ACCURACY COMPARISON OF SOME 2D NUMERICAL INVERSE LAPLACE TRANSFORM METHODS

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Abstract: Numerical Laplace inversions are essential to use when the analytical manipulations of the Laplace transform tables are not possible to apply, and this becomes more and more difficult for applications with higher number of variables. In this paper we describe three methods for 2D numerical inverse Laplace transforms and analyse the methods as for their accuracy and calculation efficiency by implementing them in the Matlab environment. The relative and absolute errors for three different testing functions with previously known originals are presented in a comparative table for the selected points, and the 3D resulting plots for the inversion of a test function are displayed.

Keywords: Numerical inversion, two-dimensional Laplace transform, test functions, Matlab language, 3D plot.

NOMENCLATURE

- f is the time domain function
- F is the Laplace domain function
- t is time and x represents space (distance)
- s is the complex frequency variable
- c is distance from the imaginary axis
- Re is the real part
- Im is the imaginary part
- α_1 and α_2 are real numbers
- M is a positive constant
- T is the period of the Fourier series function, sec.
- t_{max} is the maximum time interval, sec.
- N is the number of terms; positive integer.
- K_{1i} , K_{2l} are the residues of approximating the exponential kernel
- z_{1i}, z_{2l} are the poles of approximating the exponential kernel
- FFT is fast Fourier transform
- T_i are sampling periods in the original time domain
- Ω_i is the frequency step
- E_r is the desired relative error

1. INTRODUCTION

Two-dimensional numerical inverse Laplace transforms (2D-NILTs) arise in electrical engineering related fields for applications that require solution of partial differential equations with two variables. Such applications are those defining transients in linear distributed-parameter systems, e.g. solving telegrapher equations, obtaining voltage, current distributions in the original time domain [1, 2], or more sophisticated such as that concerning slightly nonlinear systems described by second-order Voltera series expansion [3]. In this paper three inversion methods are described, these methods are based on using Fourier series techniques or the Padé approximation. These methods have proved to give good results and to be relatively universal for a wide range of applications [4]. Each method has different free parameters which could be optimized for higher accuracy results. The three methods in the next section are introduced in the following order, the first method is the

2D NILT based on Fourier series representation by Moorthy [4]. The second method is the 2D NILT by Singhal, Vlach and Vlach [5], and the third method is the accelerated 2D-FFT NILT by Brančik [6]. The numerical methods start with the main 2D inverse Laplace transform formula described as the two-fold Bromwich integral namely,

$$f(t,x) = -\frac{1}{4\pi^2}\int_{c_1-j\infty}^{c_1+j\infty}\int_{c_2-j\infty}^{c_2+j\infty} F(s_1,s_2)e^{s_1 t+s_2 x}ds_1 ds_2, \tag{1}$$

where t and x are generally any independent variables, but due to our interest in engineering related applications, we consider them to be time and space variables respectively. The Laplace variable is defined as $s = c + j\omega$.

2. 2D NUMERICAL INVERSE LAPLACE TRANSFORM METHODS

2.1. 2D NILT BASED ON FOURIER SERIES REPRESENTATION

The method was developed by M. Moorthy in[4], it is an expansion of a 1D method based on a representation of the inverse transform by Fourier series, which was introduced by Dubner and Abate [7]. The inversion formula consists of a combination of two infinite sums and a double infinite sum and is listed as follows,

$$\tilde{f}(t,x) = e^{c_1 t+c_2 x}g(t,x), \tag{2}$$

such that,

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} g^{jk}(t,x) = \frac{1}{2T^2}\Big[\frac{1}{2}F1_{c_1,c_2} + \sum_{m=1}^{\infty}\big[Re\{F2_{c_1,c_2}\}Z1 - Im\{F2_{c_1,c_2}\}Z2\big] + \sum_{n=1}^{\infty}\big[Re\{F3_{c_1,c_2}\}Z3 - Im\{F3_{c_1,c_2}\}Z4\big] + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\big[Re\{F4_{c_1,c_2}\}Z5 + Re\{F5_{c_1,c_2}\}Z6 - Im\{F4_{c_1,c_2}\}Z7 - Im\{F5_{c_1,c_2}\}Z8\big]\Big], \tag{3}$$

where the F annotations are functions with different approximations of s_1 and s_2 and the Z are different sinusoidal functions. For the detailed derivation and annotations the reader is advised to check reference [4].

For the derivation of the final formula the following assumptions were considered,

$$|f(t,x)| \le Me^{\alpha_1 t+\alpha_2 x}, \tag{4}$$

and the Laplace domain of the function $F(s_1,s_2)$ is analytic for $Re\{s_1\} > \alpha_1$ and $Re\{s_2\} > \alpha_2$.

To obtain an optimum choice of the parameters used, an error analysis done in [4] suggests that the choice of c_1 and c_2 to be larger than α_1 and α_2 respectively; for example by first choosing $c_1 > \alpha_1$, then [4],

$$c_2 = \alpha_2 - \frac{1}{2T}\ln\left(\frac{\varepsilon - e^{-2T(c_1-\alpha_1)}}{1-e^{-2T(c_1-\alpha_1)}}\right), \tag{5}$$

to choose the parameter T, after considering $t_{max} < 2T$, and after running different tests the author suggests the choice of $0.5t_{max} \le T \le 0.8t_{max}$ [4].

2.2. 2D NILT ROPOSED BY SINGHAL, J. VLACH, AND M. VLACH

In this 2D numerical inversion method the algorithm is an expansion of a 1D NILT based on the Zakian numerical scheme[5, 8]. The inversion technique uses the Padé approximation and the residual theorem. The formulae is described as follows [5],

$$\hat{f}(t,x) = \frac{1}{t*x}\sum_{i=1}^{M_1}\sum_{l=1}^{M_2} K_{1i}K_{2l}\, F\left(\frac{z_{1i}}{t_1},\frac{z_{2l}}{t_2}\right), \tag{6}$$

z_{1i}, z_{2l} are the poles of the rational Padé approximation, shown as follows [5],

$$e^{s_1 t} \approx e^{s_2 x} \approx \xi N_k, M_k(z_k) = \frac{\sum_{i=0}^{N_k}(M_k+N_k-i)!\binom{N_k}{i}z_k{}^i}{\sum_{i=0}^{M_k}(-1)^i(M_k+N_k-i)!\binom{M_k}{i}z_k{}^i}, \tag{7}$$

where $z_{k=1} = s_1 t$, and $z_{k=2} = s_2 x$.

Tests performed on the method show that the method performs well especially for sufficiently smooth functions. In the tables and figures following in the paper the method is denoted as "Singhal-2D NILT".

2.3. ACCELERATED 2D FFT-BASED NILT

In this numerical procedure described in [6] the inversion is based on fast Fourier transform algorithms *FFT*. Basically, the method uses the sum of two-dimensional complex Fourier series calculated by *FFT* algorithms. Furthermore, the ε-algorithm of Wynn is combined into the method to further give a higher accuracy and improvement to the result. Following in the text the term FFT-ε-2D NILT will be used to refer to this method.

The basic inversion algorithm namely is [6],

$$\tilde{f}^{k_1,k_2} = C^{k_1,k_2}\{2Re[\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} F_{-n_1,-n_2}E_{-n_1,-n_2}^{k_1,k_2} + \sum_{n_1=0}^{\infty}(\sum_{n_2=0}^{\infty} F_{-n_1,n_2}E_{n_2}^{k_2})E_{-n_1}^{k_1} - \sum_{n_1=0}^{\infty} F_{-n_1,0}E_{-n_1}^{k_1} - \sum_{n_2=0}^{\infty} F_{0,-n_2}E_{-n_2}^{k_2}] + F_{0,0}\}, \tag{8}$$

where $\tilde{f}^{k_1,k_2} = \tilde{f}(k_1T_1, k_2T_2)$, is the discrete form of the approximate formula. The symbols used are given as follows [6],

$$F_{n_1,n_2} = F(c_1 + jn_1\Omega_1, c_2 + jn_2\Omega_2), \tag{9}$$

$$E_{n_1,n_2}^{k_1,k_2} = e^{jk_1T_1n_1\Omega_1 + jk_2T_2n_2\Omega_2} = E_{n_1}^{k_1}E_{n_2}^{k_2}, \tag{10}$$

$$C^{k_1,k_2} = \frac{\Omega_1\Omega_2}{4\pi^2}e^{c_1k_1T_1+c_2k_2T_2} = C^{k_1}C^{k_2}, \tag{11}$$

where $\Omega_i = \frac{2\pi}{N_iT_i}$, $i = 1,2$, the discrete points given as $t_{k_i} = k_iT_i$, and $k_i = 0,1,\cdots,N_i - 1$.

The error analysis in [6], provides us with the ideal choice of c_i , that is to say by choosing E_r as the desired relative error then the choice of c_i can be given as,

$$c_i \approx \alpha_i - \frac{\Omega_i}{2\pi}\ln\frac{E_r}{2},\ i = 1,2. \tag{12}$$

The parameter N is introduced as a free parameter, such that the infinite sums in (8) are calculated up to $N - 1$ terms; choosing $N = 512$ has shown to give good results, the terms above N are integrated into the ε-algorithm for higher accuracy. Generally, the ε-algorithm is considered as a nonlinear algorithm used to speed up the convergence of the series and it is equivalent to a rational Padé approximation. The ε-algorithm is known to give good results when applied to power series; this is possible in the current case, since with some mathematical manipulations the complex Fourier series becomes as such a power series. In [6] the ε-algorithm diagram is shown with the detailed description of its application to the numerical method.

3. 2D NILT METHODS ACCURACY TESTS

After describing the 2D NILT methods and their inversion algorithms, here the absolute errors and relative errors for test functions with previously known originals are examined on the methods described in this paper.

The three methods were implemented by using the universal mathematical language Matlab and the test functions used here have been endorsed to be used as smooth test functions in both variables [4, 5]. The first test function in the Laplace domain with its known original is listed as follows,

$$F1(s_1, s_2) = \frac{1}{(s_1+1)(s_2+2)}, \tag{13}$$

$$f1(t, x) = e^{-1t-2x}, \tag{14}$$

The absolute errors and relative errors are calculated at the specific point $(t, x) = (1,1)$, the results are listed in an ascending order starting with the method of higher accuracy in Table 1, bellow.

2D NILT method	Absolute error	Relative error
FFT-ε-2D NILT	$2.9698 * 10^{-10}$	$5.9650 * 10^{-9}$
Singhal-2D NILT	$1.338 * 10^{-8}$	$2.687 * 10^{-7}$
Moorthy-2D NILT	$4.257 * 10^{-7}$	$8.55 * 10^{-6}$

Table 1: Accuracy tests for 2D NILT, function 1

The second function to be tested is described as,

$$F2(s_1, s_2) = \frac{1}{(s_1-1)\cdot(s_2-1)\cdot(s_1+s_2-1)}, \tag{15}$$

$$f2(t, x) = \begin{cases} e^x(e^t-1), & x \geq t \\ e^t(e^x-1), & x < t \end{cases}, \tag{16}$$

In Table 2, the absolute and relative errors are shown in an ascending order for the inversion methods tested on $f2(t, x)$ at $(t, x) = (1,1)$.

2D NILT method	Absolute error	Relative error
FFT_ε-2D NILT	$1.07288 * 10^{-3}$	$2.2970 * 10^{-4}$
Moorthy-2D NILT	$1.330 * 10^{-2}$	$2.85 * 10^{-3}$
Singhal-2D NILT	$5.320 * 10^{-2}$	$1.139 * 10^{-2}$

Table 2: Accuracy tests for 2D NILT, function 2

The results shown in the tables above have the FFT-ε-2DNILT method with the relatively highest accuracy among the three methods tested with an accuracy improvement in average of about 2 orders. In Table 3, the different parameters used to implement the results of the methods are listed for each method respectively [4-6].

2D NILT Method	Parameters used to test the functions
FFT_ε-2D NILT	$\alpha_1 = 0, \alpha_2 = 0, E_r = 1e - 7$. Same values for both functions.
Moorthy-2D NILT	For function1, $c_1 = 1, c_2 = 0, T = 3.5.$ For function2, $c_1 = 15, c_2 = 10.21$, $T = 1.0.$
Singhal-2D NILT	$M = 10, N = 8$. Same values for both functions used.

Table 3: Parameters used for different NILT methods

The results of inverting function 1, with the three 2D NILT methods are shown as 3D plot results in Figure 1. The 3D plots were determined through a grid of 256x256.

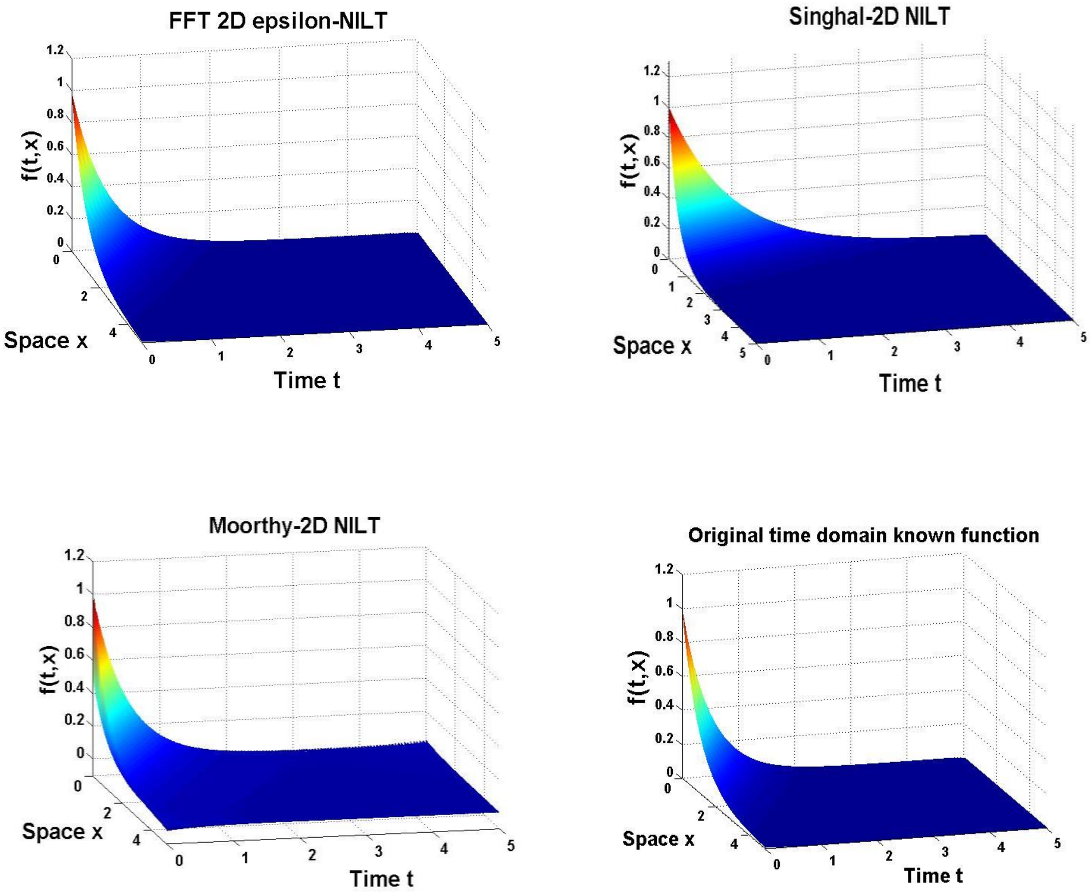

Figure 1: Results of the inverted function 1 with different NILTs and the known original

4. CONCLUSION

Three methods for 2D NILTs are described, implemented and compared in the paper. The NILT methods presented are devised based on using Fourier series techniques or Padé approximations which are classified as methods with good performance. The accuracy of the methods is emphasized by testing them with two functions that have previously known originals. We have programmed the methods and implemented them by using the universal mathematical language Matlab, and the results were shown in two tables and presented with the 3D plots of the inverted test function. The work is currently continued further by testing more methods with a larger number of functions of interest in electrical engineering along with the absolute error 3D plots and with further application of the methods on continuous space-time systems.

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