FUNDAMENTALS OF AN OPTIMAL MULTIRATE SUBBAND CODING OF CYCLOSTATIONARY SIGNALS

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Abstract

A consistent theory of optimal subband coding of zero mean wide-sense cyclostationary signals, with N-periodic statistics, is presented in this article. An M-channel orthonormal uniform filter bank, employing N-periodic analysis and synthesis filters, is used to form the subband coder. A dynamic scheme involving N-periodic bit allocation is used, while an average variance condition is applied to evaluate the output distortion. In three lemmas and final theorem, the necessity of decorrelation of blocked subband signals and requirement of specific ordering of power spectral densities are proven.

Keywords

Coding Gain, Decimation, Filter Bank, Polyphase Representation, Power Spectral Density Matrix, Multirate System, Decorrelation, Subbabd Coding

1. Introduction

A structure of the Multirate Subband Coder for Wide-sense Cyclostationary Signals, that is to be treated hereinafter is depicted in Fig. 1. Set of filters $H_i(k,z^{-1})$ and $F_i(k,z^{-1})$, represent the Analysis Bank and the Synthesis Bank respectively. The index k indicates time varying nature of these blocks, since each of them in fact consists of a sequence of N LTI (Linear Time-Invariant) filters, where N is assigned to the periodicity of cyclostationarity of an LPTV (Linear Periodically Time-Varying) structure. For the sake of simplicity we assume that period of time variation of the structure in Fig. 1 is the same as that of input signal. Blocks to the left of the analysis bank are M-fold decimators that discard all but every M-th sample. Blocks to the right of the synthesis bank are M-fold interpolators that raise the sampling rate by a factor of M, by inserting *(M-1)* zero samples between two consecutive samples of an incoming stream. Blocks Q_i stand for A-D converters, communication channel and D-A converters all together.

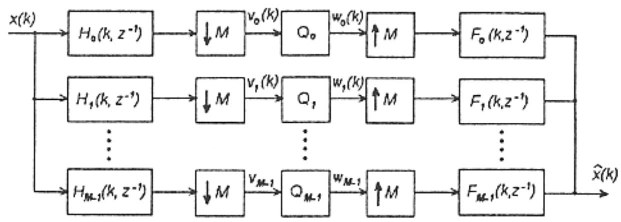

Fig. 1: Multirate Subband Coder for WSCS Signals

Solely for the purpose of following derivation, they are assumed to be a source of additive, uncorrelated, zero mean quantizing noise $q_i(k)$ (see Fig.2), such that

$$w_i(k) = v_i(k) + q_i(k) \ , \tag{1}$$

and the variance of which is

$$\sigma_{qi}^2 = c2^{-2b_i}\sigma_{vi}^2 . \tag{2}$$

Here σ_{vi}^2 is the variance of i-th subband signal and c is a constant determined by the signal distribution.

The WSCS (Wide Sense CycloStationary) signal to be coded is denoted *x(k)*, subband signals are $v_i(k)$'s (for *i* running from *0* to *(M-1)*), $\hat{x}(k)$ is the reconstruction of *x(k)*, synthesized in the output of the synthesis bank. Matrices $E(k,z^{-1})$ and $R(k,z^{-1})$ in Fig.2 are polyphase representations of the analysis and synthesis filter banks, respectively. They are introduced to simplify mathematical treatment of the coder. Since both analysis and synthesis filter banks are LPTV structures, these polyphase matrices are also LPTV, as denoted by time index k.

The goal to be achieved is to bring the synthesized output $\hat{x}(k)$ as close to *x(k)* as possible, by minimizing distortion introduced by quantizers, i.e. maximizing so called *Coding Gain* of the coder. This is done by suitable choice of the analysis and synthesis filter banks, as well as by proper allocation of bits b_i to individual quantizers Q_i.

The overall bitrate

$$b = \frac{1}{M}\sum_{i=0}^{M-1} b_i(k) \tag{3}$$

is kept constant for all k.

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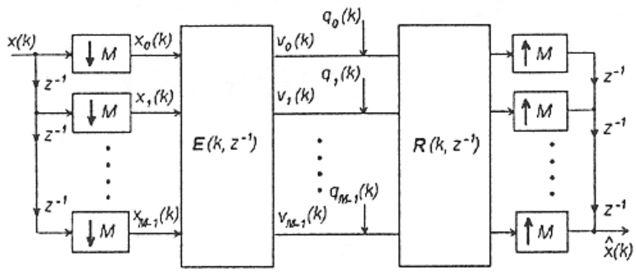

Fig. 2: Polyphase Decomposition of Multirate Coder

A definition of the Coding Gain, that follows naturally from the theory of WSS signals [2],[4] is

$$CG = \frac{E([\tilde{x}(k) - x(k)]^2)}{E([\hat{x}(k) - x(k)]^2)}, \tag{4}$$

where $\tilde{x}(k)$ in the numerator is a reference output of the direct quantizer depicted in Fig.3, and where the denominator can be expressed after using AM-GM inequality [2] as

$$J_{CG} = \frac{c2^{-2b}}{MN} \sum_{j=0}^{N-1} \left(\prod_{i=0}^{M-1} \sigma_{vi}^2(j) \right)^{1/M} . \tag{5}$$

Expression (5) is the function to be minimized to achieve optimum performance of the coder. Especially for *M>2* it is not a trivial task, and the rest of this article is mostly devoted to the solution to this problem.

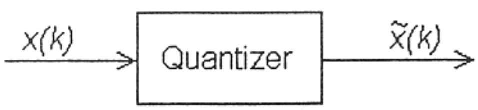

Fig. 3: Direct b-bit Quantizer (PCM)

To overcome problems brought to the analysis of the proposed structure by LPTV nature of the filter banks, an equivalent Blocked LTI system is to be analysed, as depicted in Fig.4. Here, the parallel (time-demultiplexed) input signals *x(k)*, subband signals $v_i(k)$ together with associated quantizing noise signals $q_i(k)$ and parallel reconstructed output $\hat{x}(k)$, can be described by NM-fold vectors $\overline{X}(k)$, $\overline{V}(k)$, $\overline{Q}(k)$ and $\hat{\overline{X}}(k)$, respectively. Matrices $\tilde{E}(z^{-1})$ and $\tilde{R}(z^{-1})$ are so called *Blocked Polyphase Representations* of the analysis and synthesis bank respectively, while the decimators, interpolators and delay-chains displayed in Fig.2, lay outside the depicted structure [5].

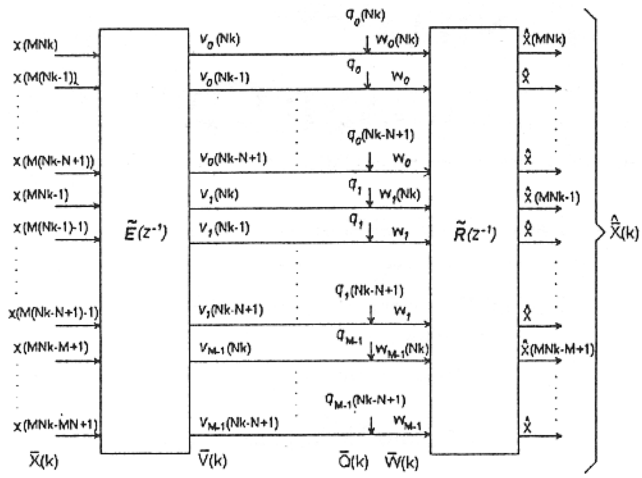

Fig. 4: Blocked Version of Multirate Subband Coder Using Polyphase Representations of the Filter Banks

2. Mathematical concept

In following definitions, lemmas and theorem, our effort is to prove that optimal performance of the LPTV Multirate Subband Coder, i.e. maximised Coding Gain as defined in (4), (5), is achieved if subband signals $v_i(k)$ are mutually decorrelated and diagonal elements of the PSD (Power Spectral Density) matrix of the vector $\overline{V}(k)$ obeys a specific ordering for all ω. This ordering shall be unique.

Definition 1:

Consider a sequence $A = \{a_i\}_{i=1}^{MN}$, such that $a_1 \geq a_2 \geq \ldots\ldots \geq a_{MN} \geq 0$, where $MN = M \times N$. Define a set of integers $S = \{1,2,3,\ldots\ldots,MN\}$, its partitioning $\{S_1, S_2, \ldots, S_N\}$, such that $card[S_i] = M$, $\forall i \in \{1,2,\ldots,N\}$.

Define a *Characteristic Sum*

$$J(A, S_1, S_2, \ldots, S_N) = \sum_{i=1}^{N} \left(\prod_{l \in S_i} a_l \right)^{1/M} , \tag{6}$$

and a *Minimum Sum*

$$J^*(A) = \min_{S_1, S_2, \ldots, S_N} J(A, S_1, S_2, \ldots, S_N) , \tag{7}$$

where the sequence-argument

$$\{S^*(A)\} = \{S_1^*(A), S_2^*(A), \ldots, S_N^*(A)\} , \tag{8}$$

represents *The Best Ordering* of a sequence *A*, that may not be unique.

Lemma 1:

Consider sequence A, and J, J^*, $\{S^*(A)\}$ as in (6) through (8). For $k \in S_i^*(A)$, $l \in S_j^*(A)$, where $k,l \in \{1,2,\ldots,MN\}$ $k \neq l$, $i,j \in \{1,2,\ldots,N\}$ $i \neq j$. Let introduce:

$$\overline{a}_{i,k} = \prod_{q \in S_i^*(A), q \neq k} a_q \ , \tag{9}$$

and

$$\overline{a}_{j,l} = \prod_{q \in S_j^*(A), q \neq l} a_q \ . \tag{10}$$

If $a_k > a_l$, then $\overline{a}_{i,k} \leq \overline{a}_{j,l}$.

Proof: By contradiction

Consider

$$J'(A) = J(A, S_1^*, S_2^*, .., \hat{S}_i^*, .., \hat{S}_j^*, .., S_N^*) \tag{11}$$

where

$$\hat{S}_i^* = \{S_i^*(A) - \{k\}\} \cup \{l\} \quad ,$$
$$\hat{S}_j^* = \{S_j^*(A) - \{l\}\} \cup \{k\} \quad .$$

Then write:

$$J'(A) - J^*(A) = (a_l \overline{a}_{i,k})^{1/M} + (a_k \overline{a}_{j,l})^{1/M} - (a_l \overline{a}_{j,l})^{1/M} - (a_k \overline{a}_{i,k})^{1/M} = (a_l^{1/M} - a_k^{1/M}) \times (\overline{a}_{i,k}^{1/M} - \overline{a}_{j,l}^{1/M})$$

Assuming $a_k > a_l$ and also $\overline{a}_{i,k} > \overline{a}_{j,l}$ we conclude that $J'(A) - J^*(A) < 0$ .Since $J^*(A)$ is *The Minimum Sum*, this establishes a contradiction.

Hence, for $a_k > a_l$,

$$\overline{a}_{i,k} \leq \overline{a}_{j,l} \tag{12}$$

must hold.

Definition 2: ***Majorization***

Lets take sequence $A = \{a_i\}_{i=1}^{MN}$ and $\Lambda = \{\lambda_i\}_{i=1}^{MN}$, with elements labeled such that $a_1 \geq a_2 \geq \ldots . \geq a_N \geq 0$, $\lambda_1 \geq \lambda_2 \geq \ldots . \geq \lambda_N \geq 0$. Suppose that A and Λ obey relation:

$$\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} \lambda_i \ , \text{ for } 1 \leq k \leq N \ , \tag{13}$$

with equality at k=N. Then we say that Λ majorizes A, or A is majorized by Λ , noting $A \prec \Lambda$.

Definition 3: ***Schur-concave function***

A real-valued function $\Phi(x_1, x_2, \ldots, x_n)$, defined on a subset $\Im \subset \Re^n$ is said to be strictly Schur-concave on this subset whenever $X \prec Y$ implies

$$\Phi(X) \geq \Phi(Y) \quad , \tag{14}$$

with equality if X=Y in terms of equal elements.

Theorem 1:

Let $\Phi(x_1, x_2, \ldots, x_n)$ be a real-valued function, where $x_1 \geq x_2 \geq \ldots \geq x_n \geq 0$, defined on $\Im \subset \Re^n$ and twice differentiable on its interior. $\Phi(x_1, x_2, \ldots, x_n)$ is symmetric on $\Im$. Denote [2] :

$$\varphi_{(k)}(X) = \frac{\partial \Phi(X)}{\partial x_k} \qquad \text{and}$$

$$\varphi_{(k,l)}(X) = \frac{\partial^2 \Phi(X)}{\partial x_k \partial x_l} .$$

Then $\Phi(X)$ is strictly Schur-concave on $\Im$ if:

1. $\varphi_{(k)}(X)$ is increasing in k
2. $\varphi_{(k)}(X) = \varphi_{(k+1)}(X)$ implies:
$$\varphi_{(k,k)}(X) - \varphi_{(k,k+1)}(X) - \varphi_{(k+1,k)}(X) + \varphi_{(k+1,k+1)}(X) < 0$$

Proof of this theorem is available in [3].

Lemma 2:

The *Minimum Sum* as presented in (7) is strictly Schur-concave function and hence for all A and Λ, $A \prec \Lambda$ implies

$$J^*(A) \geq J^*(\Lambda), \tag{15}$$

with equality if $A = \Lambda$ in terms of equal elements.

Proof:

From definition of $J^*(A)$ we have $J^*(A) = J^*(\overline{\Pi} A)$ for any permutation matrix $\overline{\Pi}$. Hence $J^*(A)$ is symmetric on $\Im$. Let's take the derivative

$$J_{(k)}^*(A) = \frac{1}{M} \left(\prod_{q \in S^*_i(A), q \neq k} a_q \right)^{1/M} a_k^{1/M-1} = \frac{1}{M} \overline{a}_{i,k}^{1/M} a_k^{1/M-1}$$

$\forall k \in \{1,2,\ldots MN\}$. Similarly for l=k+1:

$$J_{(k+1)}^*(A) = \frac{1}{M} \overline{a}_{j,l}^{1/M} a_l^{1/M-1} .$$

Under the assumption $a_k > a_{k+1} \geq 0$, is $a_k^{1/M-1} < a_{k+1}^{1/M-1}$ for $M \geq 2$. From Lemma 1, we get $0 \leq \overline{a}_{i,k} \leq \overline{a}_{j,l}$. Obviously $J_{(k)}^*(A)$ is increasing in k as required by Theorem 1.

If it happens that $\varphi_{(k)}(A) = \varphi_{(k+1)}(A)$ then requirement #2 of Theorem 1 is satisfied as well, since $\varphi_{(k,k)}(A)$ and $\varphi_{(k+1,k+1)}(A)$ are negative in sign (recall that $(1/M - 1) < 0$) and remaining two partial derivatives are positive in sign or equal to zero, depending on whether k and k+1 are in the same $S_i^*(A)$ or not. Lemma 2 then follows from definition of Schur-concave function. Hence the result.

Proposition 1:

Consider a $MN \times MN$ positive semidefinite, Hermitian symmetric matrix $\overline{S}$, its diagonal elements $A = \{a_i\}_{i=1}^{MN}$, such that $a_1 \geq a_2 \geq \ldots \geq a_{MN} \geq 0$.

Consider sequence of eigenvalues $\Lambda = \{\lambda_i\}_{i=1}^{MN}$, such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N \geq 0$. Then:

$$A \prec \Lambda \quad . \tag{16}$$

3. Existence and uniqueness of the optimal solution

The assumed all-pass nature of the Analysis and synthesis filter bank allows us to take an advantage of the results gained above. Perfect reconstruction calls for

$$\widetilde{R}(z^{-1}) = \widetilde{E}^{-1}(z^{-1}). \tag{17}$$

Orthonormality (the all-pass property) requires $\widetilde{E}(e^{-j\omega})$, $\widetilde{R}(e^{-j\omega})$ being unitary for all ω, i.e.

$$\widetilde{R}(e^{-j\omega})[\widetilde{R}(e^{-j\omega})]'^{*} = \widetilde{E}(e^{-j\omega})[\widetilde{E}(e^{-j\omega})]'^{*} = \bar{I}. \tag{18}$$

Moreover PSD matrices $S_{\overline{X}}(\omega)$ of $\overline{X}(k)$ and $S_V(\omega)$ of $\overline{V}(k)$ are both positive definite Hermitian symmetric matrices, thus the sum of their diagonal elements is majorized by the sum of their eigenvalues. We can write [2]:

$$S_{\overline{X}} = \widetilde{U}(e^{-j\omega})\widetilde{\Lambda}(\omega)\left[\widetilde{U}(e^{-j\omega})\right]'^{*} \tag{19}$$

where $\widetilde{U}(e^{-j\omega})$ is unitary at all ω, and

$$\widetilde{\Lambda}(\omega) = diag\left\{\widetilde{\lambda}_0(\omega), \ldots, \widetilde{\lambda}_{MN}(\omega)\right\} \tag{20}$$

obeying $\widetilde{\lambda}_i(\omega) \geq \widetilde{\lambda}_{i+1}(\omega) > 0$. Also follows from Fig. 4, that PSD matrix of $\overline{V}(k)$ obeys

$$S_{\overline{V}} = \widetilde{E}(e^{-j\omega})S_{\overline{X}}(\omega)\left[\widetilde{E}(e^{-j\omega})\right]'^{*} \quad . \tag{21}$$

Let's return to the denominator of the Coding Gain (5) that concerns the subband variances $\sigma_{vi}(j)$ for $i \in \{0,..,M-1\}$, $j \in \{0,\ldots,N-1\}$. Obviously the ordering of the sequence

$$\Sigma_{\overline{V}} = \left\{\sigma_{vi}(j)\right\}_{ij} = \left\{\frac{1}{2\pi}\left(\int_0^{2\pi} S_{\overline{V}}(\omega)d\omega\right)_{kk} \middle| 0 \leq k \leq MN-1\right\}$$

plays an important part in minimizing (5).

Lemma 3:

Consider sequence $\Sigma_{\overline{V}}$ as above and $\widetilde{\lambda}_i(\omega)$ as in (20). Then under (18):

$$\Sigma_{\overline{V}} \prec \left\{\frac{1}{2\pi}\left(\int_0^{2\pi} \widetilde{\lambda}_k(\omega)d\omega\right)_{kk} \middle| 0 \leq k \leq MN-1\right\}. \tag{22}$$

Proof:

Suppose that the elements of $S_V(\omega)$ are not ordered in a consistent way at all ω, e.g. around some frequency ω_1 $\left(S_{\overline{V}}(\omega_1)\right)_{ii} > \left(S_{\overline{V}}(\omega_1)\right)_{ll}$, but around another frequency ω_2 $\left(S_{\overline{V}}(\omega_1)\right)_{ii} < \left(S_{\overline{V}}(\omega_1)\right)_{ll}$. Then at each ω there exists a permutation matrix $P(\omega)$ such that for all ω,

$$S^{*}_{\overline{V}} = P(\omega)S_{\overline{X}}(\omega)P'(\omega) \tag{23}$$

obeys

$$\left(S^{*}_{\overline{V}}(\omega_1)\right)_{ll} \geq \left(S_{\overline{V}}(\omega_1)\right)_{ll} \quad . \tag{24}$$

Define

$$\Sigma^{*}_{\overline{V}} = \left\{\frac{1}{2\pi}\left(\int_0^{2\pi} S^{*}_{\overline{V}}(\omega)d\omega\right)_{kk} \middle| 0 \leq k \leq MN-1\right\} \tag{25}$$

Clearly, for all $0 \leq l \leq MN-1$ and ω

$$\sum_{k=0}^{l}\left(S_{\overline{V}}(\omega)\right)_{kk} \leq \sum_{k=0}^{l}\left(S^{*}_{\overline{V}}(\omega)\right)_{kk}, \tag{26}$$

with equality at $l = MN-1$. Hence:

$$\sum_{k=0}^{l}\int_0^{2\pi}\left(S_{\overline{V}}(\omega)\right)_{kk}d\omega \leq \sum_{k=0}^{l}\int_0^{2\pi}\left(S^{*}_{\overline{V}}(\omega)\right)_{kk}d\omega. \tag{27}$$

From (25) then

$$\Sigma_{\overline{V}} \prec \Sigma^{*}_{\overline{V}} \quad . \tag{28}$$

Also, $P(\omega)$ is a unitary permutation matrix. Thus from (18), (19)-(21), (23) $\widetilde{\lambda}_i(\omega)$ are the eigenvalues of $S^{*}_{\overline{X}}(\omega)$. Then from the Proposition 1, for all ω and $0 \leq l \leq MN-1$,

$$\sum_{k=0}^{l}\left(S^{*}_{\overline{V}}(\omega)\right)_{kk} \leq \sum_{k=0}^{l}\widetilde{\lambda}_k(\omega) \tag{29}$$

with equality at $l = MN-1$. Result (22) follows directly from (29) that is the definition of majorization.

Theorem 2:

The optimum coding, i.e. the maximum coding gain is attained if both of the following hold:

1. The subband signals $v_i(k)$ are totally decorrelated for all *k*, i.e. the blocked PSD matrix $S_V(\omega)$ is diagonal.
2. The diagonal elements obey a specific ordering at each ω. The ordering itself is not unique, but results in a unique minimum value, the *Minimum Sum.*

Proof:

First requirement follows from Lemma 2 and Proposition 1, since diagonal PSD matrix $S_V(\omega)$, which was obtained by unitary transform (filtering by all-pass filter bank) indeed contains its eigenvalues on the principal diagonal. Letting $\widetilde{E}(e^{-j\omega}) = \widetilde{U}(e^{-j\omega})$ is one possible solution.

Second requirement is a consequence of Lemma 3, the existence of a specific permutation matrix $P(\omega)$ that sorts the eigenvalues of $S_V(\omega)$ in such a way, that $\sigma_{vi}(j)$'s will minimize (5).

Hence the result.

4. Conclusion

The goal of this short insight into Multirate Subband Coding was to establish requirements for optimum subband coding of WSCS signals using uniform orthonormal filter banks. The results stated in Theorem 2 are similar to those obtained in [2] or even [4] as for the requirement of decorrelation of subband variances.

However the ordering of diagonal elements of PSD matrix or even those of the correlation matrix of subband variances is much more difficult task. Despite of the fact, that there exists an ordering that leads to the *Minimum sum*, thus to the optimum solution, there is no specific rule how to order these variances in general. The *Minimum Ordering* itself depends on the values of individual variances. From this result the only way to solve the problem, is to charge the computer with finding the minimum solution, once decorrelation is entered as a necessary condition.

The previous paragraphs treated almost solely the filter bank design. Once the filters $H_i(k,z^{-1})$'s and $F_i(k,z^{-1})$'s are obtained by deblocking and polyphase decomposition of the matrices $\widetilde{E}(z^{-1})$ and $\widetilde{R}(z^{-1})$, the only remaining task is allocating bits to individual subband quantizers. From AM-GM inequality, that was used to express (5), follows the condition for minimized mean square distortion at the output of the coder. If for all $k,l \in \{0,1,\dots,M-1\}$ and $j \in \{0,1,\dots,N-1\}$ holds

$$2^{-2b_k(j)}\sigma_{vk}^2(j) = 2^{-2b_l(j)}\sigma_{vl}^2(j), \qquad (30)$$

then the output distortion reaches its minimum. Since $\sigma_{vi}^2(j)$'s will be know from the filter bank design, bit allocation becomes a trivial task.

Suitable ways how to actually design a uniform orthonormal or orthogonal filter bank, knowing the PSD matrix of the input WSCS signal, are of persistent author's interest.

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Acknowledgement

The Author owes special thanks to Prof. Soura Dasgupta from Dept. of ECE, The University of Iowa, for his kind supervision during author's graduate stay at Iowa and for many fruitful ideas considering the subject presented above.

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