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GAME THEORY IN WASTE MANAGEMENT
VYUŽITÍ TEORIE HER V ODPADOVÉM HOSPODÁŘSTVÍ

MASTER'S THESIS
DIPLOMOVÁ PRÁCE

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As provided for by the Act No. 111/98 Coll. on higher education institutions and the BUT Study and Examination Regulations, the director of the Institute hereby assigns the following topic of Master's Thesis:

Game Theory in Waste Management

Brief description:

Application of game-theoretic approaches to engineering problem solving. Especially, cooperative games with complete as well as incomplete coalition structure are expected.

Master's Thesis goals:

Study of advanced game-theoretic techniques. Building a game-theoretic model for a particular problem in waste management.

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Students are required to submit the thesis within the deadlines stated in the schedule of the academic year 2015/16.

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Summary
In this thesis, a game-theoretic model representing a decision-making situation in the waste management is created as a noncooperative game representing the conflict of waste processors and a cooperative game representing the conflict of waste producers. For the conflict of waste processors, the Nash equilibria are used to find stable strategies on gate fee values, which serve as a good prediction for the future. To specify the strategy sets, a lower bound and an upper bound are determined. For the conflict of waste producers, assuming a cooperation among all of them, a cost distribution is determined using the Shapley value and the nucleolus. For more producers, approximation algorithms for the Shapley value and the nucleolus are developed. These algorithms are based on an assumption that distant producers can not influence each other. The model is applied to a situation in the Czech Republic. For the conflict of waste processors, one Nash equilibrium is found. For the conflict of waste producers, some producers with high potential in cooperation are recognized.

Abstrakt
V této práci je vytvořen model rozhodovací situace v odpadovém hospodářství využívající metody teorie her. Model tvoří nekooperativní hra pro reprezentaci konfliktu zpracovatelů odpadu a kooperativní hra pro reprezentaci konfliktu producentů odpadu. Pro konflikt zpracovatelů odpadu je k nalezení strategií při volbě cen na bráně využit koncept Nashovy rovnováhy, takto nalezené stabilní strategie mohou sloužit jako předpověď budoucí situace. Pro zpřesnění množin strategií jsou určeny dolní a horní meze. Pro konflikt producentů odpadu se uvažuje spolupráce všech producentů a určuje se pro ni přerozdělení nákladů pomocí Shapleyho hodnoty a nucleolu. Pro konflikt více producentů jsou vyvinuty aproximační algoritmy pro Shapleyho hodnotu i nucleolus. Tyto algoritmy jsou založeny na předpokladu, že se vzdálení hráči vzájemně neovlivňují. Model je aplikován na situaci v České republice. Pro konflikt zpracovatelů odpadu je nalezen jeden bod Nashovy rovnováhy. Pro konflikt producentů odpadu jsou určeni někteří producenti s vysokým kooperativním potenciálem.

Keywords
game theory, waste management, Nash equilibrium, Shapley value, nucleolus

Klíčová slova
teorie her, odpadové hospodářství, Nashova rovnováha, Shapleyho hodnota, nucleolus

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I hereby certify that this thesis is the result of my own work and I have properly cited all sources used in the thesis.

Bc. Ondřej Osička

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Introduction

The waste management deals with situations in which waste producers, waste processors or both are involved. Every human being is a waste producer. In this thesis, they will be considered mainly on the level of administrative units. Among waste processors in the Czech Republic, landfills or incinerators can be found. Nevertheless, according to [CR14], starting from 2024, the Czech government is most likely going to ban the landfilling. Insufficient capacity of the already standing incinerators causes that radical changes are expected in following years.

New incinerators need to be built and, before it can be done, investors demand an analysis of the potential constructions. At the Institute of Process Engineering of Brno University of Technology, there were several mathematical models on this topic using mathematical optimization.

This thesis presents a game-theoretic model of a situation in which the incinerators are already built and their decisions on the charges for waste disposal need to be determined. From the producers' point of view, their strategies on coalition formations and choices of incinerators also require attention.

In the first chapter of this thesis, all the game-theoretic instruments necessary for understanding of the developed models are explained. Firstly, a description of the noncooperative games is provided with an approach called the Nash equilibrium representing a possible outcome. For the cooperative games, besides the description, several concepts for the total profit or cost division are presented.

The game-theoretic formulation of the waste management situation, the waste management game, is presented in the second chapter. A description of two conflicts and roles of their participants is provided.

The third chapter focuses on the conflict of waste processors, a noncooperative game in which the processors make decisions on the charge for the waste processing. Instruments from the first chapter are applied as well as original algorithms to lower the computation time.

The cooperative game of waste producers is studied in the fourth chapter. Again, for the computation time reasons, with respect to the cost allocations presented in the first chapter, algorithms for their approximations are developed.

And finally, the models for both conflicts are applied to the situation in the Czech Republic. This application and its results are provided in the fifth chapter.

Appendices contain a list of symbols and input data as well as complete results for the waste management problem in the chapter 5. Corresponding cross-references occur within the text.

Several computation tests were run to compare computation times of different approaches and algorithms. All such computations were realized on the computer with Microsoft Windows 10 Home 64-bit, quad-core Intel Core i5-6300HQ at frequency 2.3 GHz and 8 GB of RAM. The algorithms were implemented exclusively in Visual Basic for Applications in MS Excel (version Professional 2016) and in MATLAB (version R2015a) with IBM ILOG CPLEX (version 12.6.3).

1 Game-Theoretic Background

According to [My91], game theory is ”the study of mathematical models of conflict and cooperation between intelligent rational decision-makers.” These models are then called games. In other words, the game is a mathematical description of a situation where decisions of several subjects are to be made. Mutual dependency of these decisions makes the search for the optimal ones in such situations impossible by using classical optimization techniques.

Only games where all players are fully aware of this dependency and all outcomes it can lead to are assumed. In game theory, this state is called a complete information.

Further, two different definitions of a game are provided regarding the cooperative or noncooperative nature of the game. For each type, approaches to obtain, in some sense, optimal decisions are also introduced.

All definitions and theorems in this chapter, if not stated otherwise, are taken exclusively from [Ow13] and [Os14].

1.1 Noncooperative Games

By a noncooperative game, a situation where no settlements among decision-makers are allowed or possible is meant.

Firstly, the mathematical representation of a noncooperative game is shown, then the way of approaching it is presented. It should be remarked that there are more options of describing noncooperative games. For the purposes of this thesis, though, the normal form representation is sufficient.

1.1.1 The Normal Form

Definition 1.1. Let $N = \{p_1, \ldots, p_n\}$ be a nonempty set with n elements representing *players*, nonempty sets $A_{p_1}, \ldots, A_{p_n}$ be their *sets of strategies*, and $A = A_{p_1} \times \cdots \times A_{p_n}$ be the Cartesian product of these sets. Finally, let $\pi\colon A \to \mathbb{R}^n$ be a function defined as $\pi(a) = (\pi_{p_1}(a), \ldots, \pi_{p_n}(a))$ for all $a \in A$, where $\pi_{p_i}\colon A \to \mathbb{R}$ denotes a *payoff* or *cost function* (according to a nature of the problem) of player p_i. The triple (N, A, π) is then called an *n-player game in normal form.*

The exact meaning of this definition will be obvious after the following example, a famous game well-known as the prisoner's dilemma.

Example 1.2. *Two persons are arrested and imprisoned. They are placed into solitary confinement with no means of communication and offered a bargain. If a prisoner betrays the other one, he will be set free and the other one will serve 10 years. If they betray each other, it will mean 5 years for both of them, but if they both remain silent, due to a lack of evidence, they will both serve only 1 year.*

Denoting the prisoners by numbers 1 and 2, the set of players is

$$N = \{1, 2\}$$

and their strategies in form of the set A are

$$A = \{(\textit{stay silent}, \textit{stay silent}), (\textit{stay silent}, \textit{betray}),\\ (\textit{betray}, \textit{stay silent}), (\textit{betray}, \textit{betray})\}.$$

Values of the cost function are

$$\begin{aligned}
&\pi_1(\textit{stay silent}, \textit{stay silent}) = 1, && \pi_2(\textit{stay silent}, \textit{stay silent}) = 1,\\
&\pi_1(\textit{stay silent}, \textit{betray}) = 10, && \pi_2(\textit{stay silent}, \textit{betray}) = 0,\\
&\pi_1(\textit{betray}, \textit{stay silent}) = 0, && \pi_2(\textit{betray}, \textit{stay silent}) = 10,\\
&\pi_1(\textit{betray}, \textit{betray}) = 5, && \pi_2(\textit{betray}, \textit{betray}) = 5,
\end{aligned}$$

or represented as Table 1.1 where the values in each cell represent the values of π_1 and π_2 respectively.

Table 1.1: The table representation of the game in Example 1.2

		Prisoner 2	
		stay silent	*betray*
Prisoner 1	*stay silent*	1, 1	10, 0
	betray	0, 10	5, 5

1.1.2 Nash Equilibrium

There are more approaches for dealing with noncooperative games. Here, however, only the domination of strategies and pure strategy Nash equilibria are shown.

Definition 1.3. Given an n-player game in normal form (N, A, π) where $N = \{p_1, \dots, p_n\}$ and $A = A_{p_1} \times \cdots \times A_{p_n}$, a strategy $\tilde{a}_{p_i} \in A_{p_i}$ is said to *dominate* a strategy $a_{p_i} \in A_{p_i}$ if

$$\pi_{p_i}(a_{p_1}, \dots, a_{p_{i-1}}, \tilde{a}_{p_i}, a_{p_{i+1}}, \dots, a_{p_n}) > \pi_{p_i}(a_{p_1}, \dots, a_{p_{i-1}}, a_{p_i}, a_{p_{i+1}}, \dots, a_{p_n})$$

for all $a_{p_1} \in A_{p_1}, \dots, a_{p_{i-1}} \in A_{p_{i-1}}, a_{p_{i+1}} \in A_{p_{i+1}}, \dots, a_{p_n} \in A_{p_n}$ and for π being a payoff function. In the case of π being a cost function, the inequality sign is reversed.

Definition 1.4. Given an n-player game in normal form (N, A, π) where $N = \{p_1, \dots, p_n\}$ and $A = A_{p_1} \times \cdots \times A_{p_n}$, a strategy n-tuple $(\tilde{a}_{p_1}, \dots, \tilde{a}_{p_n}) \in A$ is called *pure strategy Nash equilibrium* if and only if for any $i \in \{1, \dots, n\}$ and $a_{p_i} \in A_{p_i}$

$$\pi_{p_i}(\tilde{a}_{p_1}, \dots, \tilde{a}_{p_n}) \geq \pi_{p_i}(\tilde{a}_{p_1}, \dots, \tilde{a}_{p_{i-1}}, a_{p_i}, \tilde{a}_{p_{i+1}}, \dots, \tilde{a}_{p_n})$$

for π being a payoff function or

$$\pi_{p_i}(\tilde{a}_{p_1}, \dots, \tilde{a}_{p_n}) \leq \pi_{p_i}(\tilde{a}_{p_1}, \dots, \tilde{a}_{p_{i-1}}, a_{p_i}, \tilde{a}_{p_{i+1}}, \dots, \tilde{a}_{p_n})$$

for π being a cost function.

It is important to note that, for a game, neither existence nor uniqueness of a pure strategy Nash equilibrium is guaranteed.

Example 1.5. In the prisoner's dilemma presented in example 1.2, the strategy *stay silent* is dominated by the strategy *betray* for both players and there is exactly one pure strategy Nash equilibrium, a pair (*betray*, *betray*).

Theorem 1.6. *All pure strategy Nash equilibria of a game obtained by removing dominated strategies are the same as those of the original game.*

Proof. The proof is obvious as the theorem follows directly from definition 1.4. □

1.2 Cooperative Games

Cooperation in game theory means a choice of a strategy in order to ensure the greatest total payoff (lowest total cost) for cooperating players. This payoff or cost then needs to be fairly redistributed among the players.

The choice of a strategy is obviously a simple problem or at least a problem which can be easily reformulated to a noncooperative game. Therefore, game theory deals with cooperative games mainly in the field of the redistribution.

1.2.1 The Characteristic Function Form

Definition 1.7. Let N be a set of n players. Any subset of N is called a *coalition*. Specifically, $\varnothing$ is denoted as the *empty coalition* and the player set N itself is denoted as the *grand coalition*. A real-valued function v, defined on the subsets of N, satisfying conditions

$$v(\varnothing) = 0$$

and

$$v(S \cup T) \geq v(S) + v(T) \qquad \text{if} \qquad S \cap T = \varnothing$$

is denoted as the *characteristic function*. The pair (N, v) is then called an *n-player game in characteristic function form.*

In the case of v representing a cost, not representing a payoff, the second condition is in form

$$v(S \cup T) \leq v(S) + v(T) \qquad \text{if} \qquad S \cap T = \varnothing.$$

Example 1.8. *The persons from Example 1.2 were not successful and were imprisoned for five years. In prison, they met an old friend that came up with an escape plan. The plan is to dig a tunnel out of the prison. Fig. 1.1 illustrates possible ways out of the prison. Each of the prisoners is able to make one metre of a tunnel per day. Spending time digging increases the chance of getting caught.*

Denoting the prisoners by numbers 1, 2 and 3, values of the characteristic function representing the cost, days spent on digging, are

$$\begin{aligned} v(\{1\}) &= 50, \\ v(\{2\}) &= 65, \\ v(\{3\}) &= 50, \\ v(\{1,2\}) &= 85, \end{aligned}$$

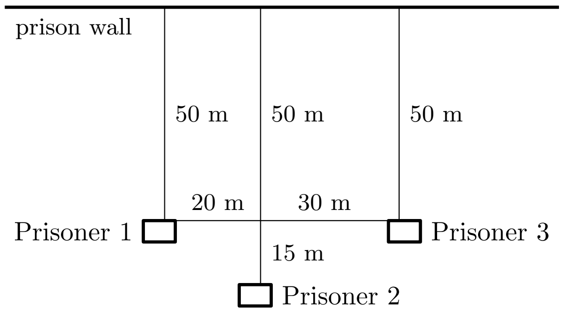

Fig. 1.1: Possible ways out of the prison from the cells

$$v(\{1,3\}) = 100,$$
$$v(\{2,3\}) = 95,$$
$$v(\{1,2,3\}) = 115.$$

Definition 1.9. An *imputation* for an n-player game (N, v) is a vector $x = (x_{p_1}, \ldots, x_{p_n})$ satisfying conditions

$$\sum_{p_i \in N} x_{p_i} = v(N)$$

and

$$x_{p_i} \geq v(\{p_i\}) \qquad \text{for all} \qquad p_i \in N.$$

For v representing a cost, the second condition is in form

$$x_{p_i} \leq v(\{p_i\}) \qquad \text{for all} \qquad p_i \in N.$$

Example 1.10. The prisoners from Example 1.8 are obviously open to a cooperation only when it allows them to get out of the prison at least as fast as on their own. For a cooperation among all of them and going the shortest way, divisions of the digging satisfying this condition are imputations of this game.

Definition 1.11. An imputation $x = (x_{p_1}, \ldots, x_{p_n})$ for an n-player game (N, v) satisfying condition

$$\sum_{p_i \in S} x_{p_i} \geq v(S) \qquad \text{for all} \qquad S \subset N$$

is called *coalitionally rational*.

In the case of v representing a cost, the condition is

$$\sum_{p_i \in S} x_{p_i} \leq v(S) \qquad \text{for all} \qquad S \subset N.$$

The choice of a reasonable imputation or a set of such imputations is a subject of the following sections.

1.2.2 The Core

The most straightforward concept seems to be a choice of an imputation from a set of all coalitionally rational imputations.

Definition 1.12. The set of all imputations $x = (x_{p_1}, \ldots, x_{p_n})$ for an n-player game (N, v) satisfying

$$\sum_{p_i \in N} x_{p_i} = v(N)$$

and

$$\sum_{p_i \in S} x_{p_i} \geq v(S) \qquad \text{for all} \qquad S \subset N$$

is called the *core*. The notation for the core is $C(N, v)$.

Clearly, for v representing a cost, the second condition is in form

$$\sum_{p_i \in S} x_{p_i} \leq v(S) \qquad \text{for all} \qquad S \subset N.$$

Despite the logic behind the definition, there is no guarantee of the core being a nonempty set. In order to recognize games with nonempty cores, the concept of balanced collections is introduced.

Definition 1.13. Let $\mathcal{C} = \{S_1, \ldots, S_m\}$ denote a collection of nonempty subsets of $N = \{p_1, \ldots, p_n\}$. Collection $\mathcal{C}$ is said to be N-*balanced* if there exist positive numbers $y_1, \ldots, y_m$ such that, for each $p_i \in N$,

$$\sum_{j \in M \colon p_i \in S_j} y_j = 1,$$

where $M = \{1, \ldots, m\}$. Then $y = (y_1, \ldots, y_m)$ is the *balancing vector* for $\mathcal{C}$. A *minimal N-balanced collection* is an N-balanced collection which is such that no proper subcollection is N-balanced.

A determination of the core can be formulated as a linear optimization program. Dual program of this formulation then leads to the following theorem.

Theorem 1.14. *A necessary and sufficient condition for the n-player game (N, v) to have a nonempty core is that, for every minimal N-balanced collection $\mathcal{C} = \{S_1, \ldots, S_m\}$ with balancing vector $y = (y_1, \ldots, y_m)$ and $M = \{1, \ldots, m\}$,*

$$\sum_{j \in M} y_j v(S_j) \leq v(N)$$

for v representing a payoff and

$$\sum_{j \in M} y_j v(S_j) \geq v(N)$$

for v representing a cost.

Proof. See [Ow13]. □

1.2.3 The Shapley Value

No nonemptiness guarantee of the core leads to study of other concepts. In [Sh53], one such concept was defined by Lloyd S. Shapley.

Definition 1.15. The *Shapley value* for an n-player game (N, v) is a vector $\varphi(N, v) = (\varphi_{p_1}(N, v), \ldots, \varphi_{p_n}(N, v))$ defined by formula

$$\varphi_{p_i}(N, v) = \sum_{S \subseteq N \colon p_i \in S} \frac{(|S| - 1)! \, (|N| - |S|)!}{|N|!} \left(v(S) - v(S \backslash \{p_i\})\right).$$

Existence of the Shapley value is guaranteed by the definition itself. This value, however, does not always belong to the core, even in cases in which the core is nonempty. For this purpose, a theorem from [Sh71] is provided.

Theorem 1.16. *The Shapley value is in the core if*

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T) \qquad \textit{for all} \qquad S, T \subseteq N$$

for v representing a payoff or

$$v(S) + v(T) \geq v(S \cup T) + v(S \cap T) \qquad \textit{for all} \qquad S, T \subseteq N$$

for v representing a cost.

Proof. See [Sh71]. □

Example 1.17. For the prison break game from Example 1.8, the Shapley value is a vector

$$\varphi = (35, 40, 40).$$

Obviously, the Shapley value belongs to the core.

1.2.4 The Bargaining Set

Next concept corresponds with an expected negotiation in a coalition. For any coalition, a player may threaten to leave and join together with another player to increase the profit or lower the cost. Other players from the original coalition may, however, oppose if they have an offer more beneficial for the player the leaving one plans to join together with. In this situation, for the leaving player, the consequences would not be any good. On the other hand, when there is nothing such to offer, there is no reason for the player to remain in the coalition.

Definition 1.18. For an n-player game (N, v), let $\mathcal{S} = \{S_1, \ldots, S_m\}$ denote a collection of nonempty subsets of $N = \{p_1, \ldots, p_n\}$ such that

$$S_i \cap S_j = \varnothing \qquad \text{for all} \qquad i, j \in M \colon i \neq j,$$

where $M = \{1, \ldots, m\}$. Collection $\mathcal{S}$ is then called a *coalition structure.*

Definition 1.19. For an n-player game (N, v), an *individually rational payoff configuration* is a pair $(x, \mathcal{S})$, where $x = (x_{p_1}, \ldots, x_{p_n})$ is an imputation and $\mathcal{S} = \{S_1, \ldots, S_m\}$ is a coalition structure. Moreover, if it is also satisfying

$$\sum_{p_i \in S} x_{p_i} \geq v(S) \qquad \text{for all} \qquad S \subseteq S_k, k \in M$$

for v representing a payoff or

$$\sum_{p_i \in S} x_{p_i} \leq v(S) \qquad \text{for all} \qquad S \subseteq S_k, k \in M$$

for v representing a cost, where $M = \{1, \ldots, m\}$, the pair $(x, \mathcal{S})$ is called a *coalitionally rational payoff configuration.*

Definition 1.20. For an n-player game (N, v), let $(x, \mathcal{U})$, $(y, \mathcal{V})$, $(z, \mathcal{W})$ be coalitionally rational payoff configurations, where $x = (x_{p_1}, \ldots, x_{p_n})$, $y = (y_{p_1}, \ldots, y_{p_n})$, $z = (z_{p_1}, \ldots, z_{p_n})$ are imputations and $\mathcal{U} = \{U_1, \ldots, U_{m_u}\}$, $\mathcal{V} = \{V_1, \ldots, V_{m_v}\}$, $\mathcal{W} = \{W_1, \ldots, W_{m_w}\}$ are coalition structures with $M_u = \{1, \ldots, m_u\}$, $M_v = \{1, \ldots, m_v\}$, $M_w = \{1, \ldots, m_w\}$, and let S and T be nonempty disjoint subsets of some $U_k \in \mathcal{U}$. A coalitionally rational payoff configuration $(y, \mathcal{V})$ is then called an *objection* of S against T if

$$\{p_i \colon p_i \in V_k, V_k \cap S = \varnothing, k \in M_v\} \cap T = \varnothing,$$

$$y_{p_i} > x_{p_i} \qquad \text{for all} \qquad p_i \in S,$$

$$y_{p_i} \geq x_{p_i} \qquad \text{for all} \qquad p_i \in \{p_i \colon p_i \in V_k, V_k \cap S = \varnothing, k \in M_v\}$$

for v representing a payoff or

$$\{p_i \colon p_i \in V_k, V_k \cap S = \varnothing, k \in M_v\} \cap T = \varnothing,$$

$$y_{p_i} < x_{p_i} \qquad \text{for all} \qquad p_i \in S,$$

$$y_{p_i} \leq x_{p_i} \qquad \text{for all} \qquad p_i \in \{p_i \colon p_i \in V_k, V_k \cap S = \varnothing, k \in M_v\}$$

for v representing a cost. A coalitionally rational payoff configuration $(z, \mathcal{W})$ is called a *counterobjection* of T against S if

$$S \not\subseteq \{p_i \colon p_i \in W_k, W_k \cap T = \varnothing, k \in M_w\},$$

$$z_{p_i} \geq x_{p_i} \qquad \text{for all} \qquad p_i \in \{p_i \colon p_i \in W_k, W_k \cap T = \varnothing, k \in M_w\},$$

$$z_{p_i} \geq y_{p_i} \qquad \text{for all} \qquad p_i \in \{p_i \colon p_i \in W_k \cap V_l, W_k \cap T = \varnothing, V_l \cap S = \varnothing, k \in M_w, l \in M_v\}.$$

for v representing a payoff or

$$S \not\subseteq \{p_i \colon p_i \in W_k, W_k \cap T = \varnothing, k \in M_w\},$$

$$z_{p_i} \leq x_{p_i} \qquad \text{for all} \qquad p_i \in \{p_i \colon p_i \in W_k, W_k \cap T = \varnothing, k \in M_w\},$$

$$z_{p_i} \leq y_{p_i} \qquad \text{for all} \qquad p_i \in \{p_i \colon p_i \in W_k \cap V_l, W_k \cap T = \varnothing, V_l \cap S = \varnothing, k \in M_w, l \in M_v\}.$$

for v representing a cost. A coalitionally rational payoff configuration $(x, \mathcal{U})$ is called *stable* if for every objection of S against T, there is a counterobjection of T against S.

Briefly, the objection of S against T represents the threat that S can obtain more by changing to a new coalitionally rational payoff configuration and their new partners would agree to this.

By the counterobjection of T against S, the members of coalition T claim that they can find another coalitionally rational payoff configuration in which they and all their partners receive at least their original payoff. If they need some of the new partners of S from the objection, they give them at least as much as in the objection coalitionally rational payoff configuration.

Definition 1.21. The *bargaining set* $\mathcal{M}$ is the set of all stable coalitionally rational payoff configurations. Dealing with individually rational payoff configurations instead of coalitionally rational payoff configurations would lead to a bargaining set denoted by $\mathcal{M}^{(i)}$.

Definition 1.22. The bargaining set $\mathcal{M}_1$ is the set of all coalitionally rational payoff configurations such that, if any coalition S has an objection against a set T, at least one member of T has a counterobjection. The same holds for individually rational payoff configurations with the bargaining set $\mathcal{M}_1^{(i)}$.

With a focus on the bargaining set $\mathcal{M}_1^{(i)}$, the nonemptiness is guaranteed from the following theorem.

Theorem 1.23. *For an n-player game (N, v) and any coalition structure $\mathcal{S}$, there is at least one imputation x such that $(x, \mathcal{S}) \in \mathcal{M}_1^{(i)}$.*

Proof. See [Pe63]. □

1.2.5 The Kernel

The kernel is a different approach. It will be, however, seen that it is closely related to the concept of bargaining sets.

Definition 1.24. The kernel of an n-player game (N, v) is the set $\mathcal{K}$ of all individually rational payoff configurations $(x, \mathcal{S})$ with $x = (x_{p_1}, \ldots, x_{p_n})$ such that, for all $S \in \mathcal{S}$, there are no $p_i, p_j \in S$ with

$$\max_{T \subseteq N:\, p_i \in T, p_j \notin T} \left(v(T) - \sum_{p_k \in T} x_{p_k} \right) > \max_{T \subseteq N:\, p_i \notin T, p_j \in T} \left(v(T) - \sum_{p_k \in T} x_{p_k} \right).$$

for v representing a payoff or

$$\min_{T \subseteq N:\, p_i \in T, p_j \notin T} \left(v(T) - \sum_{p_k \in T} x_{p_k} \right) < \min_{T \subseteq N:\, p_i \notin T, p_j \in T} \left(v(T) - \sum_{p_k \in T} x_{p_k} \right).$$

for v representing a cost.

The nonemptiness is again guaranteed from the following theorem.

Theorem 1.25. *For any coalition structure $\mathcal{S}$, there exists a vector x such that $(x, \mathcal{S}) \in \mathcal{K}$.*

Proof. See [MP66]. □

The following theorems explain the relation between the kernel and the other concepts.

Theorem 1.26. *For any game,* $\mathcal{K} \subseteq \mathcal{M}_1^{(i)}$.

Proof. See [MM65]. □

Theorem 1.27. *The kernel always intersects the core of the game, if the core is not empty.*

Proof. See [MP66]. □

1.2.6 The Nucleolus

Last concept here presented is the nucleolus.

Definition 1.28. The vector $x = (x_1, \dots, x_n)$ is said to be lexicographically smaller than the vector $y = (y_1, \dots, y_n)$ if there is some integer $i \in \{1, \dots, n\}$ such that

$$x_j = y_j \qquad \text{for all} \qquad j \in \{1, \dots, n\} \colon j < i,$$

$$x_i < y_i.$$

Definition 1.29. For an n-player game (N, v), defining the *excess vector* at imputation $x = (x_{p_1}, \dots, x_{p_n})$ as

$$\varepsilon(x) = \left(v(S_1) - \sum_{p_i \in S_1} x_{p_i}, \dots, v(S_m) - \sum_{p_i \in S_m} x_{p_i} \right),$$

where $S_1, \dots, S_m \subset N$ are all coalitions except for the empty coalition and the grand coalition, the *nucleolus* is the imputation $\varrho = (\varrho_{p_1}, \dots, \varrho_{p_n})$ for which $\varepsilon(\varrho)$ is lexicographically smaller or equal than $\varepsilon(x)$ for any imputation x (lexicographical minimum).

In the case of v representing a cost, not representing a payoff, the nucleolus realizes the lexicographical maximum, not the lexicographical minimum.

Theorem 1.30. *For any game* (N, v)*, the nucleolus* ϱ *exists uniquely and* $(\varrho, \{N\}) \in \mathcal{K}$*. Moreover, for any game with a nonempty core, the nucleolus belongs to the core.*

Proof. See [Sc69]. □

The computation of the nucleolus can be formulated as a sequence of optimization problems introduced in [Fr97]. For an n-player game (N, v), using the same notation as in [GJ15], the nucleolus $\varrho = (\varrho_{p_1}, \dots, \varrho_{p_n})$ is determined by $\varrho_{p_i} = x_{p_i}^{k'}$, where

$$\begin{aligned}
\{\varepsilon_k,\, x_{p_i}^k \colon p_i \in N\} = \underset{\varepsilon \in \mathbb{R},\, x_{p_i} \in \mathbb{R} \colon p_i \in N}{\arg\min} \quad & \varepsilon, \\
\text{s. t.} \qquad & \varepsilon + \sum_{p_i \in S} x_{p_i} \geq v(S) \qquad \forall S \subset N, S \neq \varnothing, S \notin \bigcup_{j \in \{0, \dots, k-1\}} F_j, \\
& \varepsilon_j + \sum_{p_i \in S} x_{p_i} = v(S) \qquad \forall S \in F_j, j \in \{0, \dots, k-1\}, \\
& \sum_{p_i \in N} x_{p_i} = v(N),
\end{aligned}$$

$\varepsilon_0 = 0$, $F_0 = \varnothing$, F_k is the set of all coalitions $S \subset N$, for which

$$\varepsilon_k + \sum_{p_i \in S} x^k_{p_i} = v(S),$$

and k' is the lowest positive integer for which the vector $(x^{k'}_{p_1}, \ldots, x^{k'}_{p_n})$ realizing the minimum is unique.

In the case of a characteristic function not representing a payoff, but a cost, the minimization should be replaced by a maximization and the inequality sign in the first constraint reversed.

Example 1.31. For the prison break game from Example 1.8, the nucleolus is a vector

$$\varrho = (35, 40, 40)\,,$$

which equals to the Shapley value for this game.

2 Description of Waste Management Game

As mentioned in the introduction, the waste management deals with situations where waste producers and waste processors are involved. The producers need to dispose of all the waste and the processors want to fill their capacity. For the efficiency of this process, the right decisions need to be made. This decision-making situation is further denoted as the waste management game with waste processors and waste producers as its players.

2.1 Decision-Making in Waste Management

Waste processors' only way of controlling their income is via gate fee, the charge for waste processing. The lower the gate fee is, the more capacity is used, but rationally, keeping the gate fee as low as possible is not the best choice when aiming for the highest income.

Waste producers, on the other hand, react to the gate fee settings and, as illustrated in Fig. 2.1, decide, with attention to the distance, the gate fee offered and the available capacity, which processor to choose. They may also divide the waste among more processors.

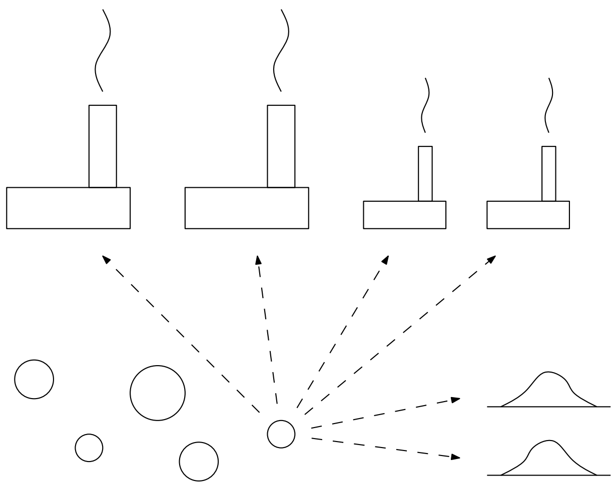

Fig. 2.1: Waste producers are choosing among waste processors, which can include incinerators or landfills, in order to minimize their total costs

The benefit of game-theoretic approach might seem questionable, two examples showing the need for game theory are therefore presented.

Example 2.1. *Let Fig. 2.2 illustrate a situation where the Processor 1 is setting a gate fee. The capacity of each incinerator in this situation is sufficient for both waste producers.*

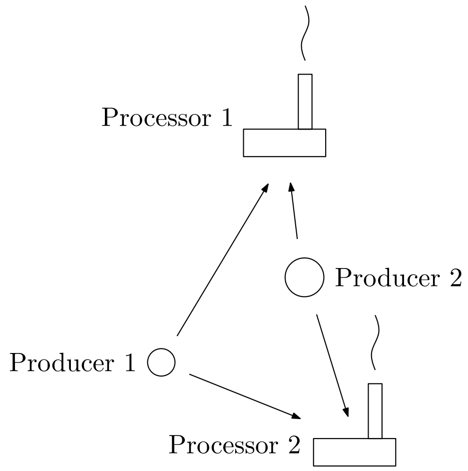

Fig. 2.2: A situation illustrating the need for game theory from the waste processors' point of view

Supposing the transportation costs equal for both choices of processors, it is obvious that for every value of gate fee that Processor 1 sets, a reaction of a slightly smaller value by Processor 2 will follow. Despite that this example is too trivial to show some important results, the need for game theory from the waste processors' point of view is obvious.

Example 2.2. *For the waste producers' point of view, the example situation differs a little bit. The gate fees are now set equally. The costs for the transportation as well as the capacities and productions are provided in the situation overview in Fig. 2.3.*

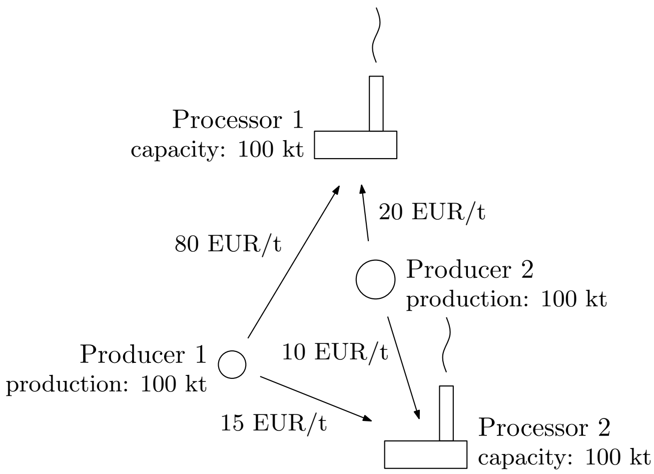

Fig. 2.3: A situation illustrating the need for game theory from the waste producers' point of view

It is easy to see that, if Producer 1 could manage to be the first one making decision, he would send all the produced waste to Processor 2 with a transportation cost

$$c_{1,2} = 100{,}000 \cdot 15 = 1{,}500{,}000 \text{ EUR}.$$

The cost for Producer 2, forced to use Processor 1, then would be

$$c_{2,1} = 100{,}000 \cdot 20 = 2{,}000{,}000 \text{ EUR}.$$

On the other hand, if Producer 2 was the first one, he would choose Processor 2 with a transportation cost

$$c_{2,2} = 100{,}000 \cdot 10 = 1{,}000{,}000 \text{ EUR}.$$

For Producer 1, the cost then would be

$$c_{1,1} = 100{,}000 \cdot 80 = 8{,}000{,}000 \text{ EUR}.$$

With no information on the order of decisions, the optimal strategy for both of them seems to be a cooperation which allows them to minimize the total cost and redistribute it. That way, they are able to reduce the total transportation cost down to

$$\min\{c_{1,1} + c_{2,2}, c_{1,2} + c_{2,1}\} = 3{,}500{,}000 \text{ EUR}.$$

For example, a distribution of 2,000,000 EUR to be paid by Producer 1 and 1,500,000 EUR to be paid by Producer 2 seems beneficial for both of them.

2.2 Goals and Strategies

Here, with reference to Example 2.1 and Example 2.2, goals of the waste management game participants and strategies to achieve them are summarized.

The objective of waste processors is to maximize their income by achieving the optimal combination of the amount of the processed waste and the charge for this processing. Assuming the waste processors already standing, and hence with no way to change the capacity, their only tool is the gate fee setting. For any setting, however, a reaction of other processors is expected. Therefore, the gate fee setting should not guarantee only high income, but also a stability.

Waste producers, on the other hand, aim to minimize their outcome. Their total cost consists of the payment of a gate fee to the chosen waste processor and of the cost for transportation of their waste to this processor. Their goal is to choose a processor with an optimal combination of the gate fee value and the transportation cost. Nevertheless, when the choice of more producers is the same processor with insufficient capacity, a cooperation might be useful, as seen in Example 2.2. In this simple example, the cooperation is natural. For a large-scale problem, the cooperation might become beneficial when the capacities of local waste processors are insufficient and the producers are forced to send their waste to more distant ones. Therefore, optimal strategies on the coalition formation require an attention too.

2.3 Separation of Conflicts

It sounds natural that firstly the waste processors make their decisions and set the gate fees and, once this is done, the waste producers come to choose their strategy. This allows the situation to be divided into two problems studied independently. The results of the conflict of processors, of course, need to be included as input data for the conflict of producers.

There exist, however, reasons why this separation could be questioned. For example, some of the producers might be decided to form a coalition already before the gate fees are set. The processors should take into account this intention too. Moreover, the income of processors depends not only on the gate fees all processors set, but might also depend on the coalition structure. For different coalitions, different processor might be chosen by a producer. Therefore, even if the producers decide once the gate fees are already set, the formed coalition structure might lead to an unstable combination of gate fees.

Even despite all that, the waste management game is further divided into two independent problems.

3 Conflict of Waste Processors

Firstly, for the conflicts of waste processors, a simple example showing a possible approach to deal with them is provided.

Example 3.1. *In the situation illustrated in Fig. 3.1, Processor 1 and Processor 2 are making decisions on a gate fee. Options of both of them are 50 EUR/t, 60 EUR/t and 70 EUR/t. The gate fee of Processor 3 is 80 EUR/t.*

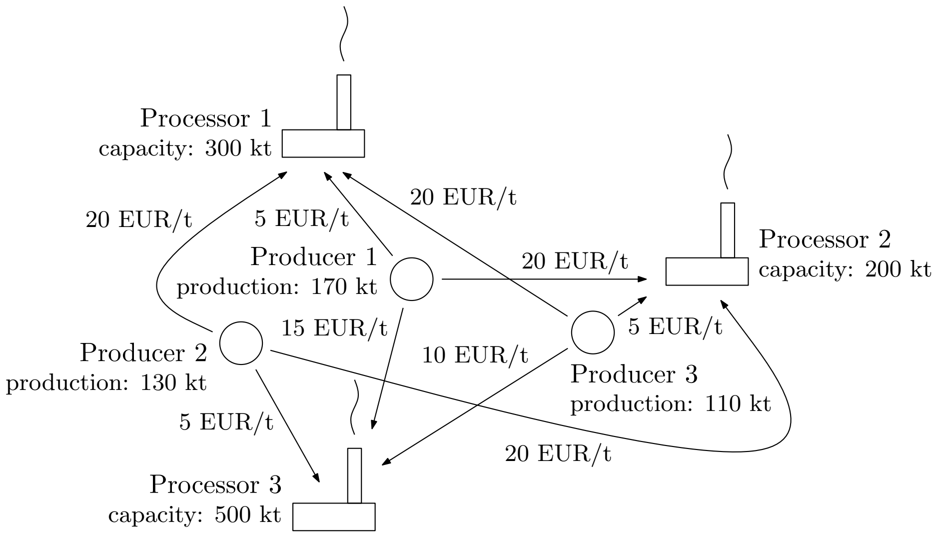

Fig. 3.1: An overview of the situation from Example 3.1

As there are only two processors making decisions, it can be approached as a game of two players, Processor 1 and Processor 2.

The payoff function is computed for each strategy combination by assuming the grand coalition formation and its choice of the optimal strategy. In other words, the grand coalition makes a decision minimizing the total cost. If there are more optimal solutions, then, for each processor, the worst solution is selected.

As already discussed in the section 2.3, the payoff can differ for another coalition structures and orders of choices. Nevertheless, by neglecting this property, the grand coalition provides the computationally easiest approach.

Denoting Processor 1 and Processor 2 by p_1 and p_2 respectively, the values of the payoff function in EUR are

$$\pi_{p_1}(50 \text{ EUR/t}, 50 \text{ EUR/t}) = 10{,}500{,}000, \quad \pi_{p_2}(50 \text{ EUR/t}, 50 \text{ EUR/t}) = 5{,}500{,}000,$$
$$\pi_{p_1}(50 \text{ EUR/t}, 60 \text{ EUR/t}) = 15{,}000{,}000, \quad \pi_{p_2}(50 \text{ EUR/t}, 60 \text{ EUR/t}) = 6{,}600{,}000,$$
$$\pi_{p_1}(50 \text{ EUR/t}, 70 \text{ EUR/t}) = 15{,}000{,}000, \quad \pi_{p_2}(50 \text{ EUR/t}, 70 \text{ EUR/t}) = 7{,}700{,}000,$$

$$\begin{aligned}
&\pi_{p_1}(60\text{ EUR/t}, 50\text{ EUR/t}) = 12{,}600{,}000, && \pi_{p_2}(60\text{ EUR/t}, 50\text{ EUR/t}) = 10{,}000{,}000,\\
&\pi_{p_1}(60\text{ EUR/t}, 60\text{ EUR/t}) = 12{,}600{,}000, && \pi_{p_2}(60\text{ EUR/t}, 60\text{ EUR/t}) = 6{,}600{,}000,\\
&\pi_{p_1}(60\text{ EUR/t}, 70\text{ EUR/t}) = 18{,}000{,}000, && \pi_{p_2}(60\text{ EUR/t}, 70\text{ EUR/t}) = 7{,}700{,}000,\\
&\pi_{p_1}(70\text{ EUR/t}, 50\text{ EUR/t}) = 11{,}900{,}000, && \pi_{p_2}(70\text{ EUR/t}, 50\text{ EUR/t}) = 10{,}000{,}000,\\
&\pi_{p_1}(70\text{ EUR/t}, 60\text{ EUR/t}) = 11{,}900{,}000, && \pi_{p_2}(70\text{ EUR/t}, 60\text{ EUR/t}) = 12{,}000{,}000,\\
&\pi_{p_1}(70\text{ EUR/t}, 70\text{ EUR/t}) = 11{,}900{,}000, && \pi_{p_2}(70\text{ EUR/t}, 70\text{ EUR/t}) = 7{,}700{,}000.
\end{aligned}$$

Table 3.1 shows a table representation where the values in each cell represent the values of π_{p_1} and π_{p_2} respectively.

Table 3.1: A table representation of the game in Example 3.1 (the payoff values in millions of EUR)

		Processor 2		
		50 EUR/t	60 EUR/t	70 EUR/t
	50 EUR/t	10.5, 5.5	15, 6.6	15, 7.7
Processor 1	60 EUR/t	12.6, 10	12.6, 6.6	18, 7.7
	70 EUR/t	11.9, 10	11.9, 12	11.9, 7.7

It is not difficult to find the Nash equilibrium of this game, which is the strategy combination (60 EUR/t, 50 EUR/t). Hence, by the choice of 60 EUR/t by Processor 1 and 50 EUR/t by Processor 2, the stability is guaranteed, as none of them has a reason to change the decision.

In the previous example, there are obviously two questionable steps. First one is the problem description itself, where only three choices for each processor are assumed. The second one is the grand coalition formation for every strategy combination. A discussion on these topics follows a little further in the section 3.2.

Firstly, a mathematical model is presented. This model approaches the conflict in the same way as it is approached in Example 3.1.

3.1 Mathematical Model

In the waste management game of n_p processors and n_r producers, the set of all processors is denoted by $P = \{p_1, \dots, p_{n_p}\}$ with the set of indices $J = \{1, \dots, n_p\}$. Their capacities are $w_1^c, \dots, w_{n_p}^c$ and the sets of strategies $C_1^g, \dots, C_{n_p}^g$ respectively. The set of all producers is denoted by $R = \{r_1, \dots, r_{n_r}\}$ with the set of indices $I = \{1, \dots, n_r\}$. Their waste productions are $w_1^p, \dots, w_{n_r}^p$ respectively. Transportation costs are represented by the matrix $\left[c_{i,j}^t\right]$, where $c_{i,j}^t$ is the cost of waste transportation from producer r_i to processor p_j.

3.1.1 Payoff Function

For each processor $p_k \in P$, the payoff function π_{p_k} for every strategy combination $(c_1^g, \dots, c_{n_p}^g) \in C_1^g \times \dots \times C_{n_p}^g$ is determined by formula

$$\pi_{p_k}(c_1^g, \dots, c_{n_p}^g) = \sum_{i \in I} c_k^g \tilde{x}_{i,k},$$

where $\tilde{x}_{i,k} \in \{\tilde{x}_{i,j}:\ i \in I, j \in J\}$, which is a set obtained as a solution of optimization problem

$$\{\tilde{x}_{i,j}:\ i \in I, j \in J\} = \underset{x_{i,j}:\, i\in I, j\in J}{\arg\min} \sum_{i\in I} \left(\left(c^t_{i,k} + c^g_k + m\right) x_{i,k} + \sum_{j \in J\setminus\{k\}} \left(c^t_{i,j} + c^g_j\right) x_{i,j} \right),$$

$$\text{s. t.} \quad \sum_{i\in I} x_{i,j} \le w^c_j \qquad \forall j \in J,$$

$$\sum_{j\in J} x_{i,j} = w^p_i \qquad \forall i \in I,$$

$$x_{i,j} \ge 0 \qquad \forall i \in I, j \in J,$$

where m is a very small positive number just to guarantee the worst optimal solution.

3.1.2 Stable Strategies

Once the payoff function is computed for all players and all combinations of strategies, the pure strategy Nash equilibria can be determined easily with Algorithm 3.1.

Algorithm 3.1: Nash equilibria determination

for all $(\tilde{c}^g_1, \dots, \tilde{c}^g_{n_p}) \in C^g_1 \times \dots \times C^g_{n_p}$ **do**
  for all $j \in J$ **do**
    if $\pi_{p_j}(\tilde{c}^g_1, \dots, \tilde{c}^g_{n_p}) \ge \pi_{p_j}(\tilde{c}^g_1, \dots, \tilde{c}^g_{j-1}, c^g_j, \tilde{c}^g_{j+1} \dots, \tilde{c}^g_{n_p})$ **for all** $j \in J$ **then**
      $(\tilde{c}^g_1, \dots, \tilde{c}^g_{n_p})$ is the Nash equilibrium
    end if
  end for
end for

It is important to remember that neither the existence nor the uniqueness of the Nash equilibrium is guaranteed. However, if there are any, they represent a stable combinations of strategies, where no processor has an intention to change the gate fee. Therefore, Nash equilibrium strategies seem to serve as predictions of probable future situations.

One more thing worth mentioning is that the model is not limited only for processors with more than one strategy. There was no such requirement on sets C^g_j. The same model can be therefore used for situations of this nature, situations where some processors are comfortable with their income and the current gate fee setting. One such situation is studied in the chapter 5 for the Czech Republic.

3.2 Strategy Set and Coalition Structure

To compute the Nash equilibria of a game, almost all strategies of all players are necessary. Therefore, it seems strange to assume, for example, the strategy sets containing only three strategies like in Example 3.1, even to assume them being finite. However, there is a reason for that. The reason is a computation time.

The computation time grows significantly with the number of processors. This is illustrated in Fig. 3.3 for the conflict of processors with three strategies and in Fig. 3.4 for the conflict of processors with five strategies.

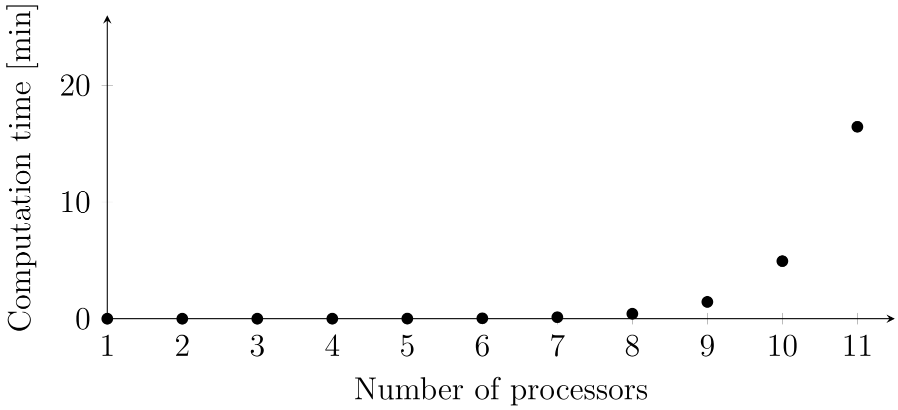

Fig. 3.3: Computation time of the Nash equilibrium determination for processors with three strategies implemented in MATLAB

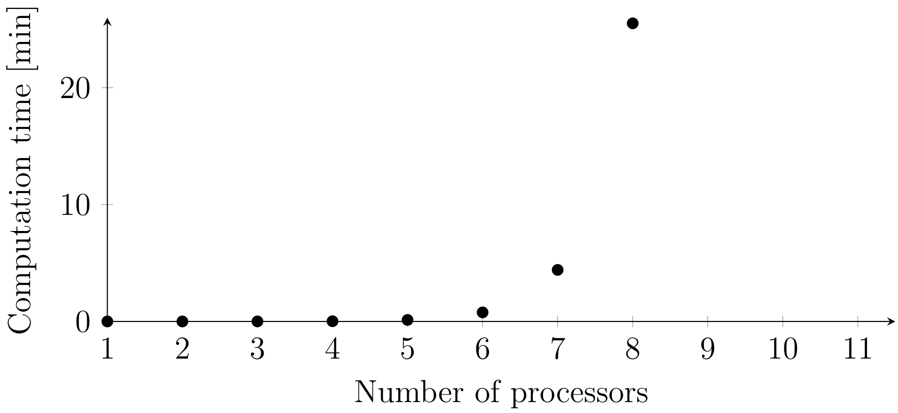

Fig. 3.4: Computation time of the Nash equilibrium determination for processors with five strategies implemented in MATLAB

In the situation in the Czech Republic studied in the chapter 5, 11 decision-making processors occur. The need for smaller strategy sets is therefore obvious.

Next question is the coalition structure and the assumption of grand coalition to be formed. Supposing only one producer to be present, there is, obviously, only one coalition structure. For two producers, if the order of coalitions matters, there are three of them. For more producers, the number is illustrated in Fig. 3.5.

Because in the situation in the Czech Republic 206 producers occur, and with attention to previous observations of the computation time, it is natural to continue in the same way of assuming only the grand coalition to form. For the solution, it could be eventually checked later, if the stability of the solution holds also for other coalition structures.

3.2.1 Bounds

The strategy sets must not be large. Therefore, they should be at least well specified. For this purpose, the bounds might be determined by following algorithms. The lower bound, in this sense, represents a strategy that dominates all strategies of a lower gate fee. Similarly, the upper bound dominates all strategies of a higher gate fee. The strategy

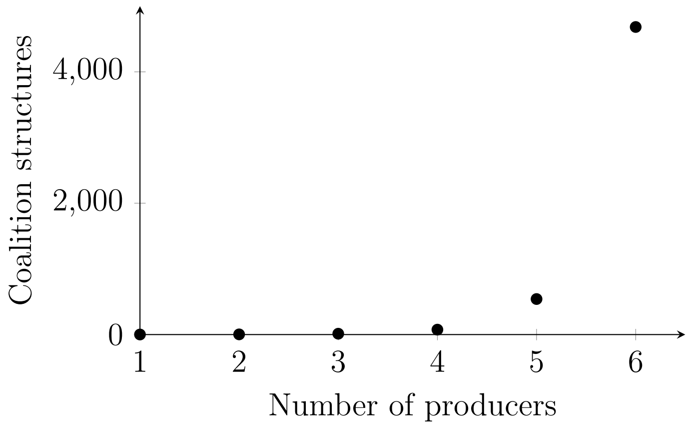

Fig. 3.5: Number of possible coalition structures in which order of coalitions matters

sets might then contain only the bounds and strategies of a gate fee between them. This follows from Theorem 1.6.

The main idea for the lower bound comes from the fact that, even for the gate fees of other processors being zero, due to transportation costs, some producers might choose a processor with a nonzero gate fee. In other words, for the gate fees of other processors being zero, the maximum value of a gate fee, for which the capacity utilization stays the same as for the gate fee equal to zero, can be computed. This value multiplied by the utilized capacity gives the income which can be obtained by any circumstances. Therefore, a choice of a gate fee which, even for the utilization of full capacity, doesn't guarantee this income makes no sense.

Mathematically, the lower bound of processor p_k can be computed by formula

$$c_k^{g,l} = \begin{cases} 0 & \text{if} \quad \sum\limits_{i \in I} x'_{i,k} = 0 \\ \dfrac{\sum\limits_{i \in I} x'_{i,k} z}{w^c_{p_k}} & \text{otherwise} \end{cases},$$

where

$$\begin{aligned} z = &\max_{y \in \mathbb{R}} y, \\ &\text{s. t.} \quad \sum_{i \in I} x'_{i,k} = \sum_{i \in I} x''_{i,k}, \end{aligned}$$

$$\begin{aligned} \{x'_{i,j} \colon i \in I, j \in J\} = &\operatorname*{arg\,min}_{x_{i,j} \colon i \in I, j \in J} \sum_{i \in I} \left(\left(c^t_{i,k} + m\right) x_{i,k} + \sum_{j \in J \setminus \{k\}} c^t_{i,j} x_{i,j} \right), \\ &\text{s. t.} \quad \sum_{i \in I} x_{i,j} \leq w^c_j \qquad \forall j \in J, \\ &\qquad\ \ \sum_{j \in J} x_{i,j} = w^p_i \qquad \forall i \in I, \\ &\qquad\ \ x_{i,j} \geq 0 \qquad \forall i \in I, j \in J \end{aligned}$$

and

$$\{x''_{i,j}\colon\ i\in I, j\in J\} = \operatorname*{arg\,min}_{x_{i,j}\colon i\in I, j\in J} \sum_{i\in I}\left(\left(c^t_{i,k}+y+m\right)x_{i,k} + \sum_{j\in J\setminus\{k\}} c^t_{i,j}x_{i,j}\right),$$

$$\begin{aligned} \text{s. t.} \qquad & \sum_{i\in I} x_{i,j} \le w^c_j && \forall j\in J,\\ & \sum_{j\in J} x_{i,j} = w^p_i && \forall i\in I,\\ & x_{i,j}\ge 0 && \forall i\in I, j\in J. \end{aligned}$$

With occurrence of processors with only one strategy, the computation changes a little, as their gate fee is not equal to zero, but to this strategy value.

For the upper bound computation, there is a requirement on the total capacity of processors with only one strategy to be sufficient for all producers. Otherwise, there would be no upper bound. The idea is that for each processor, even for the gate fees of other processors being too high, there is a gate fee value beyond which all production is obtained by the processors with only one strategy.

Denoting the set of processors with only one strategy by $P_0 \subset P$ and the set of their indices by $J_0 \subset J$, the upper bound of processor p_k with more than one strategy can be achieved by formula

$$\begin{aligned} c^{g,u}_k = & \min_{y\in\mathbb{R}} y,\\ & \text{s. t.} \qquad \sum_{i\in I} x'_{i,k} = 0, \end{aligned}$$

where

$$\{x'_{i,j}\colon\ i\in I, j\in J_0\cup\{k\}\} = \operatorname*{arg\,min}_{x_{i,j}\colon i\in I, j\in J_0\cup\{k\}} \sum_{i\in I}\left(\left(c^t_{i,k}+y+m\right)x_{i,k} + \sum_{j\in J_0} \left(c^t_{i,j}+c^g_j\right)x_{i,j}\right),$$

$$\begin{aligned} \text{s. t.} \qquad & \sum_{i\in I} x_{i,j} \le w^c_j && \forall j\in J_0\cup\{k\},\\ & \sum_{j\in J_0\cup\{k\}} x_{i,j} = w^p_i && \forall i\in I,\\ & x_{i,j}\ge 0 && \forall i\in I, j\in J_0\cup\{k\}. \end{aligned}$$

Example 3.2. Applied to Example 3.1, these algorithms produce lower bounds of approximately 8.5 EUR/t and 8.2 EUR/t and upper bounds of 90 EUR/t and 85 EUR/t for Processor 1 and Processor 2 respectively.

4 Conflict of Waste Producers

The conflict of waste producers is modeled as a cooperative game in which the benefit of cooperation among its players, the waste producers, is investigated. According to [GR16], the Shapley value and the nucleolus are commonly used in collaborative transportation. In many applications, however, they are computed only for games with few players. In the waste management, mostly, many producers are involved, as seen, for example, in the chapter 5. Such big coalitions might not be always easy, or even possible, to maintain, but the Shapley value and the nucleolus can always serve as benchmarks for other solutions, showing the potential in cooperation.

4.1 Mathematical Model

For this model, all the notation stays the same as for the model in the chapter 3, as a reminder, see appendix A.

4.1.1 Characteristic Function

For the empty coalition, the characteristic function is set equal to zero by definition. For all other coalitions of waste producers $S \subseteq R$ with related sets of indices $I_S \subseteq I$, the characteristic function is computed as optimization problem

$$
\begin{aligned}
v(S) = \min_{x_{i,j}:\, i \in I_S, j \in J} & \sum_{i \in I_S} \sum_{j \in J} \left(c^t_{i,j} + c^g_j\right) x_{i,j}, \\
\text{s. t.} \quad & \sum_{i \in I_S} x_{i,j} \leq w^c_j - \sum_{i \in I \setminus I_S} x'_{i,j} && \forall j \in J, \\
& \sum_{j \in J} x_{i,j} = w^p_i && \forall i \in I_S, \\
& x_{i,j} \geq 0 && \forall i \in I_S, j \in J,
\end{aligned}
$$

where

$$
\begin{aligned}
\{x'_{i,j}:\ i \in I \setminus I_S, j \in J\} = \operatorname*{arg\,min}_{x_{i,j}:\, i \in I \setminus I_S, j \in J} & \sum_{i \in I \setminus I_S} \sum_{j \in J} \left(c^t_{i,j} + c^g_j\right) x_{i,j}, \\
\text{s. t.} \quad & \sum_{i \in I \setminus I_S} x_{i,j} \leq w^c_j && \forall j \in J, \\
& \sum_{j \in J} x_{i,j} = w^p_i && \forall i \in I \setminus I_S, \\
& x_{i,j} \geq 0 && \forall i \in I \setminus I_S, j \in J.
\end{aligned}
$$

This computation of the characteristic function ensures that the game has a really useful property.

Theorem 4.1. *A core of this game is nonempty.*

Proof. For any R-balanced collection $\mathcal{C} = \{S_1, \ldots, S_{n_m}\}$ with balancing vector $y = (y_1, \ldots, y_{n_m})$ and $M = \{1, \ldots, n_m\}$, let

$$x^m_{i,j} = \begin{cases} \tilde{x}_{i,j} & \text{for} \quad i \in I_{S_m}, j \in J \\ \tilde{x}'_{i,j} & \text{for} \quad i \in I \setminus I_{S_m}, j \in J \end{cases},$$

where $\tilde{x}_{i,j}$ and $\tilde{x}'_{i,j}$ are values of $x_{i,j}$ and $x'_{i,j}$ determining $v(S_m)$. Obviously,

$$v(S_m) = \sum_{i \in I_{S_m}} \sum_{j \in J} \left(c^t_{i,j} + c^g_j\right) x^m_{i,j}.$$

Denoting

$$x^*_{i,j} = \sum_{m \in M\colon p_i \in S_m} y_m x^m_{i,j},$$

clearly, for all $j \in J$,

$$\sum_{i \in I} x^*_{i,j} = \sum_{i \in I} \sum_{m \in M\colon p_i \in S_m} y_m x^m_{i,j} = \sum_{m \in M} y_m \sum_{i \in I_{S_m}} x^m_{i,j} \le \sum_{m \in M} y_m \left(w^c_j - \sum_{i \in I \setminus I_{S_m}} x^m_{i,j} \right) =$$
$$= \sum_{m \in M} y_m w^c_j - \sum_{m \in M} y_m \sum_{i \in I \setminus I_{S_m}} x^m_{i,j} = n_m w^c_j - \sum_{i \in I} \sum_{m \in M\colon p_i \notin S_m} y_m x^m_{i,j} =$$
$$= n_m w^c_j - (n_m - 1) \sum_{i \in I} \sum_{m \in M\colon p_i \in S_m} y_m x^m_{i,j} = n_m w^c_j - (n_m - 1) \sum_{i \in I} x^*_{i,j}$$

and thus

$$\sum_{i \in I} x^*_{i,j} \le w^c_j.$$

Then, for all $i \in I$,

$$\sum_{j \in J} x^*_{i,j} = \sum_{j \in J} \sum_{m \in M\colon p_i \in S_m} y_m x^m_{i,j} = \sum_{m \in M\colon p_i \in S_m} y_m \sum_{j \in J} x^m_{i,j} = \sum_{m \in M\colon p_i \in S_m} y_m w^p_i = w^p_i$$

and, for all $i \in I, j \in J$,

$$x^*_{i,j} = \sum_{m \in M\colon p_i \in S_m} y_m x^m_{i,j} \ge 0.$$

It means that, for $\{x^*_{i,j}\colon\ i \in I, j \in J\}$, all constraints of optimization problem determining $v(R)$ are satisfied. Hence,

$$v(R) \le \sum_{i \in I} \sum_{j \in J} \left(c^t_{i,j} + c^g_j\right) x^*_{i,j} = \sum_{i \in I} \sum_{j \in J} \left(c^t_{i,j} + c^g_j\right) \sum_{m \in M\colon p_i \in S_m} y_m x^m_{i,j} =$$
$$= \sum_{m \in M} y_m \sum_{i \in I_{S_m}} \sum_{j \in J} \left(c^t_{i,j} + c^g_j\right) x^m_{i,j} = \sum_{m \in M} y_m v(S_m).$$

Thus, by Theorem 1.14, the core is nonempty. □

As the core is nonempty, then, by Theorem 1.30, the nucleolus belongs to the core.

The author did not find it easy to prove or disprove that

$$v(S) + v(T) \ge v(S \cup T) + v(S \cap T) \qquad \text{for all} \qquad S, T \subseteq R.$$

Therefore, the question, if also the Shapley value belongs to the core, remains unanswered.

4.1.2 Cost Allocation

Next step of the model is the cost allocation. The Shapley value $\varphi = (\varphi_{r_1}, \ldots, \varphi_{r_{n_r}})$ is determined by formula

$$\varphi_{r_i} = \sum_{S \subseteq R \colon r_i \in S} \frac{(|S|-1)!\,(|R|-|S|)!}{|R|!} \left(v(S) - v(S \setminus \{r_i\})\right)$$

and the nucleolus $\varrho = (\varrho_{r_1}, \ldots, \varrho_{r_{n_r}})$ by $\varrho_{r_i} = x_{r_i}^{k'}$, where

$$\begin{aligned}
\{\varepsilon_k,\, x_{r_i}^k \colon r_i \in R\} = \underset{\varepsilon \in \mathbb{R},\, x_{r_i} \in \mathbb{R} \colon r_i \in R}{\arg\max} \quad & \varepsilon, \\
\text{s. t.} \qquad & \varepsilon + \sum_{r_i \in S} x_{r_i} \le v(S) && \forall S \subset R, S \neq \varnothing, S \notin \bigcup_{j \in \{0,\ldots,k-1\}} F_j, \\
& \varepsilon_j + \sum_{r_i \in S} x_{r_i} = v(S) && \forall S \in F_j, j \in \{0, \ldots, k-1\}, \\
& \sum_{r_i \in R} x_{r_i} = v(R),
\end{aligned}$$

$\varepsilon_0 = 0$, $F_0 = \varnothing$, F_k is the set of all coalitions $S \subset R$, for which

$$\varepsilon_k + \sum_{r_i \in S} x_{r_i}^k = v(S),$$

and k' is the lowest positive integer for which the vector $(x_{r_1}^{k'}, \ldots, x_{r_n}^{k'})$ realizing the minimum is unique.

Finally, the potential in cooperation is for each producer r_i analyzed by comparison of values φ_{r_i} and ϱ_{r_i} with $v(r_i)$.

4.1.3 Computation Time

The computation time of this model is growing significantly with more producers involved. To determine the Shapley value or the nucleolus, values of the characteristic function for all coalitions $S \subseteq R$ are needed. For an n_r-player game, it means the characteristic function values for 2^{n_r} coalitions. This is illustrated in Fig. 4.1.

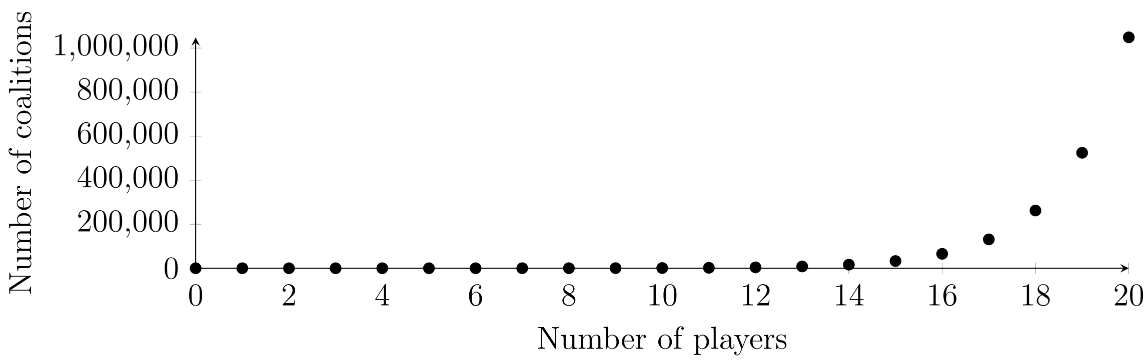

Fig. 4.1: Number of formable coalitions

The combination of this and the characteristic function values being determined as solutions of minimization problems makes the computation time very long. Fig. 4.2

and Fig. 4.3 illustrate the impact of the number of players on the computation time of the Shapley value and the nucleolus determination respectively, both implemented in MATLAB.

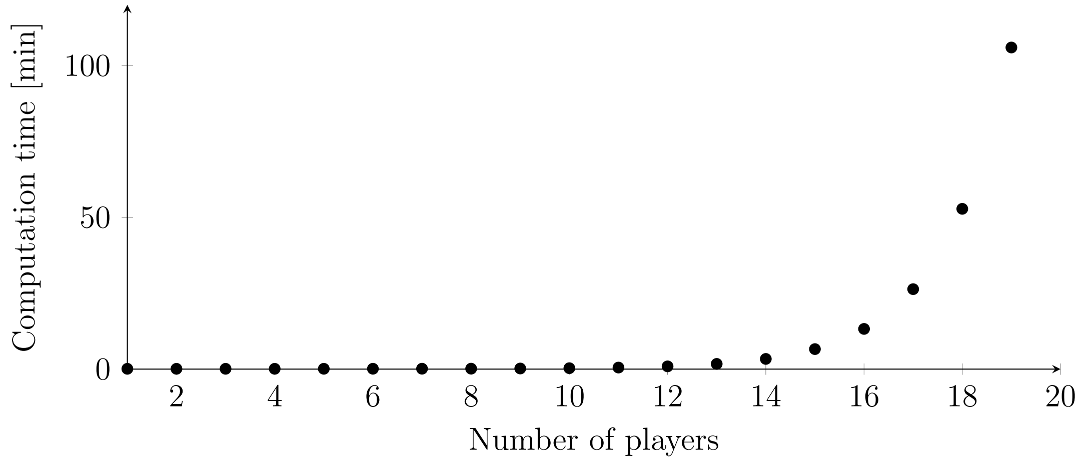

Fig. 4.2: Computation time of the Shapley value determination

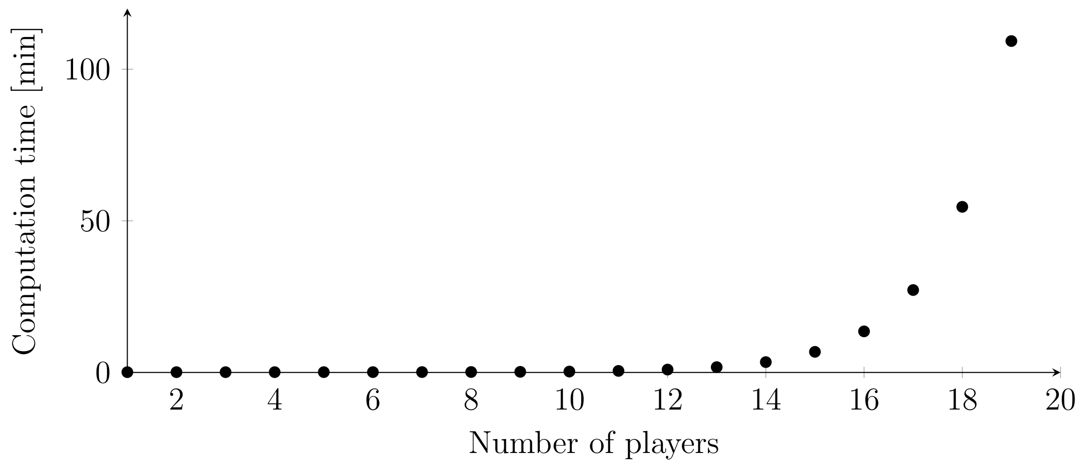

Fig. 4.3: Computation time of the nucleolus determination

The waste management game in the chapter 5 is a game of 206 players, the characteristic function values for approximately $1.03 \cdot 10^{62}$ coalitions would be therefore needed and, according to Fig. 4.2, the Shapley value computation would take approximately $3.95 \cdot 10^{52}$ years. Being able to omit some coalitions would therefore be helpful.

4.2 Cost Allocation Approximations

Algorithms for a Shapley value approximation and a nucleolus approximation were developed. These algorithms can be used for any cooperative game where, for any two players, the efficiency of a cooperation between them can be predicted. This can be obviously said about games in which players are placed in a space and their cooperation is as effective as they are close to each other.

For the waste management game, the cooperation might become beneficial when the capacities of local waste incinerators are insufficient and the municipalities are forced to send their waste to distant ones. It is natural to assume that too distant players are unlikely to influence each other, hence any coalition of them seems worthless.

4.2.1 Shapley Value Approximation

This algorithm serves to compute the Shapley value approximation $\psi = (\psi_{p_1}, \ldots, \psi_{p_n})$ for an n-player game (N, v), where $N = \{p_1, \ldots, p_n\}$. For this purpose, two inputs are needed. First of them is a critical distance, beyond which a cooperation between two players is expected to have no impact, and the other one is a maximum number of cooperating players, which is natural for the already mentioned reason that big coalitions are not always easy, or even possible, to maintain.

The characteristic function v must be in a form where $v(\{p_i\}) = 0$ for all $p_i \in N$. This prerequisite condition is not restrictive, because any characteristic function $\tilde{v}$ can be easily reformulated to this form by formula

$$v(S) = \tilde{v}(S) - \sum_{p_i \in S} \tilde{v}(\{p_i\}),$$

the computed approximation $\psi = (\psi_{p_1}, \ldots, \psi_{p_n})$ is then only modified by formula

$$\tilde{\psi}_{p_i} = \psi_{p_i} + \tilde{v}(\{p_i\})$$

and $\tilde{\psi} = (\tilde{\psi}_{p_1}, \ldots, \tilde{\psi}_{p_n})$ is then the approximation for this game.

Step 1. Given a distance matrix $D = [d_{i,j}]$, where $d_{i,j}$ represents the distance between players p_i and p_j, and a critical distance d_{crit}, beyond which the cooperation is considered worthless, a matrix $A = [a_{i,j}]$ is created by formula

$$a_{i,j} = \begin{cases} 1 & \text{if} \quad d_{i,j} \leq d_{crit} \\ 0 & \text{otherwise} \end{cases} .$$

Step 2. Using the matrix A and given a maximum number of cooperating players c_{max}, a set of coalitions C is created. For this purpose, two approaches are used. A question rises, if a coalition $\{p_i, p_j, p_k\}$ should be included in this set when $a_{i,j} = 1$, $a_{j,k} = 1$, but $a_{i,k} = 0$. For a positive answer, Algorithm 4.1 is used. And for a negative one, Algorithm 4.2 is used.

Step 3. The value $\psi' = (\psi'_{p_1}, \ldots, \psi'_{p_n})$ is computed by formula

$$\psi'_{p_i} = \sum_{S \in C:\, p_i \in S} \frac{(|S|-1)!\,(|N|-|S|)!}{|N|!} \left(v(S) - v'(S \backslash \{p_i\}) \right),$$

where

$$v'(S \backslash \{p_i\}) = \begin{cases} v(S \backslash \{p_i\}) & \text{if} \quad S \backslash \{p_i\} \in C \\ v_{min}(S \backslash \{p_i\}) & \text{otherwise} \end{cases} ,$$

Algorithm 4.1: Determination of the set of coalitions C (including $\{p_i, p_j, p_k\}$ for $a_{i,j} = 1$, $a_{j,k} = 1$, $a_{i,k} = 0$)

```
if c_max ≥ 1 then
  for j = 1 to c_max do
    set C_j = ∅
  end for
  for i = 1 to number of players do
    add {p_i} to C_1
  end for
  for j = 2 to c_max do
    for all S ∈ C_{j−1} do
      for i = 1 to number of players do
        if p_i ∉ S and Σ_{k: p_k ∈ S} a_{i,k} ≥ 1 then
          add S ∪ {p_i} to C_j
        end if
      end for
    end for
  end for
  set C = ⋃_{j=1}^{c_max} C_j
end if
add ∅ and N to C
```

where $v_{min}(S\backslash\{p_i\})$ is a solution of the following optimization problem. This approach is similar to the one used for a Shapley value refinement presented in [My77]. Denoting $C = \{T_1, \ldots, T_{|C|}\}$ and $J = \{1, \ldots, |C|\}$, the integer programming problem is in form

$$v_{min}(S\backslash\{p_i\}) = \min_{x_j:\, j\in J} \sum_{j\in J} v(T_j)x_j, \tag{4.1}$$

$$\text{s. t.} \quad \bigcup_{j\in J:\, x_j=1} T_j = S\backslash\{p_i\}, \tag{4.2}$$

$$\bigcap_{j\in J:\, x_j=1} T_j = \varnothing, \tag{4.3}$$

$$x_j \in \{0,1\} \qquad \forall j \in J. \tag{4.4}$$

In the case of a characteristic function not representing the cost, but the payoff, the minimization should be replaced by a maximization.

Optimization problems of this type are commonly recognized as the assignment problems.

Step 4. The final step's only purpose is to preserve an assumption that the profit is completely divided among the players. Therefore, the final form of the Shapley value approximation here presented is a vector $\psi = (\psi_{p_1}, \ldots, \psi_{p_n})$, where

$$\psi_{p_i} = \psi'_{p_i} + \frac{v(N) - \sum\limits_{p_i\in N} \psi'_{p_i}}{n}. \tag{4.5}$$

Algorithm 4.2: Determination of the set of coalitions C (not including $\{p_i, p_j, p_k\}$ for $a_{i,j} = 1$, $a_{j,k} = 1$, $a_{i,k} = 0$)

---

if $c_{max} \geq 1$ **then**
  for $j = 1$ **to** c_{max} **do**
    set $C_j = \varnothing$
  end for
  for $i = 1$ **to** number of players **do**
    add $\{p_i\}$ to C_1
  end for
  for $j = 2$ **to** c_{max} **do**
    for all $S \in C_{j-1}$ **do**
      for $i = 1$ **to** number of players **do**
        if $p_i \notin S$ **and** $\prod_{k:\, p_k \in S} a_{i,k} = 1$ **then**
          add $S \cup \{p_i\}$ to C_j
        end if
      end for
    end for
  end for
  set $C = \bigcup_{j=1}^{c_{max}} C_j$
end if
add $\varnothing$ and N to C

4.2.2 Nucleolus Approximation

This algorithm serves to compute the nucleolus approximation $\gamma = (\gamma_{p_1}, \ldots, \gamma_{p_n})$ for an n-player game (N, v), where $N = \{p_1, \ldots, p_n\}$. The same two inputs are needed as for the case of the Shapley value approximation, the critical distance d_{crit} and the maximum number of cooperating players c_{max}. Also the first steps of the algorithm are the same.

Step 1. The matrix A is created using the distance matrix D and the critical distance d_{crit} through the same approach as in the Shapley value approximation.

Step 2. The set of coalitions C is determined using the matrix A and the maximum number of cooperating players c_{max} in the same way as for the Shapley value approximation.

Step 3. The value $\gamma = (\gamma_{p_1}, \ldots, \gamma_{p_n})$ is computed by formula $\gamma_{p_i} = x_{p_i}^{k'}$, where

$$
\begin{aligned}
\{\varepsilon_k,\, x_{p_i}^k :\, p_i \in N\} = \underset{\varepsilon \in \mathbb{R},\, x_{p_i} \in \mathbb{R}:\, p_i \in N}{\arg\min}\ & \varepsilon, \\
\text{s. t.} \qquad & \varepsilon + \sum_{p_i \in S} x_{p_i} \geq v(S) \qquad \forall S \in C, S \neq N, S \notin \bigcup_{j \in \{0,\ldots,k-1\}} F_j, \\
& \varepsilon_j + \sum_{p_i \in S} x_{p_i} = v(S) \qquad \forall S \in F_j, j \in \{0, \ldots, k-1\}, \\
& \sum_{p_i \in R} x_{p_i} = v(N),
\end{aligned}
$$

$\varepsilon_0 = 0$, $F_0 = \varnothing$, F_k is the set of all coalitions $S \in C$, for which

$$\varepsilon_k + \sum_{p_i \in S} x_{p_i}^k = v(S),$$

and k' is the lowest positive integer for which the vector $(x_{p_1}^{k'}, \ldots, x_{p_n}^{k'})$ realizing the minimum is unique.

In the case of a characteristic function not representing a payoff, but a cost, the minimization should be replaced by a maximization and the inequality sign in the first constraint reversed.

4.2.3 Computation Time

Steps 1 and 2 of the presented algorithms were implemented in MS Excel and serve as input data for the next steps implemented in MATLAB. The algorithms were run for the waste management game in the chapter 5 with 206 producers. For the Shapley value approximation, computation times for multiple choices of d_{crit} and c_{max} are shown in Table 4.1 for the choice of Algorithm 4.1 in step 2 and in Table 4.2 for the choice of Algorithm 4.2. For the nucleolus approximation, Table 4.3 shows the times for the choice of Algorithm 4.1 and Table 4.4 the times for the choice of Algorithm 4.2.

For the difference in results for the waste management game in the Czech Republic from the chapter 5 for some of the combinations, see Table 4.5 and Table 4.6.

Table 4.1: Computation times of the Shapley value approximation with the choice of Algorithm 4.1 (Combinations marked with '–' were unable to be computed due to insufficient memory of the MS Excel implementation.)

		c_{max}		
		5	6	7
d_{crit}	0	1 min 15 s	1 min 12 s	1 min 15 s
	10	1 min 17 s	1 min 18 s	1 min 18 s
	20	3 min 52 s	4 min 13 s	4 min 35 s
	30	38 min 3 s	1 h 30 min 39 s	4 h 23 min 9 s
	40	5 h 3 min 7 s	16 h 10 min 32 s	59 h 5 min 13 s
	50	24 h 4 min 47 s	–	–

These algorithms for the cost allocation approximations make it possible to obtain a solution within a reasonable time. The accuracy of such solution depends mainly on the game itself. However, for games in which the threshold of beneficial coalitions cannot be determined, these approximations are useless.

Table 4.5 and Table 4.6 show that, for the waste management game, it is not easy to choose the appropriate algorithm and set the exact threshold value, but the algorithms can be repeated until the result seems sufficient.

Table 4.2: Computation times of the Shapley value approximation with the choice of Algorithm 4.2

		c_{max}		
		5	10	15
d_{crit}	0	1 min 9 s	1 min 7 s	1 min 9 s
	10	1 min 11 s	1 min 11 s	1 min 11 s
	20	1 min 56 s	1 min 57 s	1 min 56 s
	30	4 min 23 s	4 min 22 s	4 min 23 s
	40	11 min 47 s	11 min 55 s	11 min 48 s
	50	38 min 17 s	45 min 47 s	45 min 11 s
	60	2 h 2 min 15 s	3 h 33 min 16 s	3 h 33 min 32 s
	70	5 h 54 min 33 s	19 h 20 min 55 s	19 h 42 min 2 s

Table 4.3: Computation times of the nucleolus approximation with the choice of Algorithm 4.1 (Combinations marked with '–' were unable to be computed due to insufficient memory of the MS Excel implementation.)

		c_{max}		
		5	6	7
d_{crit}	0	1 min 11 s	1 min 12 s	1 min 13 s
	10	1 min 17 s	1 min 18 s	1 min 18 s
	20	5 min 50 s	6 min 33 s	6 min 47 s
	30	39 min 9 s	1 h 29 min 50 s	4 h 17 min 31 s
	40	4 h 51 min 27 s	20 h 35 min 25 s	75 h 12 min 21 s
	50	30 h 37 min 48 s	–	–

Table 4.4: Computation times of the nucleolus approximation with the choice of Algorithm 4.2

		c_{max}		
		5	10	15
d_{crit}	0	1 min 17 s	1 min 17 s	1 min 16 s
	10	1 min 21 s	1 min 21 s	1 min 21 s
	20	2 min 36 s	2 min 35 s	2 min 35 s
	30	8 min 39 s	8 min 47 s	8 min 46 s
	40	23 min 4 s	23 min 33 s	23 min 25 s
	50	1 h 2 min 37 s	1 h 54 min 12 s	1 h 51 min 13 s
	60	2 h 48 min 2 s	5 h 42 min 19 s	5 h 44 min 35 s
	70	8 h 24 min 13 s	30 h 57 min 1 s	31 h 31 min 43 s

Table 4.5: A change in the value of the Shapley value approximation assigned to ten randomly chosen players by using different algorithms and values of c_{max} and d_{crit} in the waste management game in the Czech Republic from the chapter 5

Algorithm	4.1	4.1	4.2	4.2	4.2
c_{max}	7	7	15	15	5
d_{crit}	20	30	30	50	70
Player 1	44,805,956	44,805,148	44,808,089	44,806,595	44,805,894
Player 2	283,942	284,043	283,849	283,897	283,885
Player 3	3,392,957	3,393,000	3,392,864	3,392,912	3,392,943
Player 4	772,790	772,832	772,697	772,745	772,739
Player 5	489,159	489,202	489,066	489,114	489,140
Player 6	1,056,604	1,056,647	1,056,511	1,056,559	1,056,562
Player 7	252,663	252,705	252,570	252,618	252,600
Player 8	309,643	309,743	309,550	309,598	309,650
Player 9	938,258	938,304	938,165	938,213	938,218
Player 10	602,744	602,845	602,651	602,699	602,706

Table 4.6: A change in the value of the nucleolus approximation assigned to ten randomly chosen players by using different algorithms and values of c_{max} and d_{crit} in the waste management game in the Czech Republic from the chapter 5

Algorithm	4.1	4.1	4.2	4.2	4.2
c_{max}	7	7	15	15	5
d_{crit}	20	30	30	50	70
Player 1	44,732,604	44,708,575	44,732,565	44,701,184	44,599,321
Player 2	293,337	305,653	293,182	297,261	302,242
Player 3	3,402,352	3,414,668	3,402,197	3,406,276	3,411,257
Player 4	782,185	794,500	782,029	786,108	791,090
Player 5	498,554	510,870	498,399	502,478	507,459
Player 6	1,065,999	1,078,315	1,065,844	1,069,923	1,074,904
Player 7	262,058	274,373	261,902	265,981	270,963
Player 8	319,038	331,354	318,882	322,962	327,943
Player 9	947,653	959,968	947,497	951,576	956,558
Player 10	612,139	624,455	611,984	616,063	621,044

5 Waste Management Game in Czech Republic

As already mentioned in Introduction, starting from 2024, landfilling is most likely going to be banned in the Czech Republic. Insufficient capacity of the already standing incinerators causes that changes are expected in following years as new incinerators need to be built.

At the Institute of Process Engineering of Brno University of Technology, several mathematical models were developed on this topic. Among others, in [SP14], the NERUDA tool was presented. This tool, using optimization techniques, determines optimal number of waste incinerators and their locations and capacities. Based on some scenarios of waste production in the Czech Republic in following years, this tool predicts waste incinerators, besides those already standing or being built in Praha, Brno, Liberec, and Plzeň, located in České Budějovice, Hradec Králové, Mělník, Most, Ústí nad Labem, Jihlava, and Otrokovice.

Besides those in the Czech Republic, waste incineration plants in other countries, which are close enough, are involved in this problem too. This holds for Austrian and German incinerators in Linz, Wels, Zwentendorf an der Donau, Zistersdorf, Wien, Schwandorf, Nürnberg, Bamberg, Coburg, Zorbau, Leuna, Lauta, Großräschen, Ingolstadt, and Burgkirchen.

In the Czech Republic, basically, there are three possible territorial divisions, into 14 districts, into 206 administrative units called *obec s rozšířenou působností* (ORP) or into 6,245 municipalities. Another division might be considered, but it could be complicated to get all the data. With respect to the numbers, for the waste management game, ORP seems to be the best choice.

The division of waste producers and waste processors within the Czech Republic is illustrated in Fig. 5.1.

The presented models and algorithms might be applied to any situation and any set of data. Even the set of processors is only a prediction. The input data of capacities and productions used in this thesis are in appendix B. The data on transportation are not included because of their size. The strategies of processors in other countries are considered being only one gate fee option of 70 EUR/t.

Stable, and therefore expected, strategies of processors require an analysis as well as the cooperation of producers. The analysis is divided into two sections in the same way like the chapters 3 and 4.

5.1 Conflict of Waste Processors

Firstly, to specify the strategy sets of the waste processors in waste management game in the Czech Republic, the bounds were computed. Due to insufficient capacity of processors

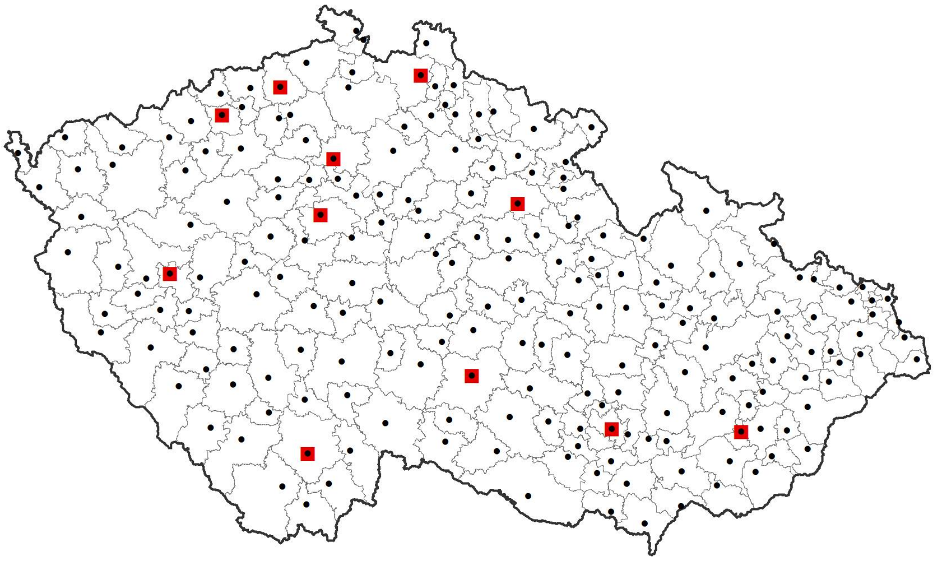

Fig. 5.1: Map of the producers (black dots) and processors (red squares) for the waste management game in the Czech Republic (source of spatial data: Arc ČR 500 v.3.2)

in other countries, however, the upper bounds could not be determined. Therefore, to compute the upper bounds, the capacity of these processors is considered double. All the computed bounds are shown in Table 5.1.

Table 5.1: Determined bounds on strategy sets for original and double capacities of processors in other countries

	Original capacities		Double capacities	
	Lower bound	Upper bound	Lower bound	Upper bound
Praha	106	–	102	126
České Budějovice	88	–	88	102
Brno	105	–	89	117
Hradec Králové	107	–	95	138
Liberec	104	–	98	122
Plzeň	95	–	92	108
Mělník	97	–	93	125
Most	91	–	88	111
Ústí nad Labem	90	–	86	116
Jihlava	108	–	104	117
Otrokovice	109	–	0	118

To determine the strategy sets, lower bounds for the original capacities and upper bounds for the double capacities were used. For each processor, the third strategy was chosen exactly in the middle of these values. Table 5.2 shows these strategy sets.

For these sets of strategies, one Nash equilibrium point was found. Strategies forming

Table 5.2: Strategy sets with the Nash equilibrium marked in bold

	First strategy	Second strategy	Third strategy
Praha	**106**	116	126
České Budějovice	**88**	95	102
Brno	**105**	111	117
Hradec Králové	**107**	122.5	138
Liberec	**104**	113	122
Plzeň	**95**	101.5	108
Mělník	**97**	111	125
Most	**91**	101	111
Ústí nad Labem	**90**	103	116
Jihlava	**108**	112.5	117
Otrokovice	109	**113.5**	118

the Nash equilibrium are in Table 5.2 marked in bold.

5.2 Conflict of Waste Producers

For the Nash equilibrium strategies of waste processors, the characteristic function values for all individual players were computed. These values represent the minimal cost the producers are able to achieve on their own.

For comparison, also the approximations of the Shapley value and the nucleolus were computed. The approximations were performed for input parameters c_{max} of 7 producers in a coalition and d_{crit} of 30 km. These values represent the minimal cost the producers are most likely able to achieve by a cooperation with all producers.

The potential in cooperation can be measured as the relative difference of these values. Sorted by this difference in percents for the nucleolus, the Table 5.3 shows five producers with the highest potential and five producers with the lowest potential. For the complete list of producers, see appendix C.

It seems that the potential is high for the producers with the lowest waste production. Hence, the approximations are probably not enough accurate. Assuming an absolute difference as a measure of the potential and sorting by this difference for the nucleolus, Table 5.4 is obtained. Again, for the complete list of producers, see appendix C.

Among the producers with high potential, for the nucleolus approximation, the absolute difference seems to be a better measure. For these producers, therefore, the cooperation seems meaningful and the potential for cooperation should be determined locally, but more accurately, by using different methods.

Table 5.3: Comparison of the characteristic function values for individual players and the divisions assigned to them according to the Shapley value approximation ψ and the nucleolus approximation γ (5 producers with the highest relative difference and 5 producers with the lowest relative difference)

	$v(r_i)$	ψ_{r_i}		γ_{r_i}	
Rýmařov	192,487	6,376	97 %	27,989	85 %
Nepomuk	274,712	88,551	68 %	110,215	60 %
Blovice	315,234	129,091	59 %	150,737	52 %
Nová Paka	320,853	134,743	58 %	156,356	51 %
Pacov	323,636	137,518	58 %	159,139	51 %
⋮	⋮	⋮		⋮	
Hradec Králové	5,693,671	5,506,497	3 %	5,514,629	3 %
Liberec	5,245,619	5,059,481	4 %	5,081,121	3 %
Plzeň	5,754,708	5,568,512	3 %	5,590,211	3 %
Brno	13,284,192	13,094,666	1 %	12,956,382	2 %
Praha	44,997,029	44,805,148	0 %	44,708,575	1 %

Table 5.4: Comparison of the characteristic function values for individual players and the differences between these and the divisions assigned to them according to the Shapley value approximation ψ and the nucleolus approximation γ (5 producers with the highest absolute difference and 5 producers with the lowest absolute difference)

	$v(r_i)$	$v(r_i) - \psi_{r_i}$	$v(r_i) - \gamma_{r_i}$
Ostrava	12,190,256	191,957	1,134,853
Frýdek-Místek	4,951,886	188,949	560,958
Olomouc	6,878,331	188,948	552,549
Havířov	4,313,879	188,227	520,260
Prostějov	4,075,878	188,432	479,842
⋮	⋮	⋮	⋮
Chomutov	2,134,450	186,306	164,497
České Budějovice	3,579,165	186,165	164,497
Liberec	5,245,619	186,138	164,497
Tábor	2,640,946	186,118	164,497
Karlovy Vary	2,498,259	186,107	164,497

Conclusion

A game-theoretic model representing the decision-making situation in the waste management was created. The model was further divided into two parts, a noncooperative game representing the conflict of waste processors and a cooperative game representing the conflict of waste producers.

For the conflict of waste processors, the Nash equilibria are used to find optimal strategies on gate fee values. The Nash equilibria guarantee a stability, the state that is likely to stay unchanged for some time. Thus, it serves as a good prediction for the future.

For the conflict of waste producers, the cooperation is assumed and a cost distribution is studied. The model determines the distribution using the Shapley value and the nucleolus. It means that the grand coalition formation is supposed. For many producers, it might seem naive, but this distribution can always serve as a benchmark for other solutions showing the potential in cooperation.

For the conflict of waste producers, the core is proved to be nonempty. Whereas the nucleolus is guaranteed to belong to the core, the same question for the Shapley value remains unanswered. This should be, however, answered in order to guarantee a stability of such solution.

With the number of players, the computation time for models of both conflicts grows significantly. Therefore, other algorithms needed to be developed.

The strategy sets of waste processors in the first conflict may not contain many strategies. Therefore, an algorithm to determine a lower bound and an upper bound was created. It specifies the strategy sets as they can contain only strategies between the bounds.

In the conflict of waste producers, the computations of the Shapley value and the nucleolus are not possible for more producers. Therefore, algorithms for approximations were developed. These algorithms are based on an assumption that distant producers can not influence each other. For different threshold values, computation tests were performed.

In the fifth chapter, the model was applied to a situation in the Czech Republic, a conflict of 11 decision-making waste processors and 206 decision-making waste producers.

For the conflict of waste processors, one Nash equilibrium was found. For the Nash equilibrium strategies, the conflict of waste producers was investigated and the approximations were computed. The results of the approximations are not much convincing. Nevertheless, at least some producers with high potential in cooperation were recognized.

The problem in the approximations was that the threshold values for the algorithms were not set correctly. Making the approximations more accurate would, however, lead to long computation times again.

To shorten the computation time, the algorithm could yet be extended by adding other conditions on reasonable coalitions. For example, assuming producers with large waste production being more likely worth cooperating with seems to be one of the possibilities for this extension.

It could be also helpful to use a different programming language for the implementation. Whereas IBM ILOG CPLEX is commonly considered as very fast, MATLAB does not belong among the fastest languages. The speed of the implementation of first steps in MS Excel might seem questionable too.

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A List of Frequently Used Symbols

$N = \{p_1, \ldots, p_n\}$	set of players
π	payoff or cost function
v	characteristic function
(N, v)	cooperative game with set of players N and characteristic function v
$\varphi = (\varphi_{p_1}, \ldots, \varphi_{p_n})$	Shapley value
$\psi = (\psi_{p_1}, \ldots, \psi_{p_n})$	Shapley value approximation
$\varrho = (\varrho_{p_1}, \ldots, \varrho_{p_n})$	nucleolus
$\gamma = (\gamma_{p_1}, \ldots, \gamma_{p_n})$	nucleolus approximation
d_{crit}	distance beyond which a cooperation between any two players is considered worthless
c_{max}	maximum number of cooperating players
n_p	number of processors
n_r	number of producers
$P = \{p_1, \ldots, p_{n_p}\}$	set of processors
$P_0 \subseteq P$	set of processors with only one strategy
$J = \{1, \ldots, n_p\}$	set of indices of processors
$J_0 \subseteq J$	set of indices of processors with only one strategy
$R = \{r_1, \ldots, r_{n_r}\}$	set of producers
$I = \{1, \ldots, n_r\}$	set of indices of producers
w_j^c	capacity of processor p_j
C_j^g	set of strategies of processor p_j
c_j^g	strategy of processor p_j (gate fee)
$c_j^{g,l}$	lower bound on strategies of processor p_j
$c_j^{g,u}$	upper bound on strategies of processor p_j
w_i^p	production of producer r_i
$c_{i,j}^t$	cost of waste transportation from producer r_i to processor p_j
m	very small positive number

B Input Data for Waste Management Game in Czech Republic

The waste incinerator's capacities data are shown in Table B.1. The ORP's productions data are shown in Table B.2 and Table B.3.

Table B.1: Yearly capacity of waste processors in kt

Praha	410,000	Zwentendorf an der Donau	262,500
České Budějovice	200,000	Zistersdorf	76,650
Brno	340,000	Wien	372,750
Hradec Králové	300,000	Schwandorf	202,500
Liberec	96,000	Nürnberg	103,500
Plzeň	95,000	Bamberg	54,900
Mělník	300,000	Coburg	58,500
Most	150,000	Zorbau	148,500
Ústí nad Labem	200,000	Leuna	175,500
Jihlava	40,000	Lauta	99,000
Otrokovice	40,000	Großräschen	90,000
Linz	124,950	Ingolstadt	108,000
Wels	157,500	Burgkirchen	103,500

Table B.2: Yearly production of waste producers in kt (part 1)

Aš	8,511	Hlinsko	5,866
Benešov	21,186	Hlučín	11,916
Beroun	17,639	Hodonín	18,888
Bílina	5,779	Holešov	7,433
Bílovec	6,052	Holice	4,097
Blansko	14,071	Horažďovice	4,318
Blatná	4,618	Horšovský Týn	4,435
Blovice	3,093	Hořice	6,120
Bohumín	11,313	Hořovice	11,285
Boskovice	14,344	Hradec Králové	51,331
Brandýs n. L.-S. Boleslav	41,797	Hranice	10,695
Brno	119,806	Humpolec	5,802
Broumov	4,473	Hustopeče	10,202
Bruntál	9,453	Cheb	16,836
Břeclav	19,561	Chomutov	22,103
Bučovice	4,147	Chotěboř	5,712
Bystřice nad Pernštejnem	4,841	Chrudim	23,157
Bystřice pod Hostýnem	4,843	Ivančice	8,108
Čáslav	7,931	Jablonec nad Nisou	14,040
Černošice	44,713	Jablunkov	5,641
Česká Lípa	24,766	Jaroměř	4,446
Česká Třebová	5,523	Jeseník	9,436
České Budějovice	40,329	Jičín	15,211
Český Brod	10,011	Jihlava	25,640
Český Krumlov	10,268	Jilemnice	5,843
Český Těšín	8,867	Jindřichův Hradec	12,617
Dačice	6,054	Kadaň	14,204
Děčín	22,885	Kaplice	5,237
Dobruška	7,216	Karlovy Vary	25,171
Dobříš	7,329	Karviná	21,874
Domažlice	9,675	Kladno	34,602
Dvůr Králové nad Labem	6,201	Klatovy	15,641
Frenštát pod Radhoštěm	5,052	Kolín	34,061
Frýdek-Místek	32,099	Konice	2,756
Frýdlant	8,079	Kopřivnice	16,721
Frýdlant nad Ostravicí	9,986	Kostelec nad Orlicí	6,735
Havířov	27,737	Kralovice	8,606
Havlíčkův Brod	17,531	Kralupy nad Vltavou	12,472

Table B.3: Yearly production of waste producers in kt (part 2)

Kraslice	3,399	Nový Bydžov	4,977
Kravaře	5,193	Nový Jičín	12,323
Králíky	2,795	Nymburk	16,659
Krnov	12,029	Nýřany	13,789
Kroměříž	18,157	Odry	5,131
Kuřim	6,250	Olomouc	50,919
Kutná Hora	20,311	Opava	34,858
Kyjov	16,045	Orlová	12,588
Lanškroun	5,124	Ostrava	82,708
Liberec	50,333	Ostrov	7,454
Lipník nad Bečvou	5,085	Otrokovice	10,346
Litoměřice	23,938	Pacov	3,138
Litomyšl	6,426	Pardubice	35,345
Litovel	7,126	Pelhřimov	13,687
Litvínov	13,171	Písek	11,683
Louny	13,842	Plzeň	60,293
Lovosice	9,881	Podbořany	6,069
Luhačovice	5,126	Poděbrady	12,347
Lysá nad Labem	8,570	Pohořelice	4,057
Mariánské Lázně	9,080	Polička	4,735
Mělník	17,996	Praha	421,456
Mikulov	4,836	Prachatice	8,108
Milevsko	4,892	Prostějov	29,322
Mladá Boleslav	36,791	Přelouč	8,529
Mnichovo Hradiště	4,793	Přerov	26,006
Mohelnice	6,042	Přeštice	6,006
Moravská Třebová	7,347	Příbram	23,618
Moravské Budějovice	5,779	Rakovník	18,238
Moravský Krumlov	6,307	Rokycany	17,998
Most	21,436	Rosice	6,974
Náchod	15,136	Roudnice nad Labem	7,595
Náměšť nad Oslavou	2,753	Rožnov pod Radhoštěm	10,416
Nepomuk	2,681	Rumburk	10,985
Neratovice	14,508	Rychnov nad Kněžnou	9,160
Nová Paka	2,794	Rýmařov	1,260
Nové Město na Moravě	4,690	Říčany	24,668
Nové Město nad Metují	3,637	Sedlčany	7,006
Nový Bor	9,279	Semily	5,987

Table B.4: Yearly production of waste producers in kt (part 3)

Slaný	12,956	Uherský Brod	15,164
Slavkov u Brna	5,520	Uničov	7,493
Soběslav	7,039	Ústí nad Labem	29,269
Sokolov	21,032	Ústí nad Orlicí	8,204
Stod	6,726	Valašské Klobouky	5,120
Strakonice	12,327	Valašské Meziříčí	12,567
Stříbro	5,592	Varnsdorf	7,067
Sušice	7,702	Velké Meziříčí	10,862
Světlá nad Sázavou	5,995	Veselí nad Moravou	9,128
Svitavy	7,745	Vimperk	5,111
Šlapanice	19,984	Vizovice	4,707
Šternberk	8,265	Vítkov	7,083
Šumperk	23,362	Vlašim	10,422
Tachov	11,774	Vodňany	3,988
Tanvald	6,046	Votice	4,519
Tábor	26,771	Vrchlabí	9,545
Telč	3,524	Vsetín	14,186
Teplice	35,241	Vysoké Mýto	8,258
Tišnov	8,029	Vyškov	14,495
Trhové Sviny	5,011	Zábřeh	10,543
Trutnov	21,281	Zlín	27,448
Třebíč	19,105	Znojmo	24,822
Třeboň	8,555	Žamberk	8,342
Třinec	14,887	Žatec	9,150
Turnov	8,719	Žďár nad Sázavou	12,097
Týn nad Vltavou	4,609	Železný Brod	3,054
Uherské Hradiště	24,179	Židlochovice	10,501

C Results for Conflict of Waste Producers in Czech Republic

Table C.1: Comparison of the characteristic function values for individual players and the divisions assigned to them according to the Shapley value approximation ψ and the nucleolus approximation γ (part 1)

	$v(r_i)$	ψ_{r_i}		γ_{r_i}	
Aš	825,577	639,411	23 %	661,079	20 %
Benešov	2,261,597	2,075,490	8 %	2,097,100	7 %
Beroun	1,868,340	1,682,141	10 %	1,703,842	9 %
Bílina	565,676	379,563	33 %	401,178	29 %
Bílovec	911,982	725,855	20 %	747,485	18 %
Blansko	1,827,069	1,640,167	10 %	1,662,571	9 %
Blatná	470,150	284,043	40 %	305,653	35 %
Blovice	315,234	129,091	59 %	150,737	52 %
Bohumín	1,743,874	1,557,287	11 %	1,579,377	9 %
Boskovice	1,907,697	1,719,650	10 %	1,743,200	9 %
Brandýs n. L.-S. Boleslav	4,674,952	4,487,045	4 %	4,386,498	6 %
Brno	13,284,192	13,094,666	1 %	12,956,382	2 %
Broumov	580,091	393,926	32 %	415,594	28 %
Bruntál	1,434,532	1,248,411	13 %	1,270,035	11 %
Břeclav	2,616,327	2,428,419	7 %	2,451,829	6 %
Bučovice	537,924	351,817	35 %	373,427	31 %
Bystřice nad Pernštejnem	590,831	404,486	32 %	426,333	28 %
Bystřice pod Hostýnem	692,757	506,636	27 %	528,260	24 %
Čáslav	933,522	747,415	20 %	769,025	18 %
Černošice	4,892,248	4,706,105	4 %	4,727,751	3 %
Česká Lípa	2,443,118	2,256,740	8 %	2,278,620	7 %
Česká Třebová	717,963	531,599	26 %	553,465	23 %

Table C.2: Comparison of the characteristic function values for individual players and the divisions assigned to them according to the Shapley value approximation ψ and the nucleolus approximation γ (part 2)

	$v(r_i)$	ψ_{r_i}		γ_{r_i}	
České Budějovice	3,579,165	3,393,000	5 %	3,414,668	5 %
Český Brod	1,145,214	958,575	16 %	980,717	14 %
Český Krumlov	958,998	772,832	19 %	794,500	17 %
Český Těšín	1,405,481	1,219,145	13 %	1,240,984	12 %
Dačice	675,367	489,202	28 %	510,870	24 %
Děčín	2,151,230	1,964,842	9 %	1,986,733	8 %
Dobruška	916,382	729,649	20 %	751,885	18 %
Dobříš	863,667	677,560	22 %	699,170	19 %
Domažlice	995,040	808,902	19 %	830,543	17 %
Dvůr Králové nad Labem	737,332	551,010	25 %	572,835	22 %
Frenštát pod Radhoštěm	761,333	575,226	24 %	596,836	22 %
Frýdek-Místek	4,951,886	4,762,937	4 %	4,390,928	11 %
Frýdlant	875,776	689,665	21 %	711,278	19 %
Frýdlant nad Ostravicí	1,527,414	1,340,987	12 %	1,362,916	11 %
Havířov	4,313,879	4,125,652	4 %	3,793,619	12 %
Havlíčkův Brod	1,973,998	1,787,829	9 %	1,809,501	8 %
Hlinsko	695,749	509,640	27 %	531,252	24 %
Hlučín	1,903,020	1,715,900	10 %	1,662,633	13 %
Hodonín	2,582,922	2,395,177	7 %	2,418,424	6 %
Holešov	1,050,995	864,803	18 %	886,498	16 %
Holice	509,831	323,182	37 %	345,333	32 %
Horažďovice	435,006	248,899	43 %	270,509	38 %
Horšovský Týn	462,108	275,970	40 %	297,611	36 %
Hořice	714,780	528,169	26 %	550,283	23 %
Hořovice	1,169,759	983,596	16 %	1,005,262	14 %
Hradec Králové	5,693,671	5,506,497	3 %	5,514,629	3 %
Hranice	1,547,600	1,361,226	12 %	1,383,103	11 %
Humpolec	675,063	488,929	28 %	510,565	24 %
Hustopeče	1,359,864	1,173,757	14 %	1,195,367	12 %
Cheb	1,562,363	1,376,256	12 %	1,397,866	11 %
Chomutov	2,134,450	1,948,144	9 %	1,969,953	8 %
Chotěboř	657,729	471,606	28 %	493,231	25 %
Chrudim	2,743,570	2,556,717	7 %	2,579,073	6 %
Ivančice	1,010,208	824,009	18 %	845,711	16 %

Table C.3: Comparison of the characteristic function values for individual players and the divisions assigned to them according to the Shapley value approximation ψ and the nucleolus approximation γ (part 3)

	$v(r_i)$	ψ_{r_i}		γ_{r_i}	
Jablonec nad Nisou	1,490,295	1,304,171	12 %	1,325,797	11 %
Jablunkov	911,054	724,936	20 %	746,557	18 %
Jaroměř	538,623	352,122	35 %	374,126	31 %
Jeseník	1,376,705	1,190,540	14 %	1,212,208	12 %
Jičín	1,719,571	1,533,462	11 %	1,555,074	10 %
Jihlava	2,781,573	2,595,408	7 %	2,617,075	6 %
Jilemnice	683,324	497,215	27 %	518,826	24 %
Jindřichův Hradec	1,242,812	1,056,647	15 %	1,078,315	13 %
Kadaň	1,414,779	1,228,331	13 %	1,250,281	12 %
Kaplice	438,871	252,705	42 %	274,373	37 %
Karlovy Vary	2,498,259	2,312,152	7 %	2,333,762	7 %
Karviná	3,420,872	3,233,221	5 %	3,168,333	7 %
Kladno	3,911,736	3,725,135	5 %	3,747,238	4 %
Klatovy	1,607,246	1,421,138	12 %	1,442,749	10 %
Kolín	3,906,750	3,720,170	5 %	3,742,252	4 %
Konice	394,239	206,858	48 %	229,741	42 %
Kopřivnice	2,522,435	2,335,408	7 %	2,338,995	7 %
Kostelec nad Orlicí	846,244	659,682	22 %	681,747	19 %
Kralovice	945,599	759,271	20 %	781,102	17 %
Kralupy nad Vltavou	1,363,172	1,176,571	14 %	1,198,675	12 %
Kraslice	324,634	138,469	57 %	160,137	51 %
Kravaře	818,470	632,349	23 %	653,973	20 %
Králíky	375,965	189,800	50 %	211,468	44 %
Krnov	1,863,284	1,677,108	10 %	1,698,787	9 %
Kroměříž	2,526,581	2,339,309	7 %	2,286,020	10 %
Kuřim	797,504	610,723	23 %	633,007	21 %
Kutná Hora	2,363,216	2,177,109	8 %	2,198,719	7 %
Kyjov	2,153,275	1,966,237	9 %	1,988,778	8 %
Lanškroun	679,997	493,343	27 %	515,500	24 %
Liberec	5,245,619	5,059,481	4 %	5,081,121	3 %
Lipník nad Bečvou	725,087	538,961	26 %	560,590	23 %
Litoměřice	2,433,327	2,247,209	8 %	2,268,830	7 %
Litomyšl	833,413	646,652	22 %	668,916	20 %
Litovel	999,131	812,926	19 %	834,634	16 %

Table C.4: Comparison of the characteristic function values for individual players and the divisions assigned to them according to the Shapley value approximation ψ and the nucleolus approximation γ (part 4)

	$v(r_i)$	ψ_{r_i}		γ_{r_i}	
Litvínov	1,255,136	1,068,872	15 %	1,090,638	13 %
Louny	1,344,888	1,158,163	14 %	1,180,391	12 %
Lovosice	995,503	809,385	19 %	831,005	17 %
Luhačovice	738,721	552,614	25 %	574,224	22 %
Lysá nad Labem	959,833	773,330	19 %	790,277	18 %
Mariánské Lázně	883,466	697,301	21 %	718,969	19 %
Mělník	1,926,492	1,739,942	10 %	1,755,162	9 %
Mikulov	629,422	443,257	30 %	464,925	26 %
Milevsko	495,851	309,743	38 %	331,354	33 %
Mladá Boleslav	3,927,460	3,740,956	5 %	3,757,903	4 %
Mnichovo Hradiště	510,196	324,087	36 %	345,698	32 %
Mohelnice	832,552	646,070	22 %	668,055	20 %
Moravská Třebová	983,750	797,249	19 %	819,253	17 %
Moravské Budějovice	680,218	494,095	27 %	515,720	24 %
Moravský Krumlov	789,665	603,558	24 %	625,168	21 %
Most	1,993,657	1,806,889	9 %	1,829,159	8 %
Náchod	1,883,329	1,696,410	10 %	1,718,832	9 %
Náměšť nad Oslavou	324,075	137,740	57 %	159,578	51 %
Nepomuk	274,712	88,551	68 %	110,215	60 %
Neratovice	1,587,869	1,401,463	12 %	1,423,372	10 %
Nová Paka	320,853	134,743	58 %	156,356	51 %
Nové Město na Moravě	561,204	374,878	33 %	396,707	29 %
Nové Město nad Metují	454,858	268,534	41 %	290,361	36 %
Nový Bor	932,049	745,661	20 %	767,552	18 %
Nový Bydžov	587,278	401,169	32 %	422,780	28 %
Nový Jičín	1,836,703	1,650,123	10 %	1,672,206	9 %
Nymburk	1,853,319	1,667,064	10 %	1,688,822	9 %
Nýřany	1,355,377	1,169,231	14 %	1,190,880	12 %
Odry	766,292	580,167	24 %	601,795	21 %
Olomouc	6,878,331	6,689,383	3 %	6,325,782	8 %
Opava	5,444,522	5,255,597	3 %	5,020,755	8 %
Orlová	1,959,325	1,772,609	10 %	1,794,828	8 %
Ostrava	12,190,256	11,998,299	2 %	11,055,403	9 %
Ostrov	755,948	569,782	25 %	591,450	22 %

Table C.5: Comparison of the characteristic function values for individual players and the divisions assigned to them according to the Shapley value approximation ψ and the nucleolus approximation γ (part 5)

	$v(r_i)$	ψ_{r_i}		γ_{r_i}	
Otrokovice	1,472,165	1,285,776	13 %	1,231,603	16 %
Pacov	323,636	137,518	58 %	159,139	51 %
Pardubice	4,049,752	3,862,899	5 %	3,870,709	4 %
Pelhřimov	1,454,925	1,268,818	13 %	1,290,428	11 %
Písek	1,124,466	938,304	17 %	959,968	15 %
Plzeň	5,754,708	5,568,512	3 %	5,590,211	3 %
Podbořany	643,447	457,143	29 %	478,950	26 %
Poděbrady	1,386,609	1,200,501	13 %	1,222,111	12 %
Pohořelice	531,052	344,945	35 %	366,555	31 %
Polička	596,417	409,886	31 %	431,920	28 %
Praha	44,997,029	44,805,148	0 %	44,708,575	1 %
Prachatice	788,952	602,845	24 %	624,455	21 %
Prostějov	4,075,878	3,887,446	5 %	3,596,036	12 %
Přelouč	1,015,421	828,985	18 %	850,924	16 %
Přerov	3,647,503	3,459,272	5 %	3,290,082	10 %
Přeštice	610,970	424,773	30 %	446,472	27 %
Příbram	2,485,810	2,299,703	7 %	2,321,313	7 %
Rakovník	1,892,740	1,706,469	10 %	1,728,243	9 %
Rokycany	1,789,335	1,603,142	10 %	1,624,837	9 %
Rosice	843,825	657,651	22 %	679,328	19 %
Roudnice nad Labem	785,708	599,347	24 %	621,211	21 %
Rožnov pod Radhoštěm	1,550,913	1,364,443	12 %	1,386,415	11 %
Rumburk	1,133,092	946,801	16 %	968,595	15 %
Rychnov nad Kněžnou	1,157,809	971,081	16 %	993,312	14 %
Rýmařov	192,487	6,376	97 %	27,989	85 %
Říčany	2,740,621	2,554,209	7 %	2,576,124	6 %
Sedlčany	750,001	563,883	25 %	585,504	22 %
Semily	668,788	482,668	28 %	504,291	25 %
Slaný	1,337,351	1,150,790	14 %	1,172,853	12 %
Slavkov u Brna	707,690	520,111	27 %	470,886	33 %
Soběslav	673,296	487,188	28 %	508,798	24 %
Sokolov	2,043,276	1,857,169	9 %	1,878,779	8 %
Stod	676,921	490,745	28 %	512,424	24 %
Strakonice	1,204,997	1,018,889	15 %	1,040,499	14 %

Table C.6: Comparison of the characteristic function values for individual players and the divisions assigned to them according to the Shapley value approximation ψ and the nucleolus approximation γ (part 6)

	$v(r_i)$	ψ_{r_i}		γ_{r_i}	
Stříbro	563,910	377,745	33 %	399,413	29 %
Sušice	794,418	608,300	23 %	629,921	21 %
Světlá nad Sázavou	689,372	503,249	27 %	524,875	24 %
Svitavy	997,552	811,126	19 %	833,055	16 %
Šlapanice	2,522,994	2,335,489	7 %	2,186,386	13 %
Šternberk	1,192,177	1,005,859	16 %	1,027,680	14 %
Šumperk	3,123,975	2,937,301	6 %	2,959,478	5 %
Tachov	1,187,962	1,001,797	16 %	1,023,465	14 %
Tanvald	656,277	470,155	28 %	491,780	25 %
Tábor	2,640,946	2,454,828	7 %	2,476,449	6 %
Telč	386,783	200,665	48 %	222,286	43 %
Teplice	3,524,052	3,336,917	5 %	3,359,555	5 %
Tišnov	1,011,278	824,977	18 %	846,780	16 %
Trhové Sviny	460,487	274,322	40 %	295,990	36 %
Trutnov	2,559,443	2,372,718	7 %	2,394,946	6 %
Třebíč	2,191,347	2,004,757	9 %	2,026,850	8 %
Třeboň	836,264	650,099	22 %	671,767	20 %
Třinec	2,375,186	2,188,168	8 %	2,210,688	7 %
Turnov	946,452	760,343	20 %	781,955	17 %
Týn nad Vltavou	444,317	258,210	42 %	279,820	37 %
Uherské Hradiště	3,377,574	3,189,534	6 %	3,041,669	10 %
Uherský Brod	2,153,343	1,966,133	9 %	1,916,540	11 %
Uničov	1,065,190	878,937	17 %	900,692	15 %
Ústí nad Labem	2,852,306	2,665,161	7 %	2,687,809	6 %
Ústí nad Orlicí	1,054,249	867,712	18 %	889,751	16 %
Valašské Klobouky	767,741	581,634	24 %	603,244	21 %
Valašské Meziříčí	1,846,677	1,660,115	10 %	1,682,179	9 %
Varnsdorf	717,270	531,152	26 %	552,773	23 %
Velké Meziříčí	1,307,798	1,121,186	14 %	1,143,301	13 %
Veselí nad Moravou	1,256,487	1,069,499	15 %	1,091,990	13 %
Vimperk	524,173	338,066	36 %	359,676	31 %
Vizovice	687,424	501,316	27 %	522,926	24 %
Vítkov	1,075,965	889,840	17 %	911,468	15 %
Vlašim	1,146,980	960,872	16 %	982,482	14 %

Table C.7: Comparison of the characteristic function values for individual players and the divisions assigned to them according to the Shapley value approximation ψ and the nucleolus approximation γ (part 7)

	$v(r_i)$	ψ_{r_i}		γ_{r_i}	
Vodňany	374,310	188,203	50 %	209,813	44 %
Votice	469,567	283,449	40 %	305,069	35 %
Vrchlabí	1,127,735	941,625	17 %	963,237	15 %
Vsetín	2,125,120	1,938,390	9 %	1,960,622	8 %
Vysoké Mýto	1,052,502	865,963	18 %	888,005	16 %
Vyškov	1,940,926	1,753,507	10 %	1,704,122	12 %
Zábřeh	1,433,871	1,246,931	13 %	1,269,374	11 %
Zlín	3,947,818	3,759,401	5 %	3,497,688	11 %
Znojmo	3,061,252	2,875,129	6 %	2,896,755	5 %
Žamberk	1,076,999	890,317	17 %	912,501	15 %
Žatec	916,726	730,418	20 %	752,228	18 %
Žďár nad Sázavou	1,421,949	1,235,420	13 %	1,257,452	12 %
Železný Brod	337,925	151,810	55 %	173,428	49 %
Židlochovice	1,361,982	1,175,433	14 %	1,171,311	14 %

Table C.8: Comparison of the characteristic function values for individual players and the differences between these and the divisions assigned to them according to the Shapley value approximation ψ and the nucleolus approximation γ (part 1)

	$v(r_i)$	$v(r_i) - \psi_{r_i}$	$v(r_i) - \gamma_{r_i}$
Aš	825,577	186,165	164,497
Benešov	2,261,597	186,107	164,497
Beroun	1,868,340	186,199	164,497
Bílina	565,676	186,112	164,497
Bílovec	911,982	186,127	164,497
Blansko	1,827,069	186,901	164,497
Blatná	470,150	186,107	164,497
Blovice	315,234	186,144	164,497
Bohumín	1,743,874	186,587	164,497
Boskovice	1,907,697	188,048	164,497
Brandýs n. L.-S. Boleslav	4,674,952	187,907	288,454
Brno	13,284,192	189,525	327,809
Broumov	580,091	186,165	164,497
Bruntál	1,434,532	186,121	164,497
Břeclav	2,616,327	187,907	164,497
Bučovice	537,924	186,107	164,497
Bystřice nad Pernštejnem	590,831	186,345	164,497
Bystřice pod Hostýnem	692,757	186,121	164,497
Čáslav	933,522	186,107	164,497
Černošice	4,892,248	186,143	164,497
Česká Lípa	2,443,118	186,377	164,497
Česká Třebová	717,963	186,364	164,497
České Budějovice	3,579,165	186,165	164,497
Český Brod	1,145,214	186,639	164,497
Český Krumlov	958,998	186,165	164,497
Český Těšín	1,405,481	186,336	164,497
Dačice	675,367	186,165	164,497
Děčín	2,151,230	186,388	164,497
Dobruška	916,382	186,733	164,497
Dobříš	863,667	186,107	164,497
Domažlice	995,040	186,138	164,497
Dvůr Králové nad Labem	737,332	186,322	164,497
Frenštát pod Radhoštěm	761,333	186,108	164,497

Table C.9: Comparison of the characteristic function values for individual players and the differences between these and the divisions assigned to them according to the Shapley value approximation ψ and the nucleolus approximation γ (part 2)

	$v(r_i)$	$v(r_i) - \psi_{r_i}$	$v(r_i) - \gamma_{r_i}$
Frýdek-Místek	4,951,886	188,949	560,958
Frýdlant	875,776	186,110	164,497
Frýdlant nad Ostravicí	1,527,414	186,427	164,497
Havířov	4,313,879	188,227	520,260
Havlíčkův Brod	1,973,998	186,170	164,497
Hlinsko	695,749	186,109	164,497
Hlučín	1,903,020	187,120	240,387
Hodonín	2,582,922	187,745	164,497
Holešov	1,050,995	186,192	164,497
Holice	509,831	186,649	164,497
Horažďovice	435,006	186,108	164,497
Horšovský Týn	462,108	186,138	164,497
Hořice	714,780	186,611	164,497
Hořovice	1,169,759	186,163	164,497
Hradec Králové	5,693,671	187,174	179,042
Hranice	1,547,600	186,374	164,497
Humpolec	675,063	186,134	164,497
Hustopeče	1,359,864	186,107	164,497
Cheb	1,562,363	186,107	164,497
Chomutov	2,134,450	186,306	164,497
Chotěboř	657,729	186,123	164,497
Chrudim	2,743,570	186,852	164,497
Ivančice	1,010,208	186,199	164,497
Jablonec nad Nisou	1,490,295	186,124	164,497
Jablunkov	911,054	186,118	164,497
Jaroměř	538,623	186,502	164,497
Jeseník	1,376,705	186,165	164,497
Jičín	1,719,571	186,109	164,497
Jihlava	2,781,573	186,165	164,497
Jilemnice	683,324	186,109	164,497
Jindřichův Hradec	1,242,812	186,165	164,497
Kadaň	1,414,779	186,448	164,497
Kaplice	438,871	186,165	164,497

Table C.10: Comparison of the characteristic function values for individual players and the differences between these and the divisions assigned to them according to the Shapley value approximation ψ and the nucleolus approximation γ (part 3)

	$v(r_i)$	$v(r_i) - \psi_{r_i}$	$v(r_i) - \gamma_{r_i}$
Karlovy Vary	2,498,259	186,107	164,497
Karviná	3,420,872	187,651	252,540
Kladno	3,911,736	186,601	164,497
Klatovy	1,607,246	186,108	164,497
Kolín	3,906,750	186,580	164,497
Konice	394,239	187,381	164,497
Kopřivnice	2,522,435	187,027	183,441
Kostelec nad Orlicí	846,244	186,562	164,497
Kralovice	945,599	186,328	164,497
Kralupy nad Vltavou	1,363,172	186,601	164,497
Kraslice	324,634	186,165	164,497
Kravaře	818,470	186,121	164,497
Králíky	375,965	186,165	164,497
Krnov	1,863,284	186,177	164,497
Kroměříž	2,526,581	187,273	240,561
Kuřim	797,504	186,781	164,497
Kutná Hora	2,363,216	186,107	164,497
Kyjov	2,153,275	187,039	164,497
Lanškroun	679,997	186,654	164,497
Liberec	5,245,619	186,138	164,497
Lipník nad Bečvou	725,087	186,125	164,497
Litoměřice	2,433,327	186,118	164,497
Litomyšl	833,413	186,762	164,497
Litovel	999,131	186,205	164,497
Litvínov	1,255,136	186,263	164,497
Louny	1,344,888	186,725	164,497
Lovosice	995,503	186,118	164,497
Luhačovice	738,721	186,107	164,497
Lysá nad Labem	959,833	186,503	169,556
Mariánské Lázně	883,466	186,165	164,497
Mělník	1,926,492	186,550	171,330
Mikulov	629,422	186,165	164,497
Milevsko	495,851	186,107	164,497

Table C.11: Comparison of the characteristic function values for individual players and the differences between these and the divisions assigned to them according to the Shapley value approximation ψ and the nucleolus approximation γ (part 4)

	$v(r_i)$	$v(r_i) - \psi_{r_i}$	$v(r_i) - \gamma_{r_i}$
Mladá Boleslav	3,927,460	186,503	169,556
Mnichovo Hradiště	510,196	186,109	164,497
Mohelnice	832,552	186,482	164,497
Moravská Třebová	983,750	186,501	164,497
Moravské Budějovice	680,218	186,123	164,497
Moravský Krumlov	789,665	186,107	164,497
Most	1,993,657	186,768	164,497
Náchod	1,883,329	186,919	164,497
Náměšť nad Oslavou	324,075	186,335	164,497
Nepomuk	274,712	186,161	164,497
Neratovice	1,587,869	186,406	164,497
Nová Paka	320,853	186,110	164,497
Nové Město na Moravě	561,204	186,326	164,497
Nové Město nad Metují	454,858	186,324	164,497
Nový Bor	932,049	186,388	164,497
Nový Bydžov	587,278	186,109	164,497
Nový Jičín	1,836,703	186,580	164,497
Nymburk	1,853,319	186,255	164,497
Nýřany	1,355,377	186,146	164,497
Odry	766,292	186,125	164,497
Olomouc	6,878,331	188,948	552,549
Opava	5,444,522	188,926	423,767
Orlová	1,959,325	186,716	164,497
Ostrava	12,190,256	191,957	1,134,853
Ostrov	755,948	186,165	164,497
Otrokovice	1,472,165	186,389	240,561
Pacov	323,636	186,118	164,497
Pardubice	4,049,752	186,853	179,042
Pelhřimov	1,454,925	186,107	164,497
Písek	1,124,466	186,162	164,497
Plzeň	5,754,708	186,196	164,497
Podbořany	643,447	186,304	164,497
Poděbrady	1,386,609	186,107	164,497

Table C.12: Comparison of the characteristic function values for individual players and the differences between these and the divisions assigned to them according to the Shapley value approximation ψ and the nucleolus approximation γ (part 5)

	$v(r_i)$	$v(r_i) - \psi_{r_i}$	$v(r_i) - \gamma_{r_i}$
Pohořelice	531,052	186,107	164,497
Polička	596,417	186,531	164,497
Praha	44,997,029	191,881	288,454
Prachatice	788,952	186,107	164,497
Prostějov	4,075,878	188,432	479,842
Přelouč	1,015,421	186,437	164,497
Přerov	3,647,503	188,231	357,422
Přeštice	610,970	186,197	164,497
Příbram	2,485,810	186,107	164,497
Rakovník	1,892,740	186,270	164,497
Rokycany	1,789,335	186,193	164,497
Rosice	843,825	186,174	164,497
Roudnice nad Labem	785,708	186,361	164,497
Rožnov pod Radhoštěm	1,550,913	186,470	164,497
Rumburk	1,133,092	186,291	164,497
Rychnov nad Kněžnou	1,157,809	186,727	164,497
Rýmařov	192,487	186,111	164,497
Říčany	2,740,621	186,413	164,497
Sedlčany	750,001	186,118	164,497
Semily	668,788	186,120	164,497
Slaný	1,337,351	186,560	164,497
Slavkov u Brna	707,690	187,578	236,804
Soběslav	673,296	186,107	164,497
Sokolov	2,043,276	186,107	164,497
Stod	676,921	186,177	164,497
Strakonice	1,204,997	186,107	164,497
Stříbro	563,910	186,165	164,497
Sušice	794,418	186,118	164,497
Světlá nad Sázavou	689,372	186,123	164,497
Svitavy	997,552	186,426	164,497
Šlapanice	2,522,994	187,505	336,608
Šternberk	1,192,177	186,318	164,497
Šumperk	3,123,975	186,674	164,497

Table C.13: Comparison of the characteristic function values for individual players and the differences between these and the divisions assigned to them according to the Shapley value approximation ψ and the nucleolus approximation γ (part 6)

	$v(r_i)$	$v(r_i) - \psi_{r_i}$	$v(r_i) - \gamma_{r_i}$
Tachov	1,187,962	186,165	164,497
Tanvald	656,277	186,122	164,497
Tábor	2,640,946	186,118	164,497
Telč	386,783	186,118	164,497
Teplice	3,524,052	187,135	164,497
Tišnov	1,011,278	186,301	164,497
Trhové Sviny	460,487	186,165	164,497
Trutnov	2,559,443	186,726	164,497
Třebíč	2,191,347	186,590	164,497
Třeboň	836,264	186,165	164,497
Třinec	2,375,186	187,018	164,497
Turnov	946,452	186,109	164,497
Týn nad Vltavou	444,317	186,107	164,497
Uherské Hradiště	3,377,574	188,040	335,905
Uherský Brod	2,153,343	187,210	236,804
Uničov	1,065,190	186,253	164,497
Ústí nad Labem	2,852,306	187,145	164,497
Ústí nad Orlicí	1,054,249	186,537	164,497
Valašské Klobouky	767,741	186,107	164,497
Valašské Meziříčí	1,846,677	186,561	164,497
Varnsdorf	717,270	186,118	164,497
Velké Meziříčí	1,307,798	186,612	164,497
Veselí nad Moravou	1,256,487	186,988	164,497
Vimperk	524,173	186,107	164,497
Vizovice	687,424	186,107	164,497
Vítkov	1,075,965	186,125	164,497
Vlašim	1,146,980	186,107	164,497
Vodňany	374,310	186,107	164,497
Votice	469,567	186,118	164,497
Vrchlabí	1,127,735	186,110	164,497
Vsetín	2,125,120	186,729	164,497
Vysoké Mýto	1,052,502	186,540	164,497
Vyškov	1,940,926	187,419	236,804

Table C.14: Comparison of the characteristic function values for individual players and the differences between these and the divisions assigned to them according to the Shapley value approximation ψ and the nucleolus approximation γ (part 7)

	$v(r_i)$	$v(r_i) - \psi_{r_i}$	$v(r_i) - \gamma_{r_i}$
Zábřeh	1,433,871	186,940	164,497
Zlín	3,947,818	188,417	450,131
Znojmo	3,061,252	186,123	164,497
Žamberk	1,076,999	186,682	164,497
Žatec	916,726	186,307	164,497
Žďár nad Sázavou	1,421,949	186,530	164,497
Železný Brod	337,925	186,115	164,497
Židlochovice	1,361,982	186,549	190,672