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SEMI-OPEN SETS IN CLOSURE SPACES

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To my parents

Abstract

The aim of this Ph.D. thesis is to introduce semi-open sets in closure spaces and study their fundamental properties. The semi-open sets are used to define semi-open maps, semi-closed maps, semi-continuous maps, contra-semi-continuous maps and semi-irresolute maps which are investigated. They are also used to introduce a new type of connectedness and compactness in closure spaces, the so-called s-connectedness and s-compactness, respectively. Further, we introduce and study generalized semi-open sets. We define generalized semi-continuous maps and generalized semi-irresolute maps by using generalized semi-open sets and study their behaviour. Another type of open sets in closure spaces, namely γ-open sets, are also introduced and some of their properties are studied, too. The concepts of γ-continuous maps and γ-irresolute maps are introduced by using γ-open sets. We also investigate the interrelation between generalized-semi-open sets and γ-open sets in closure spaces. We define a notion of semi-open sets in biclosure spaces and investigate its behaviour. Finally, the concepts of semi-open maps, semi-closed maps, semi-continuous maps, semi-irresolute maps and pre-semi-open maps of biclosure spaces are introduced and studied.

Key Words

Closure operator; closure space; semi-open set; semi-continuous map; contra-semi-continuous map; semi-irresolute map; generalized semi-open sets; generalized semi-continuous map; generalized semi-irresolute map; γ-open sets; γ-continuous map; γ-irresolute map; biclosure space; pre semi-open map

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Contents

Chapter 1

Introduction

The study of semi-open sets in topological spaces was initiated in 1963 by N. Levine in [16]. If (X,τ) is a topological space and $A \subseteq X$, then A is *semi-open* if there exists $O \in \tau$ such that $O \subseteq A \subseteq Cl(O)$, where $Cl(O)$ denotes the closure of O in (X,τ). Furthermore, Levine used semi-open sets to define semi-continuous maps in topological spaces. By utilizing the concept of semi-open sets, semi-closed sets in topological spaces were introduced by N. Biswas [3] in 1969 as complements of semi-open sets. Further, semi-open sets and semi-closed sets were used to define semi-open and semi-closed maps in topological spaces. After the work of Levine and Biswas, various mathematicians turned their attention to the generalizations of topology obtained by considering semi-open sets instead of open sets and many interesting results have been obtained - see for instance, [1], [4], [11] and [15].

In 1969, bitopological spaces were introduced by J. C. Kelly [14] as triples (X,τ_1,τ_2) where X is a set and τ_1 and τ_2 are topologies defined on X. After that, a number of papers have been written to generalize topological concept to the bitopological setting-see for instance, [2], [12] and [13].

The purpose of this thesis is to study the concept of semi-open sets in closure spaces and in biclosure spaces. Closure spaces were introduced by E.Čech [6] in 1966 and then studied by many mathematicians, see e.g. [7], [8], [9], [10], [17] and [18]. Closure spaces are sets endowed with a grounded, extensive and monotone closure operator. Biclosure spaces were introduced in [5] as triples (X,u_1,u_2) where X is a set and u_1, u_2 are closure operators on X.

In this thesis, we divide the text into six chapters.

The first chapter is an introduction which contains some remarks about the past research. We also explain our motivations and outline the aims of the thesis.

In the second chapter, we recall some definitions concerning closure spaces and some of their properties. Among others, unions, intersections and Cartesian product of closure spaces are investigated.

In the third chapter, we introduce semi-open sets in a closure space and study their unions, intersections and Cartesian product. These semi-open sets are used to define a certain types of maps, namely semi-continuous maps and semi-irresolute maps which will also be studied. Further, semi-open sets are used to introduce and study a new type of connectedness (respectively, compactness) called s-connectedness (respectively, s-compactness).

In the fourth chapter, we introduce two new types of open sets in closure space called generalized semi-open and γ-open. We then study some of their basic properties. As an application, three new kinds of closure spaces, namely T_{gs}-spaces, T_{γ}-spaces and $T_{s\gamma}$-spaces are introduced and some of their characterizations are studied. Further, we introduce generalized semi-continuous and γ-continuous maps by using generalized semi-open sets and γ-open sets, respectively. Moreover, two new types of maps called generalized semi-irresolute and γ-irresolute are introduced and studied. Further, we study the interrelation among generalized-semi-continuous, γ-continuous, generalized semi-irresolute and γ-irresolute maps.

In the fifth chapter, we recall the concepts of biclosure spaces and investigate some of their properties, e.g. unions, intersections and subspaces. We introduce and study semi-open sets in biclosure spaces. The notions of semi-continuous maps and semi-irresolute maps of biclosure spaces are defined and investigated. At the end of the chapter, we introduce pre-semi-open maps and pre-semi-closed maps obtained by using semi-open sets and semi-closed sets, respectively. We then study some of their properties.

In the last chapter, we make conclusions of the obtained results and we outline the direction of the further research.

Chapter 2

Closure Spaces

In this chapter, we study some fundamental properties of closure spaces. First, we recall some basic definitions.

A map $u : P(X) \to P(X)$ defined on the power set $P(X)$ of a set X is called a *closure operator* on X and the pair (X,u) is called a *closure space* if the following axioms are satisfied:

(A1) $u\phi = \phi$,

(A2) $A \subseteq uA$ for every $A \subseteq X$,

(A3) $A \subseteq B \Rightarrow uA \subseteq uB$ for all $A, B \subseteq X$.

A closure operator u on a set X is called *additive* (respectively, *idempotent*) if $A, B \subseteq X \Rightarrow u(A \cup B) = uA \cup uB$ (respectively, $A \subseteq X \Rightarrow uuA = uA$).

A subset $A \subseteq X$ is *closed* in the closure space (X,u) if $uA = A$. It is called *open* if its complement in X is closed. The empty set and the whole space are both open and closed.

A closure space (Y,v) is said to be a *subspace* of (X,u) if $Y \subseteq X$ and $vA = uA \cap Y$ for each subset $A \subseteq Y$.

Let (X,u) and (Y,v) be closure spaces. A map $f : (X,u) \to (Y,v)$ is called *open* (respectively, *closed*) if the image of every open (respectively, closed) set in (X,u) is open (respectively, closed) in (Y,v).

A map $f : (X,u) \to (Y,v)$ is said to be *continuous* if $f(uA) \subseteq vf(A)$ for every $A \subseteq X$. One can see that if f is continuous, then the inverse image under f of every open (respectively, closed) set in (Y,v) is open (respectively, closed) in (X,u).

A closure space (X,u) is said to be *connected* if ϕ and X are the only subsets of X which are both closed and open.

A collection $\{G_\alpha\}_{\alpha \in J}$ of sets in a closure space X is called a *cover* of a subset B of X if $B \subseteq \bigcup_{\alpha \in J} G_\alpha$ holds, and an *open cover* if G_α is open for each $\alpha \in J$.

Furthermore, a cover $\{G_\alpha\}_{\alpha \in J}$ of a subset B contains a *finite subcover,* if there exists a finite subset J_0 of J such that $B \subseteq \bigcup_{\alpha \in J_0} G_\alpha$.

A subset A of a closure space (X,u) is *compact* if every open cover of A contains a finite subcover.

The following statement is evident:

Proposition 2.1. *If* $\{A_\alpha\}_{\alpha \in J}$ *is a collection of subsets in a closure space* (X,u), *then* $\bigcup_{\alpha \in J} uA_\alpha \subseteq u \bigcup_{\alpha \in J} A_\alpha$ *and* $u \bigcap_{\alpha \in J} A_\alpha \subseteq \bigcap_{\alpha \in J} uA_\alpha$.

The following example shows that the inclusions of Proposition 2.1 cannot be replaced by equalities in general.

Example 2.2. Let $X=\{1,2,3\}$ and define a closure operator u on X by $u\phi=\phi$, $u\{1\}=\{1\}$, $u\{2\}=\{2\}$, $u\{3\}=\{3\}$ and $u\{1,2\}=u\{1,3\}=u\{2,3\}=uX=X$. It is easy to see that $u\{1\}\cup u\{2\}=\{1,2\}$ but $u(\{1\}\cup\{2\})=X$. It follows that $u\{1\}\cup u\{2\}\neq u(\{1\}\cup\{2\})$. Further, we have $u(\{1\}\cap\{2,3\})=u\phi=\phi$ but $u\{1\}\cap u\{2,3\}=\{1\}$. Hence, $u(\{1\}\cap\{2,3\})\neq u\{1\}\cap u\{2,3\}$.

Proposition 2.3. *Let $\{A_\alpha\}_{\alpha\in J}$ be a collection of closed sets in a closure space (X,u). Then $\bigcap_{\alpha\in J} A_\alpha$ is a closed set.*

Proof. Since $\bigcap_{\alpha\in J} A_\alpha \subseteq A_\alpha$ for each $\alpha\in J$, $u\bigcap_{\alpha\in J} A_\alpha \subseteq uA_\alpha$ for all $\alpha\in J$. Since $\{A_\alpha\}_{\alpha\in J}$ is a collection of closed sets, $uA_\alpha=A_\alpha$ for all $\alpha\in J$. Hence, $u\bigcap_{\alpha\in J} A_\alpha \subseteq A_\alpha$ for each $\alpha\in J$. Thus, $u\bigcap_{\alpha\in J} A_\alpha \subseteq \bigcap_{\alpha\in J} A_\alpha$. Since $\bigcap_{\alpha\in J} A_\alpha \subseteq u\bigcap_{\alpha\in J} A_\alpha$, $\bigcap_{\alpha\in J} A_\alpha = u\bigcap_{\alpha\in J} A_\alpha$. Therefore, $\bigcap_{\alpha\in J} A_\alpha$ is a closed set. ❒

If $\{A_\alpha\}_{\alpha\in J}$ is a collection of open sets in a closure space (X,u), then $\bigcap_{\alpha\in J} A_\alpha$ need not be an open set as shown in the following example.

Example 2.4. In the closure space from Example 2.2, it is easy to see that $\{1,2\}$ and $\{1,3\}$ are open but $\{1,2\}\cap\{1,3\}=\{1\}$ is not open in (X,u).

As a direct consequence of Proposition 2.3, we have:

Proposition 2.5. *Let $\{A_\alpha\}_{\alpha\in J}$ be a collection of closed sets in a closure space (X,u). Then $u\bigcap_{\alpha\in J} A_\alpha = \bigcap_{\alpha\in J} uA_\alpha$.*

Proposition 2.6. *Let $\{A_\alpha\}_{\alpha\in J}$ be a collection of open sets in a closure space (X,u). Then $\bigcup_{\alpha\in J} A_\alpha$ is an open set.*

Proof. Clearly, the complement of $\bigcup_{\alpha\in J} A_\alpha$ is $\bigcap_{\alpha\in J}(X-A_\alpha)$. Since A_α is open for each $\alpha\in J$, $X-A_\alpha$ is closed for all $\alpha\in J$. But $\bigcap_{\alpha\in J}(X-A_\alpha)$ is a closed set by Proposition 2.3. Therefore, $\bigcup_{\alpha\in J} A_\alpha$ is open. ❒

Let $\{A_\alpha\}_{\alpha\in J}$ be a collection of closed sets in a closure space (X,u). Then $\bigcup_{\alpha\in J} A_\alpha$ need not be closed set as shown in the following example.

Example 2.7. In the closure space from Example 2.2, it is easy to see that $\{1\}$ and $\{2\}$ are closed but $\{1\}\cup\{2\}=\{1,2\}$ is not closed in (X,u).

Proposition 2.8. *Let (X,u) be a closure space. If G is a subset of (X,u), then $uG-G$ has no nonempty open subset.*

Proof. Let G be a subset in (X,u) and H be a nonempty open subset of $uG-G$. Then there is $x \in H \subseteq uG-G \subseteq uG$ but $x \notin X-H$. Since H is open, we have $X-H = u(X-H)$. Hence $x \notin u(X-H)$, i.e., uG is not contained in $u(X-H)$. But $H \subseteq uG-G$, hence $G \subseteq uG-H \subseteq X-H$. It follows that uG is contained in $u(X-H)$, which is a contradiction. Therefore, $uG-G$ contains no nonempty open set. ❐

Proposition 2.9. *Let (Y,v) be a closure subspace of (X,u). If G is an open set in X, then $G \cap Y$ is an open set in (Y,v).*

Proof. Let G is an open subset of (X,u). Then $v(Y-(G\cap Y)) = u(Y-(G\cap Y)) \cap Y \subseteq u(X-G)\cap Y = (X-G)\cap Y = Y-(G\cap Y)$. Therefore, $G\cap Y$ is open in (Y,v). ❐

The converse of Proposition 2.9 is not true as shown in the following example.

Example 2.10. Let $X = \{1,2,3\}$ and define a closure operator u on X by $u\phi = \phi$, $u\{1\} = \{1,2\}$, $u\{2\} = \{2,3\}$, $u\{3\} = \{3\}$ and $u\{1,2\} = u\{1,3\} = u\{2,3\} = uX = X$. Thus, there are only three open subsets of (X,u), namely ϕ, $\{1,2\}$ and X. Let $Y = \{1,2\}$ and (Y,v) be a closure subspace of (X,u). Then $v\phi = \phi$, $v\{2\} = \{2\}$ and $v\{1\} = vY = Y$. We can see that $\{1\}$ is an open subset of (Y,v) but there is no any open set G in (X,u) such that $\{1\} = G\cap Y$.

Proposition 2.11. *Let (X,u) and (Y,v) be closure spaces. If $f:(X,u)\to(Y,v)$ is a continuous map, then the inverse image under f of each open set in (Y,v) is open in (X,u).*

Proof. Let H be open in (Y,v). Then $Y-H$ is closed in (Y,v). Hence, $f^{-1}(Y-H) \subseteq X$. But f is continuous, so we have $f\left(uf^{-1}(Y-H)\right) \subseteq vf\left(f^{-1}(Y-H)\right) \subseteq v(Y-H)$. Hence, $f^{-1}\left(f\left(uf^{-1}(Y-H)\right)\right) \subseteq f^{-1}(v(Y-H))$. Thus, $uf^{-1}(Y-H) \subseteq f^{-1}(v(Y-H)) = f^{-1}(Y-H)$. Consequently, $f^{-1}(Y-H)$ is a closed set in (X,u). But $f^{-1}(Y-H) = f^{-1}(Y) - f^{-1}(H) = X - f^{-1}(H)$, hence $f^{-1}(H)$ is open in (X,u). ❐

Remark 2.12. In a closure space, if a map $f:(X,u)\to(Y,v)$ is such that the inverse image under f of each open set in (Y,v) is open in (X,u), then f need not be continuous as shown in the following example.

Example 2.13. Let $X = \{1,2,3\}$, $Y = \{a,b,c\}$ and define a closure operator u on X by $u\phi = \phi$, $u\{1\} = u\{3\} = u\{1,2\} = u\{1,3\} = u\{2,3\} = uX = X$ and $u\{2\} = \{2\}$. Define a closure operator v on Y by $v\phi = \phi$, $v\{a\} = \{a,c\}$, $v\{b\} = \{b\}$, $v\{c\} = \{a,c\}$ and $v\{a,b\} = v\{a,c\} = v\{b,c\} = vY = Y$. Let $f:(X,u)\to(Y,v)$ be defined by $f(1) = a$, $f(2) = b$ and $f(3) = c$. It is easy to see that there are only three open sets, ϕ, Y and $\{a,c\}$ in Y. We have that $f^{-1}(\phi) = \phi$, $f^{-1}(Y) = X$ and $f^{-1}(\{a,c\}) = \{1,3\}$ are open sets in (X,u). But f is not continuous because $f(u\{3\}) = f(X) = Y$ is not contained in $vf(\{3\}) = v\{c\} = \{a,c\}$.

The following statement is evident:

Proposition 2.14. *Let* (X,u), (Y,v) *and* (Z,w) *be closure spaces, let* $f:(X,u)\to(Y,v)$ *and* $g:(Y,v)\to(Z,w)$ *be maps. Then:*

a) *If* f *and* g *are open, then so is* $g\circ f$.
b) *If* $g\circ f$ *is open and* f *is a continuous surjection, then* g *is open.*
c) *If* $g\circ f$ *is open and* g *is a continuous injection, then* f *is open.*

Proposition 2.15. *Let* (X,u) *and* (Y,v) *be closure spaces and let* $f:(X,u)\to(Y,v)$ *be a map. If* f *is open, then for every* $y\in Y$ *and every closed subset* F *of* (X,u) *such that* $f^{-1}(\{y\})\subseteq F$, *there exists a closed subset* K *of* (Y,v) *such that* $y\in K$ *and* $f^{-1}(K)\subseteq F$.

Proof. Suppose that f is open. Let $y\in Y$ and F be a closed subset of (X,u) such that $f^{-1}(\{y\})\subseteq F$. Then $f(X-F)$ is an open set in (Y,v). Let $K=\ Y-f(X-F)$. Then K is closed in (Y,v) and $f^{-1}(K)=f^{-1}(Y-f(X-F))=X-f^{-1}(f(X-F))\subseteq\ X-(X-F)=F$. But $y\in Y-(Y-\{y\})\subseteq\ Y-f\left(f^{-1}(Y-\{y\})\right)=Y-f\left(X-f^{-1}(\{y\})\right)\subseteq Y-f(X-F)$, hence $y\in K$. Therefore, K is a closed subset of (Y,v) such that $y\in K$ and $f^{-1}(K)\subseteq F$.

The converse of Proposition 2.15 is not true in general as can be seen from the following example.

Example 2.16. Let $X=\{1,2,3\}=Y$ and define a closure operator u on X by $u\phi=\phi$, $u\{1\}=u\{2\}=u\{1,2\}=\{1,2,\}$ and $u\{3\}=u\{1,3\}=u\{2,3\}=uX=X$. Define a closure operator v on Y by $v\phi=\phi$, $v\{1\}=\{1\}$, $v\{2\}=\{2\}$, $v\{3\}=v\{1,2\}=v\{1,3\}=v\{2,3\}=vY=Y$. Let $f:(X,u)\to(Y,v)$ be the identity map. Then there are only three closed subset of (X,u), namely ϕ, $\{1,2\}$ and X . Moreover, there are only four closed subset of (Y,v), namely ϕ, $\{1\}$, $\{2\}$ and Y. Then for every $y\in Y$ and every closed subset F of (X,u) such that $f^{-1}(\{y\})\subseteq F$, there exists a closed subset K of (Y,v) such that $y\in K$ and $f^{-1}(K)\subseteq F$. But f is not open because $\{3\}$ is open in (X,u) but $f(\{3\})$ is not open in (Y,v).

In [1], E.Čech defines the product $\prod_{\alpha\in J}(X_\alpha,u_\alpha)$ of a family $\{(X_\alpha,u_\alpha):\alpha\in J\}$ of closure spaces to be the closure space $\left(\prod_{\alpha\in J}X_\alpha,u\right)$ where $\prod_{\alpha\in J}X_\alpha$ denotes the Cartesian product of the sets X_α, $\alpha\in J$ and u is the closure operator generated by the projections $\pi_\alpha:\prod_{\alpha\in J}X_\alpha\to X_\alpha$, $\alpha\in J$, i.e., defined by $uA=\prod_{\alpha\in\alpha}u_\alpha\pi_\alpha(A)$ for each $A\subseteq\prod_{\alpha\in J}X_\alpha$. Clearly, π_α is continuous for each $\alpha\in J$.

Proposition 2.17. *Let* $\{(X_\alpha,u_\alpha):\alpha\in J\}$ *be a family of closure spaces. Then* F_α *is closed in* (X_α,u_α) *for all* $\alpha\in J$ *if and only if* $\prod_{\alpha\in J}F_\alpha$ *is closed in* $\prod_{\alpha\in J}(X_\alpha,u_\alpha)$.

Proof. Let $\alpha \in J$ and F_α be a closed subset of (X_α, u_α). Then $F_\alpha = u_\alpha F_\alpha$. But $\pi_\alpha\left(\prod_{\alpha\in J} F_\alpha\right) = F_\alpha$, hence, $\prod_{\alpha\in J} F_\alpha = \prod_{\alpha\in J} u_\alpha F_\alpha = \prod_{\alpha\in J} u_\alpha \pi_\alpha\left(\prod_{\alpha\in J} F_\alpha\right)$. Therefore, $\prod_{\alpha\in J} F_\alpha$ is closed in $\prod_{\alpha\in J}(X_\alpha, u_\alpha)$.

Conversely, let $\alpha \in J$ and $F_\alpha \subseteq X_\alpha$. Suppose that $\prod_{\alpha\in J} F_\alpha$ is closed in $\prod_{\alpha\in J}(X_\alpha, u_\alpha)$. Then $\prod_{\alpha\in J} F_\alpha = \prod_{\alpha\in J} u_\alpha \pi_\alpha\left(\prod_{\alpha\in J} F_\alpha\right)$. Hence, $\pi_\alpha\left(\prod_{\alpha\in J} F_\alpha\right) = \pi_\alpha\left(\prod_{\alpha\in J} u_\alpha \pi_\alpha\left(\prod_{\alpha\in J} F_\alpha\right)\right)$. Thus, $F_\alpha = u_\alpha F_\alpha$. Therefore, F_α is closed in (X_α, u_α) for all $\alpha \in J$. ❒

We will need the following, quite obvious properties of the Cartesian product of sets:

Lemma 2.18. *Let $\{(X_\alpha, u_\alpha) : \alpha \in J\}$ be a collection of closure spaces, $\beta \in J$ and π_β be a projection map. If $G \subseteq \prod_{\alpha\in J} X_\alpha$ and $(x_\alpha)_{\alpha\in J} \in G$, then $\{c_\beta\} \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in J}} \{\pi_\alpha((x_\alpha)_{\alpha\in J})\} \subseteq \prod_{\alpha\in J} X_\alpha - G$ for all $c_\beta \in X_\beta - \pi_\beta(G)$.*

Proof. Let $G \subseteq \prod_{\alpha\in J} X_\alpha$ and $(x_\alpha)_{\alpha\in J} \in G$. Let $\beta \in J$ and $c_\beta \in X_\beta - \pi_\beta(G)$. Then $c_\beta \notin \pi_\beta(G)$. Hence, $\{c_\beta\} \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in J}} \{\pi_\alpha((x_\alpha)_{\alpha\in J})\} \not\subset \pi_\beta(G) \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in J}} \pi_\alpha(G) = \prod_{\alpha\in J} \pi_\alpha(G)$. Clearly, $G \subseteq \prod_{\alpha\in J} \pi_\alpha(G)$. Consequently, $\{c_\beta\} \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in J}} \{\pi_\alpha((x_\alpha)_{\alpha\in J})\} \not\subset G$. But $\{c_\beta\} \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in J}} \{\pi_\alpha((x_\alpha)_{\alpha\in J})\}$ is singleton, thus $\{c_\beta\} \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in J}} \{\pi_\alpha((x_\alpha)_{\alpha\in J})\} \subseteq \prod_{\alpha\in J} X_\alpha - G$. ❒

Lemma 2.19. *Let $\{(X_\alpha, u_\alpha) : \alpha \in J\}$ be a collection of closure spaces and let $\beta \in J$. If $G \subseteq \prod_{\alpha\in J} X_\alpha$ and π_β is a projection map, then $X_\beta - \pi_\beta(G) \subseteq \pi_\beta\left(\prod_{\alpha\in J} X_\alpha - G\right)$.*

Proof. Let $\beta \in J$ and $G \subseteq \prod_{\alpha\in J} X_\alpha$.

If $G = \phi$, then $\pi_\beta\left(\prod_{\alpha\in J} X_\alpha - G\right) = X_\beta$. Thus, $X_\beta - \pi_\beta(G) \subseteq \pi_\beta\left(\prod_{\alpha\in J} X_\alpha - G\right)$.

If $G \neq \phi$, i.e. there exists $(x_\alpha)_{\alpha\in J} \in G$. Let $c_\beta \in X_\beta - \pi_\beta(G)$. By Lemma 2.18, $\{c_\beta\} \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in J}} \{\pi_\alpha((x_\alpha)_{\alpha\in J})\} \subseteq \prod_{\alpha\in J} X_\alpha - G$. It follows that $c_\beta \in \pi_\beta\left(\prod_{\alpha\in J} X_\alpha - G\right)$. Therefore, $X_\beta - \pi_\beta(G) \subseteq \pi_\beta\left(\prod_{\alpha\in J} X_\alpha - G\right)$. ❒

Proposition 2.20. *Let* $\{(X_\alpha,u_\alpha):\alpha\in J\}$ *be a collection of closure spaces. If* G *is an open subset of* $\prod_{\alpha\in J}(X_\alpha,u_\alpha)$*, then* $\pi_\alpha(G)$ *is open in* (X_α,u_α) *for every projection map* π_α*,* $\alpha\in J$ *.*

Proof. Let G be an open subset of $\prod_{\alpha\in J}(X_\alpha,u_\alpha)$. Suppose that there exists $\beta\in J$ such that $\pi_\beta(G)$ is not open in X_β. Since X_β is open, $\pi_\beta(G)\neq X_\beta$, i.e. $X_\beta-\pi_\beta(G)\neq\phi$. Hence, there exists $m_\beta\in X_\beta-\pi_\beta(G)$. By the hypothesis, $u_\beta(X_\beta-\pi_\beta(G))$ is not contained in $X_\beta-\pi_\beta(G)$. Thus, there exists $x_\beta^*\in u_\beta(X_\beta-\pi_\beta(G))$ but $x_\beta^*\notin X_\beta-\pi_\beta(G)$, i.e. $x_\beta^*\in\pi_\beta(G)$. Hence, there exists $(x_\alpha)_{\alpha\in J}\in G$ such that $\pi_\beta((x_\alpha)_{\alpha\in J})=x_\beta^*$. Since $m_\beta\in X_\beta-\pi_\beta(G)$, $\{m_\beta\}\times\prod_{\substack{\alpha\neq\beta\\ \alpha\in J}}\{\pi_\alpha((x_\alpha)_{\alpha\in J})\}\subseteq\prod_{\alpha\in J}X_\alpha-G$ by Lemma 2.18. Consequently, $\pi_\alpha((x_\alpha)_{\alpha\in J})\in\pi_\alpha\left(\prod_{\alpha\in J}X_\alpha-G\right)\subseteq u_\alpha\pi_\alpha\left(\prod_{\alpha\in J}X_\alpha-G\right)$ for all $\alpha\neq\beta$, $\alpha\in J$. Since $x_\beta^*\in u_\beta(X_\beta-\pi_\beta(G))$, $x_\beta^*\in u_\beta\pi_\beta\left(\prod_{\alpha\in J}X_\alpha-G\right)$ by Lemma 2.19. It follows that, $\{x_\beta^*\}\times\prod_{\substack{\alpha\neq\beta\\ \alpha\in J}}\{\pi_\alpha((x_\alpha)_{\alpha\in J})\}\subseteq\prod_{\alpha\in J}u_\alpha\pi_\alpha\left(\prod_{\alpha\in J}X_\alpha-G\right)$. But $\pi_\beta((x_\alpha)_{\alpha\in J})=x_\beta^*$, hence $\{x_\beta^*\}\times\prod_{\substack{\alpha\neq\beta\\ \alpha\in J}}\{\pi_\alpha((x_\alpha)_{\alpha\in J})\}=\{\pi_\beta((x_\alpha)_{\alpha\in J})\}\times\prod_{\substack{\alpha\neq\beta\\ \alpha\in J}}\{\pi_\alpha((x_\alpha)_{\alpha\in J})\}=\prod_{\alpha\in J}\{\pi_\alpha((x_\alpha)_{\alpha\in J})\}=\{(x_\alpha)_{\alpha\in J}\}$.

Consequently, $\{(x_\alpha)_{\alpha\in J}\}\subseteq\prod_{\alpha\in J}u_\alpha\pi_\alpha\left(\prod_{\alpha\in J}X_\alpha-G\right)$, i.e. $(x_\alpha)_{\alpha\in J}\in\prod_{\alpha\in J}u_\alpha\pi_\alpha\left(\prod_{\alpha\in J}X_\alpha-G\right)$. But $(x_\alpha)_{\alpha\in J}\in G$, i.e. $(x_\alpha)_{\alpha\in J}\notin\prod_{\alpha\in J}X_\alpha-G$. Hence, $\prod_{\alpha\in J}u_\alpha\pi_\alpha\left(\prod_{\alpha\in J}X_\alpha-G\right)$ is not contained in $\prod_{\alpha\in J}X_\alpha-G$. Thus, G is not open in $\prod_{\alpha\in J}(X_\alpha,u_\alpha)$, which is a contradiction. Therefore, $\pi_\beta(G)$ is open in X_β for all $\beta\in J$. ❒

The converse of Proposition 2.20 is not true as shown by the following example.

Example 2.21. Let $X_1=\{1,2\}$, $X_2=\{a,b\}$ and define a closure operator u_1 on X_1 by $u_1\phi=\phi$, $u_1\{1\}=\{1\}$ and $u_1\{2\}=u_1X_1=X_1$. Define a closure operator u_2 on X_2 by $u_2\phi=\phi$, $u_2\{a\}=\{a\}$ and $u_2\{b\}=u_2X_2=X_2$. Let $\pi_1:X_1\times X_2\to X_1$ and $\pi_2:X_1\times X_2\to X_2$ be the projection maps. Then $\{(2,b)\}\subseteq X_1\times X_2$ such that $\pi_1(\{(2,b)\})=\{2\}$ and $\pi_2(\{(2,b)\})=\{b\}$ are open in (X_1,u_1) and (X_2,u_2), respectively. But $\{(2,b)\}$ is not open in $(X_1,u_1)\times(X_2,u_2)$.

Remark 2.22. By Proposition 2.20, a projection map $\pi_\alpha:\prod_{\alpha\in J}(X_\alpha,u_\alpha)\to(X_\alpha,u_\alpha)$ is open for all $\alpha\in J$.

Chapter 3

Semi-open Sets in Closure Spaces

3.1 Semi-Open Sets

In this section, we introduce a new class of open sets in closure spaces, called semi-open sets, and we study some of its properties.

Definition 3.1.1. Let (X,u) be a closure space. A subset A of X is called a *semi-open set* if there exists an open set G in (X,u) such that $G \subseteq A \subseteq uG$. A subset $A \subseteq X$ is called a *semi-closed set* if its complement is semi-open.

Clearly, if A is open (respectively, closed) in (X,u), then A is semi-open (respectively, semi-closed) in (X,u). The converse is not true as shown in the following example.

Example 3.1.2. Let $X = \{1,2,3\}$ and define a closure operator u on X by $u\phi = \phi$, $u\{1\} = \{1\}$, $u\{3\} = u\{1,3\} = \{1,3\}$ and $u\{2\} = u\{1,2\} = u\{2,3\} = uX = X$. It is easy to see that $\{1,2\}$ is semi-open because there is an open set $\{2\}$ such that $\{2\} \subseteq \{1,2\} \subseteq u\{2\}$. But $\{1,2\}$ is not open. And we also see that $\{3\}$ is semi-closed but not closed.

Regarding the union of semi-open sets and the intersection of semi-closed sets we have the following:

Proposition 3.1.3. *Let $\{A_\alpha\}_{\alpha \in J}$ be a collection of semi-open sets in a closure space (X,u). Then $\bigcup_{\alpha \in J} A_\alpha$ is a semi-open subset of (X,u).*

Proof. For each $\alpha \in J$, we have an open set G_α such that $G_\alpha \subseteq A_\alpha \subseteq uG_\alpha$. Thus, $\bigcup_{\alpha \in J} G_\alpha \subseteq \bigcup_{\alpha \in J} A_\alpha \subseteq \bigcup_{\alpha \in J} uG_\alpha$. By Proposition 2.1, $\bigcup_{\alpha \in J} uG_\alpha \subseteq u \bigcup_{\alpha \in J} G_\alpha$. Therefore, $\bigcup_{\alpha \in J} G_\alpha \subseteq \bigcup_{\alpha \in J} A_\alpha \subseteq u \bigcup_{\alpha \in J} G_\alpha$. But $\bigcup_{\alpha \in J} G_\alpha$ is open, hence $\bigcup_{\alpha \in J} A_\alpha$ is a semi-open set. ❒

The intersection of two semi-open sets need not be a semi-open set as can be seen from Example 2.2: $\{1,2\} \cap \{1,3\}$ is not semi-open while $\{1,2\}$ and $\{1,3\}$ are semi-open in (X,u).

Proposition 3.1.4. *Let $\{A_\alpha\}_{\alpha \in J}$ be a collection of semi-closed sets in a closure space (X,u). Then $\bigcap_{\alpha \in J} A_\alpha$ is semi-closed.*

Proof. Let A_α be a semi-closed set in (X,u) for all $\alpha \in J$. Then $X - A_\alpha$ is semi-open for each $\alpha \in J$. By Proposition 3.1.3, $\bigcup_{\alpha\in J}(X - A_\alpha)$ is semi-open. But $\bigcup_{\alpha\in J}(X - A_\alpha) = X - \bigcap_{\alpha\in J} A_\alpha$, thus $\bigcap_{\alpha\in J} A_\alpha$ is semi-closed. ❒

The union of two semi-closed sets need not be a semi-closed set as can be seen from Example 2.2: $\{1\}\cup\{2\}$ is not semi-closed but $\{1\}$ and $\{2\}$ are semi-closed sets in (X,u).

Proposition 3.1.5. *Let (X,u) be a closure space and u be idempotent. If A is semi-open in (X,u) and $A \subseteq B \subseteq uA$, then B is semi-open.*

Proof. Let A be semi-open in (X,u). Then there exists an open set G in (X,u) such that $G \subseteq A \subseteq uG$, hence $uA \subseteq uuG$. Since u is idempotent, $uA \subseteq uG$. Thus, $G \subseteq A \subseteq B \subseteq uA \subseteq uG$. Therefore, B is semi-open. ❒

Proposition 3.1.6. *Let (Y,v) be a closure subspace of (X,u) and $A \subseteq Y$. If A is a semi-open set in (X,u), then A is a semi-open set in (Y,v).*

Proof. Let A be semi-open in (X,u). Then there exists an open set G in (X,u) such that $G \subseteq A \subseteq uG$. Since $A \subseteq Y$, $G \subseteq Y$ and $G = G \cap Y \subseteq A \cap Y \subseteq uG \cap Y = vG$. But $A \cap Y = A$, thus $G \subseteq A \subseteq vG$. Since $G = G \cap Y$, G is open in (Y,v). Hence, A is a semi-open set in (Y,v). ❒

The converse of Proposition 3.1.6 need not be true as can be seen from the following example.

Example 3.1.7. Let $X = \{1,2,3\}$ and define a closure operator u on X by $u\phi = \phi$, $u\{1\} = \{1,3\}$, $u\{2\} = \{2,3\}$ and $u\{3\} = u\{1,2\} = u\{1,3\} = u\{2,3\} = uX = X$. Let $Y = \{1,2\}$ and (Y,v) be a closure subspace of (X,u). Then $v\phi = \phi$, $v\{1\} = \{1\}$, $v\{2\} = \{2\}$ and $vY = Y$. It is easy to see that $\{1\}$ is semi-open in (Y,v) but is not semi-open in (X,u).

Proposition 3.1.8. *Let (X,u) be a closure space and let $A \subseteq X$. Then A is semi-closed if and only if there exists a closed set F in (X,u) such that $X - u(X - F) \subseteq A \subseteq F$.*

Proof. Let A be semi-closed. Then there exists an open set G in (X,u) such that $G \subseteq X - A \subseteq uG$. Thus, there exists a closed set F in (X,u) such that $G = X - F$ and $X - F \subseteq X - A \subseteq u(X - F)$. Therefore, $X - u(X - F) \subseteq A \subseteq F$.

Conversely, by the assumption, there is a closed set F in (X,u) such that $X - u(X - F) \subseteq A \subseteq F$. Thus, there exists an open set G in (X,u) such that $F = X - G$ and $X - uG \subseteq A \subseteq X - G$. It follows that $G \subseteq X - A \subseteq uG$. Therefore, A is semi-closed in (X,u). ❒

Definition 3.1.9. Let (X,u) and (Y,v) be closure spaces. A map $f:(X,u) \to (Y,v)$ is called *semi-open* (respectively, *semi-closed*) if the image of every open set (respectively, closed set) in (X,u) is semi-open (respectively, semi-closed) in (Y,v).

Clearly, if f is open (respectively, closed), then f is semi-open (respectively, semi-closed). The converse is not true as can be seen from the following example.

Example 3.1.10. Let $X=\{1,2,3\}$ and $Y=\{a,b,c\}$. Define a closure operator u on X by $u\phi=\phi$, $u\{1\}=\{1\}$ and $u\{2\}=u\{3\}=u\{1,2\}=u\{1,3\}=u\{2,3\}=uX=X$. Define a closure operator v on Y by $v\phi=\phi$, $v\{a\}=v\{c\}=v\{a,c\}=\{a,c\}$ and $v\{b\}=v\{a,b\}=v\{b,c\}=vY=Y$. Let $f:(X,u)\to(Y,v)$ be defined by $f(1)=a$, $f(2)=c$ and $f(3)=b$. It is easy to see that f is semi-open but not open because $f(\{2,3\})=\{b,c\}$ is not open in (Y,v) while $\{2,3\}$ is open in (X,u). And we also see that f is semi-closed but not closed.

Proposition 3.1.11. *Let (X,u), (Y,v) and (Z,w) be closure spaces, let $f:(X,u)\to(Y,v)$ and $g:(Y,v)\to(Z,w)$ be maps. Then:*

(i) If f is open and g is semi-open, then $g\circ f$ is semi-open.

(ii) If $g\circ f$ is semi-open and f is a continuous surjection, then g is semi-open.

Proof. (i) Let G be an open subset of (X,u). Since f is open, $f(G)$ is open in (Y,v). Hence $g(f(G))$ is semi-open in (Z,w). Thus, $g\circ f$ is semi-open.

(ii) Let G be an open subset of (Y,v). Since f is a continuous map, $f^{-1}(G)$ is open in (X,u). Since $g\circ f$ is semi-open, $g\circ f\left(f^{-1}(G)\right)=g\left(f\left(f^{-1}(G)\right)\right)$ is semi-open in (Z,w). But f is surjection, so that $g\circ f\left(f^{-1}(G)\right)=g(G)$. Hence, $g(G)$is semi-open in (Z,w). Therefore, g is semi-open. ❒

Proposition 3.1.12. *Let $\{(X_\alpha,u_\alpha):\alpha\in J\}$ be a collection of closure spaces. Let $\alpha\in J$ and $A\subseteq\prod_{\alpha\in J}X_\alpha$. If A is semi-open in $\prod_{\alpha\in J}(X_\alpha,u_\alpha)$ and π_α is a projection map, then $\pi_\alpha(A)$ is semi-open in (X_α,u_α).*

Proof. Let $\alpha\in J$ and A be a semi-open subset of $\prod_{\alpha\in J}(X_\alpha,u_\alpha)$. Then there exists an open subset G of $\prod_{\alpha\in J}(X_\alpha,u_\alpha)$ such that $G\subseteq A\subseteq\prod_{\alpha\in J}u_\alpha\pi_\alpha(G)$. It follows that $\pi_\alpha(G)\subseteq \pi_\alpha(A)\subseteq\pi_\alpha\left(\prod_{\alpha\in J}u_\alpha\pi_\alpha(G)\right)=u_\alpha\pi_\alpha(G)$. By Proposition 2.20, $\pi_\alpha(G)$ is open in (X_α,u_α). Therefore, $\pi_\alpha(A)$ is semi-open in (X_α,u_α).

❒

Remark 3.1.13. The converse of Proposition 3.1.12 need not be true in general. By Example 2.21, $\pi_1(\{(2,b)\})$ and $\pi_2(\{(2,b)\})$ are semi-open sets in (X_1,u_1) and (X_2,u_2), respectively. But $\{(2,b)\}$ is not semi-open in $(X_1,u_1)\times(X_2,u_2)$.

Proposition 3.1.14. *Let $\{(X_\alpha,u_\alpha):\alpha\in J\}$ be a collection of closure spaces. Let $\beta\in J$ and $A_\beta\subseteq X_\beta$. Then A_β is semi-open in (X_β,u_β) if and only if $A_\beta\times\prod_{\substack{\alpha\neq\beta\\ \alpha\in J}}X_\alpha$ is semi-open in $\prod_{\alpha\in J}(X_\alpha,u_\alpha)$.*

Proof. Let $\alpha \in J$ and π_β be a projection map. Let A_β be a semi-open subset of (X_β, u_β). Then there exists an open subset G_β of (X_β, u_β) such that $G_\beta \subseteq A_\beta \subseteq u_\beta G_\beta$. Hence, $G_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha \subseteq A_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha \subseteq u_\beta G_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha = \prod_{\alpha \in J} u_\alpha \pi_\alpha \left(G_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha \right)$. As π_β is continuous, $\pi_\beta^{-1}(G_\beta) = G_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha$ is open in $\prod_{\alpha \in J} (X_\alpha, u_\alpha)$. Therefore, $A_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha$ is semi-open.

The converse follows immediately from Proposition 3.1.12.

❒

Theorem 3.1.15. *Let $\{(X_\alpha, u_\alpha) : \alpha \in J\}$ be a family of closure spaces and let $\beta \in J$. Then A_β is a semi-closed subset of (X_β, u_β) if and only if $A_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha$ is semi-closed in $\prod_{\alpha \in J} (X_\alpha, u_\alpha)$.*

Proof. Let $\beta \in J$ and A_β be a semi-closed subset of (X_β, u_β). Then $X_\beta - A_\beta$ is semi-open in (X_β, u_β). By Proposition 3.1.14, $(X_\beta - A_\beta) \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha$ is a semi-open subset of $\prod_{\alpha \in J} (X_\alpha, u_\alpha)$. But $(X_\beta - A_\beta) \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha = \prod_{\alpha \in J} X_\alpha - A_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha$. It follows that $A_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha$ is semi-closed in $\prod_{\alpha \in J} (X_\alpha, u_\alpha)$.

Conversely, suppose that $A_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha$ is a semi-closed subset of $\prod_{\alpha \in J} (X_\alpha, u_\alpha)$. Then $\prod_{\alpha \in J} X_\alpha - A_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha = (X_\beta - A_\beta) \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha$ is semi-open in $\prod_{\alpha \in J} (X_\alpha, u_\alpha)$. By Proposition 3.1.14, $X_\beta - A_\beta$ is semi-open in (X_β, u_β). Therefore, A_β is semi-closed.

❒

3.2 Semi-continuous Maps

Definition 3.2.1. Let (X, u) and (Y, v) be closure spaces. A map $f : (X, u) \to (Y, v)$ is called *semi-continuous* if the inverse image under f of every open set in (Y, v) is semi-open in (X, u).

Clearly, if f is continuous, then f is semi-continuous. The converse is not true in general as shown in the following example.

Example 3.2.2. Let $X = \{1,2,3\}$, $Y = \{a,b,c\}$ and define a closure operator u on X by $u\phi = \phi$, $u\{1\} = u\{1,3\} = \{1,3\}$, $u\{2\} = u\{2,3\} = \{2,3\}$, $u\{3\} = \{3\}$ and $u\{1,2\} = uX = X$ .Define a closure operator v on Y by $v\phi = \phi$, $v\{a\} = v\{a,b\} = \{a,b\}$, $v\{b\} = \{b\}$, $v\{c\} = \{c\}$,

$v\{b,c\}=\{b,c\}$ and $v\{a,c\}=vY=Y$. Let $f:(X,u)\to(Y,v)$ be defined by $f(1)=a$, $f(2)=c$ and $f(3)=b$. It is easy to see that f is semi-continuous, but f is not continuous because $\{a,b\}$ is open in (Y,v) and $f^{-1}(\{a,b\})=\{1,3\}$ but $\{1,3\}$ is not open in (X,u).

Proposition 3.2.3. *Let (X,u) and (Y,v) be closure spaces and let $f:(X,u)\to(Y,v)$ be a map. Then f is semi-continuous if and only if the inverse image under f of every closed subset of (Y,v) is semi-closed in (X,u).*

Proof. Let F be a closed subset in (Y,v). Then $Y-F$ is open in (Y,v). Since f is semi-continuous, $f^{-1}(Y-F)$ is semi-open. But $f^{-1}(Y-F)=X-f^{-1}(F)$, thus $f^{-1}(F)$ is semi-closed in (X,u).

Conversely, let G be an open subset in (Y,v). Then $Y-G$ is closed in (Y,v). Since the inverse image of each closed subset in (Y,v) is semi-closed in (X,u), $f^{-1}(Y-G)$ is semi-closed in (X,u). But $f^{-1}(Y-G)=X-f^{-1}(G)$, thus $f^{-1}(G)$ is semi-open. Therefore, f is semi-continuous. ❐

Proposition 3.2.4. *Let (X,u), (Y,v) and (Z,w) be closure spaces, let $f:(X,u)\to(Y,v)$ and $g:(Y,v)\to(Z,w)$ be maps. If $g\circ f$ is open and g is a semi-continuous injection, then f is semi-open.*

Proof. Let G be an open subset of (X,u). Since $g\circ f$ is open, $g(f(G))$ is open in (Z,w). As g is semi-continuous, $g^{-1}(g(f(G)))$ is semi-open in (Y,v). But g is injective, so that $g^{-1}(g(f(G)))=f(G)$ is semi-open in (Y,v). Therefore, f is semi-open. ❐

The following statement is evident:

Proposition 3.2.5. *Let (X,u) and (Y,v) be closure spaces. If $f:(X,u)\to(Y,v)$ is a bijection, then the following statements are equivalent:*

(i) The inverse map $f^{-1}:(Y,v)\to(X,u)$ is semi-continuous.

(ii) f is a semi-open map.

(iii) f is a semi-closed map.

Proposition 3.2.6. *Let (X,u), (Y,v) and (Z,w) be closure spaces. If $f:(X,u)\to(Y,v)$ is semi-continuous and $g:(Y,v)\to(Z,w)$ is continuous, then $g\circ f$ is semi-continuous.*

Proof. Let H be an open subset of (Z,w). Since g is continuous, $g^{-1}(H)$ is open in (Y,v). Since f is semi-continuous, $f^{-1}(g^{-1}(H))$ is semi-open in (X,u). But $f^{-1}(g^{-1}(H))=(g\circ f)^{-1}(H)$. Therefore, $g\circ f$ is semi-continuous. ❐

The following example shows that Proposition 3.2.6 need not be true if g is not continuous.

Example 3.2.7. Let $X=\{1,2,3\}$, $Y=\{a,b,c\}$, $Z=\{x,y,z\}$ and define a closure operator u on X by $u\phi=\phi$, $u\{1\}=\{1,3\}$, $u\{2\}=u\{3\}=u\{2,3\}=\{2,3\}$ and $u\{1,2\}=u\{1,3\}=uX=X$. Define closure operator v on Y by $v\phi=\phi$, $v\{b\}=v\{c\}=v\{b,c\}=\{b,c\}$ and $v\{a\}=v\{a,b\}=$ $v\{a,c\}=$ $vY=Y$. Define closure operator w on Z by $w\phi=\phi$, $w\{z\}=\{z\}$ and $w\{x\}=w\{y\}=$ $w\{x,y\}=w\{x,z\}=w\{y,z\}=wZ=Z$. Let $f:(X,u)\to(Y,v)$ be defined by $f(1)=a$, $f(2)=$ b and $f(3)=c$. Let $g:(Y,v)\to(Z,w)$ be defined by $g(a)=x$, $g(b)=y$ and $g(c)=z$. It is easy to see that f is continuous, hence f is semi-continuous. And we also see that g is only semi-continuous but not continuous. Since $\{x,y\}$ is open in (Z,w) but $(g\circ f)^{-1}(\{x,y\})$ is not semi-open in (X,u), $g\circ f$ is not semi-continuous.

Definition 3.2.8. A closure space (X,u) is said to be a T_s-*space* if every semi-open set in (X,u) is open. The closure space (X,u) in Example 2.2 is a T_s-space.

Proposition 3.2.9. *Let (X,u) and (Z,w) be closure spaces and (Y,v) be a T_s-space . If $f:(X,u)\to(Y,v)$ and $g:(Y,v)\to(Z,w)$ are semi-continuous, then $g\circ f$ is semi-continuous.*

Proof. Let H be open in (Z,w). Since g is semi-continuous, $g^{-1}(H)$ is semi-open in (Y,v). But (Y,v) is a T_s-*space*, hence $g^{-1}(H)$ is open in (Y,v). Thus, $f^{-1}(g^{-1}(H))=$ $(g\circ f)^{-1}(H)$ is semi-open in (X,u). Therefore, $g\circ f$ is semi-continuous. ❐

Theorem 3.2.10. *Let (X,u) be a closure space, $\{(Y_\alpha,v_\alpha):\alpha\in J\}$ be a family of closure spaces and $f:(X,u)\to\prod_{\alpha\in J}(Y_\alpha,v_\alpha)$ be a map. If f is semi-continuous and π_α is a projection map, then $\pi_\alpha\circ f$ is semi-continuous for each $\alpha\in J$.*

Proof. Assume that $f:(X,u)\to\prod_{\alpha\in J}(Y_\alpha,v_\alpha)$ is semi-continuous for all $\alpha\in J$. Since π_α is continuous, $\pi_\alpha\circ f$ is semi-continuous for each $\alpha\in J$ by Proposition 3.2.6. ❐

Definition 3.2.11. Let (X,u) and (Y,v) be closure spaces. A map $f:(X,u)\to(Y,v)$ is called *contra-semi-continuous* if the inverse image under f of every open set in (Y,v) is semi-closed in (X,u).

Remark 3.2.12. The concepts of a semi-continuous map and a contra-semi-continuous map are independent as shown by two following examples.

Example 3.2.13. Let $X=\{1,2\}=Y$ and define a closure operator u on X by $u\phi=\phi$, $u\{2\}=\{2\}$ and $u\{1\}=uX=X$. Define closure operator v on Y by $v\phi=\phi$, $v\{1\}=\{1\}$ and $v\{2\}=vY=Y$. Let $f:(X,u)\to(Y,v)$ be the identity map. It is easy to see that f is contra-semi-continuous but not semi-continuous because $\{2\}$ is open in (Y,v) but $f^{-1}(\{2\})$ is not semi-open in (X,u).

Example 3.2.14. In Example 3.2.7, the map f is semi-continuous but not contra-semi-continuous because $\{a\}$ is open in (Y,v) but $f^{-1}(\{a\})$ is not semi-closed in (X,u).

Proposition 3.2.15. *Let* (X,u) *and* (Y,v) *be closure spaces and let* $f:(X,u)\to(Y,v)$ *be a map. Then* f *is contra-semi-continuous if and only if the inverse image under* f *of every closed subset of* (Y,v) *is semi-open in* (X,u).

Proof. Let F be a closed subset in (Y,v). Then $Y-F$ is open in (Y,v). Since f is contra-semi-continuous, $f^{-1}(Y-F)$ is semi-closed. But $f^{-1}(Y-F)=X-f^{-1}(F)$, thus $f^{-1}(F)$ is semi-open in (X,u).

Conversely, let G be an open subset in (Y,v). Then $Y-G$ is closed in (Y,v). Since the inverse image of each closed subset in (Y,v) is semi-open in (X,u), $f^{-1}(Y-G)$ is semi-open in (X,u). But $f^{-1}(Y-G)=X-f^{-1}(G)$, thus $f^{-1}(G)$ is semi-closed. Therefore, f is contra-semi-continuous. ❐

Proposition 3.2.16. *Let* (X,u), (Y,v) *and* (Z,w) *be closure spaces, let* $f:(X,u)\to(Y,v)$ *and* $g:(Y,v)\to(Z,w)$ *be maps. If* $g\circ f$ *is contra-semi-continuous and* g *is a closed injection, then* f *is contra-semi-continuous.*

Proof. Let H be a closed subset of (Y,v). Since g is closed, $g(H)$ is closed in (Z,w). As $g\circ f$ is contra-semi-continuous, $(g\circ f)^{-1}(g(H))=f^{-1}(g^{-1}(g(H)))$ is semi-open in (X,u) by Proposition 3.2.15. But g is injective, hence $f^{-1}(g^{-1}(g(H)))=f^{-1}(H)$. Therefore, f is contra-semi-continuous. ❐

Proposition 3.2.17. *Let* (X,u) *and* (Z,w) *be closure spaces and* (Y,v) *be a* T_s*-space . If* $f:(X,u)\to(Y,v)$ *and* $g:(Y,v)\to(Z,w)$ *are contra-semi-continuous maps, then* $g\circ f$ *is semi-continuous.*

Proof. Let H be closed in (Z,w). Since g is contra-semi-continuous, $g^{-1}(H)$ is semi-open in (Y,v). But (Y,v) is a T_s-space, hence $g^{-1}(H)$ is open in (Y,v). As f is contra-semi-continuous by Proposition 3.2.15, $f^{-1}(g^{-1}(H))=(g\circ f)^{-1}(H)$ is semi-closed in (X,u). Therefore, $g\circ f$ is semi-continuous by Proposition 3.2.3. ❐

The following statement is evident:

Proposition 3.2.18. *Let* (X,u), (Y,v) *and* (Z,w) *be closure spaces and let* $f:(X,u)\to(Y,v)$ *and* $g:(Y,v)\to(Z,w)$ *be maps. If* f *is contra-semi-continuous and* g *is continuous, then* $g\circ f$ *is contra-semi-continuous.*

As a direct consequence of Proposition 3.2.18, we have:

Proposition 3.2.19. *Let* (X,u) *be a closure space,* $\{(Y_\alpha,v_\alpha):\alpha\in J\}$ *be a family of closure spaces and* $f:(X,u)\to\prod_{\alpha\in J}(Y_\alpha,v_\alpha)$ *be a map. If* f *is contra-semi-continuous and* π_α *is a projection map, then* $\pi_\alpha\circ f$ *is contra-semi-continuous for each* $\alpha\in J$.

3.3 Semi-irresolute Maps

Definition 3.3.1. Let (X,u) and (Y,v) be closure spaces. A map $f:(X,u)\to(Y,v)$ is called *semi-irresolute* if $f^{-1}(G)$ is semi-open in (X,u) for every semi-open set G in (Y,v).

Proposition 3.3.2. *Let (X,u) and (Y,v) be closure spaces and $f:(X,u)\to(Y,v)$ be a map. Then f is semi-irresolute if and only if $f^{-1}(B)$ is semi-closed in (X,u), whenever B is semi-closed in (Y,v).*

Proof. Let B be a semi-closed subset of (Y,v). Then $Y-B$ is semi-open in (Y,v). Since $f:(X,u)\to(Y,v)$ is semi-irresolute, $f^{-1}(Y-B)$ is semi-open in (X,u). But $f^{-1}(Y-B)= X-f^{-1}(B)$, so that $f^{-1}(B)$ is semi-closed in (X,u).

Conversely, let A be a semi-open subset in (Y,v). Then $Y-A$ is semi-closed in (Y,v). By the assumption, $f^{-1}(Y-A)$ is semi-closed in (X,u). But $f^{-1}(Y-A)=X-f^{-1}(A)$, thus $f^{-1}(A)$ is semi-open in (X,u). Therefore, f is semi-irresolute. ❒

Clearly, every semi-irresolute map is semi-continuous. The converse need not be true as can be seen from the following example.

Example 3.3.3. Let $X=\{1,2,3\}=Y$ and define a closure operator u on X by $u\phi=\phi$, $u\{1\}=\{1,2\}$, $u\{2\}=u\{3\}=u\{2,3\}=\{2,3\}$and $u\{1,2\}=u\{1,3\}=uX=X$. Define closure operator v on Y by $v\phi=\phi$, $v\{1\}=\{1,3\}$, $v\{3\}=\{3\}$, $v\{2\}=v\{2,3\}=\{2,3\}$ and $v\{1,2\}=$ $v\{1,3\}=vY=Y$. Let $f:(X,u)\to(Y,v)$ be the identity map. Then f is semi-continuous but not semi-irresolute because $\{1,3\}$ is semi-open in (Y,v) but $f^{-1}(\{1,3\})=$ $\{1,3\}$ is not semi-open in (X,u).

Proposition 3.3.4. *Let (X,u), (Y,v) and (Z,w) be closure spaces. If $f:(X,u)\to(Y,v)$ is a semi-irresolute map and $g:(Y,v)\to(Z,w)$ is a semi-continuous map, then the composition $g\circ f:(X,u)\to(Z,w)$ is semi-continuous.*

Proof. Let G be an open subset of (Z,w). Then $g^{-1}(G)$ is a semi-open subset of (Y,v) as g is semi-continuous. Hence, $f^{-1}(g^{-1}(G))$ is semi-open in (X,u) because f is semi- irresolute. Thus, $g\circ f$ is semi-continuous. ❒

The following statements are evident:

Proposition 3.3.5. *Let (X,u), (Y,v) and (Z,w) be closure spaces. If $f:(X,u)\to(Y,v)$ and $g:(Y,v)\to(Z,w)$ are semi-irresolute, then $g\circ f:(X,u)\to(Z,w)$ is semi-irresolute.*

Proposition 3.3.6. *Let (X,u) and (Z,w) be closure spaces and (Y,v) be a T_s-space. If $f:(X,u)\to(Y,v)$ is a semi-continuous map and $g:(Y,v)\to(Z,w)$ is a semi-irresolute map, then the composition $g\circ f:(X,u)\to(Z,w)$ is semi-irresolute.*

Proposition 3.3.7. *Let (X,u) and (Y,v) be closure spaces and $f:(X,u)\to(Y,v)$ be a bijective map. If f and f^{-1} are continuous, then f and f^{-1} are semi-irresolute.*

Proof. Let B be a semi-open subset of (Y,v). Then there exists an open set H in (Y,v) such that $H\subseteq B\subseteq vH$, hence $f^{-1}(H)\subseteq f^{-1}(B)\subseteq f^{-1}(vH)$. Since f^{-1} is continuous, $f^{-1}(vH)\subseteq uf^{-1}(H)$. But f is continuous, thus $f^{-1}(H)$ is open in (X,u). Hence, $f^{-1}(B)$ is semi-open in (X,u). Therefore, f is semi-irresolute.

Let A be a semi-open subset of (X,u). Then there exists an open set G in (X,u) such that $G\subseteq A\subseteq uG$. Hence, $f(G)\subseteq f(A)\subseteq f(uG)$. As f is continuous, $f(uG)\subseteq vf(G)$. Since f^{-1} is continuous and $f(G)$ is the inverse image of G under f^{-1}, $f(G)$ is open in (Y,v). Thus, $f(A)$ is semi-open in (Y,v). But $f(A)$ is the inverse image of A under f^{-1}, therefore f^{-1} is semi-irresolute. ❐

3.4 S-connectedness

Definition 3.4.1. A closure space (X,u) is said to be *s-connected* if ϕ and X are the only subsets of X which are both semi-open and semi-closed.

Clearly, if (X,u) is s-connected, then (X,u) is connected. The converse is not true as can be seen from the following example.

Example 3.4.2. Let $X=\{1,2,3\}$ and define a closure operator u on X by $u\phi=\phi$, $u\{1\}=u\{1,3\}=\{1,3\}$, $u\{2\}=u\{2,3\}=\{2,3\}$, $u\{3\}=\{3\}$, and $u\{1,2\}=uX=X$. We have that (X,u) is connected but not s-connected because $\{1,3\}$ is both semi-closed and semi-open in (X,u).

Proposition 3.4.3. *Let (X,u) be a closure space. Then the following statements are equivalent*:

a) *X is s-connected.*
b) *X cannot be expressed as the union of two disjoint, non-empty, semi-closed subsets.*
c) *X cannot be expressed as the union of two disjoint, non-empty, semi-open subsets.*

Proof. Statement (a) implies statement (b): Suppose that $X=U\cup V$ where U and V are non-empty, disjoint, semi-closed subsets of (X,u). Then $U=X-V$ and U is semi-open. Thus, U is a subset of X which is both semi-open and semi-closed but U is neither X nor ϕ. Hence, (X,u) is not s-connected.

Statement (b) implies statement (c): Suppose that $X=A\cup B$ where A and B are disjoint non-empty semi-open subsets of (X,u). Then $X-A=B$ and $X-B=A$ are both complements of semi-open sets and hence are semi-closed. Thus, $X=A\cup B$ is an expression of X as the union of two disjoint, non-empty, semi-closed subset of (X,u), which contradicts (b).

Statement (c) implies statement (a): Suppose that A is a subset of X which is both semi-open and semi-closed but A is neither X nor ϕ. Then $X-A$ is also semi-closed, semi-open and non-empty. Thus, $X=(X-A)\cup A$ is the expression of X as the union of two disjoint, non-empty semi-open subsets, which contradicts (c). ❐

The following statement is evident:

Proposition 3.4.4. *Let (X,u) be a T_s-space. Then (X,u) is connected if and only if (X,u) is s-connected.*

Proposition 3.4.5. *Let (X,u) be a closure space and let $Y=\{0,1\}$ and v be a closure operator on Y defined by $v\phi=\phi$, $v\{0\}=\{0\}$, $v\{1\}=\{1\}$ and $vY=Y$. Then the following statements are equivalent*:

a) *The only contra-semi-continuous maps $f:(X,u)\to(Y,v)$ are the constant maps.*
b) *A closure space (X,u) is s-connected.*

Proof. Statement (a) implies statement (b): Suppose that there is a non-empty subset A of (X,u) such that $A\neq X$ and A is both semi-open and semi-closed. Then $X-A$ is both semi-open and semi-closed in (X,u). Define a map $f:(X,u)\to(Y,v)$ by $f(x)=0$ if $x\in A$ and $f(x)=1$ if $x\in X-A$. Consequently, $f^{-1}(\phi)=\phi$, $f^{-1}(\{0\})=A$, $f^{-1}(\{1\})=B$ and $f^{-1}(Y)=X$. Since there are only four closed subsets of (Y,v), namely ϕ, $\{0\}$, $\{1\}$ and Y, the inverse image under f of any closed subset in (Y,v) is semi-open in (X,u). Thus, f is contra-semi-continuous but non-constant, which a contradiction. Therefore, (X,u) is s-connected.

Statement (b) implies statement (a): Suppose that a contra-semi-continuous map $f:(X,u)\to(Y,v)$, where the closure operator v on Y is defined by $v\phi=\phi$, $v\{0\}=\{0\}$, $v\{1\}=\{1\}$ and $vY=Y$, is non-constant. Then $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ are non-empty. Further, neither $f^{-1}(\{0\})$ nor $f^{-1}(\{1\})$ are equal to X. Since $\{0\}$ and $\{1\}$ are closed subset of (Y,v) and f is contra-semi-continuous, $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ are semi-open subsets of (X,u). But $f^{-1}(\{0\})=X-f^{-1}(\{1\})$, hence $f^{-1}(\{0\})$ is both semi-closed and semi-open. Consequently, X is not s-connected, which a contradiction ❒

Theorem 3.4.6. *Let (X,u) and (Y,v) be closure spaces and $f:(X,u)\to(Y,v)$ be a map.*

(i) If f is a contra-semi-continuous map from (X,u) onto (Y,v) and (X,u) is s-connected, then (Y,v) is connected.

(ii) If f is a semi-irresolute map from (X,u) onto (Y,v) and (X,u) is s-connected, then (Y,v) is s-connected.

Proof. (ii) Suppose that (Y,v) is not s-connected. Then there is a non-empty subset A of Y, $A\neq Y$ such that A is both semi-open and semi-closed. Since f is semi-irresolute, the set $f^{-1}(A)$ is both semi-open and semi-closed. Since f is an onto map and A is a non-empty subset of Y with $A\neq Y$, it follows that $f^{-1}(A)$ is a non-empty subset of X with $f^{-1}(A)\neq X$. Hence, (X,u) is not s-connected which a contradiction. Therefore, (Y,v) is s-connected.

The proof of (i) is similar to that of (ii). ❒

3.5 S-compactness

As another application of semi-open sets, a new kind of compactness, namely s-compactness, is introduced.

Definition 3.5.1. A collection $\{G_\alpha\}_{\alpha\in J}$ of semi-open sets in a closure space (X,u) is called a *semi-open cover* of a subset B of X if $B \subseteq \underset{\alpha\in J}{\cup} G_\alpha$ holds.

Definition 3.5.2. A subset A of a closure space (X,u) is *s-compact* if every semi-open cover of A contains a finite subcover. Cleary, if (X,u) is s-compact, then it is compact.

The following statements are evident:

Proposition 3.5.4. *Let (X,u) be a closure space. If X is s-compact and B is a semi-closed subset of X, then B is s-compact.*

Proof. Let $\{G_\alpha\}_{\alpha\in J}$ be a collection of semi-open subsets of X such that $B \subseteq \underset{\alpha\in J}{\cup} G_\alpha$. It follows that $X = \underset{\alpha\in J}{\cup} G_\alpha \cup (X-B)$. Since B is semi-closed, $X-B$ is semi-open. Consequently, $\underset{\alpha\in J}{\cup} G_\alpha \cup (X-B)$ is a semi-open cover of X. But X is s-compact, so $\underset{\alpha\in J}{\cup} G_\alpha \cup (X-B)$ contains a finite subcover, i.e. there exits a finite subset J_0 of J such that $X = \underset{\alpha\in J_0}{\cup} G_\alpha \cup (X-B)$. Since B and $X-B$ are disjoint, $B \subseteq \underset{\alpha\in J_0}{\cup} G_\alpha$. Thus, any semi-open cover $\{G_\alpha\}_{\alpha\in J}$ of B contains a finite subcover. Therefore, B is s-compact

❐

Proposition 3.5.4. *Let (X,u) and (Y,v) be closure spaces and $f:(X,u)\to(Y,v)$ be a map. If f is semi-irresolute and a subset B of X is s-compact, then the image $f(B)\subseteq Y$ is s-compact.*

Proof. Let $\{G_\alpha\}_{\alpha\in J}$ be a collection of semi-open subsets of Y such that $f(B) \subseteq \underset{\alpha\in J}{\cup} G_\alpha$. It follows that $B \subseteq f^{-1}(f(B)) \subseteq f^{-1}\left(\underset{\alpha\in J}{\cup} G_\alpha\right) = \underset{\alpha\in J}{\cup} f^{-1}(G_\alpha)$. But f is semi-irresolute, so $\{f^{-1}(G_\alpha)\}_{\alpha\in J}$ is a semi-open cover of B. Since B is s-compact, there exists a finite subset J_0 of J such that $B \subseteq \underset{\alpha\in J_0}{\cup} f^{-1}(G_\alpha)$. It follows that $f(B) \subseteq \underset{\alpha\in J_0}{\cup} (G_\alpha)$. Thus, any semi-open cover $\{G_\alpha\}_{\alpha\in J}$ of $f(B)$ contains a finite subcover. Therefore, $f(B)$ is s-compact.

❐

Proposition 3.5.5. *Let (X,u) and (Y,v) be closure spaces and $f:(X,u)\to(Y,v)$ be a map. If f is a semi-continuous surjection and X is s-compact, then Y is compact.*

Proof. Let $\{G_\alpha\}_{\alpha\in J}$ be a collection of open subsets of Y such that $Y \subseteq \underset{\alpha\in J}{\cup} G_\alpha$. It follows that $X = f^{-1}(Y) \subseteq f^{-1}\left(\underset{\alpha\in J}{\cup} G_\alpha\right) = \underset{\alpha\in J}{\cup} f^{-1}(G_\alpha)$. But f is semi-continuous, so $\{f^{-1}(G_\alpha)\}_{\alpha\in J}$ is a semi-open cover of X. Since X is s-compact, there exists a finite subset J_0 of J such that $X = \underset{\alpha\in J_0}{\cup} f^{-1}(G_\alpha)$. It follows that $f(X) = f\left(\underset{\alpha\in J_0}{\cup} f^{-1}(G_\alpha)\right) = f\left(f^{-1}\left(\underset{\alpha\in J_0}{\cup}(G_\alpha)\right)\right)$. Since f is a surjection, $Y = \bigcup_{\alpha \in J_0} G_\alpha$. Thus, any open cover $\{G_\alpha\}_{\alpha \in J}$ of Y contains a finite subcover. Therefore, Y is compact ❐

Proposition 3.5.6. *Let (X,u) and (Y,v) be closure spaces and $f:(X,u) \to (Y,v)$ be a map. If f is a semi-irresolute surjection and X is s-compact, then Y is s-compact.*

Proof. Let $\{A_\alpha\}_{\alpha \in J}$ be a collection of semi-open subsets of Y such that $Y \subseteq \bigcup_{\alpha \in J} A_\alpha$. It follows that $X = f^{-1}(Y) \subseteq f^{-1}\left(\bigcup_{\alpha \in J} A_\alpha\right) = \bigcup_{\alpha \in J} f^{-1}(A_\alpha)$. But f is semi-irresolute, hence $\{f^{-1}(A_\alpha)\}_{\alpha \in J}$ is a semi-open cover of X. Since X is s-compact, there exists a finite subset J_0 of J such that $X = \bigcup_{\alpha \in J_0} f^{-1}(A_\alpha)$. Hence, $f(X) = f\left(\bigcup_{\alpha \in J_0} f^{-1}(A_\alpha)\right) = f\left(f^{-1}\left(\bigcup_{\alpha \in J_0} (A_\alpha)\right)\right)$. Since f is a surjection, $Y = \bigcup_{\alpha \in J} A_\alpha$. Thus, any semi-open cover $\{A_\alpha\}_{\alpha \in J}$ of Y contains a finite subcover. Therefore, Y is s-compact ❐

Chapter 4

Generalized Semi-Open and γ-open Sets in Closure Spaces

4.1 Generalized Semi-open Sets

In this section, we introduce and study generalized-semi-open sets, briefly g-semi-open sets, in order to extend some properties of semi-open sets to a larger family of sets.

Definition 4.1.1. A subset B of a closure space (X,u) is called *generalized-semi-open*, briefly *g-semi-open*, if there exists a semi-open subset A of (X,u) such that $A \subseteq B \subseteq uA$. A subset B of X is called *generalized-semi-closed*, briefly *g-semi-closed*, if its complement is g-semi-open.

Remark 4.1.2. If A is semi-open (respectively, semi-closed) in a closure space (X,u), then A is g-semi-open (respectively, g-semi-closed). But the converse is not true as can be seen from the following example.

Example 4.1.3. Let $X = \{1,2,3,4\}$ and define closure operator u on X by $u\phi = \phi$, $u\{1\} = \{1,2\}$, $u\{2\} = u\{3\} = \{2,3\}$, $u\{4\} = \{2,4\}$, $u\{1,2\} = u\{1,3\} = \{1,2,3\}$, $u\{1,4\} = \{1,2,4\}$, $u\{2,3\} = u\{2,4\} = u\{3,4\} = u\{2,3,4\} = \{2,3,4\}$ and $u\{1,2,3\} = u\{1,2,4\} = u\{1,3,4\} = uX = X$. It is easy to see that $\{1,2,3\}$ is g-semi-open but not semi-open in (X,u). And we also see that $\{4\}$ is g-semi-closed but not semi-closed in (X,u).

Proposition 4.1.4. *Let $\{B_\alpha\}_{\alpha\in J}$ be a collection of g-semi-open sets in a closure space (X,u). Then $\bigcup_{\alpha\in J} B_\alpha$ is a g-semi-open set in (X,u).*

Proof. By the assumption, we have a semi-open set A_α such that $A_\alpha \subseteq B_\alpha \subseteq uA_\alpha$ for each $\alpha \in J$. Thus, $\bigcup_{\alpha\in J} A_\alpha \subseteq \bigcup_{\alpha\in J} B_\alpha \subseteq \bigcup_{\alpha\in J} uA_\alpha$. By Proposition 2.1, $\bigcup_{\alpha\in J} uA_\alpha \subseteq u \bigcup_{\alpha\in J} A_\alpha$. Hence, $\bigcup_{\alpha\in J} A_\alpha \subseteq \bigcup_{\alpha\in J} B_\alpha \subseteq u \bigcup_{\alpha\in J} A_\alpha$. By Proposition 3.1.3, $\bigcup_{\alpha\in J} A_\alpha$ is semi-open. Therefore, $\bigcup_{\alpha\in J} B_\alpha$ is g-semi-open. ❐

Proposition 4.1.5. *Let $\{B_\alpha\}_{\alpha\in J}$ be a collection of g-semi-closed sets in a closure space (X,u). Then $\bigcap_{\alpha\in J} B_\alpha$ is a g-semi-closed set in (X,u).*

Proof. Let A_α be g-semi-closed in (X,u) for all $\alpha \in J$. Then $X - A_\alpha$ is g-semi open for each $\alpha \in J$. By Proposition 4.1.4, $\bigcup_{\alpha\in J}(X - A_\alpha)$ is g-semi open. But $\bigcup_{\alpha\in J}(X - A_\alpha) = X - \bigcap_{\alpha\in J} A_\alpha$, hence $\bigcap_{\alpha\in J} A_\alpha$ is g-semi-closed. ❒

Proposition 4.1.6. *Let (X,u) be a closure space and let $B \subseteq X$. Then B is g-semi-closed if and only if there exists a semi-closed subset $E \subseteq X$ such that $X - u(X - E) \subseteq B \subseteq E$.*

Proof. Let B be a g-semi-closed subset of (X,u). Then there exists a semi-open subset A of (X,u) such that $A \subseteq X - B \subseteq uA$. Put $E = X - A$. It follows that E is semi-closed in (X,u) and $X - E \subseteq X - B \subseteq u(X - E)$. Therefore, $X - u(X - E) \subseteq B \subseteq E$.

Conversely, by the assumption, there is a semi-closed subset E of (X,u) such that $X - u(X - E) \subseteq B \subseteq E$. Put $A = X - E$. Consequently, A is semi-open in (X,u) and $X - uA \subseteq B \subseteq X - A$. It follows that $A \subseteq X - B \subseteq uA$. Therefore, $X - B$ is g-semi-open in (X,u), i.e. B is g-semi-closed. ❒

Proposition 4.1.7. *Let (X,u) be a closure space and u be idempotent. If B is a g-semi-open subset of (X,u) and $B \subseteq C \subseteq uB$, then C is g-semi-open.*

Proof. Let B be a g-semi-open subset of (X,u). Then there exists a semi-open subset A of (X,u) such that $A \subseteq B \subseteq uA$, hence $uB \subseteq uuA$. Since u is idempotent, $uB \subseteq uA$. Thus, $A \subseteq B \subseteq C \subseteq uB \subseteq uA$. Therefore, C is g-semi-open. ❒

Proposition 4.1.8. *Let (X,u) be a closure space. If G is a subset of (X,u), then $uG - G$ has no nonempty g-semi-open subset.*

Proof. Let G be a subset of (X,u) and $B \subseteq uG - G$. Let B be nonempty g-semi-open. Then there is a nonempty semi-open subset A of (X,u) such that $A \subseteq B \subseteq uA$. Hence, there exists a nonempty open subset H of (X,u) such that $H \subseteq A \subseteq uH$. Since $B \subseteq uG - G$, $G \subseteq uG - B$. But $H \subseteq B$, thus $uG - B \subseteq uG - H \subseteq X - H$, i.e. $G \subseteq X - H$. It follows that $uG \subseteq u(X - H)$. Since $H \neq \phi$, there exists $x \in H \subseteq A \subseteq B \subseteq uG - G \subseteq uG$. As $x \in H$, $x \notin X - H$. But H is open, hence $u(X - H) = X - H$. Thus, $x \notin u(X - H)$. Consequently, uG is not contained in $u(X - H)$, which is a contradiction. Therefore, $uG - G$ has no nonempty g-semi-open subset. ❒

Proposition 4.1.9. *Let (X,u) be a closure space and B be a subset of X. If (X,u) is a T_s-space and B is g-semi-open, then B is open.*

Proof. Let (X,u) be a T_s-space and B be a g-semi-open subset of (X,u). Then there exists a semi-open subset A of (X,u) such that $A \subseteq B \subseteq uA$. Since (X,u) is a T_s–space, A is open. Hence, B is semi-open. But (X,u) is a T_s-space, thus B is open. ❒

Proposition 4.1.10. *Let $\{(X_\alpha,u_\alpha) : \alpha \in J\}$ be a collection of closure spaces. Let $\alpha \in J$ and $B \subseteq \prod_{\alpha\in J} X_\alpha$. If B is g-semi-open in $\prod_{\alpha\in J}(X_\alpha,u_\alpha)$ and π_α is a projection map, then $\pi_\alpha(B)$ is g-semi-open in (X_α,u_α).*

Proof. Let $\alpha \in J$ and B be a g-semi-open subset of $\prod_{\alpha \in J}(X_\alpha, u_\alpha)$. Then there exists a semi-open subset A of $\prod_{\alpha \in J}(X_\alpha, u_\alpha)$ such that $A \subseteq B \subseteq \prod_{\alpha \in J} u_\alpha \pi_\alpha(A)$. It follows that $\pi_\alpha(A) \subseteq \pi_\alpha(B) \subseteq \pi_\alpha\left(\prod_{\alpha \in J} u_\alpha \pi_\alpha(A)\right) = u_\alpha \pi_\alpha(A)$. By Proposition 2.20, $\pi_\alpha(A)$ is semi-open. Therefore, $\pi_\alpha(B)$ is g-semi-open in (X_α, u_α). ❐

Proposition 4.1.11. *Let* $\{(X_\alpha, u_\alpha) : \alpha \in J\}$ *be a collection of closure spaces. Let* $\beta \in J$ *and* $B_\beta \subseteq X_\beta$. *Then* B_β *is g-semi-open in* (X_β, u_β) *if and only if* $B_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha$ *is g-semi-open in* $\prod_{\alpha \in J}(X_\alpha, u_\alpha)$.

Proof. Let $\alpha \in J$ and π_β be a projection map. Let B_β be a g-semi-open subset of (X_β, u_β). Then there exists a semi-open subset A_β of (X_β, u_β) such that $A_\beta \subseteq B_\beta \subseteq u_\beta A_\beta$. Hence, $A_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha \subseteq B_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha \subseteq u_\beta A_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha = \prod_{\alpha \in J} u_\alpha \pi_\alpha\left(A_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha\right)$. By Proposition 3.1.15, $A_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha$ is semi-open in $\prod_{\alpha \in J}(X_\alpha, u_\alpha)$. Therefore, $B_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha$ is g-semi-open.

The converse follows immediately from Proposition 4.1.10. ❐

4.2 *γ*-open Sets in Closure Spaces

Now, we introduce and study a new type of sets lying, as for generality, between the class of open sets and the class of semi-open sets.

Definition 4.2.1. A subset B of a closure space (X, u) is called *γ-open* if there exists an open subset G of X such that $G \subseteq B$ and $uG = uB$. A subset B of X is called *γ-closed* if its complement is *γ*-open.

Remark 4.2.2. If G is open (respectively, closed) in (X, u), then G is *γ*-open (respectively, *γ*-closed) in (X, u). But the converse is not true as shown in the following example.

Example 4.2.3. Let $X = \{1,2,3\}$ and define closure operator u on X by $u\phi = \phi$, $u\{2\} = u\{3\} = u\{2,3\} = \{2,3\}$ and $u\{1\} = u\{1,2\} = u\{1,3\} = uX = X$. It is easy to see that $\{1,2\}$ is *γ-open* but not open in (X, u). And we also see that $\{3\}$ is *γ*-closed but not closed in (X, u)

Remark 4.2.4. If B is γ-open (respectively, γ-closed) in a closure space (X,u), then B is semi-open (respectively, semi-closed) in (X,u). But the converse is not true as shown in the following example.

Example 4.2.5. Let $X=\{1,2,3\}$ and define closure operator u on X by $u\phi=\phi$, $u\{1\}=\{1,2\}$, $u\{2\}=u\{3\}=u\{2,3\}=\{2,3\}$ and $u\{1,2\}=u\{1,3\}=uX=X$. It is easy to see that $\{1,2\}$ is semi-open but not γ-open in (X,u). And we also see that $\{3\}$ is semi-closed but not γ-closed.

Remark 4.2.6. It follows from Remark 4.1.2, 4.2.2 and 4.2.4 that, for a subset G of a closure space (X,u), we have the following implications:

G is open $\rightarrow G$ is γ-open $\rightarrow G$ is semi-open $\rightarrow G$ is g-semi-open

Proposition 4.2.7. If (X,u) is a closure space where $u\bigcup_{\alpha\in J}A_\alpha=\bigcup_{\alpha\in J}uA_\alpha$ for all subsets A_α of X and $\{B_\alpha\}_{\alpha\in J}$ is a collection of γ-open sets in (X,u), then $\bigcup_{\alpha\in J}B_\alpha$ is γ-open.

Proof. Let $\alpha\in J$ and B_α be γ-open in (X,u). Then there exists an open set G_α such that $G_\alpha\subseteq B_\alpha$ and $uG_\alpha=uB_\alpha$. Hence $\bigcup_{\alpha\in J}G_\alpha\subseteq\bigcup_{\alpha\in J}B_\alpha$ and $\bigcup_{\alpha\in J}uG_\alpha=\bigcup_{\alpha\in J}uB_\alpha$. By the assumption, $\bigcup_{\alpha\in J}uG_\alpha=u\bigcup_{\alpha\in J}G_\alpha$ and $\bigcup_{\alpha\in J}uB_\alpha=u\bigcup_{\alpha\in J}B_\alpha$. Consequently, $u\bigcup_{\alpha\in J}G_\alpha=u\bigcup_{\alpha\in J}B_\alpha$. By Proposition 2.6, $\bigcup_{\alpha\in J}G_\alpha$ is open in (X,u). Therefore, $\bigcup_{\alpha\in J}B_\alpha$ is γ-open. ❐

Remark 4.2.8. If $\{B_\alpha\}_{\alpha\in J}$ is a collection of γ-open sets in a closure space (X,u) ,then $\bigcup_{\alpha\in J}B_\alpha$ need not be γ-open in (X,u) as shown by the following example.

Example 4.2.9. Let $X=\{1,2,3,4\}$ and define closure operator u on X by $u\phi=\phi$, $u\{2\}=\{2\}$, $u\{3\}=\{3\}$, $u\{1\}=u\{1,2\}=u\{1,3\}=\{1,2,3\}$, $u\{4\}=u\{2,3\}=u\{2,4\}=u\{3,4\}=u\{2,3,4\}=\{2,3,4\}$, $u\{1,4\}=u\{1,2,3\}=u\{1,2,4\}=u\{1,3,4\}=uX=X$. It is easy to see that $\{1,2\}$ and $\{1,3\}$ are γ-open but $\{1,2\}\cup\{1,3\}$ is not γ-open in (X,u).

Proposition 4.2.10. If (X,u) is a closure space where $u\bigcup_{\alpha\in J}A_\alpha=\bigcup_{\alpha\in J}uA_\alpha$ for all subsets A_α of X and $\{B_\alpha\}_{\alpha\in J}$ is a collection of γ- closed sets in (X,u), then $\bigcap_{\alpha\in J}B_\alpha$ is γ-closed.

Proof. Let $\alpha\in J$ and B_α be γ-closed in (X,u). Consequently, $X_\alpha-B_\alpha$ is γ-open. By Proposition 4.2.7, $\bigcup_{\alpha\in J}(X-B_\alpha)$ is γ-open . But $\bigcup_{\alpha\in J}(X-B_\alpha)=X-\bigcap_{\alpha\in J}B_\alpha$, hence $\bigcap_{\alpha\in J}B_\alpha$ is γ-closed. ❐

Remark 4.2.11. If $\{B_\alpha\}_{\alpha\in J}$ is a collection of γ-closed sets in a closure space (X,u) ,then $\bigcap_{\alpha\in J}B_\alpha$ need not be γ-closed in (X,u) as shown in the following example.

Example 4.2.12. In the closure space from Example 4.2.9, it is easy to see that $\{2,4\}$ and $\{3,4\}$ are γ-closed but $\{2,4\}\cap\{3,4\}=\{4\}$ is not γ-closed in (X,u).

Proposition 4.2.13. *Let (X,u) be a closure space and let $B \subseteq X$. Then B is γ-closed if and only if there exists a closed set E in (X,u) such that $B \subseteq E$ and $u(X-B)=u(X-E)$.*

Proof. Let B be a γ-closed subset of (X,u). Then there exists an open subset G of (X,u) such that $G \subseteq X-B$ and $uG=u(X-B)$. Put $E=X-G$. Then E is closed in (X,u). Thus, $X-E \subseteq X-B$ and $u(X-E)=u(X-B)$. Therefore, $B \subseteq E$ and $u(X-B)=u(X-E)$.

Conversely, by the assumption, there is a closed subset E of (X,u) such that $B \subseteq E$ and $u(X-B)=u(X-E)$. Put $G=X-E$. Then G is open in (X,u) and such that $G \subseteq X-B$ and $uG=u(X-B)$. It follows that $X-B$ is γ-open in (X,u). Therefore, B is γ-closed. ❒

Proposition 4.2.14. *Let (X,u) be a closure space and u be idempotent. If B is a γ-open subset of (X,u) and $B \subseteq C \subseteq uB$, then C is γ-open.*

Proof. Let B be a γ-open subset of (X,u). Then there exists an open subset G of (X,u) such that $G \subseteq B$ and $uG=uB$. But $B \subseteq C$, hence $G \subseteq C$. It follows that $uG \subseteq uC$. Since $C \subseteq uB$, $uC \subseteq uuB$. As u is idempotent, $uC \subseteq uB$. But $uB=uG$, thus $uC \subseteq uG$. Consequently, $uG=uC$. Therefore, C is γ-open. ❒

Proposition 4.2.15. *Let (X,u) be a closure space. If G is a subset of (X,u), then $uG-G$ has no nonempty γ-open subset.*

Proof. Let G be a subset of (X,u) and $B \subseteq uG-G$. Let B be γ-open. Then B is g-semi-open in (X,u). By Proposition 4.1.6, B is empty. Therefore, $uG-G$ has no nonempty γ-open subset. ❒

Proposition 4.2.16. *Let $\{(X_\alpha,u_\alpha):\alpha \in J\}$ be a collection of closure spaces. Let $\alpha \in J$ and $B \subseteq \prod_{\alpha\in J} X_\alpha$. If B is γ-open in $\prod_{\alpha\in J}(X_\alpha,u_\alpha)$ and π_α is a projection map, then $\pi_\alpha(B)$ is γ-open in (X_α,u_α).*

Proof. Let $\alpha \in J$ and B be a γ-open subset of $\prod_{\alpha\in J}(X_\alpha,u_\alpha)$. Then there exists an open subset G of $\prod_{\alpha\in J}(X_\alpha,u_\alpha)$ such that $G \subseteq B$ and $\prod_{\alpha\in J} u_\alpha\pi_\alpha(G)=\prod_{\alpha\in J} u_\alpha\pi_\alpha(B)$. It follows that $\pi_\alpha(G) \subseteq \pi_\alpha(B)$ and $u_\alpha\pi_\alpha(G)=\pi_\alpha\left(\prod_{\alpha\in J} u_\alpha\pi_\alpha(G)\right)=\pi_\alpha\left(\prod_{\alpha\in J} u_\alpha\pi_\alpha(B)\right)=u_\alpha\pi_\alpha(B)$. By Proposition 2.20, $\pi_\alpha(G)$ is open in (X_α,u_α). Therefore, $\pi_\alpha(B)$ is γ-open in (X_α,u_α). ❒

Proposition 4.2.17. *Let $\{(X_\alpha,u_\alpha):\alpha \in J\}$ be a collection of closure spaces. Let $\beta \in J$ and $B_\beta \subseteq X_\beta$. Then B_β is γ-open in (X_β,u_β) if and only if $B_\beta \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in J}} X_\alpha$ is γ-open in $\prod_{\alpha\in J}(X_\alpha,u_\alpha)$.*

Proof. Let $\alpha \in J$ and π_β be a projection map. Let B_β be a γ-open subset of (X_β,u_β). Then there exists an open subset G_β of (X_β,u_β) such that $G_\beta \subseteq B_\beta$ and $u_\beta G_\beta = u_\beta B_\beta$. Hence,

$G_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha \subseteq A_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha$ and $\prod_{\alpha \in J} u_\alpha \pi_\alpha \left(G_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha \right) = u_\beta G_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha = u_\beta B_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha =$ $\prod_{\alpha \in J} u_\alpha \pi_\alpha \left(B_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha \right)$. As π_β is continuous, $\pi_\beta^{-1}(G_\beta) = G_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha$ is open in $\prod_{\alpha \in J} (X_\alpha, u_\alpha)$. Therefore, $B_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha$ is γ-open.

The converse follows immediately from Proposition 4.2.16. ❐

4.3. T_γ -spaces, $T_{s\gamma}$ -spaces and T_{gs} -spaces

As applications of g-semi-open sets and γ-open sets, three new kinds of closure spaces are introduced, namely T_γ -spaces, $T_{s\gamma}$ -spaces and T_{gs} -spaces.

Definition 4.3.1. A closure space (X,u) is said to be a T_γ-*space* if every γ-open subset of (X,u) is open.

Definition 4.3.2. A closure space (X,u) is said to be a $T_{s\gamma}$-*space* if every semi-open subset of (X,u) is γ-open.

Definition 4.3.3. A closure space (X,u) is said to be a T_{gs}-*space* if every g-semi-open subset of (X,u) is semi-open.

Remark 4.3.4. The concepts of a T_γ-space and a $T_{s\gamma}$-space are independent, as can be seen from the following examples. By Example 4.2.5, (X,u) is a T_γ-space but not a $T_{s\gamma}$-space because $\{1,2\}$ is semi-open but not γ-open. By Example 4.2.3, (X,u) is a $T_{s\gamma}$-space but not a T_γ-space because $\{1,2\}$ is γ-open but not open.

Remark 4.3.5. The concepts of a T_γ-space and a T_{gs}-space are also independent. By Example 4.1.3, (X,u) is a T_γ-space but not a T_{gs}-space because $\{1,2,3\}$ is g-semi-open but not semi-open. By Example 4.2.3, (X,u) is a T_{gs}-space but not a T_γ –space.

Proposition 4.3.6. *Let (X,u) be a closure space and B be a subset of X. If (X,u) is a $T_{s\gamma}$-space and B is g-semi-open, then B is γ-open.*

Proof. Let (X,u) be a $T_{s\gamma}$-space and B be a g-semi-open subset of (X,u). Then there exists a semi-open subset A of (X,u) such that $A \subseteq B \subseteq uA$. Since (X,u) is a $T_{s\gamma}$-space, A is γ-open. Hence, there exists an open subset G of (X,u) such that $G \subseteq A \subseteq B \subseteq uA = uG$. Thus, B is semi-open. But (X,u) is a $T_{s\gamma}$-space, thus B is γ-open. ❐

Proposition 4.3.7. *Let* (X,u) *be a closure space. If* (X,u) *is a* $T_{s\gamma}$ *–space, then* (X,u) *is a* T_{gs}*-space.*

Proof. Let (X,u) be a $T_{s\gamma}$-space and let B be a g-semi-open subset of (X,u). Then B is γ-open by Proposition 4.3.6. It follows that B is semi-open in (X,u). Therefore, (X,u) is a T_{gs}-space. ❒

The converse of Proposition 4.3.7 need not be true in general. By Example 4.2.5, (X,u) is a T_{gs}-space but (X,u) is not a $T_{s\gamma}$-space because $\{1,2\}$ is semi-open but not γ-open.

Proposition 4.3.8. *Let* (X,u) *be a closure space. If* (X,u) *is a* T_s*-space, then* (X,u) *is a* T_{gs}*-space.*

Proof. Let (X,u) be a T_s-space and let B be a g-semi-open subset of (X,u). By Proposition 4.3.7, B is open in (X,u). It follows that B is semi-open in (X,u). Therefore, (X,u) is a T_{gs}-space. ❒

The converse of Proposition 4.3.8 need not be true in general. By Example 4.2.3, (X,u) is a T_{gs}-space but (X,u) is not a T_s-space because $\{1,2\}$ is semi-open but not open.

Proposition 4.3.9. *Let* (X,u) *be a closure space. Then* (X,u) *is a* T_s*-space if and only if* (X,u) *is both a* T_γ*-space and a* $T_{s\gamma}$*-space.*

Proof. Assume that (X,u) is a T_s-space. Let A be a γ-open subset of (X,u). Then A is semi-open in (X,u). But (X,u) is a T_s-space, hence A is open. Thus, (X,u) is a T_γ-space. Let B be a semi-open subset of (X,u). Since (X,u) is a T_s-space, B is open. Consequently, B is γ-open. Therefore, (X,u) is a $T_{s\gamma}$-space.

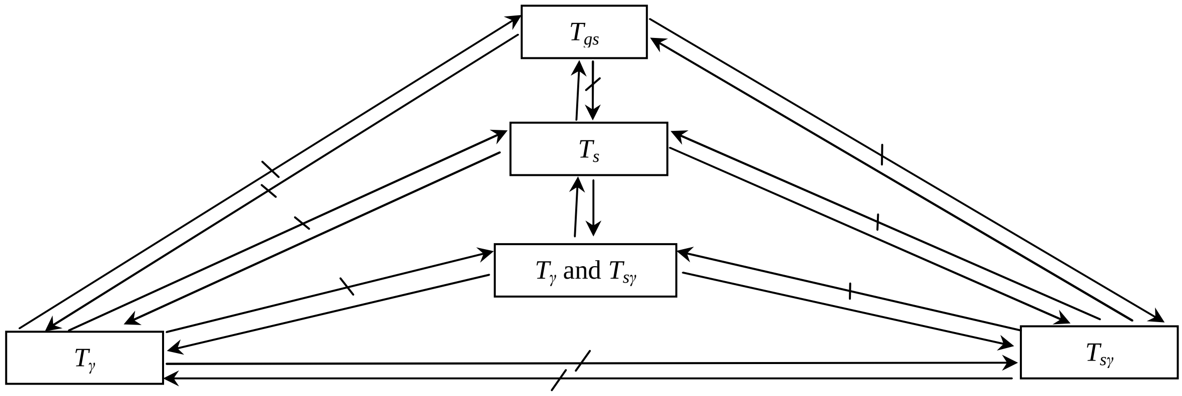
Conversely, suppose that (X,u) is both a T_γ-space and a $T_{s\gamma}$-space. Let A be a semi-open subset of (X,u). Since (X,u) is a $T_{s\gamma}$-space, A is γ-open. But (X,u) is a T_γ-space, thus A is open in (X,u). Therefore, (X,u) is a T_s-space. ❒

Remark 4.3.10. Let (X,u) be a closure space.

(i) (X,u) need not be a T_s-space even if (X,u) is a T_γ-space. By Example 4.2.5, (X,u) is a T_γ-space but not a T_s-space.

(ii) (X,u) need not be a T_s-space even if (X,u) is a $T_{s\gamma}$-space. By Example 4.2.3, (X,u) is a $T_{s\gamma}$-space but not a T_s-space.

Remark 4.3.11. The interrelation among T_s-spaces, T_{gs}-spaces, T_γ-spaces and $T_{s\gamma}$-spaces can be shown by the following diagram.



Here, $A \longrightarrow B$ means A implies B and $A \not\longrightarrow B$ means A does not necessarily imply B.

Proposition 4.3.12. *Let $\{(X_\alpha, u_\alpha) : \alpha \in J\}$ be a collection of closure spaces.*

(i) *If $\prod_{\alpha\in J}(X_\alpha, u_\alpha)$ is a T_s-space, then (X_α, u_α) is a T_s-space for all $\alpha \in J$.*

(ii) *If $\prod_{\alpha\in J}(X_\alpha, u_\alpha)$ is a T_{gs}-space, then (X_α, u_α) is a T_{gs}-space for all $\alpha \in J$.*

(iii) *If $\prod_{\alpha\in J}(X_\alpha, u_\alpha)$ is a T_γ-space, then (X_α, u_α) is a T_γ-space for all $\alpha \in J$.*

(iv) *If $\prod_{\alpha\in J}(X_\alpha, u_\alpha)$ is a $T_{s\gamma}$-space, then (X_α, u_α) is a $T_{s\gamma}$-space for all $\alpha \in J$.*

Proof. Let $\beta \in J$ and $\pi_\beta : \prod_{\alpha\in J} X_\alpha \to X_\beta$ be a projection map.

(i) Assume that $\prod_{\alpha\in J}(X_\alpha, u_\alpha)$ is a T_s-space. Let A_β be a semi-open subset of (X_β, u_β). By Proposition 3.1.15, $A_\beta \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in J}} X_\alpha$ is semi-open. But $\prod_{\alpha\in J}(X_\alpha, u_\alpha)$ is a T_s-space, hence $A_\beta \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in J}} X_\alpha$ is open. By Proposition 2.20, A_β is open in (X_β, u_β). Therefore, (X_β, u_β) is a T_s-space.

(ii) The proof is similar to that of (i) by utilizing Propositions 4.1.11 and 3.1.15.

(iii) The proof is similar to that of (i) by utilizing Propositions 4.2.17 and 2.20.

(iv) The proof is similar to that of (i) by utilizing Propositions 3.1.15 and 4.2.17. ❐

4.4 Generalized-semi-continuous and γ-continuous Maps

In this section, we introduce and study two new types of maps called generalized-semi-continuous maps and γ-continuous maps.

Definition 4.4.1. Let (X, u) and (Y, v) be closure spaces. A map $f : (X, u) \to (Y, v)$ is said to be *generalized-semi-continuous*, briefly *g-semi-continuous*, if $f^{-1}(G)$ is a g-semi-open subset of (X, u) for every open subset G of (Y, v).

Proposition 4.4.2. *A map* $f:(X,u)\to(Y,v)$ *is g-semi-continuous if and only if the inverse image under* f *of every closed subset of* (Y,v) *is g-semi-closed in* (X,u).

Proposition 4.4.3. *Let* (X,u), (Y,v) *and* (Z,w) *be closure spaces. Let* $f:(X,u)\to(Y,v)$ *and* $g:(Y,v)\to(Z,w)$ *be maps. If* f *is g-semi-continuous and* g *is continuous, then* $g\circ f$ *is g-semi-continuous.*

Definition 4.4.4. Let (X,u) and (Y,v) be closure spaces. A map $f:(X,u)\to(Y,v)$ is said to be *γ-continuous* if $f^{-1}(G)$ is a γ-open subset of (X,u) for every open subset G of (Y,v).

Proposition 4.4.5. *A map* $f:(X,u)\to(Y,v)$ *is γ-continuous if and only if the inverse image under* f *of every closed subset of* (Y,v) *is γ-closed in* (X,u).

Proposition 4.4.6. *Let* (X,u), (Y,v) *and* (Z,w) *be closure spaces. Let* $f:(X,u)\to(Y,v)$ *and* $g:(Y,v)\to(Z,w)$ *be maps. If* f *is γ-continuous and* g *is continuous, then* $g\circ f$ *is γ-continuous.*

Remark 4.4.7. For a map $f:(X,u)\to(Y,v)$, when (X,u) and (Y,v) are closure spaces, the following implications hold:

f is continuous $\to$ f is γ-continuous $\to$ f is semi-continuous $\to$ f is g-semi-continuous

Moreover, none of these implications is reversible as can be seen from the following examples.

Example 4.4.8. Let $Y=\{1,2,3\}$. Define a closure operator v on Y by $v\phi=\phi$, $v\{3\}=\{3\}$ and $v\{1\}=v\{2\}=v\{1,2\}=v\{1,3\}=v\{2,3\}=vY=Y$. Let $f:(X,u)\to(Y,v)$ be the identity map where (X,u) is the closure space from Example 4.2.3. It is easy to see that f is γ-continuous but not continuous because $\{1,2\}$ is open in (Y,v) but $f^{-1}(\{1,2\})$ is not open in (X,u).

Example 4.4.9. Let $f:(X,u)\to(Y,v)$ be the identity map where (X,u) and (Y,v) are the closure spaces from Example 4.2.5 and 4.4.8, respectively. It is easy to see that f is semi-continuous but not γ-continuous because $\{1,2\}$ is open in (Y,v) but $f^{-1}(\{1,2\})$ is not γ-open in (X,u).

Example 4.4.10. Let $f:(X,u)\to(Y,v)$ be a map where (X,u) and (Y,v) are the closure spaces from Example 4.1.3 and 4.4.8, respectively. Let f be defined by $f(1)=1$, $f(2)=2$, $f(3)=2$ and $f(4)=3$. It is easy to see that f is g-semi-continuous but not semi-continuous because $\{1,2\}$ is open in (Y,v) but $f^{-1}(\{1,2\})=\{1,2,3\}$ is not semi-open in (X,u).

Proposition 4.4.11. Let (X,u) *and* (Y,v) *be closure spaces and* $f:(X,u)\to(Y,v)$ *be a map.*

(i) *If* (X,u) *is a* T_{gs}*-space and* f *is g-semi-continuous, then* f *is semi-continuous.*
(ii) *If* (X,u) *is a* T_{γ}*-space and* f *is γ-continuous, then* f *is continuous.*
(iii) *If* (X,u) *is a* T_s*-space and* f *is g-semi-continuous, then* f *is continuous*
(iv) *If* (X,u) *is a* $T_{s\gamma}$*-space and* f *is g-semi-continuous, then* f *is γ-continuous.*

Proof. (i) Let H be an open subset of (Y,v). Since f is g-semi-continuous, $f^{-1}(H)$ is g-semi-open in (X,u). But (X,u) is a T_{gs}-space, thus $f^{-1}(H)$ is semi-open. Therefore, f is semi-continuous.

(ii) The proof is similar to that of (i).

(iii) Let H be an open subset in (Y,v). Since f is g-semi-continuous, $f^{-1}(H)$ is g-semi-open in (X,u). But (X,u) is a T_s-space, thus $f^{-1}(H)$ is open by Proposition 4.1.9. Therefore, f is continuous.

(iv) The proof is similar to that of (iii) by utilizing Proposition 4.3.6. ❒

Proposition 4.4.12. *Let (X,u) and (Z,w) be closure spaces and (Y,v) be a T_s-space. Let $f:(X,u)\to(Y,v)$ and $g:(Y,v)\to(Z,w)$ be maps.*

(i) If f and g are g-semi-continuous, then $g\circ f$ is also g-semi-continuous.

(ii) If f and g are γ-continuous, then $g\circ f$ is also γ-continuous.

Proof. (i) Since g is g-semi-continuous and (Y,v) is a T_s-space, g is continuous by Proposition 4.4.11 (iii). As f is g-semi-continuous, $g\circ f$ is also g-semi-continuous by Proposition 4.4.3.

(ii) Since (Y,v) is a T_s-space, (Y,v) is a T_γ-space by Proposition 4.3.9. As g is γ-continuous, g is continuous by Proposition 4.4.11 (iii). Since f is γ-continuous, $g\circ f$ is γ-continuous by Proposition 4.4.6. ❒

4.5 Generalized-semi-irresolute and γ-irresolute Maps

In this section, we introduce and study two new types of maps called generalized-semi-irresolute maps and γ-irresolute maps.

Definition 4.5.1. Let (X,u) and (Y,v) be closure spaces. A map $f:(X,u)\to(Y,v)$ is said to be *generalized-semi-irresolute,* briefly *g-semi-irresolute,* if $f^{-1}(B)$ is a g-semi-open subset of (X,u) for every g-semi-open subset B of (Y,v).

Proposition 4.5.2. *A map $f:(X,u)\to(Y,v)$ is g-semi-irresolute if and only if the inverse image under f of every g-semi closed subset of (Y,v) is g-semi-closed in (X,u).*

Proposition 4.5.3. *Let (X,u), (Y,v) and (Z,w) be closure spaces. Let $f:(X,u)\to(Y,v)$ and $g:(Y,v)\to(Z,w)$ be maps. If f and g are g-semi-irresolute, then $g\circ f$ is also g-semi-irresolute.*

Remark 4.5.4. The concepts of semi-irresolute maps and g-semi-irresolute maps are independent as shown by the following examples.

Example 4.5.5. Let $Y=\{1,2,3,4\}$ and define a closure operator v on Y by $v\phi=\phi$, $v\{1\}=\{1,2\}$, $v\{2\}=v\{4\}=\{2,4\}$, $v\{3\}=\{2,3\}$, $v\{1,2\}=v\{1,4\}=\{1,2,4\}$, $v\{1,3\}=\{1,2,3\}$, $v\{2,3\}=v\{2,4\}=v\{3,4\}=v\{2,3,4\}=\{2,3,4\}$ and $v\{1,2,3\}=v\{1,2,4\}=v\{1,3,4\}=vY=Y$. Let a map $f:(X,u)\to(Y,v)$ be identity where (X,u) is the closure space from Example 4.1.3. It is easy to see that f is semi-irresolute but f is not g-semi-irresolute because $\{1,2,4\}$ is g-semi-open in (Y,v) but $f^{-1}(\{1,2,4\})$ is not g-semi-open in (X,u).

Example 4.5.6. Let $f:(X,u)\to(Y,v)$ be a map where (X,u) and (Y,v) are the closure spaces from Example 4.1.3 and Example 4.2.5, respectively. Let f be defined by $f(1)=1$, $f(2)=2$, $f(3)=2$ and $f(4)=3$. It is easy to see that f is g-semi-irresolute but not semi-irresolute because $\{1,2\}$ is semi-open in (Y,v) but $f^{-1}(\{1,2\})=\{1,2,3\}$ is not semi-open in (X,u).

Remark 4.5.7. If $f:(X,u)\to(Y,v)$ is a g-semi-irresolute map, then f is g-semi-continuous. The converse is not true in general. By Example 4.5.5, the map f is g-semi-continuous but not g-semi-irresolute.

Remark 4.5.8. The concepts of g-semi-irresolute maps and γ-continuous maps are independent. By Example 4.4.10, f is g-semi-irresolute, but f is not γ-continuous. By Example 4.5.5, the map f is γ-continuous, but not g-semi-irresolute.

Definition 4.5.9. Let (X,u) and (Y,v) be closure spaces. A map $f:(X,u)\to(Y,v)$ is said to be *γ-irresolute* if $f^{-1}(B)$ is a γ-open subset of (X,u) for every γ-open subset B of (Y,v).

Proposition 4.5.10. *A map $f:(X,u)\to(Y,v)$ is γ-irresolute if and only if the inverse image under f of every γ-closed subset of (Y,v) is γ-closed in (X,u).*

Proposition 4.5.11. *Let (X,u), (Y,v) and (Z,w) be closure spaces and let $f:(X,u)\to(Y,v)$ and $g:(Y,v)\to(Z,w)$ be maps. If f and g are γ-irresolute, then $g\circ f$ is also γ-irresolute.*

Remark 4.5.12. The concepts of g-semi-irresolute maps and γ-irresolute maps are independent of each other. By Example 4.5.5, f is γ-irresolute but not g-semi-irresolute. By Example 4.4.10, f is g-semi-irresolute but not γ-irresolute.

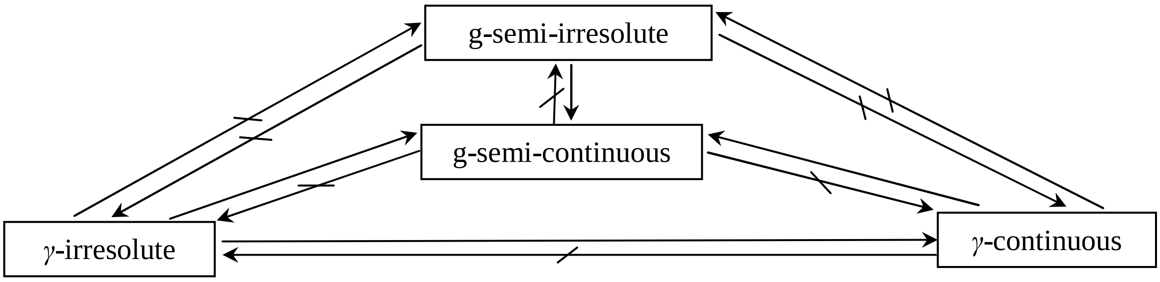
Remark 4.5.13. If $f:(X,u)\to(Y,v)$ is a γ-irresolute map, then f is g-semi-continuous. The converse is not true. By Example 4.4.10, f is g-semi-continuous but not γ-irresolute.

Remark 4.5.14. If $f:(X,u)\to(Y,v)$ is a γ-irresolute map, then f is γ-continuous. The converse is not true as can be seen from the following example.

Example 4.5.15. Let $Y=\{1,2,3\}$ and define a closure operator v on Y by $v\phi=\phi$, $v\{2\}=v\{3\}=v\{2,3\}=\{2,3\}$ and $v\{1\}=v\{1,2\}=v\{1,3\}=vY=Y$. Let $f:(X,u)\to(Y,v)$ be an identity map where (X,u) is the closure space in Example 4.2.5. It is easy to see that f is γ-

continuous but not γ-irresolute because $\{1,3\}$ is γ-open in (Y,v) but $f^{-1}(\{1,3\})$ is not γ-open in (X,u).

By Remarks 4.4.7, 4.5.7, 4.5.8, 4.5.12, 4.5.13 and 4.5.14, the interrelation among g-semi-continuous maps, g-semi-irresolute maps, γ-continuous maps and γ-irresolute maps can be shown by following diagram:



Proposition 4.5.16. *Let (X,u) be a closure space, (Y,v) be a T_s-space. If $f:(X,u)\to(Y,v)$ is a g-semi-continuous map, then f is g-semi-irresolute.*

Proof. Let B be a g-semi-open subset of (Y,v). Since (Y,v) is a T_s-space, B is open in (Y,v) by Proposition 4.1.9. As f is g-semi-continuous, $f^{-1}(B)$ is g-semi-open in (X,u). Therefore, f is g-semi-irresolute. ❒

Proposition 4.5.17. *Let (X,u) be a closure space, (Y,v) be a T_γ-space. If $f:(X,u)\to(Y,v)$ is a γ-continuous map, then f is γ-irresolute.*

Proof. Let B be a γ-open subset of (Y,v). Since (Y,v) is a T_γ-space, B is open in (Y,v). As f is γ-continuous, $f^{-1}(B)$ is γ-open in (X,u). Therefore, f is γ-irresolute. ❒

Proposition 4.5.18. *Let (X,u) be a $T_{s\gamma}$-space and (Y,v) be a T_γ-space. If $f:(X,u)\to(Y,v)$ is a g-semi-continuous map, then f is γ-irresolute.*

Proof. Let B be a γ-open subset of (Y,v). Since (Y,v) is a T_γ-space, B is open in (Y,v). As f is g-semi-continuous, $f^{-1}(B)$ is g-semi-open in (X,u). But (X,u) is a $T_{s\gamma}$-space, hence $f^{-1}(B)$ is γ-open by Proposition 4.3.6. Therefore, f is γ-irresolute. ❒

Proposition 4.5.19. *Let (X,u) and (Y,v) be closure spaces and $f:(X,u)\to(Y,v)$ be a map.*

(i) If (X,u) is a $T_{s\gamma}$-space and f is g-semi-irresolute, then f is γ-irresolute.

(ii) If (Y,v) is a $T_{s\gamma}$-space and f is γ-irresolute, then f is g-semi-irresolute.

Proof. (i) Let B be a γ-open subset of (Y,v). Then B is also g-semi-open in (Y,v). As f is g-semi-irresolute, $f^{-1}(B)$ is g-semi-open in (X,u). Since (X,u) is a $T_{s\gamma}$-space, $f^{-1}(B)$ is γ-open in (X,u) by Proposition 4.3.6. Therefore, f is γ-irresolute.

(ii) Let B be a g-semi-open subset of (Y,v). Since (Y,v) is a $T_{s\gamma}$-space, B is γ-open in (Y,v) by Proposition 4.3.6. As f is γ-irresolute, $f^{-1}(B)$ is γ-open in (X,u). Consequently, $f^{-1}(B)$ is also g-semi-open in (X,u). Therefore, f is g-semi-irresolute. ❒

Proposition 4.5.20. *Let (X,u) and (Y,v) be closure spaces and $f:(X,u)\to(Y,v)$ be a map.*

(i) If (X,u) is a $T_{s\gamma}$-space and f is g-semi-irresolute, then f is γ-continuous.

(ii) If (Y,v) is a T_s-space and f is γ-continuous, then f is g-semi-irresolute.

Proof. (i) Since f is g-semi-irresolute and (X,u) is a $T_{s\gamma}$-space, f is γ-irresolute by Proposition 4.5.19 (i). It follows that f is γ-continuous.

(ii) Let B be a g-semi-open subset of (Y,v). Since (Y,v) is a T_s-space, B is open in (Y,v) by Proposition 4.1.9. As f is γ-continuous, $f^{-1}(B)$ is γ-open in (X,u). It follows that $f^{-1}(B)$ is also g-semi-open in (X,u). Therefore, f is g-semi-irresolute.
❐

Proposition 4.5.21. *Let (X,u) and (Y,v) be closure spaces and $f:(X,u)\to(Y,v)$ be a semi-open, g-semi-irresolute and surjective map. Then (Y,v) is a T_{gs}-space if (X,u) is a T_s-space.*

Proof. Let (X,u) be a T_s-space and let B be a g-semi-open subset of (Y,v). Since f is g-semi-irresolute, $f^{-1}(B)$ is g-semi-open in (X,u). As (X,u) is a T_s-space, $f^{-1}(B)$ is open in (X,u) by Proposition 4.1.9. Since f is semi-open, $f\left(f^{-1}(B)\right)$ is semi-open in (Y,v). But f is a surjection, thus $f\left(f^{-1}(B)\right)=B$. Therefore, (Y,v) is a T_{gs}-space. ❐

Proposition 4.5.22. *Let (X,u) and (Y,v) be closure spaces and $f:(X,u)\to(Y,v)$ be a γ-open, semi-irresolute and surjective map. Then (Y,v) is a $T_{s\gamma}$-space if (X,u) is a T_s-space.*

Proof. The proof is similar to that of Proposition 4.5.21. ❐

Proposition 4.5.23. *Let (X,u) and (Y,v) be closure spaces and let $f:(X,u)\to(Y,v)$ be an open, γ-irresolute and surjective map. Then (Y,v) is a T_γ-space if (X,u) is a T_γ-space.*

Proof. The proof is similar to that of Proposition 4.5.21 ❐

Proposition 4.5.24. *Let (X,u), (Y,v) and (Z,w) be closure spaces and let $f:(X,u)\to(Y,v)$ and $g:(Y,v)\to(Z,w)$ be maps. Then*

(i) $g\circ f$ is g-semi-continuous if f is g-semi-irresolute and g is g-semi-continuous,

(ii) $g\circ f$ is γ-continuous if f is γ-irresolute and g is γ-continuous.

Proposition 4.5.25. *Let (X,u), (Y,v) and (Z,w) be closure spaces and let $f:(X,u)\to(Y,v)$ and $g:(Y,v)\to(Z,w)$ be maps.*

(i) If (X,u) is a $T_{s\gamma}$-space, f is g-semi-irresolute and g is γ-irresolute, then $g\circ f$ is γ-irresolute

(ii) If (Z,w) is a $T_{s\gamma}$-space, f is g-semi-irresolute and g is γ-irresolute, then $g\circ f$ is g-semi-irresolute.

(iii) If (Y,v) is a $T_{s\gamma}$-space, f is γ-irresolute and g is g-semi-irresolute, then $g\circ f$ is both g-semi-irresolute and γ-irresolute.

Proof. (i) Let B be a γ-open subset of (Z,w). Since g is γ-irresolute, $g^{-1}(B)$ is γ-open in (Y,v). Consequently, $g^{-1}(B)$ is also g-semi-open in (Y,v). As f is g-semi-irresolute, $f^{-1}(g^{-1}(B))=$ $(g\circ f)^{-1}(B)$ is g-semi-open in (X,u). But (X,u) is a $T_{s\gamma}$-space, hence $(g\circ f)^{-1}(B)$ is γ-open in (X,u) by Proposition 4.3.6. Therefore, $g\circ f$ is γ-irresolute.

(ii) Let B be a g-semi-open subset of (Z,w). Since (Z,w) is a $T_{s\gamma}$-space, B is γ-open by Proposition 4.3.6. As g is γ-irresolute, $g^{-1}(B)$ is γ-open in (Y,v). Consequently, $g^{-1}(B)$ is also g-semi-open in (Y,v). But f is g-semi-irresolute, hence $f^{-1}(g^{-1}(B))=(g\circ f)^{-1}(B)$ is g-semi-open in (X,u). Therefore, $g\circ f$ is g-semi-irresolute.

(iii) Let B be a g-semi-open subset of (Z,w). Since g is g-semi-irresolute, $g^{-1}(B)$ is g-semi-open in (Y,v). As (Y,v) is a $T_{s\gamma}$-space, $g^{-1}(B)$ is γ-open by Proposition 4.3.6. But f is γ-irresolute, hence $f^{-1}(g^{-1}(B))=(g\circ f)^{-1}(B)$ is γ-open in (X,u). Consequently, $(g\circ f)^{-1}(B)$ is g-semi-open in (X,u). Thus, $g\circ f$ is g-semi-irresolute.

Let C be a γ-open subset of (Z,w). It follows that C is g-semi-open in (Z,w). As g is g-semi-irresolute, $g^{-1}(C)$ is g-semi-open in (Y,v). As (Y,v) is a $T_{s\gamma}$-space, $g^{-1}(C)$ is γ-open by Proposition 4.3.6. Since f is γ-irresolute, $f^{-1}(g^{-1}(B))=(g\circ f)^{-1}(B)$ is γ-open in (X,u). Thus, $g\circ f$ is γ-irresolute. Therefore, $g\circ f$ is both g-semi-irresolute and γ-irresolute. ❒

Theorem 4.5.26. *Let $\{(X_\alpha,u_\alpha):\alpha\in J\}$ and $\{(Y_\alpha,v_\alpha):\alpha\in J\}$ be collections of closure spaces. Let $\alpha\in J$, $f_\alpha:(X_\alpha,u_\alpha)\to(Y_\alpha,v_\alpha)$ be a map and let $f:\prod_{\alpha\in J}(X_\alpha,u_\alpha)\to\prod_{\alpha\in J}(Y_\alpha,v_\alpha)$ be defined by $f\left((x_\alpha)_{\alpha\in J}\right)=\left(f_\alpha(x_\alpha)\right)_{\alpha\in J}$.*

(i) If f is g-semi-irresolute, then f_α is also g-semi-irresolute.

(ii) If f is γ-irresolute, then f_α is also γ-irresolute.

Proof. (i) Let $\beta\in J$ and B_β be a g-semi-open subset of (Y_β,v_β). By Proposition 4.1.11, $B_\beta\times\prod_{\substack{\alpha\neq\beta\\ \alpha\in J}}Y_\alpha$ is g-semi-open in $\prod_{\alpha\in J}(Y_\alpha,v_\alpha)$. As f is g-semi-irresolute, $f^{-1}\left(B_\beta\times\prod_{\substack{\alpha\neq\beta\\ \alpha\in J}}Y_\alpha\right)=$ $f_\beta^{-1}(B_\beta)\times\prod_{\substack{\alpha\neq\beta\\ \alpha\in J}}X_\alpha$ is g-semi-open in $\prod_{\alpha\in J}(X_\alpha,u_\alpha)$. By Proposition 4.1.11, $f_\beta^{-1}(B_\beta)$ is g-semi-open in (X_β,u_β). Therefore, f_α is g-semi-irresolute

(ii) The proof is similar to that of (i) by utilizing Proposition 4.2.17. ❒

Proposition 4.5.27. *Let $\{(X_\alpha,u_\alpha):\alpha\in J\}$ be a collection of closure spaces and $\beta\in J$. If $\pi_\beta:\prod_{\alpha\in J}(X_\alpha,u_\alpha)\to(X_\beta,u_\beta)$ is a projection map, then*

(i) π_β is g-semi-irresolute,

(ii) π_β is γ-irresolute.

Proof. (i) Let $\beta \in J$ and A_β be a g-semi-open subset of (X_β, u_β). Then $\pi_\beta^{-1}(A_\beta) = A_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha$. By Proposition 4.1.11, $A_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha$ is g-semi-open in $\prod_{\alpha \in J}(X_\alpha, u_\alpha)$. Therefore, π_β is g-semi-irresolute.

(ii) The proof is similar to that of (i) by utilizing Proposition 4.2.17. ❐

Chapter 5

Semi-Open Sets in Biclosure Spaces

5.1 Biclosure Spaces

In this section, we recall some basic definitions concerning biclosure spaces and study some of its fundamental properties.

A *biclosure space* is a triple (X, u_1, u_2) where X is a set and u_1, u_2 are two closure operators on X. A subset A of a biclosure space (X, u_1, u_2) is called *closed* if $u_1u_2A = A$. The complement of a closed set is called *open*.

Let (X, u_1, u_2) be a biclosure space. A biclosure space (Y, v_1, v_2) is called a *subspace* of (X, u_1, u_2) if $Y \subseteq X$ and $v_iA = u_iA \cap Y$ for all $i \in \{1,2\}$ and every subset A of Y.

Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces and let $i \in \{1,2\}$. Then a map $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ is called:

(i) i-*open* (respectively, i-*closed*) if the map $f : (X, u_i) \to (Y, v_i)$ is open (respectively, closed).

(ii) *open* (respectively, *closed*) if f is i-open (respectively, i-closed) for all $i \in \{1,2\}$.

(iii) *biopen* (respectively, *biclosed*) if the map $f : (X, u_1) \to (Y, v_2)$ is open (respectively, closed).

(iv) i-*continuous* if the map $f : (X, u_i) \to (Y, v_i)$ is continuous.

(v) *continuous* if f is i-continuous for all $i \in \{1,2\}$.

Remark 5.1.1. Let A be a subset of a biclosure space (X, u_1, u_2).

(i) A is open in (X, u_1, u_2) if and only if A is open in both (X, u_1) and (X, u_2)

(ii) If A is an open subset of (X, u_1, u_2), then $u_1u_2(X - A) = u_2u_1(X - A)$.

The converse of the statement (ii) in Remark 5.1.1 need not be true as can be seen from the following example.

Example 5.1.2. Let $X = \{1,2,3\}$ and define a closure operator u_1 on X by $u_1\phi = \phi$, $u_1\{1\} = \{1\}$, $u_1\{2\} = \{2\}$, $u_1\{3\} = \{3\}$, $u_1\{1,3\} = \{1,3\}$ and $u_1\{1,2\} = u_1\{2,3\} = u_1X = X$. Define a closure operator u_2 on X by $u_2\phi = \phi$, $u_2\{1\} = \{1,3\}$, $u_2\{2\} = \{2\}$, $u_2\{3\} = \{3\}$ and

$u_2\{1,2\} = u_2\{1,3\} = u_2\{2,3\} = u_2X = X$. We can see that $u_1u_2(X-\{1\}) = u_2u_1(X-\{1\}) = X$ but $\{1\}$ is not open in (X,u_1,u_2).

Proposition 5.1.3. *Let $\{A_\alpha\}_{\alpha\in J}$ be a collection of open sets in a biclosure space (X,u_1,u_2). Then $\bigcup_{\alpha\in J} A_\alpha$ is an open set.*

Proof. Let A_α be an open subset of (X,u_1,u_2). Then $X - A_\alpha$ is closed for all $\alpha\in J$. Since $\bigcap_{\alpha\in J}(X-A_\alpha) \subseteq X - A_\alpha$ for all $\alpha\in J$, $u_1u_2 \bigcap_{\alpha\in J}(X-A_\alpha) \subseteq u_1u_2(X-A_\alpha)$ for each $\alpha\in J$. But $X - A_\alpha = u_1u_2(X-A_\alpha)$ for all $\alpha\in J$. Thus, $u_1u_2 \bigcap_{\alpha\in J}(X-A_\alpha) \subseteq X - A_\alpha$ for all $\alpha\in J$. Consequently, $u_1u_2 \bigcap_{\alpha\in J}(X-A_\alpha) \subseteq \bigcap_{\alpha\in J}(X-A_\alpha) \subseteq u_1u_2 \bigcap_{\alpha\in J}(X-A_\alpha)$, i.e. $u_1u_2 \bigcap_{\alpha\in J}(X-A_\alpha) = \bigcap_{\alpha\in J}(X-A_\alpha)$. Thus, $\bigcap_{\alpha\in J}(X-A_\alpha) = X - \bigcup_{\alpha\in J} A_\alpha$ is closed in (X,u_1,u_2). Therefore, $\bigcup_{\alpha\in J} A_\alpha$ is open in (X,u_1,u_2). ❐

The intersection of two open sets in a biclosure space (X,u_1,u_2) need not be an open set as can be seen from Example 5.1.2 where $\{1,2\}$ and $\{1,3\}$ are open in (X,u_1,u_2) but $\{1,2\}\cap\{1,3\}$ is not open.

Proposition 5.1.4. *If $\{A_\alpha\}_{\alpha\in J}$ is a collection of subsets in a biclosure space (X,u_1,u_2), then $u_1u_2 \bigcap_{\alpha\in J} A_\alpha \subseteq \bigcap_{\alpha\in J} u_1u_2A_\alpha$.*

By Example 5.1.2, $u_1u_2\{1,2\}\cap u_1u_2\{1,3\}$ is not contained in $u_1u_2(\{1,2\}\cap\{1,3\})$, i.e. the inclusion in Proposition 5.1.4 cannot be replaced by equality in general.

Proposition 5.1.5. *If $\{A_\alpha\}_{\alpha\in J}$ is a collection of closed subsets in a biclosure space (X,u_1,u_2), then $u_1u_2 \bigcap_{\alpha\in J} A_\alpha = \bigcap_{\alpha\in J} u_1u_2A_\alpha$.*

Proof. Let A_α be closed in (X,u_1,u_2) for all $\alpha\in J$. Then $X - A_\alpha$ is open and $A_\alpha = u_1u_2A_\alpha$ for each $\alpha\in J$. By Proposition 5.1.3, $\bigcup_{\alpha\in J}(X-A_\alpha)$ is open. But $\bigcup_{\alpha\in J}(X-A_\alpha) = X - \bigcap_{\alpha\in J} A_\alpha$, hence $\bigcap_{\alpha\in J} A_\alpha$ is closed in (X,u_1,u_2), i.e. $u_1u_2 \bigcap_{\alpha\in J} A_\alpha = \bigcap_{\alpha\in J} A_\alpha = \bigcap_{\alpha\in J} u_1u_2A_\alpha$. ❐

The converse of Proposition 5.1.5 is not true in general as shown in the following example.

Example 5.1.6. Let $X = \{1,2,3\}$ and define a closure operator u_1 on X by $u_1\phi = \phi$, $u_1\{2\} = u_1\{3\} = u_1\{2,3\} = \{2,3\}$ and $u_1\{1\} = u_1\{1,2\} = u_1\{1,3\} = u_1X = X$. Define a closure operator u_2 on X by $u_2\phi = \phi$, $u_2\{1\} = u_2\{2\} = u_2\{1,2\} = \{1,2\}$ and $u_2\{3\} = u_2\{1,3\} = u_2\{2,3\} = u_2X = X$. It is easy to see that $u_1u_2(\{1,2\}\cap\{1,3\}) = u_1u_2\{1,2\}\cap u_1u_2\{1,3\}$ but neither $\{1,2\}$ nor $\{1,3\}$ is closed in (X,u_1,u_2).

Proposition 5.1.7. *Let (X,u_1,u_2) be a biclosure space. If G is a subset of X, then $u_1u_2G - G$ has no nonempty open subset of (X,u_1,u_2).*

Proof. Let G be a subset of X and H be a nonempty open subset of (X,u_1,u_2) such that $H \subseteq u_1u_2G - G$. Since H is nonempty, there is $x \in H \subseteq u_1u_2G - G$, i.e. $x \notin X - H$. Thus, u_1u_2G is not contained in $X - H$. Since $H \subseteq u_1u_2G - G$, $G \subseteq u_1u_2G - H \subseteq X - H$. It follows that $u_1u_2G \subseteq u_1u_2(X - H)$. But H is open in (X,u_1,u_2), hence $u_1u_2(X - H) = X - H$. Consequently, $u_1u_2G \subseteq X - H$, which is a contradiction. Therefore, $u_1u_2G - G$ contains no nonempty open set of (X,u_1,u_2). ❒

Remark 5.1.8. The following statement is equivalent to Proposition 5.1.7:

Let (X,u_1,u_2) be a biclosure space and G be a subset of X. If H is an open subset of (X,u_1,u_2) with $H \subseteq u_1u_2G - G$, then H is an empty set.

Moreover, if the subset H is an open subset in (X,u_1) but not in (X,u_2), then H need not be empty. And if the subset H is an open subset in (X,u_2) but not in (X,u_1), then H need not be empty. By Example 5.1.6, $\{2\}$ is a subset of X such that $\{1\}$ and $\{3\}$ are nonempty subsets of $u_1u_2\{2\} - \{2\}$. We can see that $\{1\}$ is open in (X,u_1) but not in (X,u_2), and $\{3\}$ is an open subset in (X,u_2) but not in (X,u_1).

Proposition 5.1.9. *If (Y,v_1,v_2) is a biclosure subspace of (X,u_1,u_2), then $G \cap Y$ is an open subset of (Y,v_1,v_2) for every open subset G of (X,u_1,u_2).*

Proof. Let G be an open subset of (X,u_1,u_2). By Remark 5.1.1 (i), G is open in both (X,u_1) and (X,u_2). Thus, $v_i(Y - (G \cap Y)) = u_i(Y - (G \cap Y)) \cap Y \subseteq u_i(X - G) \cap Y = (X - G) \cap Y = Y - (G \cap Y)$ for each $i \in \{1,2\}$. Consequently, $G \cap Y$ is open in both (Y,v_1) and (Y,v_2). Therefore, $G \cap Y$ is open in (Y,v_1,v_2).
❒

Remark 5.1.10. By Proposition 5.1.9, if $E \subseteq Y$ and $E = G \cap Y$ for some open subset G of (X,u_1,u_2), then E is an open subset of (Y,v_1,v_2). The converse is not true as can be seen from the following example.

Example 5.1.11. Let $X = \{1,2,3\}$ and define a closure operator u_1 on X by $u_1\phi = \phi$, $u_1\{1\} = \{1,3\}$, $u_1\{2\} = u_1\{2,3\} = \{2,3\}$, $u_1\{3\} = \{3\}$ and $u_1\{1,2\} = u_1\{1,3\} = u_1X = X$. Define a closure operator u_2 on X by $u_2\phi = \phi$, $u_2\{1\} = \{1,2\}$, $u_2\{2\} = \{2,3\}$, $u_2\{3\} = \{3\}$ and $u_2\{1,2\} = u_2\{1,3\} = u_2\{2,3\} = u_2X = X$. Thus, there are only three open subsets of (X,u_1,u_2), namely ϕ, $\{1,2\}$ and X. Let $Y = \{1,2\}$ and (Y,v_1,v_2) be a biclosure subspace of (X,u_1,u_2). Then $v_1\phi = \phi$, $v_1\{1\} = \{1\}$, $v_1\{2\} = \{2\}$ $v_1Y = Y$, $v_2\phi = \phi$, $v_2\{2\} = \{2\}$ and $v_2\{1\} = v_2Y = Y$. We can see that $\{1\}$ is an open subset of (Y,v_1,v_2) but there is no open set G in (X,u_1,u_2) such that $\{1\} = G \cap Y$.

Proposition 5.1.12. *Let (X,u_1,u_2), (Y,v_1,v_2) and (Z,w_1,w_2) be biclosure spaces, let $f : (X,u_1,u_2) \to (Y,v_1,v_2)$ and $g : (Y,v_1,v_2) \to (Z,w_1,w_2)$ be maps.*

(i) If f is 1-open and g is biopen, then $g \circ f$ is biopen.

(ii) If f is biopen and g is 2-open, then $g \circ f$ is biopen.

Proof. (i) Let G be an open subset of (X,u_1). Since f is 1-open, $f(G)$ is open in (Y,v_1). As g is biopen, $g(f(G)) = g \circ f(G)$ is open in (Z,w_2). Thus, $g \circ f$ is biopen.

(ii) Let G be an open subset of (X,u_1). Since f is biopen, $f(G)$ is open in (Y,v_2). And since g is 2-open, $g(f(G)) = g \circ f(G)$ is open in (Z,w_2). Thus, $g \circ f$ is biopen. ❐

The composition of two biopen maps need not be a biopen map as can be seen from the following example.

Example 5.1.13. Let $X = Y = Z = \{1,2\}$ and define a closure operator u_1 on X by $u_1\phi = \phi$, $u_1\{2\} = \{2\}$, and $u_1\{1\} = u_1 X = X$. Define a closure operator u_2 on X by $u_2\phi = \phi$ and $u_2\{1\} = u_2\{2\} = u_2 X = X$. Define a closure operator v_1 on Y by $v_1\phi = \phi$, $v_1\{1\} = \{1\}$ and $v_1\{2\} = v_1 Y = Y$ and define a closure operator v_2 on Y by $v_2\phi = \phi$, $v_2\{1\} = \{1\}$, $v_2\{2\} = \{2\}$ and $v_2 Y = Y$. Define a closure operator w_1 on Z by $w_1\phi = \phi$ and $w_1\{1\} = w_1\{2\} = w_1 Z = Z$ and define a closure operator w_2 on Z by $w_2\phi = \phi$, $w_2\{1\} = \{1\}$ and $w_2\{2\} = w_2 Z = Z$. Let $f : (X,u_1,u_2) \to (Y,v_1,v_2)$ and $g : (Y,v_1,v_2) \to (Z,w_1,w_2)$ be the identity maps. We can see that f and g are biopen. But $g \circ f$ is not biopen because $\{1\}$ is open in (X,u_1) but $g \circ f(\{1\})$ is not open in (Z,w_2).

Proposition 5.1.14. *Let (X,u_1,u_2), (Y,v_1,v_2) and (Z,w_1,w_2) be biclosure spaces and let f : $(X,u_1,u_2) \to (Y,v_1,v_2)$ and $g : (Y,v_1,v_2) \to (Z,w_1,w_2)$ be maps.*

(i) If $g \circ f$ is biopen and f is a 1-continuous surjection, then g is biopen.

(ii) If $g \circ f$ is biopen and g is a 2-continuous injection, then f is biopen.

Proof. (i) Let H be an open subset of (Y,v_1). Since f is 1-continuous, $f^{-1}(H)$ is open in (X,u_1). But $g \circ f$ is biopen, hence $g \circ f\left(f^{-1}(H)\right)$ is open in (Z,w_2). As f is a surjection, $g \circ f\left(f^{-1}(H)\right) = g(H)$. Therefore, g is biopen.

(ii) Let G be an open subset of (X,u_1). Since $g \circ f$ is biopen, $g \circ f(G)$ is open in (Z,w_2). But g is 2-continuous, hence $g^{-1}(g \circ f(G))$ is open in (Y,v_2). As g is an injection, $g^{-1}(g \circ f(G)) = f(G)$. Therefore, f is biopen. ❐

Proposition 5.1.15. *Let (X,u_1,u_2) and (Y,v_1,v_2) be biclosure spaces and let f : $(X,u_1,u_2) \to (Y,v_1,v_2)$ be a map. If f is open, then $f(G)$ is open in (Y,v_1,v_2) for every open subset G of (X,u_1,u_2).*

Proof. Let G be an open subset of (X,u_1,u_2). By Remark 5.1.1(i), G is open in both (X,u_1) and (X,u_2). Since f is open, f is both 1-open and 2-open. It follows that $f(G)$ is open in both (Y,v_1) and (Y,v_2). Consequently, $f(G)$ is open in (Y,v_1,v_2) by Remark 5.1.1(i). ❐

Remark 5.1.16. Let (X,u_1,u_2) and (Y,v_1,v_2) be biclosure spaces and let $f : (X,u_1,u_2) \to$ (Y,v_1,v_2) be a map. If $f(G)$ is open in (Y,v_1,v_2) for every open subset G of (X,u_1,u_2), then f need not be open as can be seen from the following example.

Example 5.1.17. Let $X=\{1,2\}=Y$ and define a closure operator u_1 on X by $u_1\phi=\phi$, $u_1\{2\}=\{2\}$, and $u_1\{1\}=u_1X=X$. Define a closure operator u_2 on X by $u_2\phi=\phi$ and $u_2\{1\}=u_2\{2\}=u_2X=X$. Define a closure operator v_1 on Y by $v_1\phi=\phi$, $v_1\{1\}=v_1\{2\}=v_1Y=Y$ and define a closure operator v_2 on Y by $v_2\phi=\phi$, $v_2\{1\}=\{1\}$ and $v_2\{2\}=v_2Y=Y$. Let $f:(X,u_1,u_2)\to(Y,v_1,v_2)$ be the identity map. It is easy to see that $f(G)$ is open in (Y,v_1,v_2) for every open subset G of (X,u_1,u_2). But f is not 1-open because $f(\{1\})$ is not open in (Y,v_1) while $\{1\}$ is open in (X,u_1). Consequently, f is not open.

Proposition 5.1.18. *Let (X,u_1,u_2) and (Y,v_1,v_2) be biclosure spaces and let $f:(X,u_1,u_2)\to(Y,v_1,v_2)$ be a map. If f is continuous, then $f^{-1}(H)$ is open in (X,u_1,u_2) for every open subset H of (Y,v_1,v_2).*

Proof. Let H be an open subset of (Y,v_1,v_2). By Remark 5.1.1 (i), H is open in both (Y,v_1) and (Y,v_2). Since f is continuous, f is both 1-continuous and 2-continuous. It follows that $f^{-1}(H)$ is open in both (X,u_1) and (X,u_2). Therefore, $f^{-1}(H)$ is open in (X,u_1,u_2) by Remark 5.1.1 (i). ❐

Remark 5.1.19. Let (X,u_1,u_2) and (Y,v_1,v_2) be biclosure spaces and let $f:(X,u_1,u_2)\to(Y,v_1,v_2)$ be a map. If $f^{-1}(H)$ is open in (X,u_1,u_2) for every open subset H of (Y,v_1,v_2), then f need not be continuous as can be seen from the following example.

Example 5.1.20. In Example 5.1.17, $f^{-1}(H)$ is open in (X,u_1,u_2) for every open subset H of (Y,v_1,v_2). But the map f is not 2-continuous because $f^{-1}(\{2\})$ is not open in (X,u_2) while $\{2\}$ is open in (Y,v_2). Consequently, f is not continuous.

Definition 5.1.21. A map $f:(X,u_1,u_2)\to(Y,v_1,v_2)$, where (X,u_1,u_2) and (Y,v_1,v_2) are biclosure spaces, is called a *homeomorphism* if f is bijective, continuous and open.

Proposition 5.1.22. *Let (X,u_1,u_2) and (Y,v_1,v_2) be biclosure spaces and $f:(X,u_1,u_2)\to(Y,v_1,v_2)$ be a map. If f is a bijective continuous map, then the following statements are equivalent:*

(i) f is a homeomorphism,
(ii) f is a closed map,
(iii) f is an open map.

Proof. (i)$\to$(ii) Since f is a homeomorphism, f is open and bijective. It follows that f is both 1-open and 2-open. Let $i\in\{1,2\}$ and let F_i be a closed subset of (X,u_i). Then $f(X-F_i)=Y-f(F_i)$ is open in (Y,v_i). Hence, $f(F_i)$ is closed in (Y,v_i). Thus, f is both 1-closed and 2-closed. Therefore, f is closed.

(ii)$\to$(iii) Let $i\in\{1,2\}$ and let G_i be an open subset of (X,u_i). Then $X-G_i$ is closed in (X,u_i). By the assumption, f is both closed and bijective. It follows that f is both 1-closed and 2-closed. Consequently, $f(X-G_i)=Y-f(G_i)$ is closed in (Y,v_i). Hence, $f(G_i)$ is open in (Y,v_i). Thus, f is both 1-open and 2-open. Therefore, f is open.

(iii) $\rightarrow$ (i) By the assumption, f is a homeomorphism. ❒

5.2 Semi-open Sets

In this section, we introduce a new type of open sets in biclosure spaces and study some of their properties.

Definition 5.2.1. A subset A of a biclosure space (X,u_1,u_2) is called *semi-open* if there exists an open subset G of (X,u_1) such that $G \subseteq A \subseteq u_2G$. The complement of a semi-open subset of X is called *semi-closed*.

Clearly, if (X,u_1,u_2) is a biclosure space and A is open (respectively, closed) in (X,u_1), then A is semi-open (respectively, semi-closed) in (X,u_1,u_2). The converse is not true as can be seen from the following example.

Example 5.2.2. Let $X = \{1,2,3\}$ and define a closure operator u_1 on X by $u_1\phi = \phi$, $u_1\{1\} = u_1\{3\} = u_1\{1,3\} = \{1,3\}$, $u_1\{2\} = \{2,3\}$ and $u_1\{1,2\} = u_1\{2,3\} = u_1X = X$. Define a closure operator u_2 on X by $u_2\phi = \phi$, $u_2\{3\} = \{3\}$ and $u_2\{1\} = u_2\{2\} = u_2\{1,2\} = u_2\{1,3\} = u_2\{2,3\} = u_2X = X$. It follows that $\{2,3\}$ is semi-open in (X,u_1,u_2) but $\{2,3\}$ is open in neither (X,u_1) nor (X,u_2). Moreover, $\{1\}$ is semi-closed in (X,u_1,u_2) but $\{1\}$ is closed in neither (X,u_1) nor (X,u_2).

Proposition 5.2.3. *Let (X,u_1,u_2) be a biclosure space and let $A \subseteq X$. Then A is semi-closed in (X,u_1,u_2) if and only if there exists a closed subset F of (X,u_1) such that $X - u_2(X-F) \subseteq A \subseteq F$.*

Proof. Let A be a semi-closed subset of (X,u_1,u_2). Then there exists an open subset G of (X,u_1) such that $G \subseteq X - A \subseteq u_2G$. Thus, there exists a closed subset F of (X,u_1) such that $G = X - F$ and $X - F \subseteq X - A \subseteq u_2(X-F)$. Therefore, $X - u_2(X-F) \subseteq A \subseteq F$.

Conversely, by the assumption, there is a closed subset F of (X,u_1) such that $X - u_2(X-F) \subseteq A \subseteq F$. Thus, there exists an open subset G of (X,u_1) such that $F = X - G$ and $X - u_2G \subseteq A \subseteq X - G$. It follows that $G \subseteq X - A \subseteq u_2G$. Therefore, A is semi-closed in (X,u_1,u_2). ❒

Proposition 5.2.4. *Let $\{A_\alpha\}_{\alpha\in J}$ be a collection of semi-open sets in a biclosure space (X,u_1,u_2). Then $\bigcup_{\alpha\in J} A_\alpha$ is a semi-open subset of (X,u_1,u_2).*

Proof. Let A_α be semi-open in (X,u_1,u_2) for all $\alpha \in J$. Hence, for each $\alpha \in J$, we have an open set G_α in (X,u_1) such that $G_\alpha \subseteq A_\alpha \subseteq u_2G_\alpha$. Thus, $\bigcup_{\alpha\in J} G_\alpha \subseteq \bigcup_{\alpha\in J} A_\alpha \subseteq \bigcup_{\alpha\in J} u_2G_\alpha$. Since $G_\alpha \subseteq \bigcup_{\alpha\in J} G_\alpha$ for each $\alpha \in J$, $u_2G_\alpha \subseteq u_2 \bigcup_{\alpha\in J} G_\alpha$ for all $\alpha \in J$. Thus, $\bigcup_{\alpha\in J} u_2G_\alpha \subseteq u_2 \bigcup_{\alpha\in J} G_\alpha$. Consequently, $\bigcup_{\alpha\in J} G_\alpha \subseteq \bigcup_{\alpha\in J} A_\alpha \subseteq u_2 \bigcup_{\alpha\in J} G_\alpha$. As G_α is open in (X,u_1) for all $\alpha \in J$,

$u_1 \bigcap_{\alpha\in J}(X - G_\alpha) \subseteq u_1(X - G_\alpha) = X - G_\alpha$ for each $\alpha \in J$. Thus, $u_1 \bigcap_{\alpha\in J}(X - G_\alpha) \subseteq \bigcap_{\alpha\in J}(X - G_\alpha)$. It follows that $\bigcap_{\alpha\in J}(X - G_\alpha)$ is closed in (X, u_1), i.e. $\bigcup_{\alpha\in J} G_\alpha$ is open in (X, u_1). Therefore, $\bigcup_{\alpha\in J} A_\alpha$ is semi-open in (X, u_1, u_2). ❐

If $\{A_\alpha\}_{\alpha\in J}$ is a collection of semi-open sets in a biclosure space (X, u_1, u_2), then $\bigcap_{\alpha\in J} A_\alpha$ need not be a semi-open subset of (X, u_1, u_2) as shown in the following example.

Example 5.2.5. In the biclosure space (X, u_1, u_2) from Example 5.1.2, we can see that $\{1,2\}$ and $\{1,3\}$ are semi-open but $\{1,2\} \cap \{1,3\}$ is not semi-open.

Proposition 5.2.6. *Let $\{A_\alpha\}_{\alpha\in J}$ be a collection of semi-closed sets in a biclosure space (X, u_1, u_2). Then $\bigcap_{\alpha\in J} A_\alpha$ is semi-closed.*

Proof. Clearly, the complement of $\bigcap_{\alpha\in J} A_\alpha$ is $\bigcup_{\alpha\in J}(X - A_\alpha)$. Since A_α is semi-closed in (X, u_1, u_2) for each $\alpha \in J$, $X - A_\alpha$ is semi-open for all $\alpha \in J$. But $\bigcup_{\alpha\in J}(X - A_\alpha)$ is a semi-open set by Proposition 5.2.4. Therefore, $\bigcap_{\alpha\in J} A_\alpha$ is semi-closed in (X, u_1, u_2). ❐

If $\{A_\alpha\}_{\alpha\in J}$ is a collection of semi-closed sets in a biclosure space (X, u_1, u_2), then $\bigcup_{\alpha\in J} A_\alpha$ need not be a semi-closed set as shown in the following example.

Example 5.2.7. In the biclosure space (X, u_1, u_2) from Example 5.1.2, we can see that $\{2\}$ and $\{3\}$ are semi-closed but $\{2\} \cup \{3\}$ is not semi-closed.

Proposition 5.2.8. *Let (X, u_1, u_2) be a biclosure space and u_2 be idempotent. If A is semi-open in (X, u_1, u_2) and $A \subseteq B \subseteq u_2 A$, then B is semi-open.*

Proof. Let A be a semi-open subset of (X, u_1, u_2). Then there exists an open set G in (X, u_1) such that $G \subseteq A \subseteq u_2 G$, hence $u_2 A \subseteq u_2 u_2 G$. Since u_2 is idempotent, $u_2 A \subseteq u_2 G$. Thus, $G \subseteq A \subseteq B \subseteq u_2 A \subseteq u_2 G$. Therefore, B is semi-open in (X, u_1, u_2). ❐

Proposition 5.2.9. *Let (Y, v_1, v_2) be a biclosure subspace of (X, u_1, u_2) and $A \subseteq Y$. If A is semi-open in (X, u_1, u_2), then A is semi-open in (Y, v_1, v_2).*

Proof. Let A be a semi-open subset of (X, u_1, u_2). Then there exists an open subset G of (X, u_1) such that $G \subseteq A \subseteq u_2 G$. It follows that $A \cap Y \subseteq u_2 G \cap Y$. But $A \cap Y = A$, hence $G \subseteq A = A \cap Y \subseteq u_2 G \cap Y = v_2 G$. Since G is open in (X, u_1), $v_1(Y - G) = u_1(Y - G) \cap Y \subseteq u_1(X - G) \cap Y = (X - G) \cap Y = Y - G$. Thus, $Y - G$ is closed in (Y, v_1), i.e. G is open in (Y, v_1). Therefore, A is a semi-open subset of (Y, v_1, v_2). ❐

The converse of Proposition 5.2.9 need not be true as can be seen from the following example.

Example 5.2.10. In the biclosure spaces (X,u_1,u_2) and (Y,v_1,v_2) from Example 5.1.11, we can see that $\{2\}\subseteq Y$ and $\{2\}$ is semi-open in (Y,v_1,v_2) but $\{2\}$ is not semi-open in (X,u_1,u_2).

Definition 5.2.11. Let (X,u_1,u_2) and (Y,v_1,v_2) be biclosure spaces. Then a map $f:(X,u_1,u_2)\to(Y,v_1,v_2)$ is called *semi-open* (respectively, *semi-closed*) if $f(A)$ is semi-open (respectively, semi-closed) in (Y,v_1,v_2) for every open (respectively, closed) subset A of (X,u_1,u_2).

Clearly, if f is open (respectively, closed), then f is semi-open (respectively, semi-closed). The converse need not be true as can be seen from the following example.

Example 5.2.12. Let $X=\{1,2\}=Y$ and define a closure operator u_1 on X by $u_1\phi=\phi$, $u_1\{1\}=\{1\}$ and $u_1\{2\}=u_1X=X$. Define a closure operator u_2 on X by $u_2\phi=\phi$, $u_2\{1\}=\{1\}$, $u_2\{2\}=\{2\}$ and $u_2X=X$. Define a closure operator v_1 on Y by $v_1\phi=\phi$, $v_1\{1\}=\{1\}$ $v_1\{2\}=v_1Y=Y$ and define a closure operator v_2 on Y by $v_2\phi=\phi$ and $v_2\{1\}=v_2\{2\}=v_2Y=Y$. Let $f:(X,u_1,u_2)\to(Y,v_1,v_2)$ be the identity map. It is easy to see that f is semi-open but not open because $f(\{2\})$ is not open in (Y,v_1,v_2) while $\{2\}$ is open in (X,u_1,u_2). Moreover, we can see that f is semi-closed but not closed because $f(\{1\})$ is not closed in (Y,v_1,v_2) while $\{1\}$ is closed in (X,u_1,u_2).

Proposition 5.2.13. *Let (X,u_1,u_2), (Y,v_1,v_2) and (Z,w_1,w_2) be biclosure spaces and let f : $(X,u_1,u_2)\to(Y,v_1,v_2)$ and $g:(Y,v_1,v_2)\to(Z,w_1,w_2)$ be maps. Then $g\circ f$ is semi-open if f is open and g is semi-open.*

Proof. Let G be an open subset of (X,u_1,u_2) and let f be open. By Proposition 5.1.15, $f(G)$ is open in (Y,v_1,v_2). As g is semi-open, $g(f(G))=g\circ f(G)$ is semi-open in (Z,w_1,w_2). Therefore, $g\circ f$ is semi-open. ❒

Proposition 5.2.14. *Let (X,u_1,u_2), (Y,v_1,v_2) and (Z,w_1,w_2) be biclosure spaces and let f : $(X,u_1,u_2)\to(Y,v_1,v_2)$ and $g:(Y,v_1,v_2)\to(Z,w_1,w_2)$ be maps. If $g\circ f$ is semi-open and f is a continuous surjection, then g is semi-open.*

Proof. Let H be an open subset of (Y,v_1,v_2) and let f be continuous. By Proposition 5.1.18, $f^{-1}(H)$ is open in (X,u_1,u_2). Since $g\circ f$ is semi-open, $g\circ f\left(f^{-1}(H)\right)$ is semi-open in (Z,w_1,w_2). But f is a surjection, hence $g\circ f\left(f^{-1}(H)\right)=g(H)$. Thus, $g(H)$ is semi-open in (Z,w_1,w_2). Therefore, g is semi-open. ❒

5.3 Semi-continuous Maps

In this section, we study the concept of semi-continuous maps obtained by using semi-open sets.

Definition 5.3.1. Let (X,u_1,u_2) and (Y,v_1,v_2) be biclosure spaces. A map $f:(X,u_1,u_2)\to(Y,v_1,v_2)$ is called *semi-continuous* if $f^{-1}(G)$ is a semi-open subset of (X,u_1,u_2) for every open subset G of (Y,v_1,v_2).

Clearly, if f is continuous, then f is semi-continuous. The converse need not be true as can be seen from the following example.

Example 5.3.2. Let $X=\{1,2\}=Y$ and define a closure operator u_1 on X by $u_1\phi=\phi$, $u_1\{1\}=\{1\}$ and $u_1\{2\}=u_1X=X$. Define a closure operator u_2 on X by $u_2\phi=\phi$ and $u_2\{1\}=u_2\{2\}=u_2X=X$. Define a closure operator v_1 on Y by $v_1\phi=\phi$, $v_1\{1\}=\{1\}$, $v_1\{2\}=\{2\}$, $v_1Y=Y$ and define a closure operator v_2 on Y by $v_2\phi=\phi$, $v_2\{1\}=\{1\}$ and $v_2\{2\}=v_2Y=Y$. Let $f:(X,u_1,u_2)\to(Y,v_1,v_2)$ be the identity map. It is easy to see that f is semi-continuous but not continuous because $f^{-1}(\{2\})$ is not open in (X,u_1,u_2) while $\{2\}$ is open in (Y,v_1,v_2).

Proposition 5.3.3. *Let (X,u_1,u_2) and (Y,v_1,v_2) be biclosure space. Then a map $f:(X,u_1,u_2)\to(Y,v_1,v_2)$ is semi-continuous if and only if $f^{-1}(F)$ is a semi-closed subset of (X,u_1,u_2) for every closed subset F of (Y,v_1,v_2).*

Proposition 5.3.4. *Let (X,u_1,u_2), (Y,v_1,v_2) and (Z,w_1,w_2) be biclosure spaces and let $f:(X,u_1,u_2)\to(Y,v_1,v_2)$ and $g:(Y,v_1,v_2)\to(Z,w_1,w_2)$ be maps. If g is continuous and f is semi-continuous, then $g\circ f$ is semi-continuous.*

Proof. Let H be an open subset of (Z,w_1,w_2) and let g be continuous. By Proposition 5.1.18, $g^{-1}(H)$ is open in (Y,v_1,v_2). As f is semi-continuous, $f^{-1}(g^{-1}(H))=(g\circ f)^{-1}(H)$ is semi-open in (X,u_1,u_2). Therefore, $g\circ f$ is semi-continuous. ❒

Definition 5.3.5. A biclosure space (X,u_1,u_2) is said to be a T_S*-space* if every semi-open set in (X,u_1,u_2) is open in (X,u_1,u_2). Clearly, the closure space (X,u_1,u_2) in Example 5.2.12 is a T_S-space.

Proposition 5.3.6. *Let (X,u_1,u_2) and (Z,w_1,w_2) be biclosure spaces and (Y,v_1,v_2) be a T_S-space and let $f:(X,u_1,u_2)\to(Y,v_1,v_2)$ and $g:(Y,v_1,v_2)\to(Z,w_1,w_2)$ be maps. If f and g are semi-continuous, then $g\circ f$ is semi-continuous.*

Proof. Let H be an open subset of (Z,w_1,w_2). Since g is semi-continuous, $g^{-1}(H)$ is semi-open in (Y,v_1,v_2). But (Y,v_1,v_2) is a T_S-space, hence $g^{-1}(H)$ is open in (Y,v_1,v_2). As f is semi-continuous, $f^{-1}(g^{-1}(H))=(g\circ f)^{-1}(H)$ is semi-open in (X,u_1,u_2). Therefore, $g\circ f$ is semi-continuous. ❒

Proposition 5.3.7. *Let (X,u_1,u_2), (Y,v_1,v_2) and (Z,w_1,w_2) be biclosure spaces, and let $f:(X,u_1,u_2)\to(Y,v_1,v_2)$ and $g:(Y,v_1,v_2)\to(Z,w_1,w_2)$ be maps.*

(i) If f is a semi-open surjection and $g \circ f$ is continuous, then g is semi- continuous.
(ii) If g is a semi-continuous injection and $g \circ f$ is open, then f is semi-open.
(iii)If g is an open injection and $g \circ f$ is semi-continuous, then f is semi-continuous.

Proof. (i) Let H be an open subset of (Z, w_1, w_2) and let $g \circ f$ be continuous. By Proposition 5.1.18, $(g \circ f)^{-1}(H)$ is open in (X, u_1, u_2). Since f is a semi-open map, $f((g \circ f)^{-1}(H)) = f(f^{-1}(g^{-1}(H)))$ is semi-open in (Y, v_1, v_2). But f is a surjection, thus $f(f^{-1}(g^{-1}(H))) = g^{-1}(H)$. Therefore, g is semi-continuous.

(ii) Let G be an open subset of (X, u_1, u_2) and let $g \circ f$ be open. By Proposition 5.1.15, $g \circ f(G)$ is open in (Z, w_1, w_2). Since g is semi-continuous, $g^{-1}(g \circ f(G))$ is semi-open in (Y, v_1, v_2). But g is an injection, hence $g^{-1}(g \circ f(G)) = f(G)$. Therefore, f is semi-open.

(iii) Let H be an open subset of (Y, v_1, v_2) and let g be open. By Proposition 5.1.15, $g(H)$ is open in (Z, w_1, w_2). Since $g \circ f$ is semi-continuous, $(g \circ f)^{-1}(g(H))$ is semi-open in (X, u_1, u_2). But g is an injection, thus $(g \circ f)^{-1}(g(H)) = f^{-1}(g^{-1}(g(H))) = f^{-1}(H)$. Therefore, f is semi-continuous. ❒

5.4 Semi-irresolute Maps

In this section, we introduce semi-irresolute maps in biclosure spaces obtained by using semi-open sets. We then study some of their basic properties.

Definition 5.4.1. Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces. A map $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ is called *semi-irresolute* if $f^{-1}(G)$ is semi-open in (X, u_1, u_2) for every semi-open set G in (Y, v_1, v_2).

It is easy to show that the composition of two semi-irresolute maps of biclosure spaces is again a semi-irresolute map.

Remark 5.4.2. If a map $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ is semi-irresolute, then f is semi-continuous. The converse need not be true as shown in the following example.

Example 5.4.3. Let $X = \{1,2\} = Y$ and define a closure operator u_1 on X by $u_1\phi = \phi$ and $u_1\{1\} = u_1\{2\} = u_1 X = X$. Define a closure operator u_2 on X by $u_2\phi = \phi$ and $u_2\{1\} = u_2\{2\} = u_2 X = X$. Define a closure operator v_1 on Y by $v_1\phi = \phi$, $v_1\{1\} = \{1\}$, $v_1\{2\} = v_1 Y = Y$ and define a closure operator v_2 on Y by $v_2\phi = \phi$ and $v_2\{1\} = v_2\{2\} = v_2 Y = Y$. Let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ be the identity map. It is easy to see that there are only two open sets in (Y, v_1, v_2), namely ϕ and Y, and their inverse images are semi-open in (X, u_1, u_2). Thus, f is semi-continuous. But f is not semi-irresolute because $f^{-1}(\{2\})$ is not semi-open in (X, u_1, u_2) while $\{2\}$ is semi-open in (Y, v_1, v_2).

Proposition 5.4.4. *Let* (X,u_1,u_2) *and* (Y,v_1,v_2) *be biclosure spaces and* $f:(X,u_1,u_2)\to (Y,v_1,v_2)$ *be a map. Then* f *is semi-irresolute if and only if* $f^{-1}(B)$ *is semi-closed in* (X,u_1,u_2) *whenever* B *is semi-closed in* (Y,v_1,v_2).

Proposition 5.4.5. *Let* (X,u_1,u_2) *and* (Y,v_1,v_2) *be biclosure spaces and* $f:(X,u_1,u_2)\to (Y,v_1,v_2)$ *be an open, semi-irresolute and surjective map. Then* (Y,v_1,v_2) *is a* T_S*-space if* (X,u_1,u_2) *is a* T_S*-space .*

Proof. Let (X,u_1,u_2) be a T_S-space and let B be a semi-open subset of (Y,v_1,v_2). Since f is semi-irresolute, $f^{-1}(B)$ is semi-open in (X,u_1,u_2). As (X,u_1,u_2) is a T_S-space, $f^{-1}(B)$ is open in (X,u_1,u_2). Since f is open, $f\left(f^{-1}(B)\right)$ is open (Y,v_1,v_2) by Proposition 5.1.15. But f is a surjection, hence $f\left(f^{-1}(B)\right)=B$. Thus, B is open in (Y,v_1,v_2). Therefore, (Y,v_1,v_2) is a T_S-space ❒

Proposition 5.4.6. *Let* (X,u_1,u_2), (Y,v_1,v_2) *and* (Z,w_1,w_2) *be biclosure spaces and let* $f:(X,u_1,u_2)\to (Y,v_1,v_2)$ *and* $g:(Y,v_1,v_2)\to (Z,w_1,w_2)$ *be maps. If* f *is semi-irresolute and* g *is semi-continuous, then* $g\circ f$ *is semi-continuous.*

Proposition 5.4.7. *Let* (X,u_1,u_2) *and* (Y,v_1,v_2) *be biclosure spaces and let* $f:(X,u_1,u_2)\to (Y,v_1,v_2)$ *be a bijective map.*

(i) If f *is 1-continuous and* f^{-1} *is 2-continuous, then* f *is semi-irresolute.*

(ii) If f *is 2-continuous and* f^{-1} *is 1-continuous, then* f^{-1} *is semi-irresolute.*

Proof. (i) Let B be semi-open in (Y,v_1,v_2). Then there exists an open set H of (Y,v_1) such that $H\subseteq B\subseteq v_2H$. Since f^{-1} is 2-continuous, $f^{-1}:(Y,v_2)\to (X,u_2)$ is continuous. Thus, $f^{-1}(v_2H)\subseteq u_2 f^{-1}(H)$, i.e. $f^{-1}(H)\subseteq f^{-1}(B)\subseteq u_2 f^{-1}(H)$. As f is 1-continuous, $f:(X,u_1)\to (Y,v_1)$ is continuous, hence $f^{-1}(H)$ is open in (X,u_1). Consequently, $f^{-1}(B)$ is semi-open in (X,u_1,u_2). Therefore, f is semi-irresolute.

(ii) Let A be semi-open in (X,u_1,u_2). Then there exists an open subset G of (X,u_1) such that $G\subseteq A\subseteq u_2G$. Since f is 2-continuous, $f:(X,u_2)\to (Y,v_2)$ is continuous. Thus, $f(u_2G)\subseteq v_2 f(G)$, i.e. $f(G)\subseteq f(A)\subseteq v_2 f(G)$. But f^{-1} is 1-continuous, hence $f^{-1}:(Y,v_1)\to (X,u_1)$ is continuous. Since $f(G)$ is the inverse image of G under f^{-1}, $f(G)$ is open in (Y,v_1). Consequently, $f(A)$ is semi-open in (Y,v_1,v_2). But $f(A)$ is the inverse image of A under f^{-1}, thus f is semi-irresolute. ❒

5.5 Pre-semi-open Maps

In this section, we introduce pre-semi-open maps obtained by using semi-open sets. We then study some of their properties.

Definition 5.5.1. Let (X,u_1,u_2) and (Y,v_1,v_2) be biclosure spaces. A map $f:$ $(X,u_1,u_2)\to(Y,v_1,v_2)$ is called *pre-semi-open* if $f(A)$ is a semi-open subset of (Y,v_1,v_2) for every semi-open subset A of (X,u_1,u_2). A map $f:$ $(X,u_1,u_2)\to(Y,v_1,v_2)$ is called *pre-semi-closed* if $f(B)$ is a semi-closed subset of (Y,v_1,v_2) for every semi-closed subset B of (X,u_1,u_2).

It is easy to show that the composition of two pre-semi-open maps in biclosure spaces is again a pre-semi-open map.

Clearly, if a map $f:(X,u_1,u_2)\to(Y,v_1,v_2)$ is pre-semi-open, then f is semi-open. The converse need not be true as shown in the following example.

Example 5.5.2. In Example 5.1.17, the map f is semi-open but f is not pre-semi-open because $\{1\}$ is semi-open in (X,u_1,u_2) but $f(\{1\})$ is not semi-open in (Y,v_1,v_2).

Proposition 5.5.3. *Let (X,u_1,u_2) and (Y,v_1,v_2) be biclosure spaces. Let $f:(X,u_1,u_2)\to$ (Y,v_1,v_2) be a map. Then the following statements are equivalent:*

(i) f is pre-semi-open

(ii) If $B\subseteq Y$ and C is a semi-closed subset of (X,u_1,u_2) such that $f^{-1}(B)\subseteq C$, then $B\subseteq E$ and $f^{-1}(E)\subseteq C$ for some semi-closed subset E of (Y,v_1,v_2).

Proof. (i)→(ii) Let B be a subset of Y and let C be a semi-closed subset of (X,u_1,u_2) such that $f^{-1}(B)\subseteq C$. Then $f(X-C)$ is a semi-open subset of (Y,v_1,v_2). Put $E=Y-f(X-C)$. Then E is semi-closed in (Y,v_1,v_2) and $X-C\subseteq X-f^{-1}(B)=f^{-1}(Y-B)$. Hence, $f(X-C)\subseteq f\left(f^{-1}(Y-B)\right)\subseteq Y-B$. Thus, $Y-(Y-B)\subseteq Y-f(X-C)$, i.e. $B\subseteq E$ and $f^{-1}(E)=f^{-1}(Y-f(X-C))=X-f^{-1}(f(X-C))\subseteq X-(X-C)=C$. Therefore, E is a semi-closed subset of (Y,v_1,v_2) such that $B\subseteq E$ and $f^{-1}(E)\subseteq C$.

(ii)→(i) Let A be a semi-open subset of (X,u_1,u_2). Then $X-A$ is semi-closed in (X,u_1,u_2) and $f^{-1}(Y-f(A))=X-f^{-1}(f(A))\subseteq X-A$ where $Y-f(A)$ is a subset of Y. By the assumption, there is a semi-closed subset E of (Y,v_1,v_2) such that $Y-f(A)\subseteq E$ and $f^{-1}(E)\subseteq X-A$. Hence, $Y-E\subseteq f(A)$ and $A\subseteq X-f^{-1}(E)$. It follows that $Y-E\subseteq$ $f(A)\subseteq f\left(X-f^{-1}(E)\right)=f\left(f^{-1}(Y-E)\right)\subseteq Y-E$, i.e. $f(A)=Y-E$. Thus, $f(A)$ is semi-open in (Y,v_1,v_2). Therefore, f is pre-semi-open. ❒

Proposition 5.5.4. *Let (X,u_1,u_2) and (Y,v_1,v_2) be biclosure spaces and let $f:$ $(X,u_1,u_2)\to$ (Y,v_1,v_2) be a pre-semi-open map. If $y\in Y$ and C is a semi-closed subset of (X,u_1,u_2) such that $f^{-1}(\{y\})\subseteq C$, then there exists a semi-closed subset E of (Y,v_1,v_2) such that $y\in E$ and $f^{-1}(E)\subseteq C$.*

Proof. Let $y\in Y$ and let C be a semi-closed subset of (X,u_1,u_2) such that $f^{-1}(\{y\})\subseteq C$. Since $\{y\}\subseteq Y$, there exists a semi-closed subset E of (Y,v_1,v_2) such that $y\in E$ and $f^{-1}(E)\subseteq C$ by Proposition 5.5.3. ❒

The converse of the previous statement is not true in general as can be seen from the following example.

Example 5.5.5. Let $X = \{1,2,3\} = Y$ and define a closure operator u_1 on X by $u_1\phi = \phi$, $u_1\{1\} = u_1\{2\} = u_1\{1,2\} = \{1,2,\}$ and $u_1\{3\} = u_1\{1,3\} = u_1\{2,3\} = u_1X = X$. Define a closure operator u_2 on X by $u_2\phi = \phi$, $u_2\{3\} = \{3\}$ and $u_2\{1\} = u_2\{2\} = u_2\{1,2\} = u_2\{1,3\} = u_2\{2,3\} = u_2X = X$. Define a closure operator v_1 on Y by $v_1\phi = \phi$, $v_1\{1\} = \{1\}$, $v_1\{2\} = \{2\}$, $v_1\{3\} = v_1\{1,2\} = v_1\{1,3\} = v_1\{2,3\} = v_1Y = Y$ and define a closure operator v_2 on Y by $v_2\phi = \phi$ and $v_2\{1\} = v_2\{2\} = v_2\{3\} = v_2\{1,2\} = v_2\{1,3\} = v_2\{2,3\} = v_2Y = Y$. Let $f : (X,u_1,u_2) \to (Y,v_1,v_2)$ be the identity map. Then there are only three semi-closed subset of (X,u_1,u_2), namely ϕ, $\{1,2\}$ and X. Moreover, there are only four semi-closed subset of (Y,v_1,v_2), namely ϕ, $\{1\}$, $\{2\}$ and Y. Then for every $y \in Y$ and every semi-closed subset C of (X,u_1,u_2) such that $f^{-1}(\{y\}) \subseteq C$, there exists a semi-closed subset E of (Y,v_1,v_2) such that $y \in E$ and $f^{-1}(E) \subseteq C$. But f is not pre-semi-open because $\{3\}$ is semi-open in (X,u_1,u_2) but $f(\{3\})$ is not semi-open in (Y,v_1,v_2).

Theorem 5.5.6. *Let (X,u_1,u_2) and (Y,v_1,v_2) be biclosure spaces and let $f : (X,u_1,u_2) \to (Y,v_1,v_2)$ be a map. Then the following statements are equivalent:*

(i) f is pre-semi-closed.

(ii) If $D \subseteq Y$ and A is a semi-open subset of (X,u_1,u_2) such that $f^{-1}(D) \subseteq A$, then $D \subseteq M$ and $f^{-1}(M) \subseteq A$ for some semi-open subset M of (Y,v_1,v_2).

(iii)If $y \in Y$ and A is a semi-open subset of (X,u_1,u_2) such that $f^{-1}(\{y\}) \subseteq A$, then $y \in M$ and $f^{-1}(M) \subseteq A$ for some semi-open subset M of (Y,v_1,v_2).

Proof. (i)→(ii) The proof is a modification of the proof (i)→(ii) of Proposition 5.5.3.

(ii)→(iii) Let $y \in Y$ and A be a semi-open subset of (X,u_1,u_2) such that $f^{-1}(\{y\}) \subseteq A$. Put $D = \{y\}$. Then there exists a semi-open subset M of (Y,v_1,v_2) such that $y \in M$ and $f^{-1}(M) \subseteq A$.

(iii)→(i) Let C be a semi-closed subset of (X,u_1,u_2). Then $X - C$ is semi-open in (X,u_1,u_2) and $f^{-1}(Y - f(C)) = X - f^{-1}(f(C)) \subseteq X - C$. Let $y \in Y - f(C) \subseteq Y$ and put $A = X - C$. Then $f^{-1}(\{y\}) \subseteq X - C = A$. By (iii), there exists a semi-open subset M_y of (Y,v_1,v_2) such that $y \in M_y$ and $f^{-1}(M_y) \subseteq A = X - C$, i.e. $C \subseteq X - f^{-1}(M_y)$. Hence, $f(C) \subseteq f(X - f^{-1}(M_y)) = f(f^{-1}(Y - M_y)) \subseteq Y - M_y$. Thus, $y \in M_y \subseteq Y - f(C)$ for all $y \in Y - f(C)$. It follows that $Y - f(C) = \bigcup_{y \in Y - f(C)} M_y$. By Proposition 5.2.4, $\bigcup_{y \in Y - f(C)} M_y$ is semi-open in (Y,v_1,v_2). Consequently, $f(C)$ is semi-closed in (Y,v_1,v_2). Therefore, f is pre-semi-closed. ❒

Proposition 5.5.7. *Let (X,u_1,u_2), (Y,v_1,v_2) and (Z,w_1,w_2) be biclosure spaces and let $f : (X,u_1,u_2) \to (Y,v_1,v_2)$* and *$g : (Y,v_1,v_2) \to (Z,w_1,w_2)$ be maps.*

(i) If f is a semi-irresolute surjection and $g \circ f$ is pre-semi-open, then g is pre-semi-open.

(ii) If g is a semi-irresolute injection and $g \circ f$ is pre-semi-open, then f is pre-semi-open.

(iii) If f is a pre-semi-open surjection and $g \circ f$ is semi-irresolute, then g is semi-irresolute.

(iv) If g is a pre-semi-open injection and $g \circ f$ is semi-irresolute, then f is semi-irresolute.

Proof. (i) Let B be a semi-open subset of (Y, v_1, v_2). Since f is semi-irresolute, $f^{-1}(B)$ is semi-open in (X, u_1, u_2). But $g \circ f$ is pre-semi-open and f is surjective, hence $g \circ f\left(f^{-1}(B)\right) = g(B)$ is semi-open in (Z, w_1, w_2). Therefore, g is pre-semi-open.

The proofs of (ii)-(iv) are just modifications that of (i). ❐

Proposition 5.5.8. *Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces and $f:(X, u_1, u_2) \to (Y, v_1, v_2)$ be a continuous, pre-semi-open and injective map. Then (X, u_1, u_2) is a T_S-space if (Y, v_1, v_2) is a T_S-space.*

Proof. Let (Y, v_1, v_2) be a T_S-space and let A be a semi-open subset of (X, u_1, u_2). Since f is pre-semi-open, $f(A)$ is semi-open in (Y, v_1, v_2). But (Y, v_1, v_2) is a T_S-space, hence $f(A)$ is open in (Y, v_1, v_2). As f is continuous, $f^{-1}(f(A))$ is open in (X, u_1, u_2) by Proposition 5.1.18. Since f is injective, $f^{-1}(f(A)) = A$. Thus, A is open in (X, u_1, u_2).Therefore, (X, u_1, u_2) is a T_S-space. ❐

Proposition 5.5.9. *Let (X, u_1, u_2)* and *(Y, v_1, v_2) be biclosure spaces and $f:(X, u_1, u_2) \to (Y, v_1, v_2)$ be a 1-open and 2-continuous map. Then f is pre-semi-open.*

Proof. Let A be a semi-open subset of (X, u_1, u_2). Then there exists an open subset G of (X, u_1) such that $G \subseteq A \subseteq u_2 G$. Consequently, $f(G) \subseteq f(A) \subseteq f(u_2 G)$. Since f is 2-continuous, $f:$ $(X, u_2) \to (Y, v_2)$ is continuous. Hence, $f(u_2 G) \subseteq v_2 f(G)$, i.e. $f(G) \subseteq f(A) \subseteq v_2 f(G)$. But f is 1-open, thus $f:(X, u_1) \to (Y, v_1)$ is open. It follows that $f(G)$ is open in (Y, v_1). Thus, $f(A)$ is a semi-open subset of (Y, v_1, v_2). Therefore, f is pre-semi-open. ❐

Definition 5.5.10. A map $f:(X, u_1, u_2) \to (Y, v_1, v_2)$, where (X, u_1, u_2) and (Y, v_1, v_2) are biclosure spaces, is called a *semi-homeomorphism* if f is bijective, semi-irresolute and pre-semi-open.

It is easy to show that the composition of two semi-homeomorphisms of biclosure spaces is again a semi-homeomorphism.

Remark 5.5.11. The concepts of a homeomorphism and a semi-homeomorphism are independent as can be seen from two following examples.

Example 5.5.12. In Example 5.2.12, the map f is a semi-homeomorphism but f is not open. Consequently, f is not a homeomorphism.

Example 5.5.13. Let $X=\{1,2,3\}=Y$ and define a closure operator u_1 on X by $u_1\phi=\phi$, $u_1\{2\}=u_1\{3\}=u_1\{2,3\}=\{2,3\}$ and $u_1\{1\}=u_1\{1,2\}=u_1\{1,3\}=u_1X=X$. Define a closure operator u_2 on X by $u_2\phi=\phi$, $u_2\{1\}=\{1,3\}$ and $u_2\{2\}=u_2\{3\}=u_2\{1,2\}=u_2\{1,3\}=u_2\{2,3\}=u_2X=X$. Define a closure operator v_1 on Y by $v_1\phi=\phi$, $v_1\{2\}=v_1\{3\}=v_1\{2,3\}=\{2,3\}$ and $v_1\{1\}=v_1\{1,2\}=v_1\{1,3\}=v_1Y=Y$. Define a closure operator v_2 on Y by $v_2\phi=\phi$ and $v_2\{1\}=v_2\{2\}=v_2\{3\}=v_2\{1,2\}=v_2\{1,3\}=v_2\{2,3\}=v_2Y=Y$. Let $f:(X,u_1,u_2)\to(Y,v_1,v_2)$ be the identity map. Then f is a homeomorphism but f is not semi-irresolute because $f^{-1}\{1,2\}$ is not semi-open in (X,u_1,u_2) while $\{1,2\}$ is semi-open in (Y,v_1,v_2), i.e. f is not semi-homeomorphism.

Proposition 5.5.14. *Let (X,u_1,u_2) and (Y,v_1,v_2) be biclosure spaces and $f:(X,u_1,u_2)\to(Y,v_1,v_2)$ be a bijective map. Then f is pre-semi-open if and only if f is pre-semi-closed.*

As a direct consequence of Proposition 5.5.14, we have:

Proposition 5.5.15. *Let (X,u_1,u_2) and (Y,v_1,v_2) be biclosure spaces. If $f:(X,u_1,u_2)\to(Y,v_1,v_2)$ is a bijective semi-irresolute map, then the following statements are equivalent:*

(i) f is a semi-homeomorphism,
(ii) f is a pre-semi-closed map,
(iii) f is a pre-semi-open map.

6 Conclusion

The outcomes of the Ph.D. Thesis are as follows:

1. Semi-open sets and semi-closed sets in closure spaces were introduced. Their fundamental properties and behaviour of unions, intersections and subspaces were described. We also showed that the openness and semi-openness are preserved under the projection maps. Semi-open sets were used to define semi-open maps, semi-closed maps, semi-continuous maps, contra-semi-continuous maps and semi-irresolute maps which were investigated. We proved that the class of all semi-open (respectively, semi-closed) maps properly contains the class of all open (respectively, closed) maps and the class of all semi-continuous maps properly contains the class of all semi-irresolute maps. Semi-open sets were also used to introduce s-connectedness and s-compactness of closure spaces. Further, we proved that s-connectedness and s-compactness are preserved under semi-irresolute surjections.

2. Two new kinds of sets called generalized semi-open sets and γ-open sets were introduced. Their basic properties were studied. We proved that the generalized semi-openness and γ-openness are preserved by projection maps. Three new kinds of spaces, namely T_{gs}-spaces, T_{γ}-spaces and $T_{s\gamma}$-spaces, were introduced and studied. Further, the interrelation among them was investigated. We then introduced generalized semi-continuous maps and generalized semi-irresolute maps by using generalized semi-open sets and studied some of their properties. Moreover, we introduced the concepts of γ-continuous maps and γ-irresolute maps by using γ-open sets and investigated their behaviour. The interrelations among generalized-semi-continuous maps, generalized semi-irresolute maps, γ-continuous maps and γ-irresolute maps in closure spaces were also studied.

3. We studied some fundamental properties of biclosure spaces. We then defined a notion of semi-open sets in biclosure spaces and investigated their behaviour. We also introduced the concepts of semi-open maps, semi-closed maps, semi-continuous maps, semi-irresolute maps, pre-semi-open maps and pre-semi-closed maps of biclosure spaces and investigate some their properties. We proved that the class of all semi-irresolute maps is properly contained in the class of all semi-continuous maps and the class of all pre-semi-open maps is properly contained in the class of all semi-open maps.

In our further research, we will introduce the notion of φ-open sets and φ-closed sets. If (X,u) is a closure space and $A \subseteq X$, then A is *φ-open* if there exists an open subset G of (X,u) such that $A \subseteq G \subseteq uA$. A subset $A \subseteq X$ is called *φ-closed* if its complement is φ-open. We will study some basic properties of φ-open and φ-closed sets. We will introduce and study φ-open maps, φ-closed maps, φ-continuous maps, contra-φ-continuous maps, φ-irresolute maps, pre-φ-open and pre-φ-closed maps by using φ-open sets and φ-closed sets. We will also study the interrelation among these concepts.

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