time. Plasma cells also have the characteristic of aging and, because of that, their amount decreases, which is included by $\mu_C(C(t) - C(0))$.

Let us to assume that the white noise influences the plasma cell coefficient μ_C by $\mu_C \to \mu_C - \vartheta\xi(t)$

$$\mathrm{d}C(t) = \alpha F(t-\tau)V(t-\tau)\mathrm{d}t - \mu_C(C(t)-C(0))\mathrm{d}t + \vartheta(C(t)-C(0))\xi(t)\mathrm{d}t,$$

where ϑ represents the intensity of the white noise and $\xi(t)$ represents white noise which can be rewritten into the form

$$\mathrm{d}C(t) = \alpha F(t-\tau)V(t-\tau)\mathrm{d}t - \mu_C(C(t)-C(0))\mathrm{d}t + \vartheta(C(t)-C(0))\mathrm{d}B(t), \quad (6.4)$$

where $B(t)$ is a Brownian motion. The equation (6.4) is the stochastic differential equation with delay which has been studied in chapter 5.

6.2.3 Antibody Dynamics

Antibodies are proteins that react with antigens and destroy them. The dynamics of antibodies is described by the equation

$$\mathrm{d}F(t) = \rho C(t)\mathrm{d}t - \mu_F F(t)\mathrm{d}t - \eta\gamma V(t)F(t)\mathrm{d}t, \quad (6.5)$$

where ρ is the antibody formation rate per one plasma cell, it is the second coefficient reflecting the strength of immune reaction, μ_F is the antibody coefficient (mortality rate), it is inversely proportional to antibody lifetime and η is the rate of antibodies necessary to neutralize one antigen.

The equation (6.5) describes the increment of antibodies, or the balance of antibodies reacting with an antigen. The amount of antibodies that are produced by plasma cells is described by $\rho C(t)\mathrm{d}t$. Reduction of antibodies due to their aging is described by $\mu_F F(t)\mathrm{d}t$. The reduction of the amount of antibodies due to their encounter with antigens is described by $\eta\gamma V(t)F(t)\mathrm{d}t$ (in the first equation, the expression $\gamma F(t)V(t)\mathrm{d}t$ describes a reduction of the amount of antigens due to their encounter with antibodies).

Let us to assume that the white noise influences the antibody production rate ρ by $\rho \to \rho + \vartheta\xi(t)$ and the antibody coefficient μ_F by $\mu_F \to \mu_F - \vartheta\xi(t)$

$$\mathrm{d}F(t) = \rho C(t)\mathrm{d}t - \mu_F F(t)\mathrm{d}t - \eta\gamma V(t)F(t)\mathrm{d}t + \vartheta[C(t)-C(0)+F(t)-F(0)]\xi(t)\mathrm{d}t,$$

where ϑ represents the intensity of the white noise and $\xi(t)$ represents white noise which can be rewritten into the form

$$\mathrm{d}F(t) = \rho C(t)\mathrm{d}t - \mu_F F(t)\mathrm{d}t - \eta\gamma V(t)F(t)\mathrm{d}t + \vartheta[C(t)-C(0)+F(t)-F(0)]\mathrm{d}B(t), \quad (6.6)$$

where $B(t)$ is a Brownian motion.

6.2.4 Relative Characteristics of the Affected Organ

Relative organ damage is described by the equation

$$\mathrm{d}m(t) = \sigma V(t)\mathrm{d}t - \mu_m m(t)\mathrm{d}t, \tag{6.7}$$

where $m(t)$ is a dimensionless quantity defined by the equation below, it characterizes the amount of damage caused in the infected organism (we assume that the plasma cells formation is not dependent on the condition of the organism), $\sigma > 0$ indicates damage to the body which is directly proportional to the number of antigens, μ_m is the coefficient of natural regeneration of an organism directly proportional to the condition of this organism (organ).

The equation describes the rate of organ damage. The increase in this characteristic depends on the amount of antigens $\sigma V(t)$ and the reduction may be caused by organism recovery or healing.

Relative characteristics of the affected organ is

$$m(t) = 1 - \frac{M_H}{M},$$

where M characterizes a healthy organism and M_H characterizes the size of the healthy part in time t. The quantity m takes values $0-1$. For $m = 0$, the organism is completely healthy. For $m = 1$, the disease leads to death.

6.2.5 Mathematical Model of Immune System Response

Deterministic model

The deterministic model of the immune system response to infection is simulated by the next system of ODEs with delay τ (6.1), (6.3), (6.5) and (6.7)

$$\begin{aligned}
\mathrm{d}V(t) &= \beta V(t)\mathrm{d}t - \gamma F(t)V(t)\mathrm{d}t, \\
\mathrm{d}C(t) &= \alpha F(t-\tau)V(t-\tau)\mathrm{d}t - \mu_C(C(t) - C(0))\mathrm{d}t, \\
\mathrm{d}F(t) &= \rho C(t)\mathrm{d}t - \mu_F F(t)\mathrm{d}t - \eta\gamma V(t)F(t)\mathrm{d}t, \\
\mathrm{d}m(t) &= \sigma V(t)\mathrm{d}t - \mu_m m(t)\mathrm{d}t.
\end{aligned} \tag{6.8}$$

Stochastic model

The deterministic model (6.8) is spread about random influence in the form of Brownian motion and there is simulated the stochastic model of the immune system response to infection by the next system of SDEs with delay τ (6.2), (6.4), (6.6) and (6.7)

$$
\begin{aligned}
\mathrm{d}V(t) &= \beta V(t)\mathrm{d}t - \gamma F(t)V(t)\mathrm{d}t + \vartheta(V(t) - V(0))\mathrm{d}B(t),\\
\mathrm{d}C(t) &= \alpha F(t-\tau)V(t-\tau)\mathrm{d}t - \mu_C(C(t) - C(0))\mathrm{d}t\\
&+ \vartheta(C(t) - C(0))\mathrm{d}B(t),\\
\mathrm{d}F(t) &= \rho C(t)\mathrm{d}t - \mu_F F(t)\mathrm{d}t - \eta\gamma V(t)F(t)\mathrm{d}t\\
&+ \vartheta[(C(t) - C(0)) + (F(t) - F(0))]\mathrm{d}B(t),\\
\mathrm{d}m(t) &= \sigma V(t)\mathrm{d}t - \mu_m m(t)\mathrm{d}t.
\end{aligned} \tag{6.9}
$$

Denote $V(t) \equiv X_1(t), V(0) \equiv X_1(0) = 0, C(t) \equiv X_2(t), C(0) \equiv X_2(0) = 0,$ $F(t) \equiv X_3(t), F(0) \equiv X_3(0) = 0, m(t) \equiv X_4(t)$. Then the system (6.9) can be rewritten into matrix form

$$
\begin{aligned}
\mathrm{d}\begin{pmatrix} X_1(t)\\ X_2(t)\\ X_3(t)\\ X_4(t)\end{pmatrix} &= \begin{pmatrix} \beta-\gamma X_3(t) & 0 & 0 & 0\\ 0 & -\mu_C & 0 & 0\\ 0 & \rho & -\mu_F-\eta\gamma X_1(t) & 0\\ \sigma & 0 & 0 & -\mu_m\end{pmatrix}\begin{pmatrix} X_1(t)\\ X_2(t)\\ X_3(t)\\ X_4(t)\end{pmatrix}\mathrm{d}t\\
&+ \begin{pmatrix} 0 & 0 & 0 & 0\\ \alpha X_3(t-\tau) & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix} X_1(t-\tau)\\ X_2(t-\tau)\\ X_3(t-\tau)\\ X_4(t-\tau)\end{pmatrix}\mathrm{d}t\\
&+ \begin{pmatrix} \vartheta & 0 & 0 & 0\\ 0 & \vartheta & 0 & 0\\ 0 & \vartheta & \vartheta & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix} X_1(t)\\ X_2(t)\\ X_3(t)\\ X_4(t)\end{pmatrix}\mathrm{d}B(t).
\end{aligned}
$$

Denote $X(t) \equiv (X_1(t), X_2(t), X_3(t), X_4(t))^T$,

$$
\begin{pmatrix} \beta-\gamma X_3(t) & 0 & 0 & 0\\ 0 & -\mu_C & 0 & 0\\ 0 & \rho & -\mu_F-\eta\gamma X_1(t) & 0\\ \sigma & 0 & 0 & -\mu_m\end{pmatrix} \equiv A
$$

is a matrix of dimension (4×4),

$$
\begin{pmatrix} 0 & 0 & 0 & 0\\ \alpha X_3(t-\tau) & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} \equiv H
$$

is a matrix of dimension (4×4) and

$$\begin{pmatrix} \vartheta & 0 & 0 & 0 \\ 0 & \vartheta & 0 & 0 \\ 0 & \vartheta & \vartheta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \equiv G$$

is a matrix of dimension (4×4), then we get the system

$$\mathrm{d}X(t) = AX(t)\mathrm{d}t + HX(t-\tau)\mathrm{d}t + GX(t)\mathrm{d}B(t),$$

the similar system has been studied in previous chapters 4 and 5.

Initial conditions

For the system (6.8) at time t, we establish $X(t) = (V(t), C(t), F(t), m(t))^T$ and determine the initial function. We assume the state of equilibrium of the system in time $t \in [-\tau, 0)$, i.e. $X(t) = (0, C(0), F(0), 0)^T$ (no antigens are present in the organism, the organism is healthy). The system lost its equilibrium at time $t = 0$, i.e. $X(0) = (V(0), C(0), F(0), 0)^T$ (the organism was infected with a population of antigens $V(0)$). The initial function $\psi(t)$ is

$$\psi(t) = \begin{cases} (0, C(0), F(0), 0)^T & t \in [-\tau, 0)\,, \\ (V(0), C(0), F(0), 0)^T & t = 0. \end{cases}$$

Due to the biological processes that describe the model, let's assume that

- there is a certain initial level of plasma cells in the organism, $C(0) \geq 0$,
- there is a certain initial level of antibodies in the organism, $F(0) \geq 0$,
- if the organism is so damaged that leads to death $m(t) = 1$, i.e. $\frac{\mathrm{d}m(t)}{\mathrm{d}t} = 0$,
- all parameters of the model are positive, stable and initial conditions are ≥ 0, i.e. for $t = 0$, applies

$$V(0) \geq 0, C(0) \geq 0, F(0) \geq 0, m(0) \geq 0.$$

For all $t \geq 0$, the solution of the model will be non-negative and continuous,

$$V(t) \geq 0, C(t) \geq 0, F(t) \geq 0, m(t) \geq 0.$$

Equilibrium states of deterministic model

Let us look for the equilibrium points of the system, i.e. values for which there is no system change, i.e. $X(t) = X^*, V(t) = V(t-\tau) = V^*, C(t) = C^*$,

$F(t) = F(t-\tau) = F^*, m(t) = m^*$ and $X^* = (V^*, C^*, F^*, m^*)^T = 0$

$$\begin{aligned} 0 &= \beta V^* - \gamma F^* V^*, \\ 0 &= \alpha F^* V^* - \mu_C (C^* - C(0)), \\ 0 &= \rho C^* - \mu_F F^* - \eta\gamma V^* F^*, \\ 0 &= \sigma V^* - \mu_m m^*. \end{aligned}$$

This system has two linear independent solutions

$$X_1^* = \left(0, C(0), \frac{\rho C(0)}{\mu_F}, 0\right)^T, \tag{6.10}$$

$$X_2^* = \left(\frac{\mu_C(\mu_F\beta - C(0)\gamma\rho)}{\beta(\alpha\rho - \mu_C\gamma\eta)}, \frac{\alpha\beta\mu_F - C(0)\gamma^2\eta\mu_C}{\gamma(\alpha\rho - \mu_C\gamma\eta)}, \frac{\beta}{\gamma}, \frac{\mu_C\sigma(\mu_F\beta - C(0)\gamma\rho)}{\beta\mu_m(\alpha\rho - \mu_C\gamma\eta)}\right)^T. \tag{6.11}$$

Numerical simulations
The following numerical simulations created in graphical environment of Simulink based on MATLAB R2018a illustrate the behavior of the solution of the immune system. Fig. 6.4 shows the basic scheme that consists of subsystems.

Numerical values used to prepare the simulations and presented in Tab. 6.1 have been taken from papers U. Foryś [49], U. Foryś and M. Bodnar [50], U. Foryś [54], and G. I. Marchuk, R. V. Petrov at al [105]. Various values of antigen reproduction rate β, initial dose of antigens $V(0)$ and stochastic coefficient ϑ are adjusted for following simulations.

6.3 Simulation of the Sub Clinical Form

The equilibrium state (6.10) corresponds to a healthy organism in which no antigens are present, the amount of plasma cells is constant, the amount of antibodies is kept at the level $\frac{\rho C(0)}{\mu_F}$ and the organism is not damaged.

The sub clinical form of disease and complete organism recovery corresponds to the solution (6.10), there must be met the condition of stability

$$\beta < \gamma F(0).$$

If this condition is met, the amount of antigens is declining monotonous to zero. This state corresponds to the gentle course, the immune system meets with this course commonly. The infected person does not observe any symptoms, probably does not feel the illness, the disease is hidden and quickly disappears.

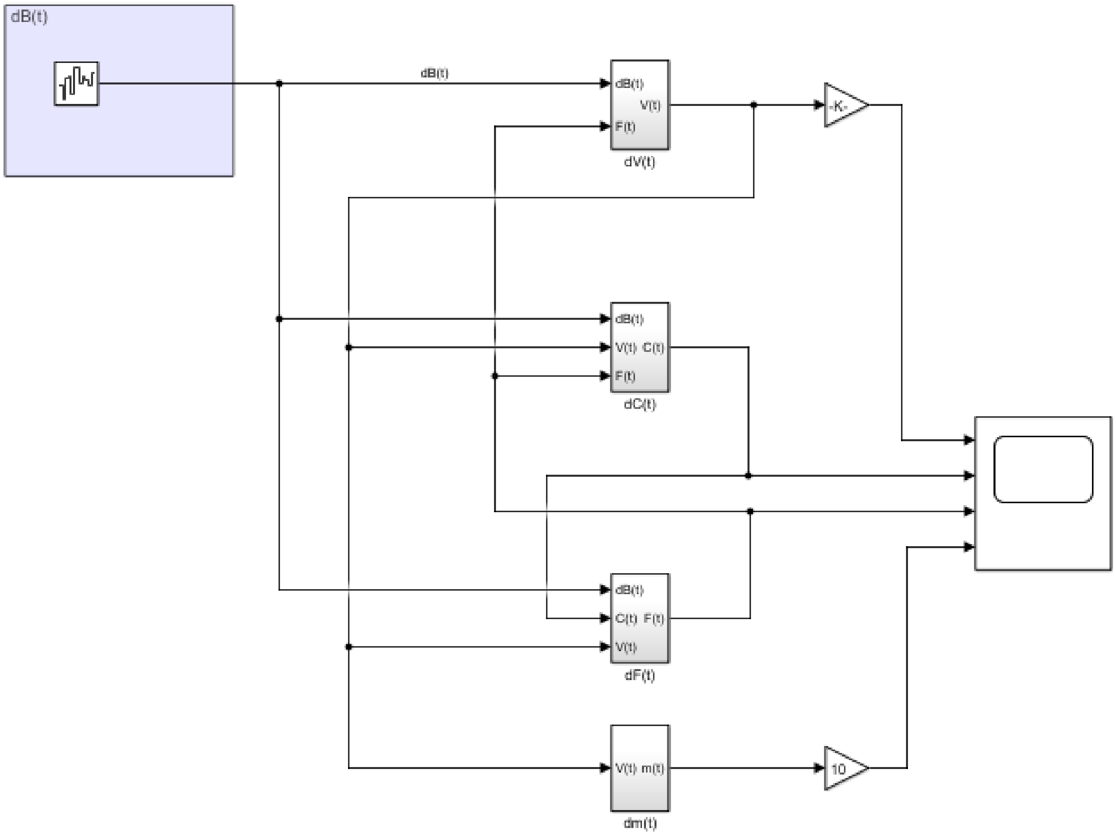

Fig. 6.4: Model in Simulink. [own source]

Tab. 6.1: Fixed parameters used in simulations.

Initial condition	**Description**	**Value**
C_0	Initial plasma cells	1
F_0	Initial antibodies	0.5
m_0	Damage of organism	0
Parameter	**Description**	**Value**
α	Immune reactivity coefficient	1000
γ	Immune system effectivity rate	0.8
ρ	Antibody production rate	0.17
η	Antibody neutralization rate	0.15
σ	Damage rate	10
μ_F	Antibody death rate	0.17
μ_C	Plasma cell coefficient	0.5
μ_m	Regeneration coefficient	0.1
τ	Time delay (changes of immune system)	5

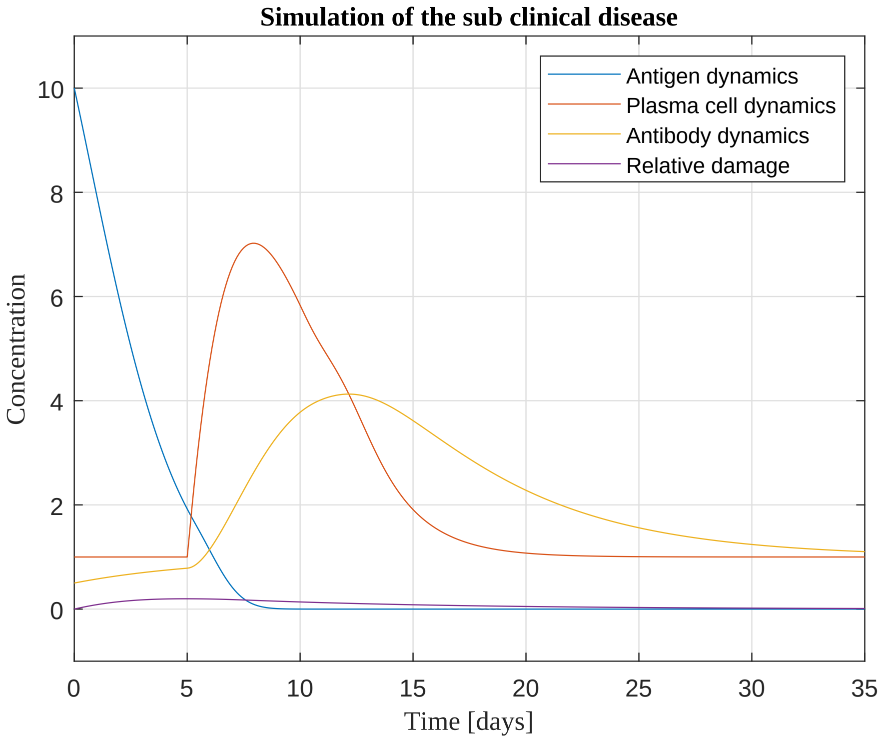

Fig. 6.5: Sub clinical form - deterministic model. [own source]

6.3.1 Simulation of the Deterministic Model

There must be met the condition $\beta < 0.4$, so we choose an antigen reproduction rate $\beta = 0.2$ and initial dose of antigens $V(0) = 0.01$ to simulate the sub clinical form of disease, see Fig. 6.5.

The immune response is sufficiently strong and all antigens that have been reached into the organism are destroyed by antibodies present in the organism (without the production of new ones). The speed of antigen reproduction is too small as compared with the neutralization of antigens by antibodies.

6.3.2 Simulation of the Stochastic Model

For simulation of the sub clinical disease, we choose an antigen reproduction rate $\beta = 0.2$, initial dose of antigens $V(0) = 0.01$ and $\vartheta = 0.2$, see Fig. 6.6.

By comparing the deterministic and stochastic model, we can see that a white noise has an effect on higher plasma cell production in the body corresponding with higher level of antibodies.

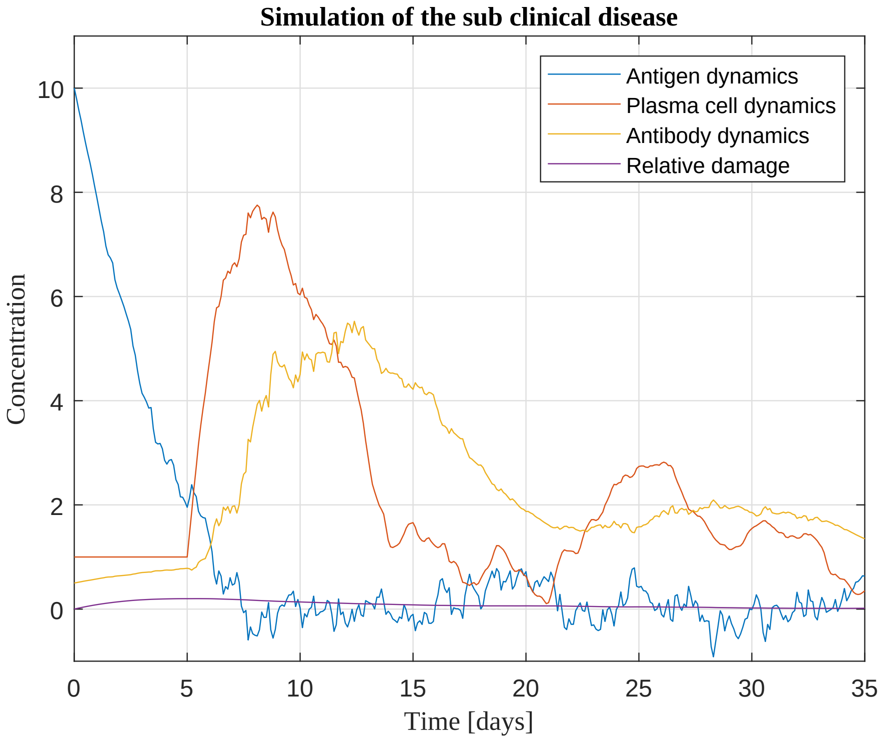

Fig. 6.6: Sub clinical form - stochastic model. [own source]

6.4 Simulation of the Acute Form

The acute form of disease and complete organism recovery corresponds to the equilibrium state (6.10). The solution (6.10) is asymptotic stable if the initial amount of antigen $V(0)$ meets the inequality

$$0 \leq V(0) \leq \frac{\mu_F(\gamma F(0) - \beta)}{\beta\eta\gamma} = V_{IB},$$

where V_{IB} is called an immunological barrier. From a biological point of view, this means that if the organism is infected by the initial amount of pathogens $V(0) < V_{IB}$, the disease does not develop, the number of antigens in the organism converges over time to 0 and the affected organ is restored and the organism is cured. This value depends on $C(0)$. If the number of plasma cells that recognize the disease is increased, the immunological barrier increases and the body gains the immunity.

6.4.1 Simulation of the Deterministic Model

In this case $\beta > 0.4$, however the immunological barrier $V_{IB} = 0.8$ is not exceeded. We simulate the acute form of disease for an antigen reproduction rate $\beta = 0.6$ and initial dose of antigens $V(0) = 0.01$, see Fig. 6.7.

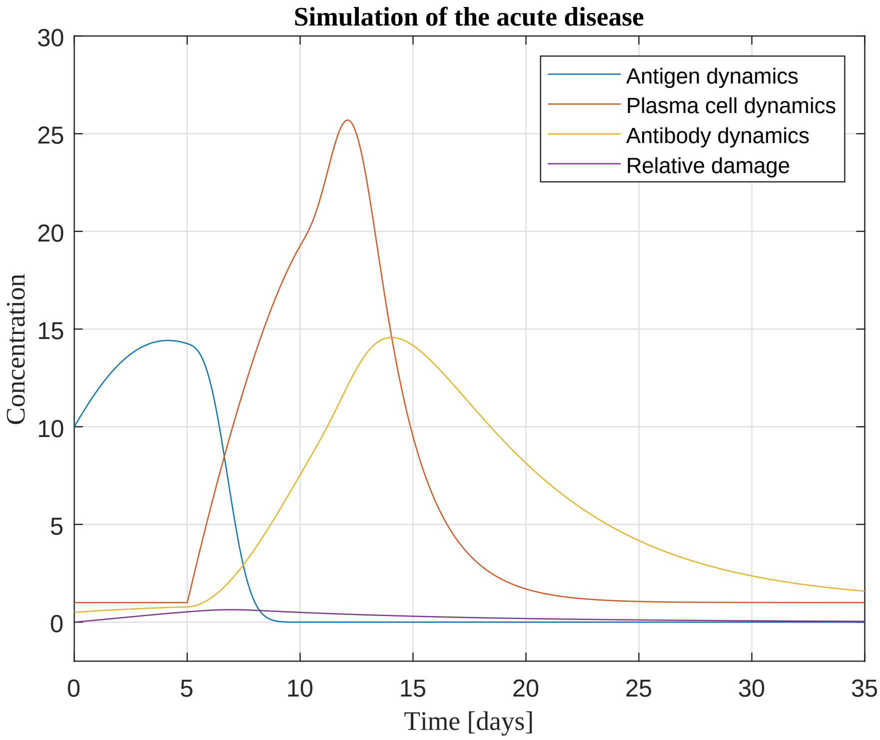

Fig. 6.7: Acute form - deterministic model. [own source]

The disease has a typical acute course, after the initial antigen growth, their population reaches the maximum in time $t = \tau = 5$ days, and then monotonous falls to zero. Antigens are not reproduced very quickly and the immune system is strong. This situation describes a healthy organism that is infected in time t_0 by not too aggressive disease. Typical example is the flu, that the infected person observes the symptoms (fever, cough, rhinitis, muscle weakness, nausea, etc.) for several days, then follows the relief that depends on the delay τ of the immune system reaction.

6.4.2 Simulation of the Stochastic Model

We simulate the acute form of disease for an antigen reproduction rate $\beta = 0.6$, initial dose of antigens $V(0) = 0.01$ and $\vartheta = 0.2$, see Fig. 6.8.

By comparing the deterministic and stochastic model, we can see as in the previous simulation that a white noise has an effect on higher plasma cell production in the body corresponding with higher level of antibodies.

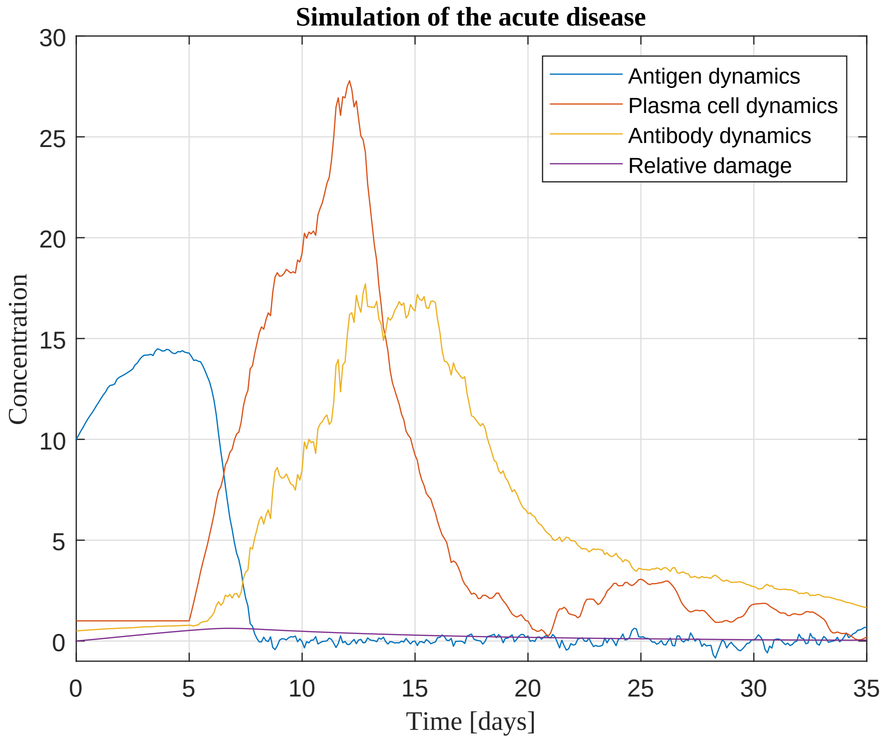

Fig. 6.8: Acute form - stochastic model. [own source]

6.5 Simulation of the Chronic Form

In order to the disease can reach a chronic stage, the growth rate of antigens must be additionally large but not over-large to avoid the death, so $\beta > \gamma F(0)$ (the solution (6.10) loses its stability).

The solution (6.11) presents a chronic disease in which a number of antigens persist in the body, causing damage to the body while the level of antibodies is kept at the level $\frac{\beta}{\gamma}$. This equilibrium exists for condition $\alpha\rho \neq \mu_C \gamma \eta$. The solution is asymptotic stable for $\alpha \to \infty$ and for the following inequalities

$$\mu_C \leq 1$$

and

$$0 < \beta - \gamma F(0) < \left(\tau + \frac{1}{\mu_C + \mu_F} \right)^{-1}.$$

The length of the period and the time for which oscillations appear is dependent on the delay τ.

For the immune system,

$$\beta - \gamma F(0) = \beta - \gamma F^* = \frac{\mathrm{d}V(t)}{\mathrm{d}t} \frac{1}{V(t)} = \frac{\mathrm{d} \ln V(t)}{\mathrm{d}t}$$

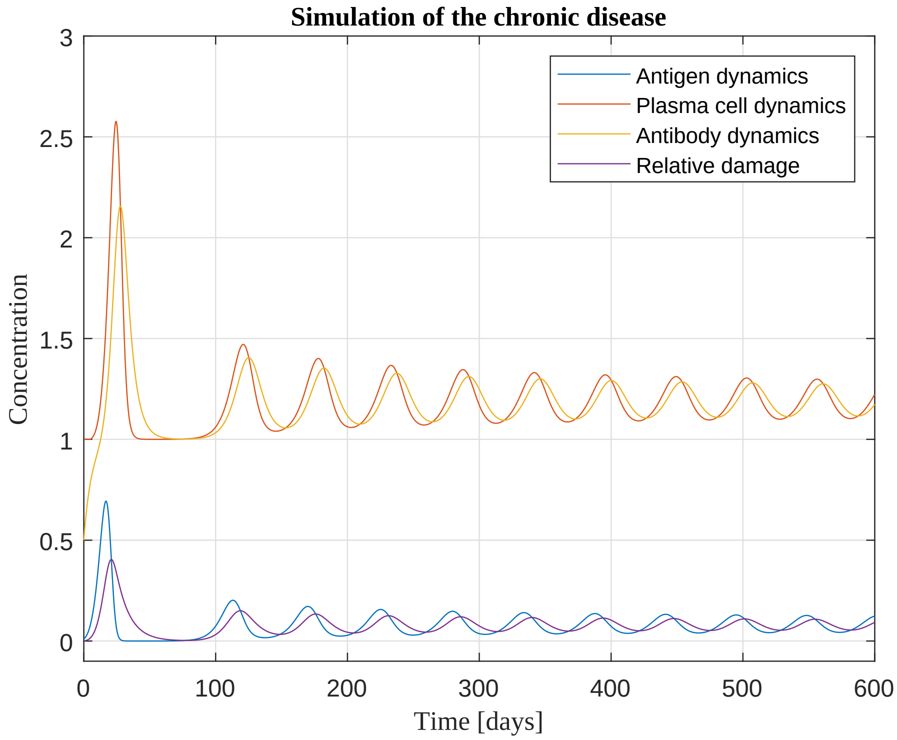

Fig. 6.9: Chronic form - deterministic model. [own source]

and simultaneously reactivity α is high. That means, the reproduction rate of antigens β is sufficiently high so that the disease is not cured but does not lead to organism failure.

6.5.1 Simulation of the Deterministic Model

In this case $\beta > V_{IB}$, however β meets the condition $0.8 < \beta < \frac{83}{87}$. We simulate the chronic form of disease for an antigen reproduction rate $\beta = 0.95$ and initial dose of antigens $V(0) = 0.00001$, see Fig. 6.9.

After the sharp initial antigen growth, most of them is exterminated. However, after a while, the disease is returned and the antigen population converges to an equilibrium state (6.11) by inhibited oscillations.

We show the case when β does not meet the condition $0.8 < \beta < \frac{83}{87}$, but the disease does not lead to organism failure. We simulate the chronic form of disease for an antigen reproduction rate $\beta = 0.99$ and initial dose of antigens $V(0) = 0.00001$, see Fig. 6.10.

If the solution (6.11) is unstable, oscillations are not inhibited.

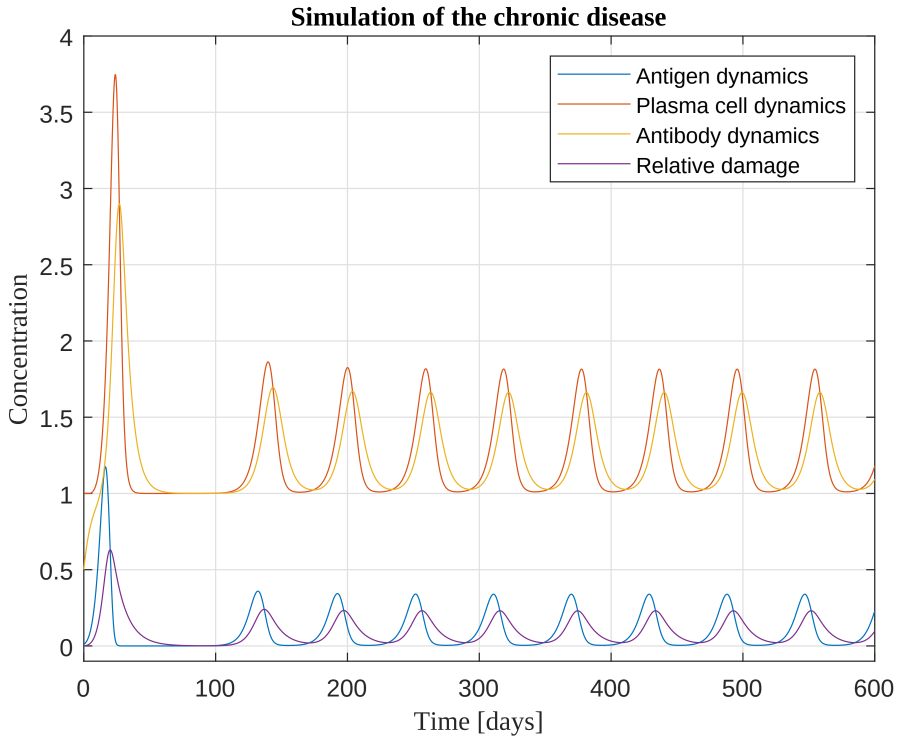

Fig. 6.10: Chronic form - deterministic model. [own source]

6.5.2 Simulation of the Stochastic Model

We simulate the chronic form of disease for an antigen reproduction rate $\beta = 0.95$, initial dose of antigens $V(0) = 0.00001$ and $\vartheta = 0.15$, see Fig. 6.11.

We simulate the unstable chronic form of disease for an antigen reproduction rate $\beta = 0.99$, initial dose of antigens $V(0) = 0.00001$ and $\vartheta = 0.05$, see Fig. 6.12.

By comparing the deterministic and stochastic model, we can see that oscillations of the stochastic model are not regular and demonstrate a more realistic state of chronic disease.

6.6 Simulation of the Lethal Form

At last, we get to the course of diseases ending lethally. The organism (organ) failure may be caused by too high initial dose of antigens, too high growth speed of antigens or lower antibody production.

6.6.1 Simulation of the Deterministic Model

The first case, we simulate the lethal form of disease for an antigen reproduction rate $\beta = 0.2$ and initial dose of antigens $V(0) = 0.1$, see Fig. 6.13.

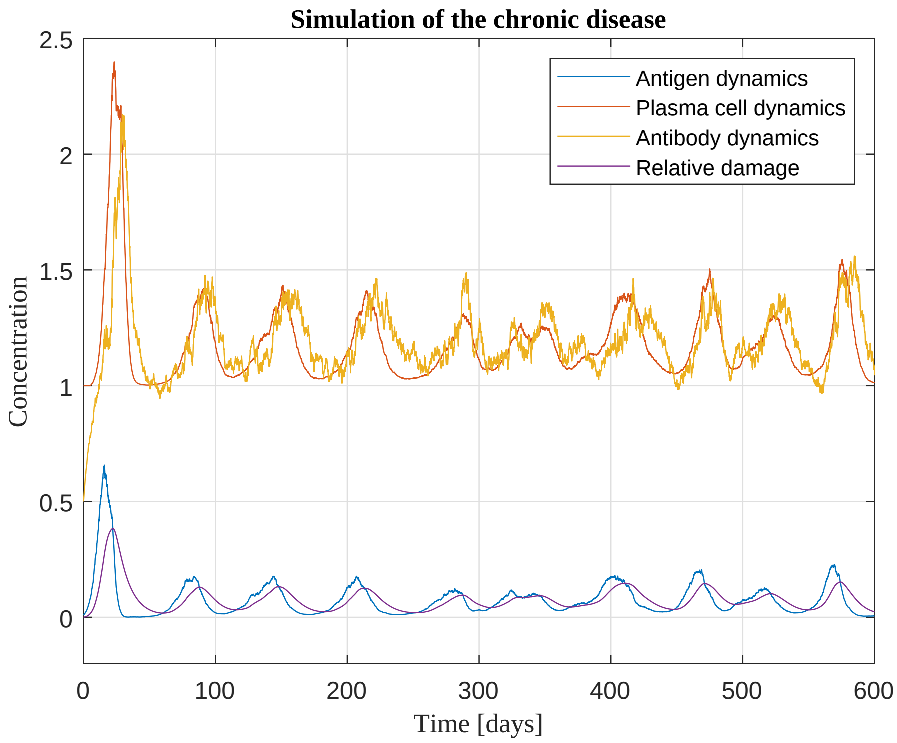

Fig. 6.11: Chronic form - stochastic model. [own source]

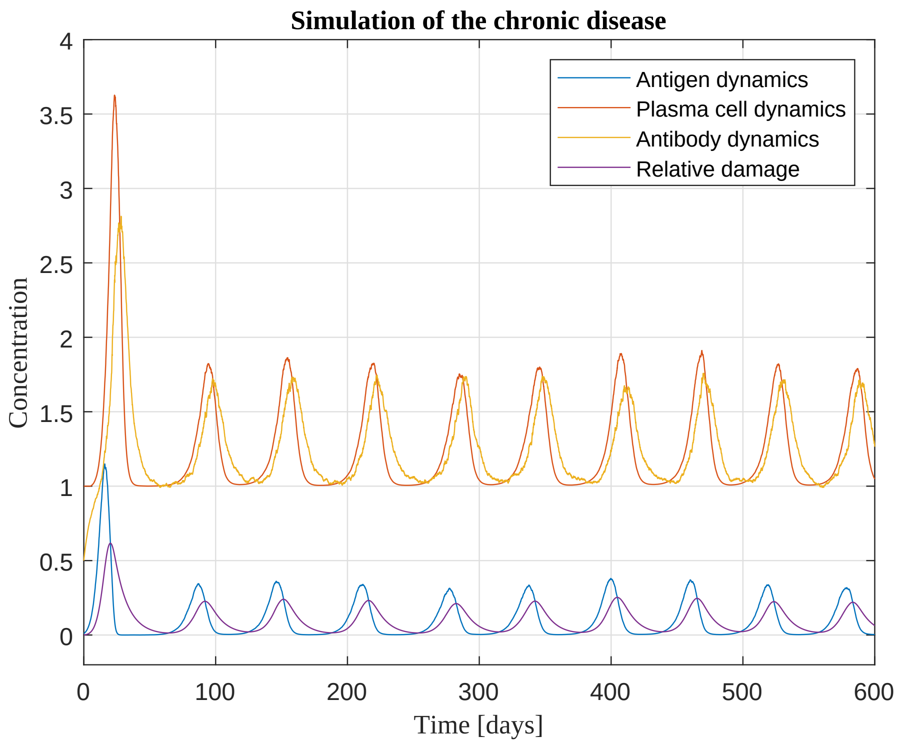

Fig. 6.12: Chronic form - stochastic model. [own source]

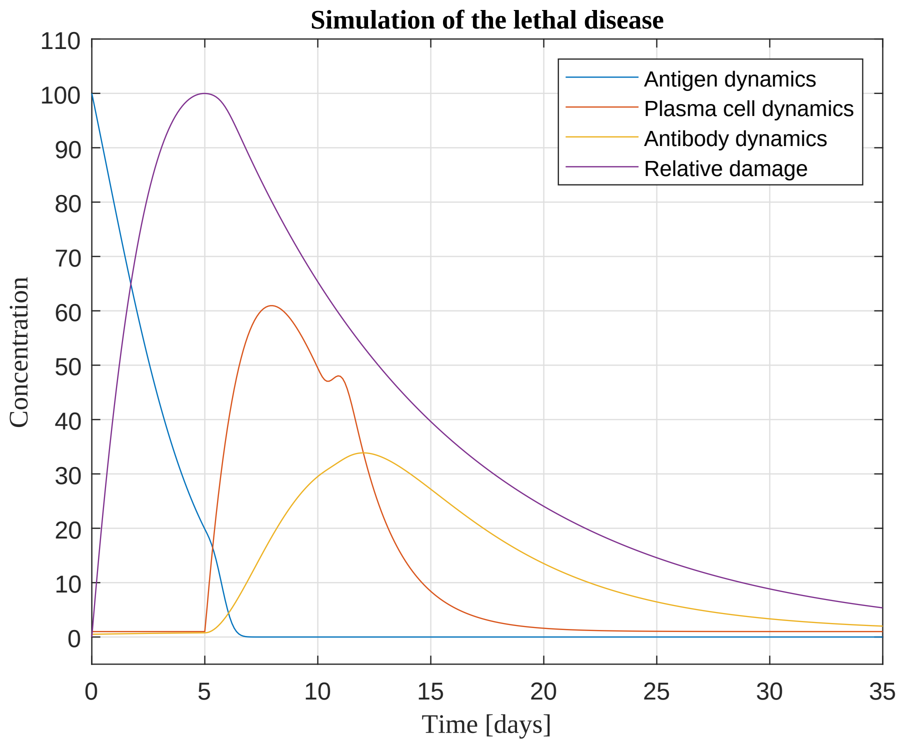

Fig. 6.13: Lethal form - deterministic model. [own source]

The organism is exposed to too high the initial dose of antigens $V(0)$, while the initial level of antibodies $F(0)$ and plasma cells $C(0)$ in the body is small compared to the antigen's dose. Antigens are not promptly removed from the organism and the immune system delay leads to organism failure.

The second case, we simulate the lethal form of disease for an antigen reproduction rate $\beta = 1.2$ and initial dose of antigens $V(0) = 0.01$, see Fig. 6.14.

An antigen reproduction rate coefficient β is enormous, the growth speed of antigens is too high. In this case, the small initial dose of antigens $V(0)$ leads to lethal organism damage.

The third case, we simulate the lethal form of disease for an antigen reproduction rate $\beta = 0.6$, initial dose of antigens $V(0) = 0.01$ and immune system reactivity $\alpha = 500$, see Fig. 6.15.

The immune system of organism is weakened. The parameter α is small, plasma cells are produced slowly, related lower antibody production. Then the commonly strong disease can cause the death of the organism.

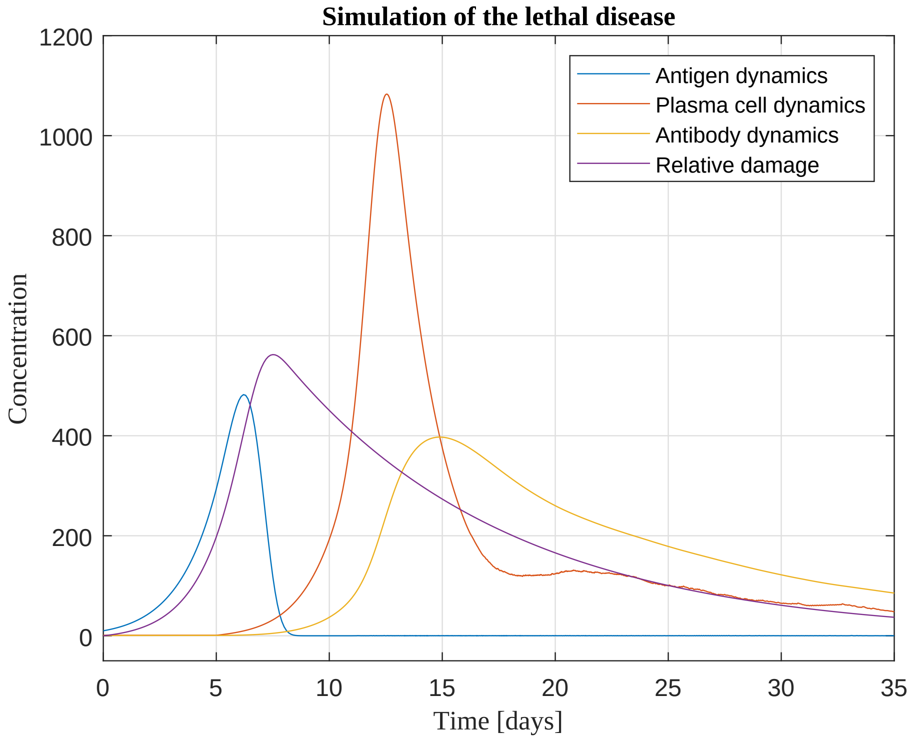

Fig. 6.14: Lethal form - deterministic model. [own source]

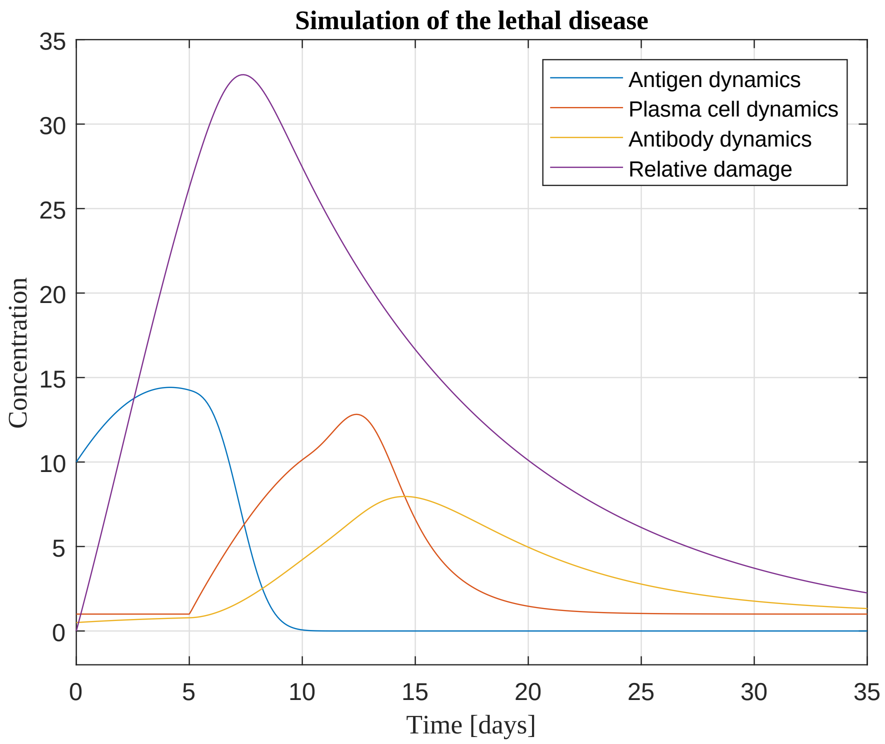

Fig. 6.15: Lethal form - deterministic model. [own source]

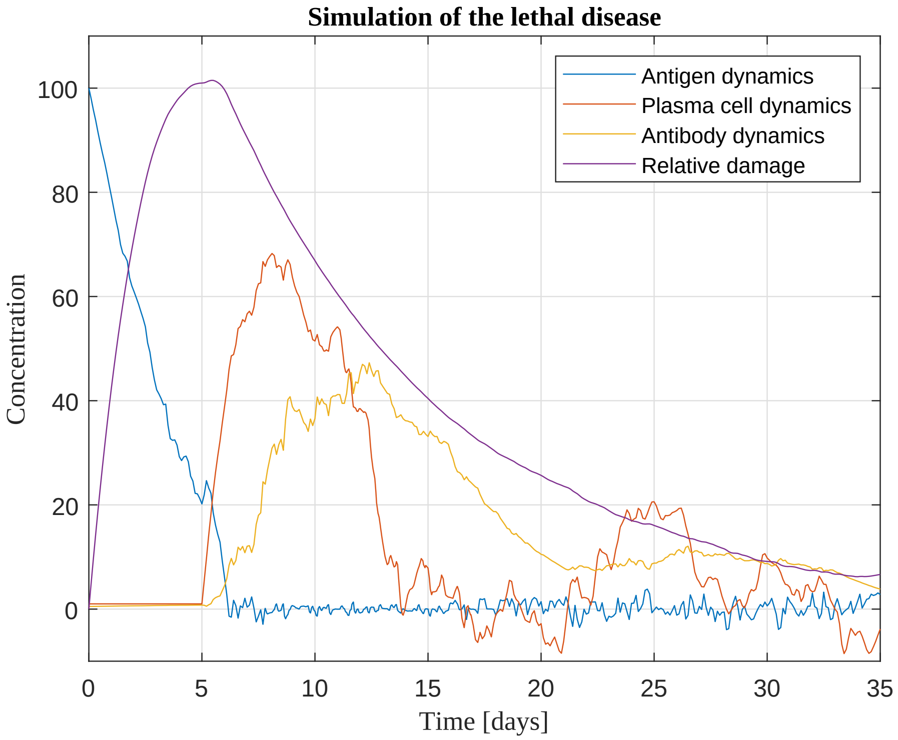

Fig. 6.16: Lethal form - stochastic model. [own source]

6.6.2 Simulation of the Stochastic Model

The first case, we simulate the lethal form of disease for an antigen reproduction rate $\beta = 0.2$, initial dose of antigens $V(0) = 0.1$ and $\vartheta = 0.2$, see Fig. 6.16.

The second case, we simulate the lethal form of disease for an antigen reproduction rate $\beta = 1.2$, initial dose of antigens $V(0) = 0.01$ and $\vartheta = 0.2$, see Fig. 6.17.

The third case, we simulate the lethal form of disease for an antigen reproduction rate $\beta = 0.6$, initial dose of antigens $V(0) = 0.01$, immune system reactivity $\alpha = 500$ and $\vartheta = 0.2$, see Fig. 6.18.

By comparing the deterministic and stochastic models, we can see that a white noise has an effect on higher antibody production in the body.

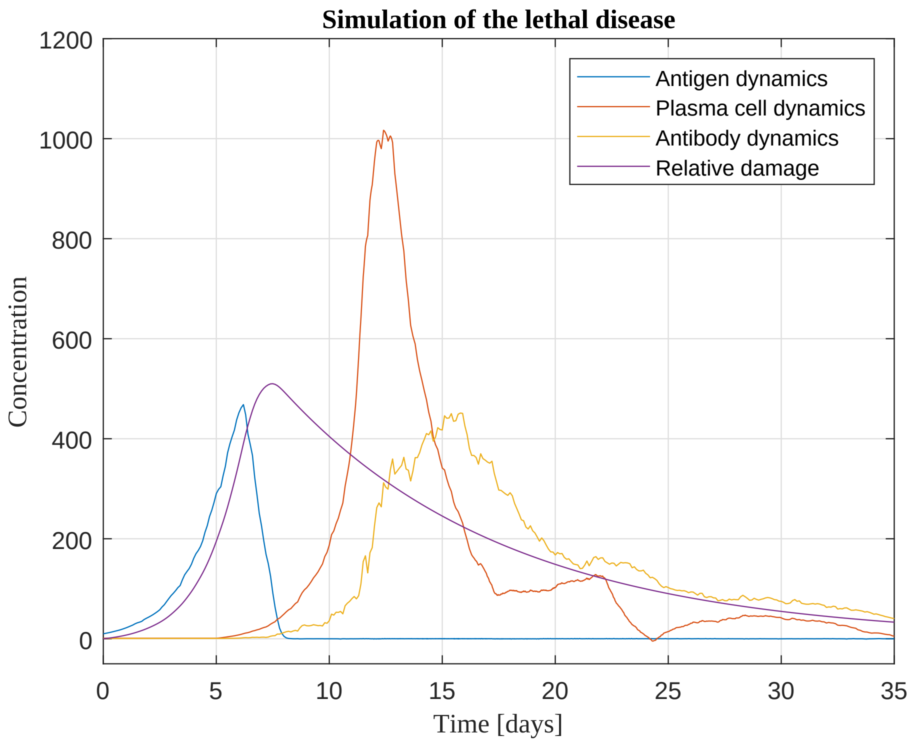

Fig. 6.17: Lethal form - stochastic model. [own source]

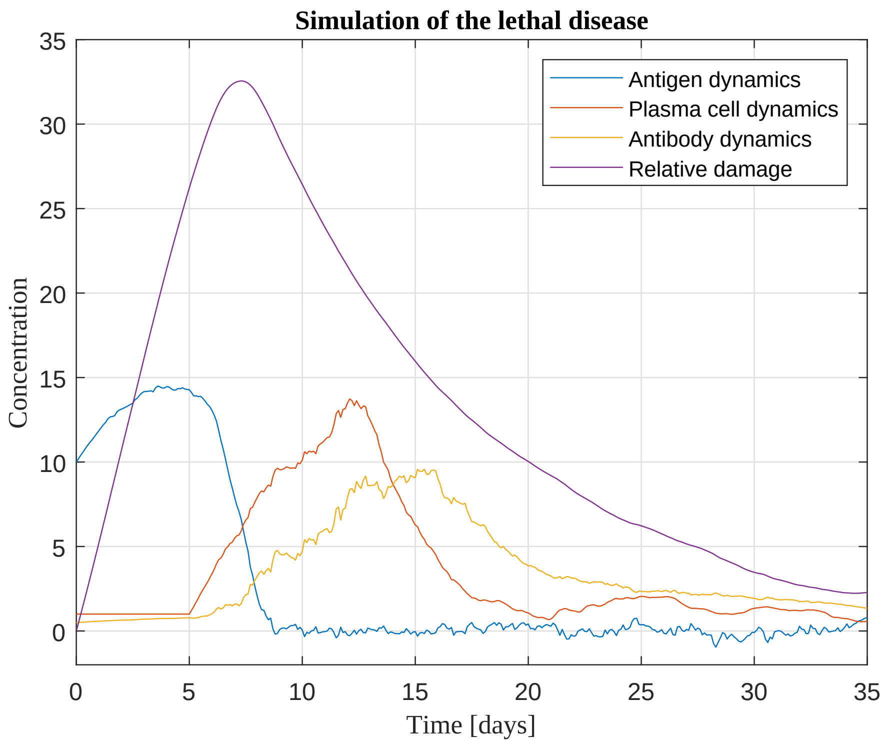

Fig. 6.18: Lethal form - stochastic model. [own source]

7 CONCLUSION

Mathematical modeling is a discipline which deals with the mathematical description of phenomena around us. If we want the model to be as faithfully as possible, we need to improve it. And it is the precise moment of stochastic modeling which can approach the reality with a certain probability by extending the deterministic system to a random process.

Using an assembled stochastic model, it is possible to simulate the course of events under different model parameters and to observe the expected behavioral of the system. Information for decision can be derived from the behavioral analysis of the model in the simulation task.

The basis of understanding the stochastic structure is to be well acquainted with the basic concepts, including Brownian motion, which was first observed at the beginning of the 19th century as a random movement of pollen grains in water. At the beginning of the 20th century the essence of this phenomenon was elucidated by Albert Einstein, based on kinetic theory of matter. Since then, stochastic theory has experienced unprecedented development, especially in the last 60 years, and today we are able to describe a random process using stochastic differential equations based on Itô integral.

Based on the theory of stochastic differential equations and systems, a solution of the stochastic equation with Brownian motion was found. A solution of stochastic modeling can be found in four positions. If the system after deviation depending on the initial conditions converge to its original position, we say that the system is stable. If the system after deviation converges to a different equilibrium position, then we say that the system is stochastically stable. However, there may also be situation when the system after deviation does not return or remain in a deviation position. Then we say that the system is unstable. The main part of the thesis was therefore not only the search for a suitable solution of the stochastic equation or the stochastic system, but also the search for a general formula for determining the stability of the solution of the given stochastic equations or systems of the orders 3 and 4. It is necessary to state that it is possible to study systems of orders higher than 4, but it is mainly a programming issue.

Stochasticity is unavoidable when considering biological systems and processes, both at the macro scale with populations surviving in rapidly and unpredictably changing environments, but also and especially at the molecular level, where entropic considerations can have significant implications. Not only must systems be robust but some systems actually rely upon Brownian motions in order to operate efficiently. Therefore, the final part of the thesis is devoted to the application of the stochastic process to the biomedical model. There is simulated the immune system's response to infection. The deterministic model was compared with the stochastic model and four types of immune response reactions were observed (sub clinical, acute, chronic and lethal form). Within all forms, there was observed that white noise significantly affects the production of plasma cells related to antibodies in the body. The subject of another study may be a simulation of a delayed model for the body's immune response to the use of drugs, which takes time to manifest. An interesting topic of another study may also be the hyper-toxic form of the viral disease and its associated epidemic.

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List of Symbols, Constants and Abbreviations

BMI	Body Mass Index
ODE	Ordinary Differential Equation
SDE	Stochastic Differential Equation
$\mathbb{R}$	Set of real numbers
$B_t, B_t(\omega)$	Brownian motion
$\mathcal{B}$	Borel σ-algebra
$(\Omega, \mathcal{F}, P)$	Probability space
$\mathbb{E}[X_t]$	Mean value (or expected value) of random variable X_t
$N(\mu, \sigma^2)$	Gaussian distribution with mean value μ and variance σ^2
$W_t, W_t(\omega)$	Wiener process
$\mathcal{F}$	σ-algebra of subsets of Ω
τ	Delay
$\int_0^t G dB_t$	Itô integral on the interval $(0, t)$
$\|\cdot\|$	Norm of vector