KOSZALIN UNIVERSITY OF TECHNOLOGY

RESEARCH AND MODELLING IN CIVIL ENGINEERING 2019

Edited by
Jacek Katzer, Krzysztof Cichocki and Jacek Domski

KOSZALIN 2019

MONOGRAPH NO 366
FACULTY OF CIVIL ENGINEERING,
ENVIRONMENTAL AND GEODETIC SCIENCES

ISSN 0239-7129
ISBN 978-83-7365-525-6

Editors
Jacek Katzer, Krzysztof Cichocki and Jacek Domski
Koszalin University of Technology, Poland

Reviewers
Jacek Gołaszewski – Silesian University of Technology, Poland
Wojciech Sumelka – Poznań University of Technology, Poland

Technical editor
Czesław Suchocki

Website editor
Mariusz Ruchwa

Linguistic consultations
Ewa Sokołowska-Katzer

Typesetting
Czesław Suchocki

Cover design
Tadeusz Walczak

www.cecem.eu

KOSZALIN UNIVERSITY OF TECHNOLOGY PUBLISHING HOUSE
75-620 Koszalin, Racławicka 15-17, Poland

Koszalin 2019, 1st edition, publisher's sheet 8,23, circulation 120 copies
Printing: INTRO-DRUK, Koszalin, Poland

Table of contents

Research and modelling in civil engineering 2019
www.cecem.eu/monographs/

ISSN 0239-7129
ISBN 978-83-7365-525-6 pp. 71-84

5. Numerical analysis of a semi-circular disc with an angled crack loaded in mixed-mode

Lucie Malíková[1], Petr Miarka[2], Hana Šimonová[3], Barbara Kucharczyková[4]

[1] *Brno University of Technology, Faculty of Civil Engineering, Institute of Structural Mechanics, Brno and Czech Republic and Institute of Physics of Materials, Brno, Czech Republic, malikova.l@fce.vutbr.cz, orcid.org/0000-0001-5868-5717*

[2] *Academy of Sciences of the Czech Republic, Institute of Physics of Materials, Brno, Czech Republic and Brno University of Technology, Faculty of Civil Engineering, Institute of Structural Mechanics, Brno, Czech Republic, petr.miarka@vut.cz, orcid.org/0000-0002-4953-4324*

[3] *Brno University of Technology, Faculty of Civil Engineering, Institute of Structural Mechanics, Brno, Czech Republic, simonova.h@vutbr.cz, orcid.org/0000-0003-1537-6388*

[4] *Brno University of Technology, Faculty of Civil Engineering, Institute of Building Testing, Brno, Czech Republic, barbara.kucharczykova@vutbr.cz, orcid.org/0000-0002-7123-5099*

Abstract: A semi-circular disc with an angled crack loaded in mixed-mode (mode I + II) was chosen for a numerical analysis of crack deflection angles. Although the mode I of loading is often dominant in engineering praxis, lots of structures are subjected at least at the beginning of the crack propagation to a mixed-mode loading. Therefore, it is necessary to study the crack behaviour under such conditions and it has been shown that the multi-parameter fracture mechanics approach can be useful in this case of loading. Thus, various cracked configurations of the semi-circular disc were simulated numerically under three-point bending and a generalized form of selected fracture criteria is applied to calculate the crack deflection angle. It is discussed, which configurations can be safely described by means of the classical single-parameter fracture mechanics (i.e. only the stress intensity factors K_{I} and K_{II} controls the crack propagation), and which ones shall be analysed via multi-parameter fracture mechanics based on the Williams expansion. As the result, several configurations of the semi-circular disc with an angled crack under three-point bending are recommended to be tested experimentally in order to prove the results of the numerical simulations based on the finite elements.

Keywords: Surface cracks, Finite element analysis, stress

5.1. Introduction

Fracture mechanics appeared in order to be able to assess the reliability and lifetime of structures with defects and cracks (Anderson, 1995) that are presented everywhere because of structures production as well as because of their handling and using.

In 1920, Griffith found a quantitative connection between fracture stress and flaw size (Griffith, 1920). He took advantage of the work of stress analysis of an elliptical hole (made by Inglis in 1913 (Inglis, 1913)) applying it to the unstable crack propagation. The progress in the fracture mechanics concepts takes place up to now. There exist still lots of problems in assessment of fracture response in various novel as well old materials. One of the directions, where the fracture mechanics is oriented, is the multi-parameter concept, what represents using of more than only one controlling parameter for the crack behavior description.

The stress tensor components were firstly expressed as a series expansion by Williams (Williams, 1957) assuming its coefficients dependent on the loading conditions. In the last years, lots of studies have proved that not only the first term of the Williams expansion (corresponding to the stress intensity factor), but also the second one (corresponding to *T*-stress) is important when the crack-tip stress states shall be properly described (Chen, 2002; Ayatollahi and Zakeri, 2007; Christopher *et al.*, 2007; Aliha, Ayatollahi and Kharazi, 2009). The necessity of considering also the higher-order terms in some practical cases has been published for instance in (Ramesh, Gupta and Kelkar, 1997).

Another example, how the multi-parameter approach can be applied, is its utilization in fracture criteria derived for estimation of the further crack propagation angle (Smith, Ayatollahi and Pavier, 2001; Malíková, 2015; Hou *et al.*, 2016). Again, the referred works show that taking into account more than only single parameter for approximation of the crack-tip stress field can be very useful for some specific geometric configuration, materials etc.

In this work, a crack deflection angle in a semi-circular disc loaded under 3-point bending is investigated. The various initial inclination angles of the crack ensure various levels of mixed-mode (I+II) conditions. A numerical parametric analysis is performed in order to study the influence of several parameters on the crack deflection angles and in order to prove the importance of the higher-order terms of the Williams expansion. Conclusions are made based on the results obtained and several particular cracked SCB configurations are recommended for future experiments on a novel quasi-brittle composite material to establish the significance of the multi-parameter fracture mechanics concept via comparison of the numerical analysis with experimental results.

5.2. Williams-expansion-based crack-tip stress field

As stated above, the stress tensor components were firstly expressed as a series expansion by Williams (Williams, 1957). The relations were derived for a homogeneous elastic isotropic cracked body subjected to an arbitrary remote loading and can be written as follows:

$$\sigma_{ij} = \sum_{n=1}^{\infty} \frac{n}{2} r^{\frac{n}{2}-1} f_{ij}(n,\theta) A_n + \sum_{m=1}^{\infty} \frac{m}{2} r^{\frac{m}{2}-1} g_{ij}(m,\theta) B_m \ . \qquad (5.1)$$

Where:

I, j	–	stress tensor components indexes: $i, j \in \{x, y\}$,
r, θ	–	polar coordinates considering the centre of the system at the crack tip,
f_{ij}, g_{ij}	–	known stress functions corresponding to the loading mode I and II, respectively,
A_n, B_m	–	unknown coefficients of the higher-order terms of the Williams expansion (WE).

A similar series can be derived for displacement vector components and it is directly applied when the over-deterministic method is used for calculation of the unknown coefficients A_n and B_m.

5.3. Estimation of the WE coefficients

There have been presented several methods for estimation of the WE coefficients. Each of them is based on a different principle. The most known methods can be the boundary collocation method, hybrid crack element method, as well as over-deterministic method etc. (Tong, Pian and Lasry, 1973; Karihaloo and Xiao, 2001; Xiao, Karihaloo and Liu, 2004; Su and Fok, 2007; Ayatollahi and Nejati, 2011). Each from the methods mentioned has its advantages and disadvantages. For purposes of this paper, the over-deterministic method (ODM) was chosen because of its simplicity. The ODM requires neither utilization of special crack elements nor special mathematical theories. Its principle consists in direct application of the WE series derived for the displacement vector components.

Practically, a common finite element analysis on the cracked specimen under study is performed and displacements of a selected set of nodes around the crack tip are obtained together with their polar coordinates during post-processing. Thus, quite a large system of equation is built and its solution is represented by the coefficients of the higher-order terms. An arbitrary number of the WE coefficients can be chosen until the following condition is fulfilled:

$$2k > N + M + 2 \ . \qquad (5.2)$$

Where:

N	–	number of mode I WE terms,
M	–	number of mode II WE terms,
k	–	number of nodes selected around the crack tip.

Several more recommendations on the use of the over-deterministic method, its accuracy etc. can be found in literature (Šestáková, 2013; Růžička, Malíková and Seitl, 2017).

5.4. Multi-parameter fracture criteria

When the WE coefficients are known an arbitrary stress component can be expressed via multi-parameter power series. This idea is used within the fracture criteria that were suggested for estimation of the further crack propagation angle. In this work, two very common criteria are mentioned. The former one is the maximum tangential stress (MTS) criterion and the latter one is the minimum strain energy density (SED) criterion. Both criteria are applied in their generalized form, which is connected to assuming more than only one parameter (i.e. the stress intensity factor) for assessment of the crack behaviour. The multi-parameter form of the fracture criteria is related to one more phenomenon and it is their dependence on a length parameter that is often called the *critical distance* (Sih and Ho, 1991; Seweryn and Lukaszewicz, 2002; Susmel and Taylor, 2008). Note that classical MTS as well as SED criteria can be expressed only via the stress intensity factors and there is not any dependence on a length parameter.

The idea of the MTS criterion (Erdogan and Sih, 1963) consists in the assumption that a crack propagates in a material in the direction where the tangential stress reaches its maximum. This condition can be mathematically written by means of derivatives:

$$\frac{\partial \sigma_{\theta\theta}}{\partial \theta} = 0 \text{ and } \frac{\partial^2 \sigma_{\theta\theta}}{\partial \theta^2} < 0 \ . \qquad (5.3)$$

Where:

$\sigma_{\theta\theta}$	–	tangential stress,
θ	–	angle of the polar coordinate system defined with its origin at the crack tip.

The multi-parameter approach consists in the approximation of the tangential stress via the WE expansion considering various numbers of the WE terms.

Similarly, the energy-based SED criterion (Sih, 1973, 1974) assumes that a crack will propagate in a material in the direction where the strain energy density reaches its minimum. Again, mathematically written:

$$\frac{\partial S}{\partial \theta} = 0 \text{ and } \frac{\partial^2 S}{\partial \theta^2} > 0 \ \text{, where } S = \frac{1}{2\mu}\left[\frac{\kappa+1}{8}(\sigma_{rr}+\sigma_{\theta\theta})^2 - \sigma_{rr}\sigma_{\theta\theta} + \sigma_{r\theta}^2\right] . \qquad (5.4)$$

Where:

S	–	strain energy density factor,
θ	–	angle of the polar coordinate system defined with its origin at the crack tip.
$\sigma_{\theta\theta}$	–	tangential stress,
σ_{rr}	–	radial stress,
$\sigma_{r\theta}$	–	shear stress in the polar coordinate system,
μ	–	shear modulus,
κ	–	Kolosov's constant.

The tangential, radial and shear stress in the expression for the strain energy density factor are approximated via WE considering various number of its initial terms.

The procedure of searching for maximum (MTS criterion) or minimum (SED criterion) is programmed in Wolfram Mathematica software ('Wolfram Mathematica Documentation Center', 2018) and the values of the further crack propagation angles of the crack in SCB specimen are presented in the section devoted to presenting of obtained results.

5.5. Semi-circular disc with and angled crack under three-point bending

The schema of the specimen selected for the analysis of investigation of the further crack deflection with impact to the importance of the higher-order terms of the WE can be seen in Fig. 5.1. In the referred figure, the P represents the loading force, S is the span between the supports, R is the radius of the disc, β is the initial crack inclination angle and a represents the crack length. Because the analysis presented in this work should be parametric and should involve more various mixed-mode levels, several parameters were not constant but varied within the analysis. The following values of the individual parameters were considered: $P = 1$ kN, $R = 50$ mm, $S = 60 \div 80$ mm, $a = 10 \div 25$ mm, $\beta = 10° \div 50°$. Combination of these values of the parameters ensures the mixed-mode level varying between $K_I/K_{II} = 0.2 \div 8.2$. Material properties of the numerical model were set up with accordance to the properties of concrete, because an experimental campaign on a novel fine-grained composite based on the alkali-activated slag is prepared: Young's modulus $E = 35$ MPa, Poisson's ratio $\nu = 0.23$.

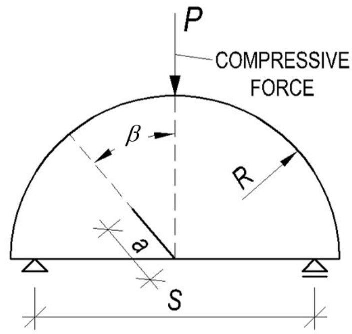


Fig. 5.1. Schema of the Semi-Circular disc under 3-point Bending (SCB) with an angled crack

A two-dimensional numerical model of the cracked SCB disc was created in a commercial finite element software ANSYS ('ANSYS Program Documentation', 2019). Plane strain boundary conditions were set up and a quadrilateral element type PLANE183 were used for meshing the cracked specimen. The crack-tip singularity was modelled through special crack elements that are implemented in the software. For evaluation of the ODM, the displacements of nodes at the radial distance of 4 mm were used and 10 coefficients of the WE were calculated from the system of equations. Then, the fracture criteria introduced above were applied at various critical distances r_c = 0.1, 0.5 and 1.0 mm in order to estimate the angle of the further crack propagation direction, see the obtained results in the following section.

5.6. Achieved results and their discussion

The dependences of the crack deflection angle are displayed as functions of the mixity parameter as has been used in several works of other authors (Aliha and Ayatollahi, 2012; Li *et al.*, 2018):

$$M^{\mathrm{e}} = \frac{2}{\pi} arctg \frac{K_{\mathrm{I}}}{K_{\mathrm{II}}} \ . \tag{5.5}$$

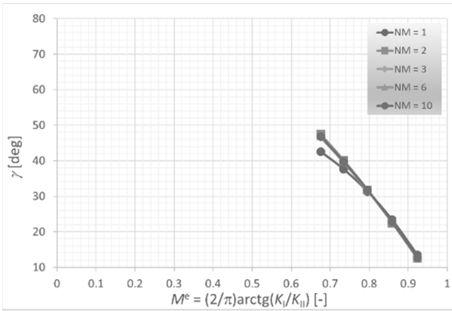
Where:

K_{I} – mode I stress intensity factor,

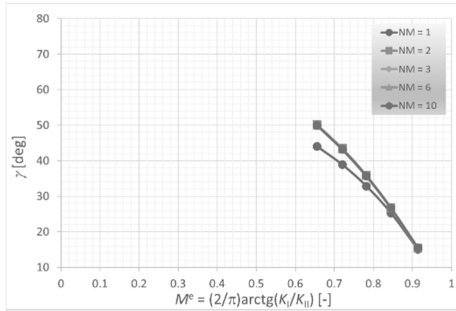
K_{II} – mode II stress intensity factor.

The mixity parameter can achieve the values between 0 (pure mode II) and 1 (pure mode I). Note that the positive crack deflection angles represent the crack propagation towards the loading force. Only selected results are presented in order to point to the most important conclusions. The basic set of result is in Figs. 5.2 and 5.3, where the dependences of the crack deflection angle γ obtained via MTS and SED criterion, respectively, on the mixity parameter M^{e} for the span

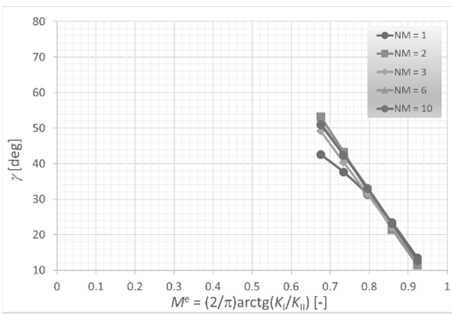
S = 80 mm, relative crack lengths a/R = 0.2 and 0.5, critical distances r_c = 0.1, 0.5 and 1.0 mm and 1, 2, 3, 6 and 10 initial WE terms considered for stress approximation are presented.



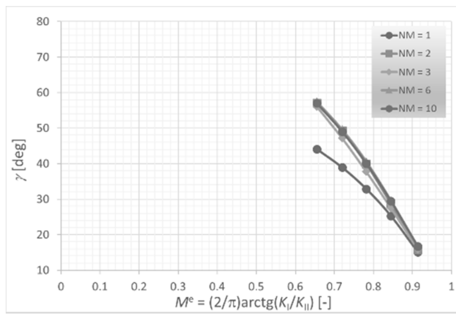
(a) a/R = 0.2, r_c = 0.1 mm



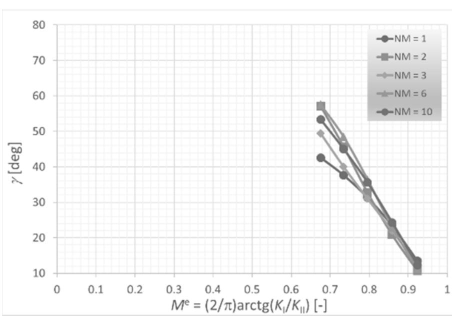
(b) a/R = 0.5, r_c = 0.1 mm



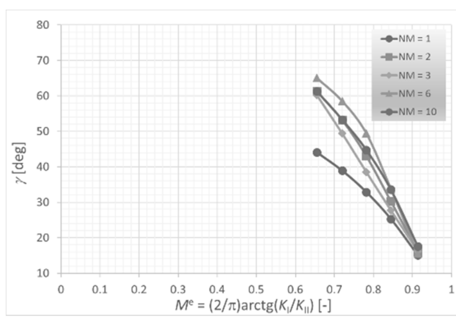
(c) a/R = 0.2, r_c = 0.5 mm



(d) a/R = 0.5, r_c = 0.5 mm

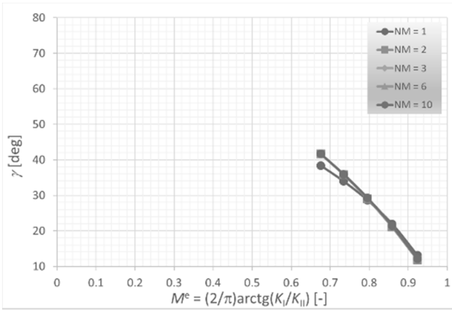


(e) a/R = 0.2, r_c = 1.0 mm

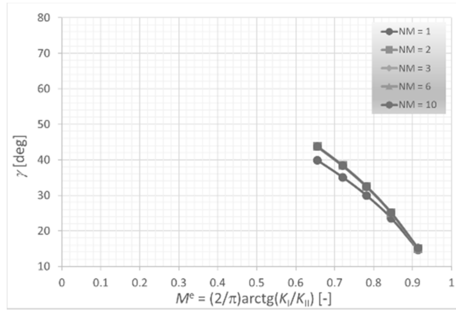


(f) a/R = 0.5, r_c = 1.0 mm

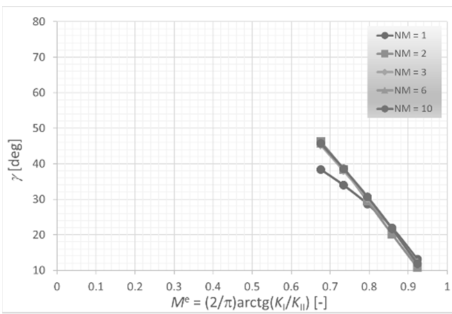
Fig. 5.2. Dependence of the crack deflection angle γ obtained via MTS criterion on the mixity parameter M^e for the span S = 80 mm, relative crack lengths a/R = 0.2 and 0.5, critical distances r_c = 0.1, 0.5 and 1.0 mm and 1, 2, 3, 6 and 10 initial WE terms considered for stress approximation.



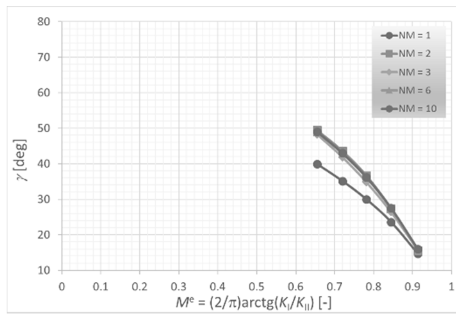
(a) a/R = 0.2, r_c = 0.1 mm



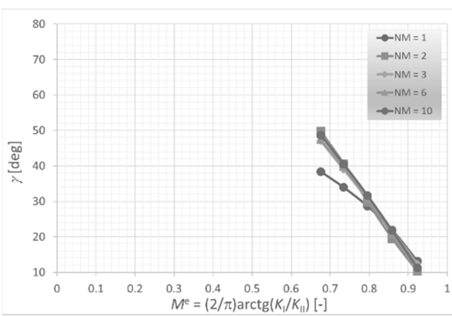
(b) a/R = 0.5, r_c = 0.1 mm



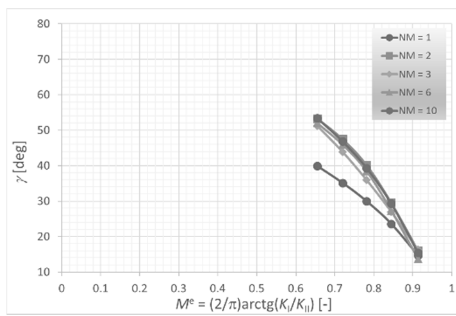
(c) a/R = 0.2, r_c = 0.5 mm



(d) a/R = 0.5, r_c = 0.5 mm



(e) a/R = 0.2, r_c = 1.0 mm



(f) a/R = 0.5, r_c = 1.0 mm

Fig. 5.3. Dependence of the crack deflection angle γ obtained via SED criterion on the mixity parameter M^e for the span S = 80 mm, relative crack lengths a/R = 0.2 and 0.5, critical distances r_c = 0.1, 0.5 and 1.0 mm and 1, 2, 3, 6 and 10 initial WE terms considered for stress approximation.

The results in Figs. 5.2 and 5.3 enables to summarize several conclusions:

- The classical single-parameter fracture mechanics concept differs from the multi-parameter one even at small critical distances from the crack tip

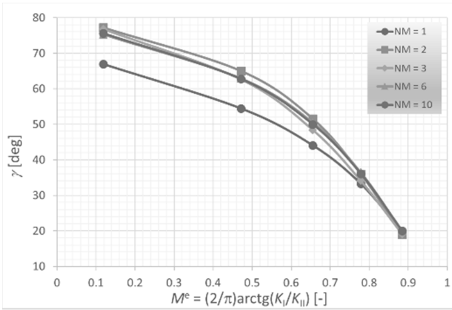
(r_c = 0.1 mm). Nevertheless, if the mode I of loading prevails, the classical form of the fracture criteria can be used safely.

- Generalized fracture criteria become more significant for cracked configuration, with increasing effect of loading mode II.
- The differences between the results, when more WE terms are considered, increase with larger distances from the crack tip, where the generalized fracture criteria are used.
- The SED criterion seems to be less sensitive to the choice of the number of the WE higher-order terms than the MTS criterion; but it surely proves the difference between the single-parameter and multi-parameter concept.
- The differences between the crack deflection angles obtained via single-parameter and multi-parameter fracture criteria are more evident in cases of the longer crack a/R = 0.5.

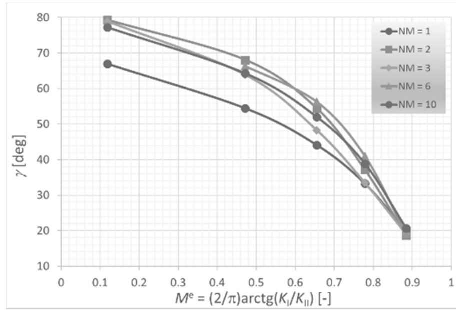
With respect to the dependences plotted in Figs. 5.2 and 5.3, further results are presented only for MTS criterion applied at larger distances for the SCB configuration with a longer crack (a/R = 0.5). Now, the effect of the various span between the supports of the SCB specimen is analysed, see Fig. 5.4, where the dependences of the crack deflection angle γ on the mixity parameter M^e for the supports span S = 60, 70 and 80 mm, critical distances r_c = 0.5 and 1.0 mm and 1, 2, 3, 6 and 10 initial WE terms considered for stress approximation are displayed.

Based on the results in Fig. 5.4, the following summary can be stated:

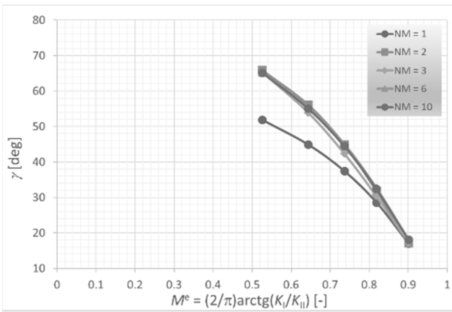
- When the distance between the supports decrease, loading mode II becomes more dominant (as show the values of the mixity parameter M^e approaching the zero value) and the crack deflection angles are generally higher.
- At larger distances from the crack tip, the multi-parameter MTS criterion gives more different values of the crack deflection angle than the single-parameter one.
- When the results obtained via the single-parameter and multi-parameter form of the MTS criterion are compared, the values of the crack deflection angle can differ by up to 20°, which is the case of the longer span between the supports (S = 70 or 80 mm) considering the larger critical distance value r_c = 1.0 mm.



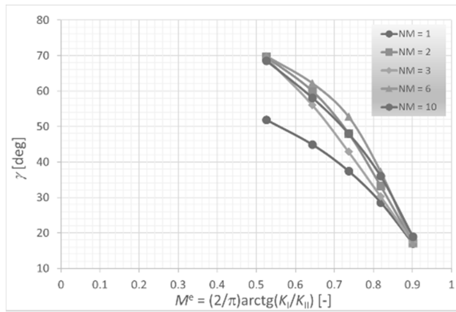
(a) S = 60 mm, r_c = 0.5 mm



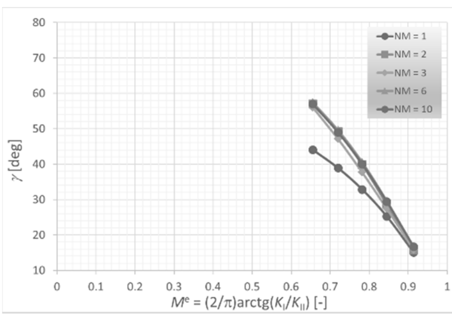
(b) S = 60 mm, r_c = 1.0 mm



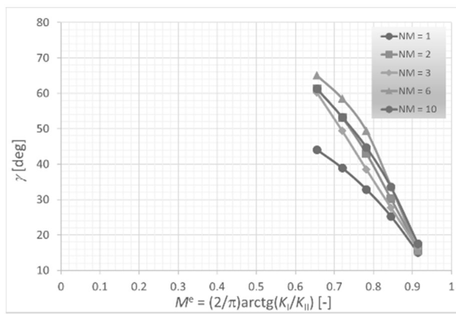
(c) S = 70 mm, r_c = 0.5 mm



(d) S = 70 mm, r_c = 1.0 mm



(e) S = 80 mm, r_c = 0.5 mm



(f) S = 80 mm, r_c = 1.0 mm

Fig. 5.4. Dependence of the crack deflection angle γ obtained via MTS criterion on the mixity parameter M^e for the span S = 60, 70 and 80 mm, relative crack length a/R = 0.5, critical distances r_c = 0.5 and 1.0 mm and 1, 2, 3, 6 and 10 initial WE terms considered for stress approximation.

5.7. Conclusions

A semi-circular disc with an angled crack under three-point bending was investigated via the multi-parameter fracture mechanics concept. Particularly, the crack deflection angle was studied and results obtained by means of the generalized form of the MTS and SED criteria taking into account various numbers of the initial WE terms were compared. It was found out that the largest differences in comparison to the classical single-parameter fracture criteria concept occur when:

- the length of the crack is larger,
- the fracture criterion is applied at larger distance from the crack tip
- the MTS criterion instead of SED criterion is applied,
- loading mode II becomes significant,
- the span between the supports of the SCB configuration is higher.

Based on the results presented in this work, a particular geometry can be chosen in order to verify the conclusions stated above and prove the significance of the higher-order terms of the WE.

Acknowledgment

Financial support from the Czech Science Foundation (project No. 18-12289Y) and from the Faculty of Civil Engineering, Brno University of Technology (project No. FAST-S-19-5896) is gratefully acknowledged. This paper has been worked out under the "National Sustainability Programme I" project "AdMaS UP – Advanced Materials, Structures and Technologies" (No. LO1408) supported by the Ministry of Education, Youth and Sports of the Czech Republic and Brno University of Technology.

References

Aliha, M. R., Ayatollahi, M. R. and Kharazi, B. (2009) 'Mode II fracture assessment using ASFPB specimen', *International Journal of Fracture*, 159, pp. 241–246.

Aliha, M. R. M. and Ayatollahi, M. R. (2012) 'Analysis of fracture initiation angle in some cracked ceramics using the generalized maximum tangential stress criterion', *International Journal of Solids and Structures*. Elsevier Ltd, 49(13), pp. 1877–1883. doi: 10.1016/j.ijsolstr.2012.03.029.

Anderson, T. L. (1995) *Fracture mechanics: fundamentals and applications*. Florida: CRC Press LLC.

'ANSYS Program Documentation' (2019). Houston: Swanson Analysis System, Inc.

Ayatollahi, M. R. and Nejati, M. (2011) 'An over-deterministic method for calculation of coefficients of crack tip asymptotic field from finite element analysis', *Fatigue and Fracture of Engineering Materials and Structures*, 34(3), pp. 159–176. doi: 10.1111/j.1460-2695.2010.01504.x.

Ayatollahi, M. R. and Zakeri, M. (2007) 'T-stress effects on isochromatic fringe patterns in mode II', *International Journal of Fracture*, 143, pp. 189–194.

Chen, Y. Z. (2002) 'Closed form solution of T-stress in plane elasticity crack problems', *International Journal of Solids and Structures*, 37, pp. 1629–1637.

Christopher, C. J. *et al.* (2007) 'Towards a new model of crack tip stress fields', *International Journal of Fracture*, 148, pp. 361–371.

Erdogan, F. and Sih, G. C. (1963) 'On the crack extension in plates under plane loading and transverse shear', *Journal of Basic Engineering*, 55(6), pp. 519–525.

Griffith, A. A. (1920) 'The phenomena of rupture and flow in solids', *Philosophical Transactions, Series A*, 221, pp. 163–198.

Hou, C. *et al.* (2016) 'Determination of fracture parameters in center cracked circular discs of concrete under diametral loading: A numerical analysis and experimental results', *Theoretical and Applied Fracture Mechanics*, 85, pp. 355–366.

Inglis, C. E. (1913) 'Stresses in a plate due to the presence of cracks and sharp corners', *Transactions of the Institute of Naval Architects*, 55, pp. 219–241.

Karihaloo, B. L. and Xiao, Q. Z. (2001) 'Accurate determination of the coefficients of elastic crack tip asymptotic field by a hybrid crack element with p-adaptivity', *Engineering Fracture Mechanics*, 68(15), pp. 1609–1630. doi: 10.1016/S0013-7944(01)00063-7.

Li, X. H. *et al.* (2018) 'Instability of cracks initiation from a mixed-mode crack tip with iso-stress intensity factors K I and K II', 96(May), pp. 262–271. doi: 10.1016/j.tafmec.2018.05.004.

Malíková, L. (2015) 'Multi-parameter fracture criteria for the estimation of crack propagation direction applied to a mixed-mode geometry', *Engineering Fracture Mechanics*, 143. doi: 10.1016/j.engfracmech.2015.06.029.

Ramesh, K., Gupta, S. and Kelkar, A. A. (1997) 'Evaluation of stress field parameters in fracture mechanics by photoelasticity-revisited', *Engineering Fracture Mechanics*, 56, pp. 25–45.

Růžička, V., Malíková, L. and Seitl, S. (2017) 'Over-deterministic method: The influence of rounding numbers on the accuracy of the values of williams' expansion terms', *Frattura ed Integrita Strutturale*, 11(42). doi: 10.3221/IGF-ESIS.42.14.

Šestáková, L. (2013) *How to enhance efficiency and accuracy of the over-deterministic method used for determination of the coefficients of the higher-order terms in williams expansion*, *Applied Mechanics and Materials*. doi: 10.4028/www.scientific.net/AMM.245.120.

Seweryn, A. and Lukaszewicz, A. (2002) 'Verification of brittle fracture criteria for elements with V-shaped notches', *Engineering Fracture Mechanics*, 69, pp. 1487–1510.

Sih, G. C. (1973) 'Some basic problems in fracture mechanics and new concepts', *Engineering Fracture Mechanics*, 5, pp. 365–377.

Sih, G. C. (1974) 'Strain energy density factor applied to mixed mode crack problems', *International Journal of Fracture Mechanics*, 10, pp. 305–321.

Sih, G. C. and Ho, J. W. (1991) 'Sharp notch fracture strength characterized by critical energy density', *Journal of Theoretical and Applied Fracture Mechanics*, 16, pp. 179–214.

Smith, D. J., Ayatollahi, M. R. and Pavier, M. J. (2001) 'The role of T-stress in brittle fracture for linear elastic materials under mixed-mode loading', *Fatigue and Fracture of Engineering Materials and Structures*, 24(2), pp. 137–150.

Su, R. K. L. and Fok, S. L. (2007) 'Determination of coefficients of the crack tip asymptotic field by fractal hybrid finite elements', 74, pp. 1649–1664. doi: 10.1016/j.engfracmech.2006.09.009.

Susmel, L. and Taylor, D. (2008) 'The theory of critical distances to predict static strength of notched brittle components subjected to mixed-mode loading', *Engineering Fracture Mechanics*, 75, pp. 534–550.

Tong, P., Pian, T. H. and Lasry, S. J. (1973) 'A hybrid-element approach to crack problems in plane elasticity', *International Journal of Numerical Methods in Engineering*, 7, pp. 297–308.

Williams, M. L. (1957) 'On the stress distribution at the base of a stationary crack', *Journal of Applied Mechanics*, 24, pp. 109–114.

'Wolfram Mathematica Documentation Center' (2018). Champaign: Wolfram Research.

Xiao, Q. Z., Karihaloo, B. L. and Liu, X. Y. (2004) 'Direct determination of SIF and higher order terms of mixed', *International Journal of Fracture*, 2(3), pp. 207–225.