Development of an Improved Frequency Limited Model Order Reduction Technique and Error Bound for Discrete-Time Systems

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Abstract. *Frequency limited model order reduction algorithm presented by Wang & Zilouchian for discrete-time systems provide unstable reduced-order models and also do not provide a priori error bound formula. Many stability-preserving model order reduction algorithms were presented; however, these methods produce significant approximation errors in the desired frequency interval. An improved algorithm of model order reduction for the discrete-time systems is presented. The proposed technique gives the stable reduced-order model and also provides less approximation error as compared with other algorithms and also provides the formula for the frequency response a priori error bound. Numerical examples provided at the end of the section show the efficacy of the proposed technique.*

Keywords

Model order reduction, controllability Gramians, observability Gramians, error bound, balanced truncation

1. Introduction

The process of model order reduction (MOR) is to reduce a system from higher-order to its lower order for ease in simulation, analysis, and design of complex systems, filters, controller and circuits, antennas, sensor networks [1–4]. Balanced truncation (BT) [5] is a common and useful scheme to get a stable reduced-order model (ROM) for an original stable system. Moreover, this scheme also has an error bounds formula. However, it uses full frequency interim to get ROM. This encourages the introduction of the frequency weights to perform MOR. Enns [6] extended the work of BT's technique [5] to incorporate frequency weights. Enns [6] method may use single-sided (input/output) and double-sided weights. It gives stable ROM when using only single side weights, whereas, with double-sided weights, stability is not guaranteed. To overcome the problem of Enns [6], many other techniques are given in the literature [7–13].

In some applications, a particular frequency range is of interest. Wang & Zilouchian (WZ) [14] proposed a limited frequency technique for discrete-time systems without explicit weights. It can yield unstable ROM, and no *a priori* error bound formula exists.

To overcome the problem of WZ's [14], Ghafoor & Sreeram (GS) [15] proposed two methods to guarantee the stability of ROM by inducing some variation in input and output related matrices to ensure positive/semi-positive definiteness of some input and output related matrices. Later on, Imran & Ghafoor (IG) [16] proposed a technique to ensure positive/semi-positive definiteness of some input and output related matrices to get a stable ROM. The work in [15] and [16] guarantees the stability of ROMs and carry *a priori* error bounds; however, due to extensive variation in input and output related matrices, these methods produce large approximation error and error bound.

In this paper, new measures are proposed to ensure the positive/semi-positive definiteness of the input and the output related matrices by introducing some modifications to the input and output related matrices for discrete-frequency limited Gramians based model reduction. These modifications equally affect the negative eigenvalues, which minimize the variation in these matrices. The stability of ROMs are guaranteed and *a priori* error bound formula exists. Simulation results show that the proposed method not only ensures the stability of ROMs but also provides better approximation results as compared with other methods in the desired frequency interval, which shows the efficacy of the proposed scheme.

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Elementary operators	Terminology
$\mathbf{G}(z)$	$\mathbf{C}(z\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D} \Leftrightarrow$ state-space realization $\{\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}\}$
ROM $\mathbf{G}_{\mathrm{tr}}(z)$	$\mathbf{C}_{\mathrm{tr}}(z\mathbf{I}-\mathbf{A}_{\mathrm{tr}})^{-1}\mathbf{B}_{\mathrm{tr}}+\mathbf{D}_{\mathrm{tr}} \Leftrightarrow$ state-space realization $\{\mathbf{A}_{\mathrm{tr}},\mathbf{B}_{\mathrm{tr}},\mathbf{C}_{\mathrm{tr}},\mathbf{D}_{\mathrm{tr}}\}$
$\bar{\mathbf{P}}_{\mathrm{c}} > 0$	Positive definite controllability matrix
$\bar{\mathbf{P}}_{\mathrm{c}} \geq 0$	Positive semi-definite controllability matrix
$\bar{\mathbf{P}}_{\mathrm{c}}^{\mathrm{T}}$	Transpose of matrix $\bar{P}_{\mathrm{c}}^{\mathrm{T}}$
$\bar{\mathbf{P}}_{\mathrm{c}}^{*}$	Complex conjugate transpose of matrix $\bar{\mathbf{P}}_{\mathrm{c}}^{*}$
$\lvert\bar{\mathbf{P}}_{\mathrm{c}}\rvert$	Modulus of matrix $\lvert\bar{\mathbf{P}}_{\mathrm{c}}\rvert$
$\bar{\lambda}_i[\bar{\mathbf{P}}_{\mathrm{c}}]$	Eigenvalues of matrix $\lvert\bar{\mathbf{P}}_{\mathrm{c}}\rvert$
$\bar{\sigma}_i[\bar{\mathbf{P}}_{\mathrm{c}}]$	Singular-values of matrix $\lvert\bar{\mathbf{P}}_{\mathrm{c}}\rvert$
$p \mapsto \infty \lVert x\rVert_\infty := \max\lvert x_i\rvert$	Infinity- norm (maximum norm)
$\sum_{j=1}^{n}\bar{\sigma}_j$	$\bar{\sigma}_1+\bar{\sigma}_2+\bar{\sigma}_3+\ldots+\bar{\sigma}_n$

Tab. 1. Elementary operators and terminologies.

2. Preliminaries

2.1 Elementary Operators and Terminologies

Table 1 briefly summarizes some elementary operators and their terminologies used in this paper.

2.2 Balancing MOR Techniques

Consider a discrete-time system be given as:

$$\begin{aligned} \mathbf{x}[k+1] &= \mathbf{A}x[k]+\mathbf{B}u[k], && (1)\\ \mathbf{y}[k] &= \mathbf{C}x[k]+\mathbf{D}u[k], &&\\ \mathbf{G}(z) &= \mathbf{C}(z\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D} && (2)\end{aligned}$$

where $\{\mathbf{A}\in\mathcal{R}^{n\times n},\mathbf{B}\in\mathcal{R}^{n\times m},\mathbf{C}\in\mathcal{R}^{p\times n},\mathbf{D}\in\mathcal{R}^{p\times m}\}$ is its n^{th} order minimal realization with m number of inputs and p numbers of outputs. The ROM

$$\begin{aligned} \mathbf{x}_{\mathrm{tr}}[k+1] &= \mathbf{A}_{\mathrm{tr}}x_{\mathrm{tr}}[k]+\mathbf{B}_{\mathrm{tr}}u[k], && (3)\\ \mathbf{y}_{\mathrm{tr}}[k] &= \mathbf{C}_{\mathrm{tr}}x_{\mathrm{tr}}[k]+\mathbf{D}_{\mathrm{tr}}u[k], &&\\ \mathbf{G}_{\mathrm{tr}}(z) &= \mathbf{C}_{\mathrm{tr}}(z\mathbf{I}-\mathbf{A}_{\mathrm{tr}})^{-1}\mathbf{B}_{\mathrm{tr}}+\mathbf{D}_{\mathrm{tr}} && (4)\end{aligned}$$

is obtained by approximating the actual large-scale system (in the desired frequency interval $[\bar{\omega}_1,\bar{\omega}_2]$) where $\bar{\omega}_2>\bar{\omega}_1$, where $\{\mathbf{A}_{\mathrm{tr}}\in\mathcal{R}^{r\times r},\mathbf{B}_{\mathrm{tr}}\in\mathcal{R}^{r\times m},\mathbf{C}_{\mathrm{tr}}\in\mathcal{R}^{p\times r},\mathbf{D}_{\mathrm{tr}}\in\mathcal{R}^{p\times m}\}$ with $r<<n$. $\bar{\mathbf{P}}_{\mathrm{c}}$ and $\bar{\mathbf{Q}}_{\mathrm{o}}$ are controllability and observability Gramians, respectively

$$\begin{aligned} \bar{\mathbf{P}}_{\mathrm{c}} &= \tfrac{1}{2\pi}\textstyle\int_{-\pi}^{\pi}(\mathrm{e}^{\mathrm{j}\bar{\omega}}\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\mathbf{B}^{\mathrm{T}}(\mathrm{e}^{-\mathrm{j}\bar{\omega}}\mathbf{I}-\mathbf{A}^{\mathrm{T}})^{-1}\mathrm{d}\bar{\omega}, && (5)\\ \bar{\mathbf{Q}}_{\mathrm{o}} &= \tfrac{1}{2\pi}\textstyle\int_{-\pi}^{\pi}(\mathrm{e}^{-\mathrm{j}\bar{\omega}}\mathbf{I}-\mathbf{A}^{\mathrm{T}})^{-1}\mathbf{C}^{\mathrm{T}}\mathbf{C}(\mathrm{e}^{\mathrm{j}\bar{\omega}}\mathbf{I}-\mathbf{A})^{-1}\mathrm{d}\bar{\omega} && (6)\end{aligned}$$

which are the solution of the following Lyapunov equations:

$$\begin{aligned} \mathbf{A}\bar{\mathbf{P}}_{\mathrm{c}}\mathbf{A}^{\mathrm{T}}-\bar{\mathbf{P}}_{\mathrm{c}}+\mathbf{B}\mathbf{B}^{\mathrm{T}} &= 0, && (7)\\ \mathbf{A}^{\mathrm{T}}\bar{\mathbf{Q}}_{\mathrm{o}}\mathbf{A}-\bar{\mathbf{Q}}_{\mathrm{o}}+\mathbf{C}^{\mathrm{T}}\mathbf{C} &= 0. && (8)\end{aligned}$$

2.2.1 WZ's Technique [14]

The pioneer discrete-frequency limited approach presented by WZ [14] offers controllability and observability Gramians at desired discrete-frequency intervals, these Gramians $\bar{\mathbf{P}}_{\mathrm{WZ}}$ and $\bar{\mathbf{Q}}_{\mathrm{WZ}}$ for the discrete-time limited frequency interval, respectively, can be defined as

$$\begin{aligned} \bar{\mathbf{P}}_{\mathrm{WZ}} &= \bar{\mathbf{P}}_{\mathrm{c}}(\bar{\omega}_2)-\bar{\mathbf{P}}_{\mathrm{c}}(\bar{\omega}_1),\\ \bar{\mathbf{Q}}_{\mathrm{WZ}} &= \bar{\mathbf{Q}}_{\mathrm{o}}(\bar{\omega}_2)-\bar{\mathbf{Q}}_{\mathrm{o}}(\bar{\omega}_1)\end{aligned}$$

where $\bar{\mathbf{P}}_{\mathrm{WZ}}$ and $\bar{\mathbf{Q}}_{\mathrm{WZ}}$ be given as

$$\begin{aligned} \bar{\mathbf{P}}_{\mathrm{WZ}} &= \frac{1}{2\pi}\int_{\delta\bar{\omega}}(\mathrm{e}^{\mathrm{j}\bar{\omega}}\mathbf{I}-\mathbf{A})^{-1}\bar{\mathbf{B}}_{\mathrm{WZ}}\bar{\mathbf{B}}_{\mathrm{WZ}}^{\mathrm{T}}(\mathrm{e}^{-\mathrm{j}\bar{\omega}}\mathbf{I}-\mathbf{A}^{\mathrm{T}})^{-1}\mathrm{d}\bar{\omega},\\ \bar{\mathbf{Q}}_{\mathrm{WZ}} &= \frac{1}{2\pi}\int_{\delta\bar{\omega}}(\mathrm{e}^{-\mathrm{j}\bar{\omega}}-\mathbf{A}^{\mathrm{T}})^{-1}\bar{\mathbf{C}}_{\mathrm{WZ}}^{\mathrm{T}}\bar{\mathbf{C}}_{\mathrm{WZ}}(\mathrm{e}^{\mathrm{j}\bar{\omega}}\mathbf{I}-\mathbf{A})^{-1}\mathrm{d}\bar{\omega}\end{aligned}$$

where $\delta\bar{\omega}$ is the interval of integration $[\bar{\omega}_1,\bar{\omega}_2]$. These Gramians $\bar{\mathbf{P}}_{\mathrm{WZ}}$ and $\bar{\mathbf{Q}}_{\mathrm{WZ}}$ satisfy the following Lyapunov equations

$$\begin{aligned} \mathbf{A}\bar{\mathbf{P}}_{\mathrm{WZ}}\mathbf{A}^{\mathrm{T}}-\bar{\mathbf{P}}_{\mathrm{WZ}}+\bar{\mathbf{X}}_{\mathrm{WZ}} &= 0, && (9)\\ \mathbf{A}^{\mathrm{T}}\bar{\mathbf{Q}}_{\mathrm{WZ}}\mathbf{A}-\bar{\mathbf{Q}}_{\mathrm{WZ}}+\bar{\mathbf{Y}}_{\mathrm{WZ}} &= 0 && (10)\end{aligned}$$

where $\bar{\mathbf{X}}_{\mathrm{WZ}} = \left(\bar{\mathbf{F}}(\bar{\omega}_2)-\bar{\mathbf{F}}(\bar{\omega}_1)\right)\ \bar{\mathbf{B}}_{\mathrm{WZ}}\bar{\mathbf{B}}_{\mathrm{WZ}}^{\mathrm{T}}+\ \bar{\mathbf{B}}_{\mathrm{WZ}}\bar{\mathbf{B}}_{\mathrm{WZ}}^{\mathrm{T}}\left(\bar{\mathbf{F}}^{*}(\bar{\omega}_2)-\bar{\mathbf{F}}^{*}(\bar{\omega}_1)\right)$ and $\bar{\mathbf{Y}}_{\mathrm{WZ}} = \left(\bar{\mathbf{F}}(\bar{\omega}_2)-\bar{\mathbf{F}}(\bar{\omega}_1)\right)\bar{\mathbf{C}}_{\mathrm{WZ}}^{\mathrm{T}}\bar{\mathbf{C}}_{\mathrm{WZ}}+\bar{\mathbf{C}}_{\mathrm{WZ}}^{\mathrm{T}}\bar{\mathbf{C}}_{\mathrm{WZ}}\left(\bar{\mathbf{F}}^{*}(\bar{\omega}_2)-\bar{\mathbf{F}}^{*}(\bar{\omega}_1)\right)$ are the input and output related matrices in the limited discrete-frequency intervals, $\bar{\mathbf{F}}(\bar{\omega})=-\frac{\bar{\omega}_2-\bar{\omega}_1}{4\pi}\mathbf{I}+\frac{1}{2\pi}\int_{\delta\bar{\omega}}(\mathrm{e}^{\mathrm{j}\bar{\omega}}\mathbf{I}-\mathbf{A})^{-1}\mathrm{d}\bar{\omega}$ and $\bar{\mathbf{F}}^{*}(\bar{\omega})$ is the conjugate transpose of $\bar{\mathbf{F}}(\bar{\omega})$. By eigenvalues decomposition of $\bar{\mathbf{X}}_{\mathrm{WZ}}$ and $\bar{\mathbf{Y}}_{\mathrm{WZ}}$ we have following

$$\bar{\mathbf{X}}_{\mathrm{WZ}} = \bar{\mathbf{U}}_{\mathrm{WZ}} \begin{bmatrix} \bar{\mathbf{S}}_{\mathrm{WZ}_1} & 0 \\ 0 & \bar{\mathbf{S}}_{\mathrm{WZ}_2} \end{bmatrix} \bar{\mathbf{U}}_{\mathrm{WZ}}^{\mathrm{T}}, \quad (11)$$

$$\bar{\mathbf{B}}_{\mathrm{WZ}} = \bar{\mathbf{U}}_{\mathrm{WZ}} \begin{bmatrix} \bar{\mathbf{S}}_{\mathrm{WZ}_1}^{1/2} & 0 \\ 0 & \bar{\mathbf{S}}_{\mathrm{WZ}_2}^{1/2} \end{bmatrix} = \bar{\mathbf{U}}_{\mathrm{WZ}}\bar{\mathbf{S}}_{\mathrm{WZ}}^{1/2}, \quad (12)$$

$$\bar{\mathbf{Y}}_{\mathrm{WZ}} = \bar{\mathbf{V}}_{\mathrm{WZ}} \begin{bmatrix} \bar{\mathbf{R}}_{\mathrm{WZ}_1} & 0 \\ 0 & \bar{\mathbf{R}}_{\mathrm{WZ}_2} \end{bmatrix} \bar{\mathbf{V}}_{\mathrm{WZ}}^{\mathrm{T}}, \quad (13)$$

$$\bar{\mathbf{C}}_{\mathrm{WZ}} = \begin{bmatrix} \bar{\mathbf{R}}_{\mathrm{WZ}_1}^{1/2} & 0 \\ 0 & \bar{\mathbf{R}}_{\mathrm{WZ}_2}^{1/2} \end{bmatrix} \bar{\mathbf{V}}_{\mathrm{WZ}}^{\mathrm{T}} = \bar{\mathbf{R}}_{\mathrm{WZ}}^{1/2}\bar{\mathbf{V}}_{\mathrm{WZ}}^{\mathrm{T}} \quad (14)$$

where

$$\bar{\mathbf{S}}_{\mathrm{WZ}_1} = \begin{bmatrix} \bar{s}_1 & 0 & \cdots & 0 \\ 0 & \bar{s}_2 & \cdots & 0 \\ 0 & 0 & \cdots & \bar{s}_{q-1} \end{bmatrix}, \ \bar{\mathbf{S}}_{\mathrm{WZ}_2} = \begin{bmatrix} \bar{s}_q & 0 & \cdots & 0 \\ 0 & \bar{s}_{q+1} & \cdots & 0 \\ 0 & 0 & \cdots & \bar{s}_n \end{bmatrix},$$

$$\bar{\mathbf{R}}_{\mathrm{WZ}_1} = \begin{bmatrix} \bar{r}_1 & 0 & \cdots & 0 \\ 0 & \bar{r}_2 & \cdots & 0 \\ 0 & 0 & \cdots & \bar{r}_{k-1} \end{bmatrix}, \ \bar{\mathbf{R}}_{\mathrm{WZ}_2} = \begin{bmatrix} \bar{r}_k & 0 & \cdots & 0 \\ 0 & \bar{r}_{k+1} & \cdots & 0 \\ 0 & 0 & \cdots & \bar{r}_n \end{bmatrix}$$

$\bar{\mathbf{X}}_{\mathrm{WZ}}$ and $\bar{\mathbf{Y}}_{\mathrm{WZ}}$ contain $(q-1)$ and $(k-1)$ number of positive eigenvalues, respectively. Let a transformation matrix $\bar{\mathbf{T}}_{\mathrm{WZ}}$ is obtained as:

$$\bar{\mathbf{T}}_{\mathrm{WZ}}^{\mathrm{T}}\bar{\mathbf{Q}}_{\mathrm{WZ}}\bar{\mathbf{T}}_{\mathrm{WZ}} = \bar{\mathbf{T}}_{\mathrm{WZ}}^{-1}\bar{\mathbf{P}}_{\mathrm{WZ}}\bar{\mathbf{T}}_{\mathrm{WZ}}^{-\mathrm{T}} = \mathrm{diag}\{\bar{\sigma}_1, \bar{\sigma}_2, \ldots, \bar{\sigma}_n\} \quad (15)$$

and the ROM $\mathbf{G}_{\mathrm{tr}}(z) = \bar{\mathbf{C}}_{\mathrm{tr}}(z\mathbf{I} - \bar{\mathbf{A}}_{\mathrm{tr}})^{-1}\bar{\mathbf{B}}_{\mathrm{tr}} + \bar{\mathbf{D}}_{\mathrm{tr}}$ obtained is

$$\bar{\mathbf{T}}_{\mathrm{WZ}}^{-1}\mathbf{A}\bar{\mathbf{T}}_{\mathrm{WZ}} = \bar{\mathbf{A}} = \begin{bmatrix} \bar{\mathbf{A}}_{\mathrm{tr}} & \bar{\mathbf{A}}_{12} \\ \bar{\mathbf{A}}_{21} & \bar{\mathbf{A}}_{22} \end{bmatrix}, \bar{\mathbf{T}}_{\mathrm{WZ}}^{-1}\mathbf{B} = \bar{\mathbf{B}} = \begin{bmatrix} \bar{\mathbf{B}}_{\mathrm{tr}} \\ \bar{\mathbf{B}}_2 \end{bmatrix} \quad (16)$$

$$\mathbf{C}\bar{\mathbf{T}}_{\mathrm{WZ}} = \bar{\mathbf{C}} = \begin{bmatrix} \mathbf{C}_{\mathrm{tr}} & \bar{\mathbf{C}}_2 \end{bmatrix}, \quad \mathbf{D} = \bar{\mathbf{D}}_{\mathrm{tr}} \quad (17)$$

where $\bar{\sigma}_j \geq \bar{\sigma}_{j+1}$, $j = 1, 2, 3, \ldots, n-1$, $\bar{\sigma}_r > \bar{\sigma}_{r+1}$ where r is the order of the ROM and $\bar{\mathbf{T}}_{\mathrm{WZ}}$ is a similarity transformation matrix used to obtain the transformed system. The ROM can be acquired by truncating the transformed system.

Remark 1 For the discrete frequency-range $[\bar{\omega}_1, \bar{\omega}_2] = [-\pi, \pi]$, $lim_{\bar{\omega}_{\mapsto [-\pi,\pi]}} \bar{\mathbf{P}}_{\mathrm{c}}[\bar{\omega}] = \bar{\mathbf{P}}_{\mathrm{c}} = \bar{\mathbf{P}}_{\mathrm{WZ}}$, $lim_{\bar{\omega}_{\mapsto [-\pi,\pi]}} \bar{\mathbf{Q}}_{\mathrm{o}}[\bar{\omega}] = \bar{\mathbf{Q}}_{\mathrm{o}} = \bar{\mathbf{Q}}_{\mathrm{WZ}}$, where $\bar{\mathbf{P}}_{\mathrm{c}}[\bar{\omega}]$ and $\bar{\mathbf{Q}}_{\mathrm{o}}[\bar{\omega}]$ are obtained using Parseval's relationship [17], the ROMs obtained using the WZ [14] and the BT [5] are same.

Remark 2 WZ [14] failed to make sure the stability of ROMs because input/output associated matrices $\bar{\mathbf{X}}_{\mathrm{WZ}}$ and $\bar{\mathbf{Y}}_{\mathrm{WZ}}$ may no longer be positive definite or semi-definite [15].

2.2.2 Existing Stability Preserving Frequency Limited Techniques

To overcome the main drawback of WZ [14], GS [15], and IG [16] proposed stability preserving MOR approaches. However, these techniques produce large approximation error and extensive *a priori* error bound formula due to large variation in input and output related matrices. Let the controllability $\bar{\mathbf{P}}_{\mathrm{E}}$ and observability $\bar{\mathbf{Q}}_{\mathrm{E}}$ Gramians, respectively, satisfy

$$\mathbf{A}\bar{\mathbf{P}}_{\mathrm{E}}\mathbf{A}^{\mathrm{T}} - \bar{\mathbf{P}}_{\mathrm{E}} + \bar{\mathbf{B}}_{\mathrm{E}}\bar{\mathbf{B}}_{\mathrm{E}}^{\mathrm{T}} = 0, \quad (18)$$

$$\mathbf{A}^{\mathrm{T}}\bar{\mathbf{Q}}_{\mathrm{E}}\mathbf{A} - \bar{\mathbf{Q}}_{\mathrm{E}} + \bar{\mathbf{C}}_{\mathrm{E}}^{\mathrm{T}}\bar{\mathbf{C}}_{\mathrm{E}} = 0 \quad (19)$$

where $\bar{\mathbf{B}}_{\mathrm{E}} \in \{\bar{\mathbf{B}}_{\mathrm{E}_1}$ [15], $\bar{\mathbf{B}}_{\mathrm{E}_2}$ [15], $\bar{\mathbf{B}}_{\mathrm{E}_3}$ [16]$\}$ and $\bar{\mathbf{C}}_{\mathrm{E}} \in \{\bar{\mathbf{C}}_{\mathrm{E}_1}$ [15], $\bar{\mathbf{C}}_{\mathrm{E}_2}$ [15], $\bar{\mathbf{C}}_{\mathrm{E}_3}$ [16]$\}$.

$$\bar{\mathbf{B}}_{\mathrm{E}_1} = \bar{\mathbf{U}}_{\mathrm{E}_1} \begin{bmatrix} \bar{\mathbf{S}}_{\mathrm{WZ}_1}^{1/2} & 0 \\ 0 & |\bar{\mathbf{S}}_{\mathrm{WZ}_2}|^{1/2} \end{bmatrix},$$

$$\bar{\mathbf{B}}_{\mathrm{E}_2} = \bar{\mathbf{U}}_{\mathrm{E}_2} \begin{bmatrix} \bar{\mathbf{S}}_{\mathrm{WZ}_1}^{1/2} & 0 \\ 0 & 0 \end{bmatrix},$$

$$\bar{\mathbf{B}}_{\mathrm{E}_3} = \begin{cases} \bar{\mathbf{U}}_{\mathrm{E}_3}(\bar{\mathbf{S}}_{\mathrm{WZ}} - \bar{s}_n I)^{1/2} & \text{for } \bar{s}_n < 0, \\ \bar{\mathbf{U}}_{\mathrm{E}_3}\bar{\mathbf{S}}_{\mathrm{WZ}}^{1/2} & \text{for } \bar{s}_n \geq 0, \end{cases}$$

$$\bar{\mathbf{C}}_{\mathrm{E}_1} = \begin{bmatrix} \bar{\mathbf{R}}_{\mathrm{WZ}_1}^{1/2} & 0 \\ 0 & |\bar{\mathbf{R}}_{\mathrm{WZ}_2}|^{1/2} \end{bmatrix} \bar{\mathbf{V}}_{\mathrm{E}_1}^{\mathrm{T}},$$

$$\bar{\mathbf{C}}_{\mathrm{E}_2} = \begin{bmatrix} \bar{\mathbf{R}}_{\mathrm{WZ}_1}^{1/2} & 0 \\ 0 & 0 \end{bmatrix} \bar{\mathbf{V}}_{\mathrm{E}_2}^{\mathrm{T}},$$

$$\bar{\mathbf{C}}_{\mathrm{E}_3} = \begin{cases} (\bar{\mathbf{R}}_{\mathrm{WZ}} - \bar{r}_n\mathbf{I})^{1/2}\bar{\mathbf{V}}_{\mathrm{E}_3}^{\mathrm{T}} & \text{for } \bar{r}_n < 0, \\ \bar{\mathbf{R}}_{\mathrm{WZ}}^{1/2}\bar{\mathbf{V}}_{\mathrm{E}_3}^{\mathrm{T}} & \text{for } \bar{r}_n \geq 0. \end{cases}$$

Let a transformation matrix $\bar{\mathbf{T}}_{\mathrm{E}}$ is obtained as:

$$\bar{\mathbf{T}}_{\mathrm{E}}^{\mathrm{T}}\bar{\mathbf{Q}}_{\mathrm{E}}\bar{\mathbf{T}}_{\mathrm{E}} = \bar{\mathbf{T}}_{\mathrm{E}}^{-1}\bar{\mathbf{P}}_{\mathrm{E}}\bar{\mathbf{T}}_{\mathrm{E}}^{-\mathrm{T}} = \begin{bmatrix} \bar{\sigma}_1 & 0 & \cdots & 0 \\ 0 & \bar{\sigma}_2 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & \bar{\sigma}_n \end{bmatrix}$$

and the ROM $\mathbf{G}_{\mathrm{tr}}(z) = \bar{\mathbf{C}}_{\mathrm{tr}}(z\mathbf{I} - \bar{\mathbf{A}}_{\mathrm{tr}})^{-1}\bar{\mathbf{B}}_{\mathrm{tr}} + \bar{\mathbf{D}}_{\mathrm{tr}}$ obtained is

$$\bar{\mathbf{T}}_{\mathrm{E}}^{-1}\mathbf{A}\bar{\mathbf{T}}_{\mathrm{E}} = \bar{\mathbf{A}} = \begin{bmatrix} \bar{\mathbf{A}}_{\mathrm{tr}} & \bar{\mathbf{A}}_{12} \\ \bar{\mathbf{A}}_{21} & \bar{\mathbf{A}}_{22} \end{bmatrix}, \bar{\mathbf{T}}_{\mathrm{E}}^{-1}\mathbf{B} = \bar{\mathbf{B}} = \begin{bmatrix} \bar{\mathbf{B}}_{\mathrm{tr}} \\ \bar{\mathbf{B}}_2 \end{bmatrix}$$

$$\mathbf{C}\bar{\mathbf{T}}_{\mathrm{E}} = \bar{\mathbf{C}} = \begin{bmatrix} \mathbf{C}_{\mathrm{tr}} & \bar{\mathbf{C}}_2 \end{bmatrix}, \quad \mathbf{D} = \bar{\mathbf{D}}_{\mathrm{tr}}$$

where $\bar{\sigma}_j \geq \bar{\sigma}_{j+1}$, $j = 1, 2, 3, \ldots, n-1$, $\bar{\sigma}_r > \bar{\sigma}_{r+1}$ where r is the order of the ROM and $\bar{\mathbf{T}}_{\mathrm{E}}$ is a similarity transformation matrix used to obtain the transformed system. The ROM can be acquired by truncating the transformed system.

Remark 3 Since $\bar{\mathbf{X}}_{\mathrm{WZ}} \leq \bar{\mathbf{B}}_{\mathrm{E}}\bar{\mathbf{B}}_{\mathrm{E}}^{\mathrm{T}} \geq 0$, $\bar{\mathbf{Y}}_{\mathrm{WZ}} \leq \bar{\mathbf{C}}_{\mathrm{E}}^{\mathrm{T}}\bar{\mathbf{C}}_{\mathrm{E}} \geq 0$, $\bar{\mathbf{P}}_{\mathrm{E}} > 0$ and $\bar{\mathbf{Q}}_{\mathrm{E}} > 0$. Therefore, the realization $(\mathbf{A}, \bar{\mathbf{B}}_{\mathrm{E}}, \bar{\mathbf{C}}_{\mathrm{E}})$ is minimal and the stability of the ROM is guaranteed.

Remark 4 Since for each input related matrix $\bar{\mathbf{B}}_{\mathrm{E}} \in \{\bar{\mathbf{B}}_{\mathrm{E}_1}$ [15], $\bar{\mathbf{B}}_{\mathrm{E}_2}$ [15], $\bar{\mathbf{B}}_{\mathrm{E}_3}$ [16]$\}$ and for each output related matrix $\bar{\mathbf{B}}_{C} \in \{\bar{\mathbf{C}}_{\mathrm{E}_1}$ [15], $\bar{\mathbf{C}}_{\mathrm{E}_2}$ [15], $\bar{\mathbf{C}}_{\mathrm{E}_3}$ [16]$\}$ ensure positive/semi-positive definiteness of input and output related matrices, which results positive/semi-positive definiteness of $\bar{\mathbf{P}}_{\mathrm{E}} \in \{\bar{\mathbf{P}}_{\mathrm{E}_1}$ [15], $\bar{\mathbf{P}}_{\mathrm{E}_2}$ [15], $\bar{\mathbf{P}}_{\mathrm{E}_3}$ [16]$\}$ and $\bar{\mathbf{Q}}_{\mathrm{E}} \in \{\bar{\mathbf{Q}}_{\mathrm{E}_1}$ [15], $\bar{\mathbf{Q}}_{\mathrm{E}_2}$ [15], $\bar{\mathbf{Q}}_{\mathrm{E}_3}$ [16]$\}$ in a unique way. This leads to different transformation matrix $\bar{\mathbf{T}}_{\mathrm{E}} \in \{\bar{\mathbf{T}}_{\mathrm{E}_1}$ [15], $\bar{\mathbf{T}}_{\mathrm{E}_2}$ [15], $\bar{\mathbf{T}}_{\mathrm{E}_3}$ [16]$\}$, which subsequently results in three existing stability preserving model order reduction techniques.

3. Main Results

The existing stability preserving techniques GS [15] and IG [16] introduced some modification in the input and the output related matrices $\bar{\mathbf{X}}_{\mathrm{WZ}}$ and $\bar{\mathbf{Y}}_{\mathrm{WZ}}$ respectively to ensure positive/semi-positive definiteness. In 1-st algorithm of GS [15] modifications are done by taking absolute of eigenvalues of the input and the output related matrices; consequently, positive/semi-positive definiteness of the input and the output related matrices which ensure the stability of ROMs; however, by taking the absolute it may cause an unequal effect on eigenvalues which results in a large variation in some eigenvalues and small variation in some eigenvalues which leads to producing a large approximation error. In 2-nd algorithm of GS [15] stability is ensured by truncating negative eigenvalues, which result in loss of information of negative eigenvalues that leads to a large approximation error, whereas, in IG [16] the stability is ensured by subtracting all the eigenvalues with least eigenvalues which made the last eigenvalue zero that leads to large approximation error. Keeping the above in view, all methods produce large variations in input/output related matrices, which may affect the input/output properties of the original system.

In this paper, new measures are proposed for the input/output related matrices by introducing the concept of the norm of the negative part of the eigenvalues of input/output related matrices without affecting the positive part of the eigenvalues. These modifications equally affect these negative eigenvalues which minimize the variation in the input and the output related matrices, consequently, ensure the positive/semi-positive definiteness the of input and the output related matrices that yield a stable ROM and produces low-frequency response error along with low error bound with minimum variation.

Let the new virtual/fictitious controllability $\bar{\mathbf{P}}_{\mathrm{SB}}$ and observability $\bar{\mathbf{Q}}_{\mathrm{SB}}$ Gramians are computed as

$$\mathbf{A}\bar{\mathbf{P}}_{\mathrm{SB}}\mathbf{A}^{\mathrm{T}} - \bar{\mathbf{P}}_{\mathrm{SB}} + \bar{\mathbf{X}}_{\mathrm{SB}} = 0, \tag{20}$$

$$\mathbf{A}^{\mathrm{T}}\bar{\mathbf{Q}}_{\mathrm{SB}}\mathbf{A} - \bar{\mathbf{Q}}_{\mathrm{SB}} + \bar{\mathbf{Y}}_{\mathrm{SB}} = 0. \tag{21}$$

where $\bar{\mathbf{X}}_{\mathrm{SB}} = \bar{\mathbf{B}}_{\mathrm{SB}}\bar{\mathbf{B}}_{\mathrm{SB}}^{\mathrm{T}}$ and $\bar{\mathbf{Y}}_{\mathrm{SB}} = \bar{\mathbf{C}}_{\mathrm{SB}}^{\mathrm{T}}\bar{\mathbf{C}}_{\mathrm{SB}}$. By eigenvalues decomposition of $\bar{\mathbf{X}}_{\mathrm{SB}}$ and $\bar{\mathbf{Y}}_{\mathrm{SB}}$ we have

$$\bar{\mathbf{X}}_{\mathrm{SB}} = \bar{\mathbf{U}}_{\mathrm{SB}}\bar{\mathbf{S}}_{\mathrm{SB}}\bar{\mathbf{U}}_{\mathrm{SB}}^{\mathrm{T}}, \tag{22}$$

$$\bar{\mathbf{Y}}_{\mathrm{SB}} = \bar{\mathbf{V}}_{\mathrm{SB}}\bar{\mathbf{R}}_{\mathrm{SB}}\bar{\mathbf{V}}_{\mathrm{SB}}^{\mathrm{T}}. \tag{23}$$

The new virtual/fictitious input and output related matrices respectively are given as $\bar{\mathbf{B}}_{\mathrm{SB}}$ and $\bar{\mathbf{C}}_{\mathrm{SB}}$ where

$$\bar{\mathbf{B}}_{\mathrm{SB}} = \bar{\mathbf{U}}_{\mathrm{SB}}\begin{bmatrix} \bar{\mathbf{S}}_{\mathrm{SB}_1}^{1/2} & 0 \\ 0 & \bar{\mathbf{S}}_{\mathrm{SB}_2}^{1/2} \end{bmatrix} = \bar{\mathbf{U}}_{\mathrm{SB}}\bar{\mathbf{S}}_{\mathrm{SB}}^{1/2}, \tag{24}$$

$$\bar{\mathbf{C}}_{\mathrm{SB}} = \begin{bmatrix} \bar{\mathbf{R}}_{\mathrm{SB}_1}^{1/2} & 0 \\ 0 & \bar{\mathbf{R}}_{\mathrm{SB}_2}^{1/2} \end{bmatrix}\bar{\mathbf{V}}_{\mathrm{SB}}^{\mathrm{T}} = \bar{\mathbf{R}}_{\mathrm{SB}}^{1/2}\bar{\mathbf{V}}_{\mathrm{SB}}^{\mathrm{T}} \tag{25}$$

where $\bar{\mathbf{S}}_{\mathrm{SB}_1} = \bar{\mathbf{S}}_{\mathrm{WZ}_1}$, $\bar{\mathbf{R}}_{\mathrm{SB}_1} = \bar{\mathbf{R}}_{\mathrm{WZ}_1}$,

$$\bar{\mathbf{S}}_{\mathrm{SB}_2} = \begin{bmatrix} \hat{s}_q & 0 & \cdots & 0 \\ 0 & \hat{s}_{q+1} & \cdots & 0 \\ 0 & 0 & \cdots & \hat{s}_n \end{bmatrix}, \quad \bar{\mathbf{R}}_{\mathrm{SB}_2} = \begin{bmatrix} \hat{r}_k & 0 & \cdots & 0 \\ 0 & \hat{r}_{k+1} & \cdots & 0 \\ 0 & 0 & \cdots & \hat{r}_n \end{bmatrix},$$

$$\begin{aligned} \hat{s}_{q+l} &= \|(\bar{s}_{q+l}, \hat{s})\|_{(n-q)} = (|\bar{s}_{q+l}|^{(n-q)} + |\hat{s}|^{(n-q)})^{1/(n-q)}, \\ \hat{s}_n &= \|(\bar{s}_n, \hat{s})\|_{(n-q)} = (|\bar{s}_n|^{(n-q)} + |\hat{s}|^{(n-q)})^{1/(n-q)}, \\ \hat{r}_{k+h} &= \|(\bar{r}_{k+h}, \hat{r})\|_{(n-k)} = (|\bar{r}_{k+h}|^{(n-k)} + |\hat{r}|^{(n-k)})^{1/(n-k)}, \\ \hat{r}_n &= \|(\bar{r}_n, \hat{r})\|_{(n-k)} = (|\bar{r}_n|^{(n-k)} + |\hat{r}|^{(n-k)})^{1/(n-k)} \end{aligned}$$

for $l = 0, 1, ..., n-q-1$, and for $h = 0, 1, ..., n-k-1$. $\bar{\mathbf{S}}_{\mathrm{SB}_2}$ and $\bar{\mathbf{R}}_{\mathrm{SB}_2}$ contain $(n-q)$ and $(n-k)$ number of modified positive eigenvalues, respectively, where $\hat{s} = \sum_{i=q}^{n} \bar{s}_i$ and $\hat{r} = \sum_{i=k}^{n} \bar{r}_i$. Let the similarity transformation matrix $\bar{\mathbf{T}}_{\mathrm{SB}}$ is calculated as

$$\bar{\mathbf{T}}_{\mathrm{SB}}^{\mathrm{T}}\bar{\mathbf{Q}}_{\mathrm{SB}}\bar{\mathbf{T}}_{\mathrm{SB}} = \bar{\mathbf{T}}_{\mathrm{SB}}^{-1}\bar{\mathbf{P}}_{\mathrm{SB}}\bar{\mathbf{T}}_{\mathrm{SB}}^{-\mathrm{T}} = \mathrm{diag}\{\bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_3 \ldots, \bar{\sigma}_n\}$$

where $\bar{\sigma}_j \geq \bar{\sigma}_{j+1}$ and $\bar{\sigma}_r \geq \bar{\sigma}_{r+1}$. The ROM $\mathbf{G}_{\mathrm{tr}}(z) = \hat{\mathbf{C}}_{\mathrm{tr}}(z\mathbf{I} - \hat{\mathbf{A}}_{\mathrm{tr}})^{-1}\hat{\mathbf{B}}_{\mathrm{tr}} + \hat{\mathbf{D}}_{\mathrm{tr}}$ is obtained as

$$\bar{\mathbf{T}}_{\mathrm{SB}}^{-1}\mathbf{A}\bar{\mathbf{T}}_{\mathrm{SB}} = \hat{\mathbf{A}} = \begin{bmatrix} \hat{\mathbf{A}}_{\mathrm{tr}} & \hat{\mathbf{A}}_{12} \\ \hat{\mathbf{A}}_{21} & \hat{\mathbf{A}}_{22} \end{bmatrix}, \bar{\mathbf{T}}_{\mathrm{SB}}^{-1}\mathbf{B} = \hat{\mathbf{B}} = \begin{bmatrix} \hat{\mathbf{B}}_{\mathrm{tr}} \\ \hat{\mathbf{B}}_2, \end{bmatrix} \tag{26}$$

$$\mathbf{C}\bar{\mathbf{T}}_{\mathrm{SB}} = \hat{\mathbf{C}} = \begin{bmatrix} \hat{\mathbf{C}}_{\mathrm{tr}} & \hat{\mathbf{C}}_2 \end{bmatrix}, \qquad \mathbf{D} = \hat{\mathbf{D}}_{\mathrm{tr}}. \tag{27}$$

Remark 5 Since $\bar{\mathbf{X}}_{\mathrm{WZ}} \leq \bar{\mathbf{B}}_{\mathrm{SB}}\bar{\mathbf{B}}_{\mathrm{SB}}^{\mathrm{T}} \geq 0$, $\bar{\mathbf{Y}}_{\mathrm{WZ}} \leq \bar{\mathbf{C}}_{\mathrm{SB}}^{\mathrm{T}}\bar{\mathbf{C}}_{\mathrm{SB}} \geq 0$, $\bar{\mathbf{P}}_{\mathrm{SB}} > 0$ and $\bar{\mathbf{Q}}_{\mathrm{SB}} > 0$. Therefore, the realization $(\mathbf{A}, \bar{\mathbf{B}}_{\mathrm{SB}}, \bar{\mathbf{C}}_{\mathrm{SB}})$ is minimal and the stability of the ROM is guaranteed.

Theorem 1 Let $\mathrm{rank}\begin{bmatrix}\bar{\mathbf{B}}_{\mathrm{SB}} & \mathbf{B}\end{bmatrix} = \mathrm{rank}\begin{bmatrix}\bar{\mathbf{B}}_{\mathrm{SB}}\end{bmatrix}$ and $\mathrm{rank}\begin{bmatrix}\bar{\mathbf{C}}_{\mathrm{SB}} \\ \mathbf{C}\end{bmatrix} = \mathrm{rank}\begin{bmatrix}\bar{\mathbf{C}}_{\mathrm{SB}}\end{bmatrix}$ following error bound holds

$$\|\mathbf{G}(z) - \mathbf{G}_{\mathrm{tr}}(z)\|_\infty \leq 2\|\bar{\mathbf{L}}_{\mathrm{SB}}\|\|\bar{\mathbf{K}}_{\mathrm{SB}}\| \sum_{j=r+1}^{n} \bar{\sigma}_j$$

where

$$\bar{\mathbf{L}}_{\mathrm{SB}} = \begin{cases} \mathbf{C}\bar{\mathbf{V}}_{\mathrm{SB}}\bar{\mathbf{R}}_{\mathrm{SB}}^{-1/2} & \text{if } \bar{\mathbf{R}}_{\mathrm{WZ}_2} \text{ exists,} \\ \mathbf{C}\bar{\mathbf{V}}_{\mathrm{WZ}}\bar{\mathbf{R}}_{\mathrm{WZ}}^{-1/2} & \text{otherwise,} \end{cases}$$

$$\bar{\mathbf{K}}_{\mathrm{SB}} = \begin{cases} \bar{\mathbf{S}}_{\mathrm{SB}}^{-1/2}\bar{\mathbf{U}}_{\mathrm{SB}}^{\mathrm{T}}\mathbf{B} & \text{if } \bar{\mathbf{S}}_{\mathrm{WZ}_2} \text{ exists,} \\ \bar{\mathbf{S}}_{\mathrm{WZ}}^{-1/2}\bar{\mathbf{U}}_{\mathrm{WZ}}^{\mathrm{T}}\mathbf{B} & \text{otherwise.} \end{cases}$$

Proof: The proof follow similar to [16]; however, it is presented here for completeness. Since $\mathrm{rank}\begin{bmatrix}\bar{\mathbf{B}}_{\mathrm{SB}} & \mathbf{B}\end{bmatrix} = \mathrm{rank}\begin{bmatrix}\bar{\mathbf{B}}_{\mathrm{SB}}\end{bmatrix}$ and $\mathrm{rank}\begin{bmatrix}\bar{\mathbf{C}}_{\mathrm{SB}} \\ \mathbf{C}\end{bmatrix} = \mathrm{rank}\begin{bmatrix}\bar{\mathbf{C}}_{\mathrm{SB}}\end{bmatrix}$, the relationships $\mathbf{B} = \bar{\mathbf{B}}_{\mathrm{SB}}\bar{\mathbf{K}}_{\mathrm{SB}}$ and $\mathbf{C} = \bar{\mathbf{L}}_{\mathrm{SB}}\bar{\mathbf{C}}_{\mathrm{SB}}$ hold. By partitioning $\bar{\mathbf{B}}_{\mathrm{SB}} = \begin{bmatrix}\bar{\mathbf{B}}_{\mathrm{SB}_1} \\ \bar{\mathbf{B}}_{\mathrm{SB}_2}\end{bmatrix}$, $\bar{\mathbf{C}}_{\mathrm{SB}} = \begin{bmatrix}\bar{\mathbf{C}}_{\mathrm{SB}_1} & \bar{\mathbf{C}}_{\mathrm{SB}_2}\end{bmatrix}$ and substituting

$\hat{\mathbf{B}}_{\rm tr} = \bar{\mathbf{B}}_{\rm SB_1}\bar{\mathbf{K}}_{\rm SB}$, $\hat{\mathbf{C}}_{\rm tr} = \bar{\mathbf{L}}_{\rm SB}\bar{\mathbf{C}}_{\rm SB_1}$, respectively, yields

$$\begin{aligned}\|\mathbf{G}(z)-\mathbf{G}_{\rm tr}(z)\|_\infty &= \|\mathbf{C}(z\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}-\mathbf{C}_{\rm tr}(z\mathbf{I}-\hat{\mathbf{A}}_{\rm tr})^{-1}\hat{\mathbf{B}}_{\rm tr}\|_\infty\\ &= \|\bar{\mathbf{L}}_{\rm SB}\bar{\mathbf{C}}_{\rm SB}(z\mathbf{I}-\mathbf{A})^{-1}\bar{\mathbf{B}}_{\rm SB}\bar{\mathbf{K}}_{\rm SB}\\ &\quad -\bar{\mathbf{L}}_{\rm SB}\bar{\mathbf{C}}_{\rm SB_1}(z\mathbf{I}-\hat{\mathbf{A}}_{\rm tr})^{-1}\bar{\mathbf{B}}_{\rm SB_1}\bar{\mathbf{K}}_{\rm SB}\|_\infty\\ &= \|\bar{\mathbf{L}}_{\rm SB}(\bar{\mathbf{C}}_{\rm SB}(z\mathbf{I}-\mathbf{A})^{-1}\bar{\mathbf{B}}_{\rm SB}\\ &\quad -\bar{\mathbf{C}}_{\rm SB_1}(z\mathbf{I}-\hat{\mathbf{A}}_{\rm tr})^{-1}\bar{\mathbf{B}}_{\rm SB_1})\bar{\mathbf{K}}_{\rm SB}\|_\infty\\ &\leq \|\bar{\mathbf{L}}_{\rm SB}\|\|(\bar{\mathbf{C}}_{\rm SB}(z\mathbf{I}-\mathbf{A})^{-1}\bar{\mathbf{B}}_{\rm SB}\\ &\quad -\bar{\mathbf{C}}_{\rm SB_1}(z\mathbf{I}-\hat{\mathbf{A}}_{\rm tr})^{-1}\bar{\mathbf{B}}_{\rm SB_1})\|_\infty\|\bar{\mathbf{K}}_{\rm SB}\|.\end{aligned}$$

If $\{\hat{\mathbf{A}}_{\rm tr}, \bar{\mathbf{B}}_{\rm SB_1}, \bar{\mathbf{C}}_{\rm SB_1}\}$ is the model obtained after reduction of the original system $\{\mathbf{A}, \bar{\mathbf{B}}_{\rm SB}, \bar{\mathbf{C}}_{\rm SB}\}$.

$$\|(\bar{\mathbf{C}}_{\rm SB}(z\mathbf{I}-\mathbf{A})^{-1}\bar{\mathbf{B}}_{\rm SB}-\bar{\mathbf{C}}_{\rm SB_1}(z\mathbf{I}-\hat{\mathbf{A}}_{\rm tr})^{-1}\bar{\mathbf{B}}_{\rm SB_1})\|_\infty \leq 2\sum_{j=r+1}^{n}\bar{\sigma}_j.$$

Therefore, $\|\mathbf{G}(z)-\mathbf{G}_{\rm tr}(z)\|_\infty \leq 2\|\bar{\mathbf{L}}_{\rm SB}\|\|\bar{\mathbf{K}}_{\rm SB}\|\sum_{j=r+1}^{n}\bar{\sigma}_j$.

Corollary 1 Theorem 1 holds true subject to the following rank conditions rank $\begin{bmatrix}\bar{\mathbf{B}}_{\rm SB} & \bar{\mathbf{B}}\end{bmatrix}$ = rank $\begin{bmatrix}\bar{\mathbf{B}}_{\rm SB}\end{bmatrix}$ and rank $\begin{bmatrix}\bar{\mathbf{C}}_{\rm SB}\\ \bar{\mathbf{C}}\end{bmatrix}$ = rank $\begin{bmatrix}\bar{\mathbf{C}}_{\rm SB}\end{bmatrix}$ (which follows from [16]) are satisfied.

Remark 6 The rank condition follows from [12].

Remark 7 When input and output related matrices $\bar{\mathbf{X}}_{\rm WZ} \geq 0$ and $\bar{\mathbf{Y}}_{\rm WZ} \geq 0$, respectively, then $\bar{\mathbf{P}}_{\rm WZ} = \bar{\mathbf{P}}_{\rm E} = \bar{\mathbf{P}}_{\rm SB}$ and $\bar{\mathbf{Q}}_{\rm WZ} = \bar{\mathbf{Q}}_{\rm E} = \bar{\mathbf{Q}}_{\rm SB}$. Otherwise, $\bar{\mathbf{P}}_{\rm WZ} < \bar{\mathbf{P}}_{\rm SB}$ and $\bar{\mathbf{Q}}_{\rm WZ} < \bar{\mathbf{Q}}_{\rm SB}$. Moreover, frequency limited Hankel values satisfy : $(\bar{\lambda}_j[\bar{\mathbf{P}}_{\rm WZ}\bar{\mathbf{Q}}_{\rm WZ}])^{1/2} \leq (\bar{\lambda}_j[\bar{\mathbf{P}}_{\rm SB}\bar{\mathbf{Q}}_{\rm SB}])^{1/2}$.

Remark 8 When the input and the output related matrices $\bar{\mathbf{X}}_{\rm WZ} \geq 0$ and $\bar{\mathbf{Y}}_{\rm WZ} \geq 0$, respectively, then approximated models obtained using [14] and proposed technique are the equivalent.

Theorem 2 The following Lyapunov equation for the proposed technique holds

$$\mathbf{A}\bar{\mathbf{P}}_{\rm (ext)}\mathbf{A}^{\rm T} - \bar{\mathbf{P}}_{\rm (ext)} + \bar{\mathbf{B}}_{\rm (ext)}\bar{\mathbf{B}}^{\rm T}_{\rm (ext)} = 0, \quad (28)$$

$$\mathbf{A}^{\rm T}\bar{\mathbf{Q}}_{\rm (ext)}\mathbf{A} - \bar{\mathbf{Q}}_{\rm (ext)} + \bar{\mathbf{C}}^{\rm T}_{\rm (ext)}\bar{\mathbf{C}}_{\rm (ext)} = 0. \quad (29)$$

Proof: Using (11), (13), (24) and (25) we have following

$$\bar{\mathbf{S}}_{\rm WZ} = {\rm diag}[\bar{\mathbf{S}}_{\rm WZ_1}, \bar{\mathbf{S}}_{\rm WZ_2}] = {\rm diag}[(\bar{s}_1, .., \bar{s}_{q-1}), (\bar{s}_q, .., \bar{s}_n)] \quad (30)$$

$$\bar{\mathbf{S}}_{\rm SB} = {\rm diag}[\bar{\mathbf{S}}_{\rm WZ_1}, \bar{\mathbf{S}}_{\rm SB_2}] = {\rm diag}[(\bar{s}_1, .., \bar{s}_{q-1}), (\hat{s}_q, .., \hat{s}_n)] \quad (31)$$

$$\bar{\mathbf{R}}_{\rm WZ} = {\rm diag}[\bar{\mathbf{R}}_{\rm WZ_1}, \bar{\mathbf{R}}_{\rm WZ_2}] = {\rm diag}[(\bar{r}_1, .., \bar{r}_{k-1}), (\bar{r}_k, .., \bar{r}_n)] \quad (32)$$

$$\bar{\mathbf{R}}_{\rm SB} = {\rm diag}[\bar{\mathbf{R}}_{\rm WZ_1}, \bar{\mathbf{R}}_{\rm SB_2}] = {\rm diag}[(\bar{r}_1, .., \bar{r}_{k-1}), (\hat{r}_k, .., \hat{r}_n)] \quad (33)$$

$\bar{\mathbf{S}}_{\rm (ext)}$ and $\bar{\mathbf{R}}_{\rm (ext)}$ are obtained by subtracting (31 − 30) and (33 − 32) respectively

$$\bar{\mathbf{S}}_{\rm (ext)} = \begin{bmatrix} 0 & 0\\ 0 & \bar{\mathbf{S}}_{\rm (ext)_2}\end{bmatrix}, \quad \bar{\mathbf{R}}_{\rm (ext)} = \begin{bmatrix} 0 & 0\\ 0 & \bar{\mathbf{R}}_{\rm (ext)_2}\end{bmatrix}$$

where $\bar{\mathbf{S}}_{\rm (ext)_2} = \bar{\mathbf{S}}_{\rm SB_2} - \bar{\mathbf{S}}_{\rm WZ_2}$ and $\bar{\mathbf{R}}_{\rm (ext)_2} = \bar{\mathbf{R}}_{\rm SB_2} - \bar{\mathbf{R}}_{\rm WZ_2}$. $\bar{\mathbf{B}}_{\rm (ext)}$ and $\bar{\mathbf{C}}_{\rm (ext)}$ are obtained by subtracting (24 − 12) and (25 − 14) respectively

$$\begin{aligned}\bar{\mathbf{B}}_{\rm (ext)} &= \bar{\mathbf{U}}_{\rm (ext)}\begin{bmatrix} 0 & 0\\ 0 & \bar{\mathbf{S}}^{1/2}_{\rm (ext)_2}\end{bmatrix} = \bar{\mathbf{U}}_{\rm (ext)}\bar{\mathbf{S}}^{1/2}_{\rm (ext)},\\ \bar{\mathbf{C}}_{\rm (ext)} &= \begin{bmatrix} 0 & 0\\ 0 & \bar{\mathbf{R}}^{1/2}_{\rm (ext)_2}\end{bmatrix}\bar{\mathbf{V}}^{\rm T}_{\rm (ext)} = \bar{\mathbf{R}}^{1/2}_{\rm (ext)}\bar{\mathbf{V}}^{\rm T}_{\rm (ext)}\end{aligned}$$

where $\bar{\mathbf{U}}_{\rm (ext)} = \bar{\mathbf{U}}_{\rm SB} = \bar{\mathbf{U}}_{\rm WZ}$ and $\bar{\mathbf{V}}_{\rm ext} = \bar{\mathbf{V}}_{\rm SB} = \bar{\mathbf{V}}_{\rm WZ}$. Since,

$$\begin{aligned}\bar{\mathbf{X}}_{\rm (ext)} &= \bar{\mathbf{B}}_{\rm (ext)}\bar{\mathbf{B}}^{\rm T}_{\rm (ext)} = \bar{\mathbf{U}}_{\rm (ext)}\bar{\mathbf{S}}^{1/2}_{\rm (ext)}\bar{\mathbf{S}}^{1/2}_{\rm (ext)}\bar{\mathbf{U}}^{\rm T}_{\rm (ext)}\\ &= \bar{\mathbf{U}}_{\rm (ext)}\bar{\mathbf{S}}_{\rm (ext)}\bar{\mathbf{U}}^{\rm T}_{\rm (ext)} = \bar{\mathbf{U}}_{\rm (ext)}(\bar{\mathbf{S}}_{\rm SB} - \bar{\mathbf{S}}_{\rm WZ})\bar{\mathbf{U}}^{\rm T}_{\rm (ext)}\\ &= \bar{\mathbf{U}}_{\rm (ext)}\bar{\mathbf{S}}_{\rm SB}\bar{\mathbf{U}}^{\rm T}_{\rm (ext)} - \bar{\mathbf{U}}_{\rm (ext)}\bar{\mathbf{S}}_{\rm WZ}\bar{\mathbf{U}}^{\rm T}_{\rm (ext)} = \bar{\mathbf{X}}_{\rm SB} - \bar{\mathbf{X}}_{\rm WZ},\end{aligned} \quad (34)$$

$$\begin{aligned}\bar{\mathbf{Y}}_{\rm (ext)} &= \bar{\mathbf{C}}^{\rm T}_{\rm (ext)}\bar{\mathbf{C}}_{\rm (ext)} = \bar{\mathbf{V}}_{\rm (ext)}\bar{\mathbf{R}}^{1/2}_{\rm (ext)}\bar{\mathbf{R}}^{1/2}_{\rm (ext)}\bar{\mathbf{V}}^{\rm T}_{\rm (ext)}\\ &= \bar{\mathbf{V}}_{\rm (ext)}\bar{\mathbf{R}}_{\rm (ext)}\bar{\mathbf{V}}^{\rm T}_{\rm (ext)} = \bar{\mathbf{V}}_{\rm (ext)}(\bar{\mathbf{R}}_{\rm SB} - \bar{\mathbf{R}}_{\rm WZ})\bar{\mathbf{V}}^{\rm T}_{\rm (ext)}\\ &= \bar{\mathbf{V}}_{\rm (ext)}\bar{\mathbf{R}}_{\rm SB}\bar{\mathbf{V}}^{\rm T}_{\rm (ext)} - \bar{\mathbf{V}}\bar{\mathbf{R}}_{\rm WZ}\bar{\mathbf{V}}^{\rm T}_{\rm (ext)} = \bar{\mathbf{Y}}_{\rm SB} - \bar{\mathbf{Y}}_{\rm WZ},\end{aligned} \quad (35)$$

using (9 and 20) in (34) and (10 and 21) in (35) we have following

$$\begin{aligned}(\mathbf{A}\bar{\mathbf{P}}_{\rm SB}\mathbf{A}^{\rm T} - \bar{\mathbf{P}}_{\rm SB}) - (\mathbf{A}\bar{\mathbf{P}}_{\rm WZ}\mathbf{A}^{\rm T} - \bar{\mathbf{P}}_{\rm WZ}) &= -\bar{\mathbf{X}}_{\rm (ext)},\\ (\mathbf{A}^{\rm T}\bar{\mathbf{Q}}_{\rm SB}\mathbf{A} - \bar{\mathbf{Q}}_{\rm SB}) - (\mathbf{A}^{\rm T}\bar{\mathbf{Q}}_{\rm WZ}\mathbf{A} - \bar{\mathbf{Q}}_{\rm WZ}) &= -\bar{\mathbf{Y}}_{\rm (ext)},\\ \mathbf{A}(\bar{\mathbf{P}}_{\rm SB} - \bar{\mathbf{P}}_{\rm WZ})\mathbf{A}^{\rm T} - (\bar{\mathbf{P}}_{\rm SB} - \bar{\mathbf{P}}_{\rm WZ}) &= -\bar{\mathbf{X}}_{\rm (ext)},\\ \mathbf{A}^{\rm T}(\bar{\mathbf{Q}}_{\rm SB} - \bar{\mathbf{Q}}_{\rm WZ})\mathbf{A} - (\bar{\mathbf{Q}}_{\rm SB} - \bar{\mathbf{Q}}_{\rm WZ}) &= -\bar{\mathbf{Y}}_{\rm (ext)}.\end{aligned}$$

If controllability Gramian $\bar{\mathbf{P}}_{\rm (ext)} = \bar{\mathbf{P}}_{\rm SB} - \bar{\mathbf{P}}_{\rm WZ}$ and observability Gramian $\bar{\mathbf{Q}}_{\rm (ext)} = \bar{\mathbf{Q}}_{\rm SB} - \bar{\mathbf{Q}}_{\rm WZ}$ then

$$\begin{aligned}\mathbf{A}\bar{\mathbf{P}}_{\rm (ext)}\mathbf{A}^{\rm T} - \bar{\mathbf{P}}_{\rm (ext)} + \bar{\mathbf{B}}_{\rm (ext)}\bar{\mathbf{B}}^{\rm T}_{\rm (ext)} &= 0,\\ \mathbf{A}^{\rm T}\bar{\mathbf{Q}}_{\rm (ext)}\mathbf{A} - \bar{\mathbf{Q}}_{\rm (ext)} + \bar{\mathbf{C}}^{\rm T}_{\rm (ext)}\bar{\mathbf{C}}_{\rm (ext)} &= 0.\end{aligned}$$

Corollary 2 Theorem 2 holds true subject to the realization $(\mathbf{A}, \bar{\mathbf{B}}_{\rm (ext)}, \bar{\mathbf{C}}_{\rm (ext)}, \mathbf{D})$ is minimal and stable.

Remark 9 For the realization $\{\mathbf{A}, \bar{\mathbf{B}}_{\rm (ext)}, \bar{\mathbf{C}}_{\rm (ext)}, \mathbf{D}\}$ to the following Lyapunov equation

$$\begin{aligned}\mathbf{A}\bar{\mathbf{P}}_{\rm (ext)}\mathbf{A}^{\rm T} - \bar{\mathbf{P}}_{\rm (ext)} + \bar{\mathbf{B}}_{\rm (ext)}\bar{\mathbf{B}}^{\rm T}_{\rm (ext)} &= 0,\\ \mathbf{A}^{\rm T}\bar{\mathbf{Q}}_{\rm (ext)}\mathbf{A} - \bar{\mathbf{Q}}_{\rm (ext)} + \bar{\mathbf{C}}^{\rm T}_{\rm (ext)}\bar{\mathbf{C}}_{\rm (ext)} &= 0\end{aligned}$$

where the input matrix $\bar{\mathbf{B}}_{\rm (ext)} \geq 0$ and the output matrix $\bar{\mathbf{C}}_{\rm (ext)} \geq 0$ ensure positive (semi-positive) definiteness of the input and the output related matrices $\bar{\mathbf{B}}_{\rm SB}$ and $\bar{\mathbf{C}}_{\rm SB}$, respectively; consequently, positive definiteness of $\bar{\mathbf{P}}_{\rm (ext)}$ and $\bar{\mathbf{Q}}_{\rm (ext)}$ in a way leads to positive definiteness of $\bar{\mathbf{P}}_{\rm SB}$ and $\bar{\mathbf{Q}}_{\rm SB}$.

Remark 10 Since the input matrix $\bar{\mathbf{B}}_{\mathrm{SB}}$ and the output matrix $\bar{\mathbf{C}}_{\mathrm{SB}}$ ensure positive (semi-positive) definiteness of the input and the output related matrices; consequently, positive definiteness of $\bar{\mathbf{P}}_{\mathrm{SB}}$ and $\bar{\mathbf{Q}}_{\mathrm{SB}}$ in a way leads to transformation matrix $\bar{\mathbf{T}}_{\mathrm{SB}}$, which subsequently, results in the stability preserving model order reduction technique. Moreover, $\bar{\mathbf{L}}_{\mathrm{SB}}$ and $\bar{\mathbf{K}}_{\mathrm{SB}}$ form bases for the derivation of *a priori* error bound for proposed technique.

Remark 11 WZ's technique [14] involves the computation of the similarity transformation matrix $\bar{\mathbf{T}}_{\mathrm{WZ}}$ employing the usage of the controllability and the observability Gramians $\bar{\mathbf{P}}_{\mathrm{WZ}}$ and $\bar{\mathbf{Q}}_{\mathrm{WZ}}$, respectively, are directly computed from the original system realization $\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$. Consequently, it requires $2 \times n^{\mathrm{th}}$ number of Lyapunov equations for n^{th} order systems to perform MOR; however, stability of the ROM obtained is not guaranteed. Whereas, the similarity transformation matrix $\bar{\mathbf{T}}_{\mathrm{SB}}$ obtained by using the proposed technique employing the usage of controllability and observability Gramians $\bar{\mathbf{P}}_{\mathrm{SB}} = \bar{\mathbf{P}}_{\mathrm{WZ}} + \bar{\mathbf{P}}_{(\mathrm{ext})}$ and $\bar{\mathbf{Q}}_{\mathrm{SB}} = \bar{\mathbf{Q}}_{\mathrm{WZ}} + \bar{\mathbf{Q}}_{(\mathrm{ext})}$, respectively, are computed from the realization $\{\mathbf{A}, \bar{\mathbf{B}}_{\mathrm{SB}}, \bar{\mathbf{C}}_{\mathrm{SB}}\}$. The relationship between Gramians matrices of WZ [14] and the proposed technique is given as:

$$
\begin{aligned}
&\mathbf{A}(\bar{\mathbf{P}}_{\mathrm{WZ}} + \bar{\mathbf{P}}_{(\mathrm{ext})})\mathbf{A}^{\mathrm{T}} - (\bar{\mathbf{P}}_{\mathrm{WZ}} + \bar{\mathbf{P}}_{(\mathrm{ext})}) \\
&\quad + (\bar{\mathbf{X}}_{\mathrm{WZ}} + \bar{\mathbf{X}}_{(\mathrm{ext})}) = 0, && \text{for } \bar{s}_n < 0, \\
&\mathbf{A}\bar{\mathbf{P}}_{\mathrm{WZ}}\mathbf{A}^{\mathrm{T}} - \bar{\mathbf{P}}_{\mathrm{WZ}} + \bar{\mathbf{X}}_{\mathrm{en}} = 0, && \text{for } \bar{s}_n \geq 0, \\
&\mathbf{A}^{\mathrm{T}}(\bar{\mathbf{Q}}_{\mathrm{WZ}} + \bar{\mathbf{Q}}_{(\mathrm{ext})})\mathbf{A} - (\bar{\mathbf{Q}}_{\mathrm{WZ}} + \bar{\mathbf{Q}}_{(\mathrm{ext})}) \\
&\quad + (\bar{\mathbf{Y}}_{\mathrm{WZ}} + \bar{\mathbf{Y}}_{(\mathrm{ext})}) = 0, && \text{for } \bar{r}_n < 0, \\
&\mathbf{A}^{\mathrm{T}}\bar{\mathbf{Q}}_{\mathrm{WZ}}\mathbf{A} - \bar{\mathbf{Q}}_{\mathrm{WZ}} + \bar{\mathbf{Y}}_{\mathrm{WZ}} = 0, && \text{for } \bar{r}_n \geq 0, \\
&\mathbf{A}\bar{\mathbf{P}}_{(\mathrm{ext})}\mathbf{A}^{\mathrm{T}} - \bar{\mathbf{P}}_{(\mathrm{ext})} + \bar{\mathbf{X}}_{(\mathrm{ext})} = 0, && \text{for } \bar{s}_n < 0, \\
&\mathbf{A}^{\mathrm{T}}\bar{\mathbf{Q}}_{(\mathrm{ext})}\mathbf{A} - \bar{\mathbf{Q}}_{(\mathrm{ext})} + \bar{\mathbf{Y}}_{(\mathrm{ext})} = 0, && \text{for } \bar{r}_n < 0.
\end{aligned}
$$

Since

$$
\begin{aligned}
\bar{\mathbf{X}}_{\mathrm{SB}} &= \bar{\mathbf{U}}_{\mathrm{SB}}\, (\bar{\mathbf{S}}_{\mathrm{SB}})^{1/2}(\bar{\mathbf{S}}_{\mathrm{SB}})^{1/2}\, \bar{\mathbf{U}}_{\mathrm{SB}}^{\mathrm{T}} \\
&= \bar{\mathbf{X}}_{\mathrm{WZ}} + \bar{\mathbf{X}}_{(\mathrm{ext})}, \quad \text{for } \bar{s}_n < 0, \\
\bar{\mathbf{X}}_{\mathrm{SB}} &= \bar{\mathbf{U}}_{\mathrm{SB}}\, (\bar{\mathbf{S}}_{\mathrm{SB}})^{1/2}(\bar{\mathbf{S}}_{\mathrm{SB}})^{1/2}\, \bar{\mathbf{U}}_{\mathrm{SB}}^{\mathrm{T}} \\
&= \bar{\mathbf{X}}_{\mathrm{WZ}}, \qquad \text{for } \bar{s}_n \geq 0, \\
\bar{\mathbf{Y}}_{\mathrm{SB}} &= \bar{\mathbf{V}}_{\mathrm{SB}}^{\mathrm{T}}\, (\bar{\mathbf{R}}_{\mathrm{SB}})^{1/2}(\bar{\mathbf{R}}_{\mathrm{SB}})^{1/2}\, \bar{\mathbf{V}}_{\mathrm{SB}} \\
&= \bar{\mathbf{Y}}_{\mathrm{WZ}} + \bar{\mathbf{Y}}_{(\mathrm{ext})}, \quad \text{for } \bar{r}_n < 0, \\
\bar{\mathbf{Y}}_{\mathrm{SB}} &= \bar{\mathbf{V}}_{\mathrm{SB}}^{\mathrm{T}}\, (\bar{\mathbf{R}}_{\mathrm{SB}})^{1/2}(\bar{\mathbf{R}}_{\mathrm{SB}})^{1/2}\, \bar{\mathbf{V}}_{\mathrm{SB}} \\
&= \bar{\mathbf{Y}}_{\mathrm{WZ}}, \qquad \text{for } \bar{r}_n \geq 0.
\end{aligned}
$$

Consequently, the proposed method requires $4 \times n^{\mathrm{th}}$ number of Lyapunov equations to obtained ROM and the stability is also guaranteed.

4. Numerical Simulations

In this section, a comparison among different techniques is presented. Reduced-order transfer functions and poles location of ([14–16]) and the proposed technique are also provided where [14] produces unstable ROMs. Figures 1 and 2 represent the frequency response Bode plot (magnitude and phase) in the given frequency range. Furthermore, Figures 3 and 5 represent the frequency response error in the entire frequency range of the approximated model obtained by using existing ([14–16]) and proposed techniques, whereas, Figures 4 and 6 signify the frequency response errors in the given limited discrete-frequency ranges of the approximated model in the given frequency interval acquired via using existing and proposed techniques. Tables 2, 3 and 4 provide the ROMs obtained by using existing ([14–16]) and proposed techniques; whereas, the Table 5 provide the location of poles by using the WZ [14] proposed technique; furthermore, the Table 6 provides the error bounds by using existing ([14–16]) and proposed techniques.

Example 1 Benchmark example of 48[th] order is presented here, the building model (the Los Angeles University Hospital) [18], it contains 8 floors each have 3 degrees of freedom, namely displacement in two different directions x_1 and x_2, and rotation. State-space form representation of given an example is available at [19], given model is discretized at sampling time $T_s = 0.001$ s, with the given frequency interval $[\bar{\omega}_1 - \bar{\omega}_2] = [0.01\pi - 0.25\pi]$ rad/s, where $\mathbf{G}_{\mathrm{tr}}(z)$ is the approximated model of 4[th] order obtained by using [14–16] and proposed techniques. Figure 1 provides a comparison for the frequency response Bode plot (magnitude, phase) in the given frequency range $[0.01\pi - 0.25\pi]$ rad/s. Reduced-order transfer functions and poles location of ([14–16]) and proposed techniques are also provided in the Tables 3 and 5, respectively, poles location of ROM obtained by using [14] are $z = 0.9945 \pm 0.0522\mathrm{i}, 0.9860 \pm 0.1340\mathrm{i}$. Whereas, poles location of ROM obtained by using the proposed technique are $z = 0.9946 \pm 0.0525\mathrm{i}, 0.9862 \pm 0.1348\mathrm{i}$. Table 6 provide a comparison for the frequency response error bounds in the given frequency range $[0.01\pi - 0.25\pi]$ rad/s. Furthermore, the proposed algorithm grants stable ROMs, minimum frequency response error along with the frequency response error bound comparable with different existing stability retaining algorithms ([14–16]) in the given frequency range.

Techniques	**Reduced order model $\mathbf{G}_{\mathrm{tr}}(z)$** — **4th Order ROMs**
WZ [14]	$\dfrac{7.69e^{-5}z^3-0.0002306z^2+0.0002313z-7.759e^{-5}}{z^4-3.961z^3+5.905z^2-3.926z+0.9821}$
GSI [15]	$\dfrac{7.781e^{-5}z^3-0.0002332z^2+0.0002338z-7.843e^{-5}}{z^4-3.961z^3+5.904z^2-3.925z+0.9818}$
GSII [15]	$\dfrac{7.773e^{-5}z^3-0.000233z^2+0.0002336z-7.836e^{-5}}{z^4-3.961z^3+5.904z^2-3.925z+0.9818}$
IG [16]	$\dfrac{6.283e^{-5}z^3-0.0001877z^2+0.0001875z-6.266e^{-5}}{z^4-3.966z^3+5.919z^2-3.941z+0.9874}$
Proposed	$\dfrac{7.673e^{-5}z^3-0.0002296z^2+0.0002297z-7.693e^{-5}}{z^4-3.962z^3+5.906z^2-3.928z+0.9829}$

Tab. 2. Reduced order models for Example-1.

Example 2 Consider a 20^{th} order band-stop Butterworth filter with normalized edge frequencies ranges [0.5 − 0.6] rad/sample, with following transfer function form (36) [under this paragraph], with the given frequency interval $[\bar{\omega}_1 - \bar{\omega}_2] = [0.5\pi - 0.9\pi]$ rad/s, where $\mathbf{G}_{\mathrm{tr}}(z)$ is the approximated model of 7^{th} order obtained by using [14–16] and proposed techniques. Figure 2 provides a comparison for the frequency response Bode plot (magnitude, phase) in the given frequency range $[0.5\pi - 0.9\pi]$ rad/s. Reduced-order transfer functions and poles location of ([14–16]) and proposed techniques are also provided in the Tabs. 3 and 5, respectively, poles location of ROM obtained by using [14] are $z = -0.0910, -0.2659 \pm 0.9534\mathrm{i}, -0.1738 \pm 0.9693\mathrm{i}, -0.0472 \pm 0.9770\mathrm{i}$. Whereas, poles location of ROM obtained by using the proposed technique are $z = -0.1309, -0.2770 \pm 0.9513\mathrm{i}, -0.1845 \pm 0.9671\mathrm{i}, -0.0673 \pm 0.9874\mathrm{i}$. Table 6 provide a comparison for the frequency response error bounds in the given frequency range $[0.5\pi - 0.9\pi]$ rad/s. Furthermore, the proposed algorithm grants stable ROMs, minimum frequency response error along with the frequency response error bound comparable with different existing stability retaining algorithms ([14–16]) in the given frequency range.

Example 3 Consider a 6^{th} order stable discrete-time system [16] having following state space representation (37), with the given frequency interval $[\bar{\omega}_1 - \bar{\omega}_2] = [0.65\pi - 0.81\pi]$ rad/s. Figures 3 and 5 provide a comparison for the frequency response error $\bar{\sigma}\,[\mathbf{G}(z) - \mathbf{G}_{\mathrm{tr}}(z)]$ for the entire frequency interval, where $\mathbf{G}_{\mathrm{tr}}(z)$ are approximated models of 4^{th} and 5^{th} order respectively obtained by using [14–16] and proposed techniques. Figures 4 and 6 provide a comparison for the frequency response error $\bar{\sigma}\,[G(z) - G_{\mathrm{tr}}(z)]$ in the given frequency range $[0.65\pi - 0.81\pi]$ rad/s. Reduced order transfer functions and poles location of ([14–16]) and proposed techniques are also provided in the Tabs. 4 and 5, respectively, [14] produces unstable 4^{th} and 5^{th} order ROMs with poles location at $z = -2.5368, -0.3400, -0.3721 \pm 0.8901\mathrm{i}$, and $z = 2.2355, -0.0368 \pm 1.1440\mathrm{i}, -0.0996 \pm 0.7056\mathrm{i}$ respectively. Whereas, the proposed technique produces stable 4^{th} and 5^{th} order ROMs with poles location at $z = 0.0139 \pm 0.9454\mathrm{i}, 0.3842 \pm 0.7131\mathrm{i}$, and $z = 0.5879, 0.0015 \pm 0.9408\mathrm{i}, 0.2884 \pm 0.7024\mathrm{i}$, respectively. Table 6 provide a comparison for the frequency response error bounds in the given frequency range $[0.65\pi - 0.81\pi]$ rad/s. Furthermore, the proposed algorithm grants stable ROMs, minimum frequency response error along with the frequency response error bound comparable with the different existing stability retaining algorithms ([14–16]) in the given frequency range.

$$G(z) = \frac{\begin{array}{r}3.62\mathrm{e}^{-9}z^{20} - 1.125\mathrm{e}^{-23}z^{19} - 3.62\mathrm{e}^{-8}z^{18} + 1.607\mathrm{e}^{-22}z^{17} + 1.629\mathrm{e}^{-7}z^{16} + 5.144\mathrm{e}^{-23}z^{15} - 4.344\mathrm{e}^{-7}z^{14} - 8.179\mathrm{e}^{-21}z^{13} \\ +7.601\mathrm{e}^{-7}z^{12} - 2.685\mathrm{e}^{-20}z^{11} - 9.122\mathrm{e}^{-7}z^{10} - 3.035\mathrm{e}^{-20}z^{9} + 7.601\mathrm{e}^{-7}z^{8} - 1.276\mathrm{e}^{-20}z^{7} - 4.344\mathrm{e}^{-7}z^{6} - 1.003\mathrm{e}^{-21}z^{5} \\ +1.629\mathrm{e}^{-7}z^{4} + 3.215\mathrm{e}^{-23}z^{3} - 3.62\mathrm{e}^{-8}z^{2} + 3.62\mathrm{e}^{-9}\end{array}}{\begin{array}{r}z^{20} + 2.85z^{19} + 11.65z^{18} + 23.34z^{17} + 53.82z^{16} + 82.43z^{15} + 135.7z^{14} + 165.3z^{13} + 210.6z^{12} + 207.8z^{11} + 212.2z^{10} \\ +169.9z^{9} + 140.8z^{8} + 90.38z^{7} + 60.64z^{6} + 30.11z^{5} + 16.07z^{4} + 5.688z^{3} + 2.321z^{2} + 0.4628z + 0.1328\end{array}} \quad (36)$$

$$\left[\begin{array}{c|c}\mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D}\end{array}\right] = \left[\begin{array}{cccccc|c} 1.4637 & -2.2838 & 2.0587 & -1.4467 & 0.6746 & -0.1825 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0.0799 & 0.1351 & 0.2388 & 0.1370 & 0.0776 & -0.0011 & 0.0107 \end{array}\right] \quad (37)$$

Techniques	Reduced order model $\mathbf{G}_{\mathrm{tr}(z)}$
	7^{th} Order ROMs
WZ [14]	$\dfrac{3.62\mathrm{e}^{-9}z^7 + 0.04341z^6 + 0.00285z^5 + 0.09095z^4 - 0.02476z^3 + 0.05937z^2 - 0.0261z + 0.01779}{z^7 + 1.065z^6 + 3.263z^5 + 2.188z^4 + 3.246z^3 + 1.189z^2 + 0.9917z + 0.08274}$
GS-I [15]	$\dfrac{0.01053z^6 - 0.02528z^5 + 0.01888z^4 - 0.05901z^3 + 0.02622z^2 - 0.03108z + 0.0211}{z^7 + 1.033z^6 + 3.264z^5 + 2.172z^4 + 3.288z^3 + 1.235z^2 + 1.028z + 0.1111}$
GS-II [15]	$\dfrac{3.62\mathrm{e}^{-9}z^7 - 0.02857z^6 - 0.0644z^5 - 0.1308z^4 - 0.1524z^3 - 0.1503z^2 - 0.08584z - 0.04303}{z^7 + 1.144z^6 + 3.365z^5 + 2.488z^4 + 3.464z^3 + 1.509z^2 + 1.103z + 0.1826}$
IG [16]	$\dfrac{3.62\mathrm{e}^{-9}z^7 + 0.01767z^6 - 0.01624z^5 + 0.01523z^4 - 0.05923z^3 - 0.01421z^2 - 0.04263z - 0.006478}{z^7 + 1.152z^6 + 3.37z^5 + 2.458z^4 + 3.44z^3 + 1.419z^2 + 1.073z + 0.1354}$
Proposed	$\dfrac{3.62\mathrm{e}^{-9}z^7 - 0.008832z^6 - 0.0339z^5 - 0.07912z^4 - 0.1026z^3 - 0.1257z^2 - 0.06966z - 0.04929}{z^7 + 1.189z^6 + 3.398z^5 + 2.52z^4 + 3.458z^3 + 1.426z^2 + 1.064z + 0.122}$

Tab. 3. Reduced order models for Example-2.

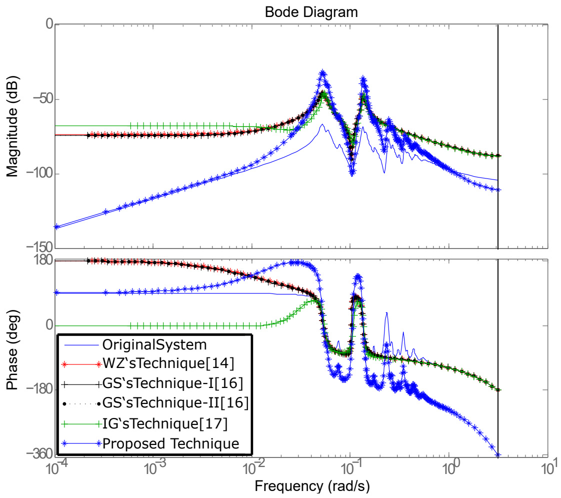

Fig. 1. Frequency response Bode plot (magnitude and phase) in the given frequency range $[\bar{\omega}_1 - \bar{\omega}_2] = [0.01\pi - 0.25\pi]$ rad/s of 4th order for Example-1.

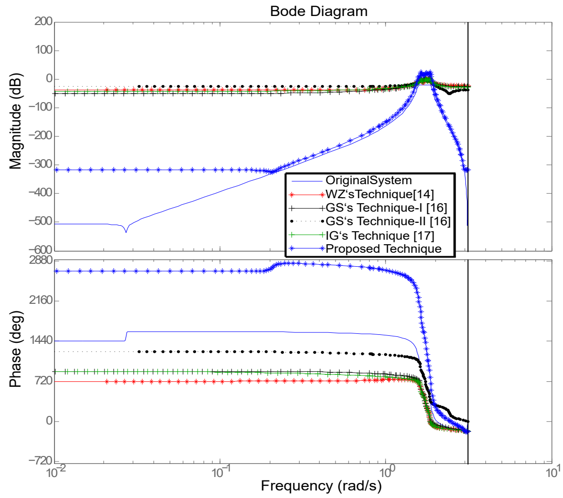

Fig. 2. Frequency response Bode plot (magnitude and phase) in the given frequency range $[\bar{\omega}_1 - \bar{\omega}_2] = [0.5\pi - 0.9\pi]$ rad/s of 7th order for Example-2.

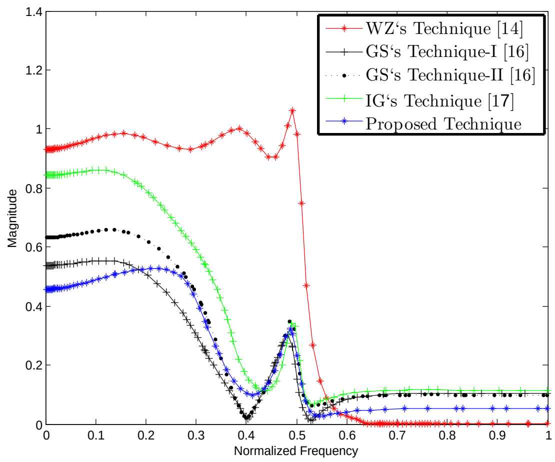

Fig. 3. Frequency response error $\bar{\sigma}\,[\mathbf{G}(z) - \mathbf{G}_{\text{tr}}(z)]$ comparison of 4th order for Example-3.

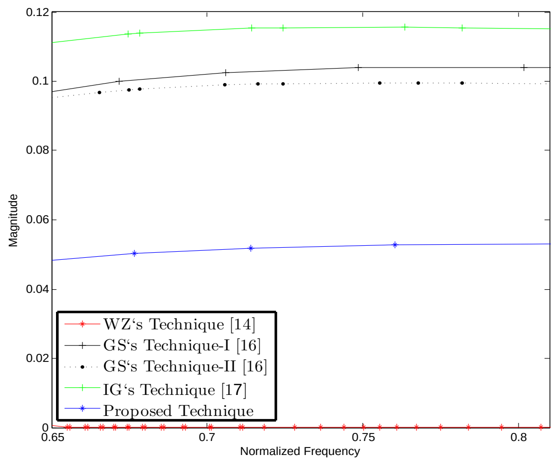

Fig. 4. $\bar{\sigma}\,[\mathbf{G}(z) - \mathbf{G}_{\text{tr}}(z)]$ in the given frequency range $[\bar{\omega}_1 - \bar{\omega}_2] = [0.65\pi - 0.81\pi]$ rad/s of 4th order for Example-3.

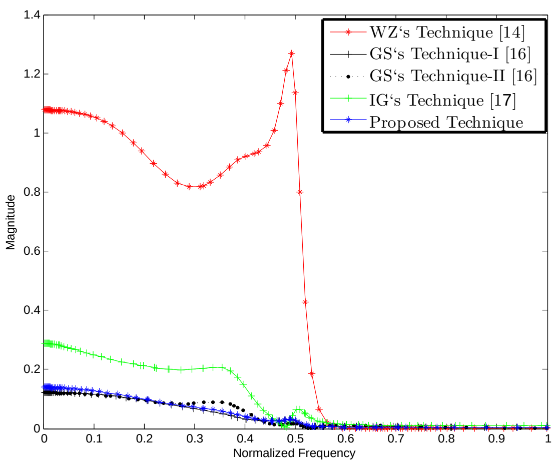

Fig. 5. Frequency response error $\bar{\sigma}\,[\mathbf{G}(z) - \mathbf{G}_{\text{tr}}(z)]$ comparison of 5th order for Example-3.

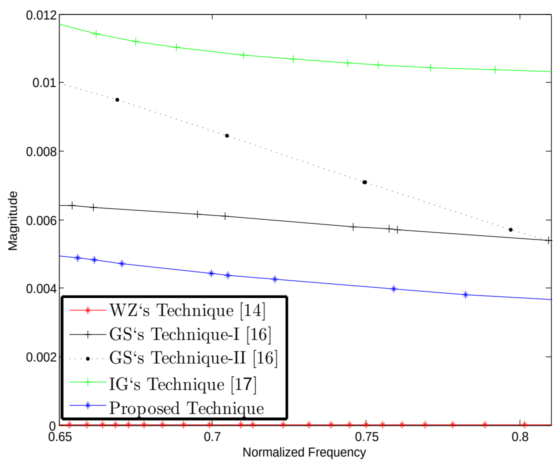

Fig. 6. $\bar{\sigma}\,[G(z) - G_{\text{tr}}(z)]$ in the given frequency range $[\bar{\omega}_1 - \bar{\omega}_2] = [0.65\pi - 0.81\pi]$ rad/s of 5th order for Example-3.

Techniques	Reduced order model $G_{tr}(z)$	
	4th Order ROMs	5th Order ROMs
WZ [14]	$\frac{0.0107z^4+0.03384z^3+0.04972z^2+0.03774z+0.01755}{z^4+3.621z^3+3.934z^2+3.319z+0.8027}$	$\frac{0.0107z^5+0.06433z^4+0.16z^3+0.2176z^2+0.1518z+0.06955}{z^5-1.781z^4+1.616z^3-3.2z^2+0.4466z-1.296}$
GS-I [15]	$\frac{-0.01431z^3+0.2195z^2+0.08892z+0.275}{z^4-0.8755z^3+1.536z^2-0.759z+0.5415}$	$\frac{0.07933z^4+0.1585z^3+0.2797z^2+0.1945z+0.1119}{z^5-1.149z^4+1.776z^3-1.341z^2+0.7839z-0.2889}$
GS-II [15]	$\frac{0.0107z^4+0.02961z^3+0.1995z^2+0.1517z+0.2255}{z^4-1.025z^3+1.604z^2-0.8832z+0.5752}$	$\frac{0.0107z^5+0.06697z^4+0.1765z^3+0.2586z^2+0.1979z+0.09983}{z^5-1.18z^4+1.804z^3-1.382z^2+0.8073z-0.3008}$
IG [16]	$\frac{0.0107z^4-0.1022z^3+0.1515z^2-0.04009z+0.1921}{z^4-0.6289z^3+1.648z^2-0.5681z+0.666}$	$\frac{0.0107z^5+0.06833z^4+0.1593z^3+0.2539z^2+0.1618z+0.1061}{z^5-1.286z^4+1.999z^3-1.645z^2+0.9858z-0.4375}$
Proposed	$\frac{0.0107z^4-0.02853z^3+0.2084z^2+0.06448z+0.2592}{z^4-0.7962z^3+1.571z^2-0.7052z+0.5865}$	$\frac{0.0107z^5+0.06427z^4+0.1814z^3+0.2585z^2+0.2048z+0.1054}{z^5-1.168z^4+1.804z^3-1.373z^2+0.8115z-0.3}$

Tab. 4. Reduced order models for Example-3.

Examples	Order of ROMs	WZ's Technique [14]	Proposed Technique
Example-1	4th order	0.9945 ± 0.0522i, 0.9860 ± 0.1340i	0.9946 ± 0.0525i, 0.9862 ± 0.1348i
Example-2	7th order	−0.0910, −0.2659 ± 0.9534i, −0.1738 ± 0.9693i, −0.0472 ± 0.9770i	−0.1309, −0.2770 ± 0.9513i, −0.1845 ± 0.9671i, −0.0673 ± 0.9874i
Example-3	4th order	−2.5368, −0.3400, −0.3721 ± 0.8901i	0.0139 ± 0.9454i, 0.3842 ± 0.7131i
	5th order	2.2355, −0.0368 ± 1.1440i, −0.0996 ± 0.7056i	0.5879, 0.0015 ± 0.9408i, 0.2884 ± 0.7024i

Tab. 5. Poles locations of reduced order models.

Examples	Order of ROMs	BT [5]	GS-I [15]	GS-II [15]	IG [16]	Proposed
Example-1	4th order	5.6190	5.8478	5.855	7.2905	5.8096
Example-2	7th order	24.115	25.022	25.064	24.937	24.884
Example-3	4th order	39.268	45.163	43.832	46.163	41.991
	5th order	40.714	55.038	53.649	46.523	44.663

Tab. 6. Theoretical and actual error bounds comparison.

5. Analysis & Discussion

From Figs. 4 and 6 it is determined that truncated systems attained from WZ's [14] method give low approximation error as in contrast to the different existing techniques, however, it occasionally yields unstable truncated systems as proven in Tab. 5. Whereas, the different existing methods (GS-I [15], GS-II [15], IG [16]) produce stable ROMs and also provide the *a priori* error bound formula; however, these methods produce large approximation errors. The proposed approach produces stable ROMs, as proven in Tab. 5. Furthermore, Table 6 provides the comparison among theoretical error bound (BT [5]) and actual error bounds of existing discrete-frequency limited stability preserving approaches (GS-I [15], GS-II [15], IG [16]), and the proposed approach. It can be observed that as compared to the existing stability retaining techniques, the proposed method yields better approximation error along with the *a priori* error bound formula.

6. Conclusion

The frequency restricted improved MOR approach is proposed for the discrete-time systems. The proposed technique produces stable ROMs and lowers approximation error along with the formula for the *a priori* error bound calculation. MOR technique presented by WZ provides unstable ROMs and does not provide the *a priori* error bound formula, whereas the different existing methods produce stable ROMs and also provide the *a priori* error bound formula; however, these methods produce large approximation errors. The frequency response error of the proposed method is well comparable with the other existing methods. Numerical examples have proven that the proposed method provides stable ROMs, lower approximation error, and the *a priori* error bound formula, which shows the efficacy of the presented method.

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