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The focal boundary value problem for strongly singular higher-order nonlinear functional-differential equations

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Abstract
The *a priori* boundedness principle is proved for the two-point right-focal boundary value problems for strongly singular higher-order nonlinear functional-differential equations. Several sufficient conditions of solvability of the two-point right-focal problem under consideration are derived from the *a priori* boundedness principle. The proof of the *a priori* boundedness principle is based on Agarwal-Kiguradze type theorems, which guarantee the existence of the Fredholm property for strongly singular higher-order linear differential equations with argument deviations under the two-point right-focal boundary conditions.
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1 Statement of the main results

1.1 Statement of the problem and the literature survey

Consider the functional differential equation

$$u^{(n)}(t) = F(u)(t) \tag{1.1}$$

with the two-point boundary conditions

$$u^{(i-1)}(a) = 0 \quad (i = 1, \dots, m), \qquad u^{(j-1)}(b) = 0 \quad (j = m+1, \dots, n). \tag{1.2}$$

Here $n \geq 2$, m is the integer part of $n/2$, $-\infty < a < b < +\infty$, and the operator F acts from the set of $(m-1)$th time continuously differentiable on $]a, b]$ functions to the set $L_{\mathrm{loc}}(]a, b])$. By $u^{(i-1)}(a)$ we denote the right limit of the function $u^{(i-1)}$ at the point a.

The problem is singular in the sense that for an arbitrary $u \in C^{m-1}(]a, b])$ the right-hand side of equation (1.1) may have nonintegrable singularities at the point a. Throughout the paper we use the following notations:

$R^+ = [0, +\infty[$; $[x]_+$ the positive part of number x, that is, $[x]_+ = \frac{x+|x|}{2}$;

$L_{\mathrm{loc}}(]a, b])$ is the space of functions $y :]a, b] \to R$, which are integrable on $[a + \varepsilon, b]$ for arbitrarily small $\varepsilon > 0$;

$L_\alpha(]a,b])$ ($L^2_\alpha(]a,b])$) is the space of integrable (square integrable) with the weight $(t-a)^\alpha$ functions $y:]a,b]\to R$ with the norm

$$\|y\|_{L_\alpha}=\int_a^b (s-a)^\alpha\big|y(s)\big|\,ds \qquad \biggl(\|y\|_{L^2_\alpha}=\biggl(\int_a^b (s-a)^\alpha y^2(s)\,ds\biggr)^{1/2}\biggr);$$

$L([a,b])=L_0(]a,b])$, $L^2([a,b])=L^2_0(]a,b])$;

$M(]a,b])$ is the set of measurable functions $\tau:]a,b]\to]a,b]$;

$\widetilde{L}^2_\alpha(]a,b])$ is the Banach space of $y\in L_{\mathrm{loc}}(]a,b])$ functions with the norm

$$\|y\|_{\widetilde{L}^2_\alpha}\equiv\max\biggl\{\biggl[\int_a^t (s-a)^\alpha\biggl(\int_s^t y(\xi)\,d\xi\biggr)^2 ds\biggr]^{1/2}:a\le t\le b\biggr\};$$

$L_n(]a,b])$ is the Banach space of $y\in L_{\mathrm{loc}}(]a,b])$ functions with the norm

$$\|y\|_{L_n}=\sup\biggl\{(s-a)^{m-1/2}\int_s^t(\xi-a)^{n-2m}\big|y(\xi)\big|\,d\xi:a<s\le t\le b\biggr\}<+\infty;$$

$\widetilde{C}^{n-1}_{\mathrm{loc}}(]a,b])$ is the space of functions $y:]a,b]\to R$, which are continuous (absolutely continuous) together with $y',y'',\dots,y^{(n-1)}$ on $[a+\varepsilon,b]$ for arbitrarily small $\varepsilon>0$;

$\widetilde{C}^{n-1,m}(]a,b])$ is the space of functions $y\in\widetilde{C}^{n-1}_{\mathrm{loc}}(]a,b])$ such that

$$\int_a^b\big|x^{(m)}(s)\big|^2\,ds<+\infty; \tag{1.3}$$

$C^{m-1}_1(]a,b])$ is the Banach space of functions $y\in C^{m-1}_{\mathrm{loc}}(]a,b])$ such that

$$\limsup_{t\to a}\frac{|x^{(i-1)}(t)|}{(t-a)^{m-i+1/2}}<+\infty \quad (i=1,\dots,m) \tag{1.4}$$

with the norm $\|x\|_{C^{m-1}_1}=\sum_{i=1}^m\sup\{\frac{|x^{(i-1)}(t)|}{(t-a)^{m-i+1/2}}:a<t\le b\}$;

$\widetilde{C}^{m-1}_1(]a,b])$ is the Banach space of functions $y\in\widetilde{C}^{m-1}_{\mathrm{loc}}(]a,b])$ such that conditions (1.3) and (1.4) hold with the norm $\|x\|_{\widetilde{C}^{m-1}_1}=\|x\|_{C^{m-1}_1}+(\int_a^b|x^{(m)}(s)|^2\,ds)^{1/2}$;

$D_n(]a,b]\times R^+)$ is the set of such functions $\delta:]a,b]\times R^+\to L_n(]a,b])$ that $\delta(t,\cdot):R^+\to R^+$ is nondecreasing for every $t\in]a,b]$, and $\delta(\cdot,\rho)\in L_n(]a,b])$ for any $\rho\in R^+$.

A solution of problem (1.1), (1.2) is sought in the space $\widetilde{C}^{n-1,m}(]a,b])$.

The principles of the theory of singular boundary value problems were built by Kiguradze in his study [1]. This theory has been intensively developed and studied with sufficient completeness both for the ordinary differential equations and the functional differential equations (see [2–28]).

But equation (1.1), even under the boundary condition (1.2), is not studied in the case when the operator F has the form $F(x)(t)=\sum_{j=1}^m p_j(t)x^{(j-1)}(\tau_j(t))+q(x)(t)$, where the singularities of the functions $p_j:L_{\mathrm{loc}}(]a,b])$ $(j=2,\dots,m)$ are such that the inequalities

$$\int_a^b(s-a)^{n-1}\big[(-1)^{n-m}p_1(s)\big]_+\,ds<+\infty, \qquad \int_a^b(s-a)^{n-j}\big|p_j(s)\big|\,ds<+\infty \tag{1.5}$$

are not fulfilled (in this case we say that the linear part of the operator F is strongly singular), the operator q continuously acts from $C^{m-1}_1(]a,b])$ to $L_{\widetilde{L}^2_{2n-2m-2}}(]a,b])$, and the inclusion

$$\sup\{q(x)(t) : \|x\|_{C_1^{m-1}} \le \rho\} \in \widetilde{L}^2_{2n-2m-2}(]a,b])$$

holds. The first step in studying the differential equations with strong singularities was made by Agarwal and Kiguradze in the article [29], where the linear ordinary differential equations under conditions (1.2), in the case when the functions p_j have strong singularities at the points a and b, are studied. Also the ordinary differential equations with strong singularities under two-point boundary conditions are studied in the articles [30, 31] by Kiguradze. In the papers [32–34] these results are generalized for a linear differential equation with deviating arguments, *i.e.*, the Agarwal-Kiguradze type theorems are proved, which guarantee the Fredholm property for the linear differential equation with deviating arguments. In this paper, on the basis of articles [33, 34], we prove the *a priori* boundedness principle for problem (1.1), (1.2) from which several sufficient conditions of the solvability of this problem follow.

Now we introduce some results from the articles [33, 34] in this section, which we need for this work. Consider the equation

$$u^{(n)}(t) = \sum_{j=1}^{m} p_j(t)u^{(j-1)}(\tau_j(t)) + q(t) \quad \text{for } a < t < b \tag{1.6}$$

with $q, p_j \in L_{\mathrm{loc}}(]a,b])$.

By $h_j :]a,b] \times]a,b] \to R_+$ and $f_j : [a,b] \times M(]a,b]) \to C_{\mathrm{loc}}(]a,b] \times]a,b])$ $(j = 1,\dots,m)$ we denote the functions and the operator, respectively, defined by the equalities

$$\begin{aligned} h_1(t,s) &= \left| \int_s^t (\xi - a)^{n-2m} \big[(-1)^{n-m} p_1(\xi)\big]_+ \, d\xi \right|, \\ h_j(t,s) &= \left| \int_s^t (\xi - a)^{n-2m} p_j(\xi)\, d\xi \right|, \end{aligned} \tag{1.7}$$

and

$$f_j(c,\tau_j)(t,s) = \left| \int_s^t (\xi - a)^{n-2m} \big|p_j(\xi)\big| \left| \int_\xi^{\tau_j(\xi)} (\xi_1 - c)^{2(m-j)}\, d\xi_1 \right|^{1/2} d\xi \right|. \tag{1.8}$$

Let also $k = 2k_1 + 1$ $(k_1 \in Z)$, then

$$k!! = \begin{cases} 1 & \text{for } k \le 0, \\ 1 \cdot 3 \cdot 5 \cdots k & \text{for } k \ge 1. \end{cases}$$

Now we can introduce the main theorem of the papers [33] and [34].

Theorem 1.1 *Let there exist the numbers* $\ell_j > 0$, $\overline{\ell}_j \ge 0$, *and* $\gamma_j > 0$ $(j = 1,\dots,m)$ *such that along with*

$$B \equiv \sum_{j=1}^{m} \left(\frac{(2m-j)2^{2m-j+1}\ell_j}{(2m-1)!!(2m-2j+1)!!} + \frac{2^{2m-j-1}(b-a)^{\gamma_j}\overline{\ell}_j}{(2m-2j-1)!!(2m-3)!!\sqrt{2\gamma_j}} \right) < 1, \tag{1.9}$$

the conditions

$$(t-a)^{2m-j}h_j(t,s) \le \ell_j, \qquad (t-a)^{m-\gamma_{0j}-1/2}f_j(a,\tau_j)(t,s) \le \overline{\ell}_j \tag{1.10}$$

hold for $a < t \le s \le b$. *Then problem* (1.6), (1.2) *is uniquely solvable in the space* $\widetilde{C}^{n-1,m}(]a,b])$.

Remark 1.1 From Lemma 2.5 it is clear that any solution of problem (1.6), (1.2) from the space $\widetilde{C}^{n-1,m}(]a,b])$ belongs also to the space $\widetilde{C}_1^{m-1}(]a,b])$.

Theorem 1.2 *Let all the conditions of Theorem* 1.1 *be satisfied. Then the unique solution* u *of problem* (1.6), (1.2) *for every* $q \in \widetilde{L}^2_{2n-2m-2}(]a,b])$ *admits the estimate*

$$\left\|u^{(m)}\right\|_{L^2} \le r\|q\|_{\widetilde{L}^2_{2n-2m-2}}, \tag{1.11}$$

with

$$r = \frac{2^{m-1}(2n-2m-1)}{(\nu_n - B)(2m-1)!!}, \qquad \nu_{2m} = 1, \qquad \nu_{2m+1} = \frac{2m+1}{2},$$

and thus constant $r > 0$ *depends only on the numbers* ℓ_j, $\overline{\ell}_j$, γ_j $(j = 1,\dots,m)$, *and* a, b, n.

Remark 1.2 Under the conditions of Theorem 1.2, for every $q \in \widetilde{L}^2_{2n-2m-2}(]a,b])$, the unique solution u of problem (1.6), (1.2) admits the estimate

$$\|u\|_{\widetilde{C}_1^{m-1}} \le r_n\|q\|_{\widetilde{L}^2_{2n-2m-2}}, \tag{1.12}$$

with $r_n = (1 + \sum_{j=1}^m \frac{(2m-2j+1)^{-1/2}}{(m-j)!})\frac{2^{m-1}(2n-2m-1)}{(\nu_n-B)(2m-1)!!}$.

1.2 Theorems on the solvability of problem (1.1), (1.2)

Define the operator $P: C_1^{m-1}(]a,b]) \times C_1^{m-1}(]a,b]) \to L_{\text{loc}}(]a,b])$ by the equality

$$P(x,y)(t) = \sum_{j=1}^{m} p_j(x)(t)y^{(j-1)}\bigl(\tau_j(t)\bigr) \quad \text{for } a < t \le b, \tag{1.13}$$

where $p_j : C_1^{m-1}(]a,b]) \to L_{\text{loc}}(]a,b])$ and $\tau_j \in M(]a,b])$. Also, for any $\gamma > 0$, define the set A_γ by the relation

$$A_\gamma = \bigl\{x \in \widetilde{C}_1^{m-1}\bigl(]a,b]\bigr) : \|x\|_{\widetilde{C}_1^{m-1}} \le \gamma\bigr\}. \tag{1.14}$$

Now, following the article [6] by Kiguradze and Půža, we introduce the following definitions.

Definition 1.1 Let γ_0 and γ be positive numbers. We say that the continuous operator $P: C_1^{m-1}(]a,b]) \times C_1^{m-1}(]a,b]) \to L_n(]a,b])$ is γ_0, γ consistent with boundary condition (1.2) if:

(i) for any $x \in A_{\gamma_0}$ and almost all $t \in]a,b]$, the inequality

$$\sum_{j=1}^{m} \left| p_j(x)(t) x^{(j-1)}\big(\tau_j(t)\big) \right| \le \delta\big(t, \|x\|_{\widetilde{C}_1^{m-1}}\big) \|x\|_{\widetilde{C}_1^{m-1}} \tag{1.15}$$

holds, where $\delta \in D_n(]a,b] \times R^+)$;

(ii) for any $x \in A_{\gamma_0}$ and $q \in \widetilde{L}^2_{2n-2m-2}(]a,b])$, the equation

$$y^{(n)}(t) = \sum_{j=1}^{m} p_j(x)(t) y^{(j-1)}\big(\tau_j(t)\big) + q(t) \tag{1.16}$$

under boundary conditions (1.2) has the unique solution y in the space $\widetilde{C}^{n-1,m}(]a,b])$ and

$$\|y\|_{\widetilde{C}_1^{m-1}} \le \gamma \|q\|_{\widetilde{L}^2_{2n-2m-2}}. \tag{1.17}$$

Definition 1.2 We say that the operator P is γ consistent with boundary condition (1.2) if the operator P is γ_0, γ consistent with boundary condition (1.2) for any $\gamma_0 > 0$.

In the sequel it will always be assumed that the operator F_p is defined by the equality

$$F_p(x)(t) = \left| F(x)(t) - \sum_{j=1}^{m} p_j(x)(t) x^{(j-1)}\big(\tau_j(t)\big)(t) \right|,$$

continuously acting from $C_1^{m-1}(]a,b])$ to $L_{\widetilde{L}^2_{2n-2m-2}}(]a,b])$, and

$$\widetilde{F}_p(t,\rho) \equiv \sup\big\{F_p(x)(t) : \|x\|_{C_1^{m-1}} \le \rho\big\} \in \widetilde{L}^2_{2n-2m-2}\big(]a,b]\big) \tag{1.18}$$

for each $\rho \in [0, +\infty[$. Then the following theorem is valid.

Theorem 1.3 *Let the operator P be γ_0, γ consistent with boundary condition* (1.2), *and let there exist a positive number $\rho_0 \le \gamma_0$ such that*

$$\big\|\widetilde{F}_p\big(\cdot, \min\{2\rho_0, \gamma_0\}\big)\big\|_{\widetilde{L}^2_{2n-2m-2}} \le \frac{\gamma_0}{\gamma}. \tag{1.19}$$

Let, moreover, for any $\lambda \in]0,1[$, an arbitrary solution $x \in A_{\gamma_0}$ of the equation

$$x^{(n)}(t) = (1-\lambda)P(x,x)(t) + \lambda F(x)(t) \tag{1.20}$$

under conditions (1.2) *admit the estimate*

$$\|x\|_{\widetilde{C}_1^{m-1}} \le \rho_0. \tag{1.21}$$

Then problem (1.1), (1.2) *is solvable in the space* $\widetilde{C}^{n-1,m}(]a,b])$.

From Theorem 1.3 with $\rho_0 = \gamma_0$, the corollary immediately follows.

Corollary 1.1 *Let the operator P be γ_0, γ consistent with boundary condition* (1.2), *and*

$$\left|F(x)(t) - \sum_{j=1}^{m} p_j(x)(t)x^{(j-1)}\big(\tau_j(t)\big)(t)\right| \le \eta\big(t, \|x\|_{\widetilde{C}_1^{m-1}}\big) \tag{1.22}$$

for $x \in A_{\gamma_0}$ and almost all $t \in]a,b]$, and

$$\big\|\eta(\cdot,\gamma_0)\big\|_{\widetilde{L}^2_{2n-2m-2}} \le \frac{\gamma_0}{\gamma}, \tag{1.23}$$

where $\eta \in D_{2n-2m-2}(]a,b] \times R^+)$. Then problem (1.1), (1.2) *is solvable in the space $\widetilde{C}^{n-1,m}(]a,b])$.*

Corollary 1.2 *Let the operator P be γ consistent with boundary condition* (1.2), *let inequality* (1.22) *hold for $x \in \widetilde{C}_1^{m-1}(]a,b])$ and almost all $t \in]a,b]$, where $\eta(\cdot,\rho) \in \widetilde{L}^2_{2n-2m-2}(]a,b])$ for any $\rho \in R^+$, and*

$$\limsup_{\rho\to+\infty} \frac{1}{\rho}\big\|\eta(\cdot,\rho)\big\|_{\widetilde{L}^2_{2n-2m-2}} < \frac{1}{\gamma}. \tag{1.24}$$

Then problem (1.1), (1.2) *is solvable in the space $\widetilde{C}^{n-1,m}(]a,b])$.*

Now define the operators $h_j : C_1^{m-1}(]a,b])\times]a,b]\times]a,b] \to L_{\mathrm{loc}}(]a,b]\times]a,b])$, $f_j : C_1^{m-1}(]a,b]) \times [a,b] \times M(]a,b]) \to C_{\mathrm{loc}}(]a,b]\times]a,b])$ ($j = 1,\dots,m$) by the equalities

$$\begin{aligned} h_1(x,t,s) &= \left|\int_s^t (\xi - a)^{n-2m}\big[(-1)^{n-m}p_1(x)(\xi)\big]_+ d\xi\right|, \\ h_j(x,t,s) &= \left|\int_s^t (\xi - a)^{n-2m} p_j(x)(\xi)\, d\xi\right| \quad (j = 2,\dots,m), \end{aligned} \tag{1.25}$$

$$f_j(x,c,\tau_j)(t,s) = \left|\int_s^t (\xi - a)^{n-2m}\big|p_j(x)(\xi)\big| \left|\int_\xi^{\tau_j(\xi)} (\xi_1 - c)^{2(m-j)}\, d\xi_1\right|^{1/2} d\xi\right| \tag{1.26}$$

and the functions $\alpha_j : [a,b] \to R_+$ by the equality $\alpha_j(t) = (t-a)^{m-j+1/2}$.

Theorem 1.4 *Let the continuous operator $P : C_1^{m-1}(]a,b]) \times C_1^{m-1}(]a,b]) \to L_n(]a,b])$ admit condition* (1.15) *where $\delta \in D_n(]a,b] \times R^+)$, $\tau_j \in M(]a,b])$, and let the numbers $\gamma_0 \in]a,b]$, $l_j > 0$, $\bar{l}_j > 0$, $\gamma_j > 0$ ($j = 1,\dots,m$) be such that the inequalities*

$$(t-a)^{2m-j}h_j(x,t,s) \le l_j, \qquad \limsup_{t\to a}(t-a)^{m-\frac{1}{2}-\gamma_j} f_j(x,a,\tau_j)(t,s) \le \bar{l}_j \tag{1.27}$$

for $a < t \le s \le b$, $\|x\|_{\widetilde{C}_1^{m-1}} \le \gamma_0$, and conditions (1.9) *hold. Let, moreover, the operator F and the function $\eta \in D_{2n-2m-2}(]a,b] \times R^+)$ be such that condition* (1.22) *and the inequality*

$$\big\|\eta(\cdot,\gamma_0)\big\|_{\widetilde{L}^2_{2n-2m-2}} < \frac{\gamma_0}{r_n}, \tag{1.28}$$

are fulfilled, where $r_n = (1 + \sum_{j=1}^m \frac{(2m-2j+1)^{-1/2}}{(m-j)!})\frac{2^{m-1}(2n-2m-1)}{(\nu_n-B)(2m-1)!!}$. Then problem (1.1), (1.2) *is solvable in the space $\widetilde{C}^{n-1,m}(]a,b])$.*

Theorem 1.5 *Let the operator F and the function η be such that conditions* (1.22), (1.24) *hold, and let the continuous operator $P : C_1^{m-1}(]a,b]) \times C_1^{m-1}(]a,b]) \to L_n(]a,b])$ admit condition* (1.15), *where $\delta \in D_n(]a,b] \times R^+)$. Let, moreover, the measurable functions $\tau_j \in M(]a,b])$ and the numbers $l_j > 0$, $\bar{l}_j > 0$, $\gamma_j > 0$ $(j = 1,\dots,m)$ be such that the inequalities*

$$(t-a)^{2m-j} h_j(x,t,s) \le l_j, \qquad \limsup_{t\to a}(t-a)^{m-\frac{1}{2}-\gamma_j} f_j(x,a,\tau_j)(t,s) \le \bar{l}_j \tag{1.29}$$

for $a < t \le s \le b$, $x \in \widetilde{C}_1^{m-1}(]a,b])$, and conditions (1.9) *hold. Then problem* (1.1), (1.2) *is solvable in the space $\widetilde{C}^{n-1,m}(]a,b])$.*

Remark 1.3 Let $\gamma_0 > 0$, let the operators $\alpha_j p_j$ $(j = 1,\dots,m)$ continuously act from the space $C_1^{m-1}(]a,b])$ to the space $L_n(]a,b])$, let there exist the function $\delta_j \in D_n(]a,b])$ such that for any $x \in A_{\gamma_0}$,

$$\bigl|p_j(x)(t)\bigr|\alpha_j(t) \le \delta_j\bigl(t, \|x\|_{\widetilde{C}_1^{m-1}}\bigr) \quad \text{for } a < t \le b, \tag{1.30}$$

and let there exist constants $\kappa > 0$, $\varepsilon > 0$ such that

$$\bigl|\tau_j(t) - t\bigr| \le \kappa(t-a) \quad (j = 1,\dots,m) \text{ for } a < t < a + \varepsilon. \tag{1.31}$$

Then the operator P defined by equality (1.13) continuously acts from A_{γ_0} to the space $L_n(]a,b])$, and there exists the function $\delta \in D_n(]a,b])$ such that item (i) of Definition 1.1 holds.

Now consider the equation with deviating arguments

$$u^{(n)}(t) = f\bigl(t, u\bigl(\tau_1(t)\bigr), u'\bigl(\tau_2(t)\bigr),\dots,u^{(m-1)}\bigl(\tau_m(t)\bigr)\bigr) \quad \text{for } a < t \le b, \tag{1.32}$$

where $-\infty < a < b < +\infty$, $f :]a,b] \times R^m \to R$ is a function satisfying the local Carathéodory conditions and $\tau_j \in M(]a,b])$ $(j = 0,\dots,n-1)$ are measurable functions.

Corollary 1.3 *Let the functions $\tau_j \in M(]a,b])$ and the numbers $\kappa \ge 0$, $\varepsilon > 0$, $l_j > 0$, $\bar{l}_j > 0$, $\gamma_j > 0$ $(j = 1,\dots,m)$ be such that conditions* (1.9), (1.10), (1.31) *and the inclusions*

$$\alpha_j p_j \in L_n\bigl(]a,b]\bigr) \quad (j = 1,\dots,m) \tag{1.33}$$

are fulfilled. Let, moreover,

$$\begin{aligned}&\left|f\bigl(t, x\bigl(\tau_1(t)\bigr), x'\bigl(\tau_2(t)\bigr),\dots,x^{(m-1)}\bigl(\tau_m(t)\bigr)\bigr) - \sum_{j=1}^{m} p_j(t)x^{(j-1)}\bigl(\tau_j(t)\bigr)(t)\right| \\ &\quad \le \eta\bigl(t, \|x\|_{\widetilde{C}_1^{m-1}}\bigr)\end{aligned}$$

for $x \in \widetilde{C}_1^{m-1}(]a,b])$ and almost all $t \in]a,b]$, where $\eta(\cdot,\rho) \in \widetilde{L}^2_{2n-2m-2}(]a,b])$ for any $\rho \in R^+$, and let condition (1.24) *hold. Then problem* (1.32), (1.2) *is solvable in the space $\widetilde{C}^{n-1,m}(]a,b])$.*

Remark 1.4 Conditions (1.5) do not follow from conditions (1.33).

Now, to illustrate our results, consider on $]a,b]$ the second-order functional-differential equations

$$u''(t) = -\frac{\lambda |u(t)|^k}{(t-a)^{2+k/2}} u\big(\tau(t)\big) + q(x)(t), \tag{1.34}$$

$$u''(t) = -\frac{\lambda |\sin u^k(t)|}{(t-a)^2} u\big(\tau(t)\big) + q(x)(t), \tag{1.35}$$

where $\lambda, k \in R^+$ the function $\tau \in M(]a,b])$, the operator $q : C_1^{m-1}(]a,b]) \to \widetilde{L}_0^2(]a,b])$ is continuous and

$$\eta(t,\rho) \equiv \sup\big\{ \big|q(x)(t)\big| : \|x\|_{\widetilde{C}_1^{m-1}} \le \rho \big\} \in \widetilde{L}_0^2\big(]a,b]\big).$$

Then, from Theorems 1.4 and 1.5 with $n = 2$, the corollary follows.

Corollary 1.4 *Let the function $\tau \in M(]a,b])$, the continuous operator $q : C_1^{m-1}(]a,b]) \to \widetilde{L}_0^2(]a,b])$, and the numbers $\gamma_0 > 0$, $\lambda \ge 0$, $k > 0$ be such that*

$$\big|\tau(t) - t\big| \le (t-a)^{3/2} \quad \textit{for } a < t \le b, \tag{1.36}$$

$$\big\|\eta(t,\gamma_0)\big\|_{\widetilde{L}_0^2} \le \frac{1 - 4\lambda\gamma_0^k(1 + [4(b-a)]^{1/4})}{2}\gamma_0, \tag{1.37}$$

and

$$\lambda < \frac{1}{4\gamma_0^k(1 + [4(b-a)]^{1/4})}. \tag{1.38}$$

Then problem (1.34), (1.2) *is solvable.*

Corollary 1.5 *Let the function $\tau \in M(]a,b])$, the continuous operator $q : C_1^{m-1}(]a,b[) \to \widetilde{L}_{0,0}^2(]a,b])$, and the number $\lambda \ge 0$ be such that inequalities* (1.24), (1.36) *and*

$$\lambda < \frac{1}{4(1 + [4(b-a)]^{1/4})}, \tag{1.39}$$

hold. Then problem (1.35), (1.2) *is solvable.*

2 Auxiliary propositions

2.1 Lemmas on some properties of the equation $x^{(n)}(t) = \lambda(t)$

First, we introduce two lemmas without proofs. The first lemma is proved in [29].

Lemma 2.1 *Let $i \in \{1,2\}$, $x \in \widetilde{C}_{\mathrm{loc}}^{m-1}(]t_0,t_1[)$ and*

$$x^{(j-1)}(t_i) = 0 \quad (j = 1, \dots, m), \qquad \int_{t_0}^{t_1} \big|x^{(m)}(s)\big|^2\, ds < +\infty. \tag{2.1}$$

Then

$$\left|\int_{t_i}^{t} \frac{(x^{(j-1)}(s))^2}{(s-t_i)^{2m-2j+2}}\,ds\right|^{1/2} \le \frac{2^{m-j+1}}{(2m-2j+1)!!}\left|\int_{t_i}^{t} \left|x^{(m)}(s)\right|^2 ds\right|^{1/2} \tag{2.2}$$

for $t_0 \le t \le t_1$.

This second lemma is a particular case of Lemma 4.1 in [35].

Lemma 2.2 *If* $x \in C_{\mathrm{loc}}^{n-1}(]a,a_1])$, *then for any* $s,t \in]a,a_1]$ *the equality*

$$(-1)^{n-m}\int_s^t (\xi-a)^{n-2m}x^{(n)}(\xi)x(\xi)\,d\xi = w_n(x)(t) - w_n(x)(s) + \nu_n\int_s^t \left|x^{(m)}(\xi)\right|^2 d\xi$$

is valid, where $\nu_{2m}=1$, $\nu_{2m+1}=\frac{2m+1}{2}$, $w_{2m}(x)(t)=\sum_{j=1}^{m}(-1)^{m+j-1}x^{(2m-j)}(t)x(t)$,

$$w_{2m+1}(x)(t) = \sum_{j=1}^{m}(-1)^{m+j}\left[(t-a)x^{(2m+1-j)}(t) - jx^{(2m-j)}(t)\right]x^{(j-1)}(t) - \frac{t-a}{2}\left|x^{(m)}(t)\right|^2.$$

Lemma 2.3 *Let the numbers* $a_1 \in]a,b[$, $t_{0,k} \in]a,a_1[$, *and* $\varepsilon_{i,k}, \varepsilon_i, \beta_k, \beta \in R^+$, $k \in N$, $i = m+1,\dots,n$, *be such that*

$$\lim_{k\to+\infty} t_{0,k} = a, \qquad \lim_{k\to+\infty}\beta_k = \beta, \qquad \lim_{k\to+\infty}\varepsilon_{i,k} = \varepsilon_i. \tag{2.3}$$

Let, moreover,

$$\lambda \in \widetilde{L}^2_{2n-2m-2}\big(]a,a_1]\big) \tag{2.4}$$

be a nonnegative function, $x_k \in \widetilde{C}^{n-1,m}(]a,a_1])$ *be a solution of the problem*

$$x^{(n)}(t) = \beta_k\lambda(t), \tag{2.5}$$

$$x^{(i-1)}(t_{0,k}) = 0 \quad (i=1,\dots,m), \qquad x^{(j-1)}(a_1) = \varepsilon_{j,k} \quad (j=m+1,\dots,n), \tag{2.6}$$

and $x \in \widetilde{C}^{n-1,m}(]a,a_1])$ *be a solution of the problem*

$$x^{(n)}(t) = \beta\lambda(t), \tag{2.7}$$

$$x^{(i-1)}(a) = 0 \quad (i=1,\dots,m), \qquad x^{(j-1)}(a_1) = \varepsilon_j \quad (j=m+1,\dots,n). \tag{2.8}$$

Then

$$\lim_{k\to+\infty} x_k^{(j-1)}(t) = x^{(j-1)}(t) \quad (j=1,\dots,n) \text{ uniformly in }]a,a_1]. \tag{2.9}$$

Proof First, let us prove our lemma under the assumption that there exists the number $r_1 > 0$ such that the estimates

$$\int_{t_{0,k}}^{a_1} \left|x_k^{(m)}(s)\right|^2 ds \le r_1, \quad k \in N \tag{2.10}$$

hold. Now, suppose that $t_1, \dots, t_n$ are such numbers that $t_{0k} < t_1 < \cdots < t_n < a_1$ $(k \in N)$, and g_i are the polynomials of $(n-1)$th degree satisfying the conditions $g_j(t_j) = 1$, $g_j(t_i) = 0$ $(i \neq j;\ i,j = 1, \dots, n)$. Then if x_k is a solution of problem (2.5), (2.6), and x is a solution of problem (2.7), (2.8), for the solution $x - x_k$ of the equation $\frac{d^n(x(t)-x_k(t))}{dt^n} = (\beta - \beta_k)\lambda(t)$, the representation

$$x(t) - x_k(t) = \sum_{j=1}^{n} \left(\big(x(t_j) - x_k(t_j)\big) - \frac{\beta - \beta_k}{(n-1)!} \int_{t_1}^{t_j} (t_j - s)^{n-1} \lambda(s)\, ds \right) g_j(t) + \frac{\beta - \beta_k}{(n-1)!} \int_{t_1}^{t} (t-s)^{n-1} \lambda(s)\, ds \quad k \in N \text{ for } t_{0,k} \le t \le a_1 \tag{2.11}$$

is valid. On the other hand, in view of inequality (2.10), the identities

$$x_k^{(i-1)}(t) = \frac{1}{(m-i)!} \int_{t_{0k}}^{t} (t-s)^{m-i} x_k^{(m)}(s)\, ds \quad (i = 1, 2, k \in N)$$

by Schwarz's inequality yield

$$\big| x_k^{(i-1)}(t) \big| \le r_2 (t-a)^{m-i-1/2} \quad \text{for } t_{0,k} \le t \le a_1 \ (i = 1, 2, k \in N), \tag{2.12}$$

where $r_2 = \frac{r_1}{(m-i)!\sqrt{2m-2i+1}}$. By virtue of the Arzela-Ascoli lemma and (2.12), the sequence $\{x_k\}_{k=1}^{+\infty}$ contains a subsequence $\{x_{k_l}\}_{l=1}^{+\infty}$ which is uniformly convergent in $]a, a_1]$. Suppose $\lim_{l \to +\infty} x_{k_l}(t) = x_0(t)$. Thus from (2.11) by (2.3) the existence of such $r_3 > 0$ that

$$\big| x_{k_l}^{(j-1)}(t) \big| \le r_3 + \big| x^{(j-1)}(t) \big| \quad (j = 1, \dots, n) \text{ for } t_{0,k_l} \le t \le a_1$$

follows, and then, without loss of generality, we can assume that

$$\lim_{l \to +\infty} x_{k_l}^{(j-1)}(t) = x_0^{(j-1)}(t) \quad (j = 1, \dots, n) \text{ uniformly in }]a, a_1]. \tag{2.13}$$

Then, by virtue of (2.3), (2.11) and (2.13), we have

$$x(t) - x_0(t) = \sum_{j=1}^{n} \big(\big(x(t_j) - x_0(t_j) \big) \big) g_j(t) \quad \text{for } a \le t \le a_1.$$

From the last two relations by (2.10) it is clear that $x^{(n)} = x_0^{(n)}$ and $x_0 \in \widetilde{C}^{n-1,m}(]a, a_1])$, *i.e.*, the function $x_0 \in \widetilde{C}^{n-1,m}(]a, a_1])$ is a solution of problem (2.7), (2.8). In view of (2.4) all the conditions of Theorem 1.1 are fulfilled, thus problem (2.7), (2.8) is uniquely solvable in the space $\widetilde{C}^{n-1,m}(]a, a_1])$ and $x = x_0$. Therefore from (2.13) it follows that

$$\lim_{l \to +\infty} x_{k_l}^{(j-1)}(t) = x^{(j-1)}(t) \quad (j = 1, \dots, n) \text{ uniformly in }]a, a_1]. \tag{2.14}$$

Now suppose that relations (2.9) are not fulfilled. Then there exist $\delta \in]0, \frac{a_1 - a}{2}[$, $\varepsilon > 0$, and the increasing sequence of natural numbers $\{k_l\}_{l=1}^{+\infty}$ such that

$$\max \left\{ \sum_{j=1}^{n} \big| x_{k_l}^{(j-1)}(t) - x^{(j-1)}(t) \big| : a + \delta \le t \le a_1 \right\} > \varepsilon \quad (l \in N). \tag{2.15}$$

By virtue of the Arzela-Ascoli lemma and condition (2.10), the sequence $\{x_{k_l}^{(j-1)}\}_{l=1}^{+\infty}$ ($j = 1,\ldots,m$), without loss of generality, can be assumed to be uniformly converging in $]a+\delta, a_1]$. Then, in view of what we have shown above, equality (2.14) holds. But this contradicts condition (2.15). Thus (2.9) holds if conditions (2.10) are fulfilled.

Now assume that conditions (2.10) are not fulfilled. Then there exists the subsequence $\{t_{0,k_l}\}_{l=1}^{+\infty}$ of the sequence $\{t_{0,k}\}_{k=1}^{+\infty}$ such that

$$\int_{t_{0,k}}^{a_1} \left|x_{k_l}^{(m)}(s)\right|^2 ds \geq l \quad (l \in N). \tag{2.16}$$

Suppose that $\widetilde{\beta}_l = (\int_{t_{0,k}}^{a_1} |x_{k_l}^{(m)}(s)|^2 ds)^{-1}$ and $v_l(t) = x_{k_l}(t)\widetilde{\beta}_l$. Thus in view of (2.16) and our notations,

$$\int_{t_{0,k_l}}^{a_1} \left|v_{k_l}^{(m)}(s)\right|^2 ds = 1 \quad (l \in N), \qquad \lim_{l\to+\infty} \widetilde{\beta}_l = 0, \tag{2.17}$$

$$v_l^{(n)}(t) = \beta_{k_l}\widetilde{\beta}_l\lambda(t), \tag{2.18}$$

$$\begin{aligned} &v_l^{(i-1)}(t_{0,k_l}) = 0 \quad (i = 1,\ldots,m), \\ &v_l^{(j-1)}(a_1) = \varepsilon_{j,k_l}\widetilde{\beta}_l \quad (j = m+1,\ldots,n, l \in N). \end{aligned} \tag{2.19}$$

From the first part of our lemma by (2.17) it follows that there exists limit $\lim_{l\to+\infty} v_l(t) \equiv v_0(t)$, and v_0 is a solution of the corresponding homogeneous problem (2.18), (2.19). Thus $v_0 \equiv 0$. On the other hand, from (2.17) it is clear that $\int_{t_{0,k_l}}^{a_1} |v_0^{(m)}(s)|^2 ds = 1$, which contradicts with $v_0 \equiv 0$. Thus our assumption is invalid and (2.10) holds. □

Lemma 2.4 *Let $a < a_1 < b$, $\varepsilon_j \in R^+$ and $\lambda \in \widetilde{L}^2_{2n-2m-2}(]a,a_1])$ be a nonnegative function. Then, for the solution $x \in \widetilde{C}^{n-1,m}(]a,a_1])$ of problem* (2.7), (2.8) *with $\beta = 1$, the estimate*

$$\int_a^{a_1} \left|x^{(m)}(s)\right|^2 ds \leq \Theta_1(x,a_1,\lambda) \quad (k \in N) \tag{2.20}$$

is valid, where

$$\Theta_1(x,a_1,\lambda) = 2\left|w_n(x)(a_1)\right| + \left(\frac{2^{m-1}(2m+1)}{(2m-1)!!}\right)^2 \|\lambda\|^2_{\widetilde{L}^2_{2n-2m-2}(]a,a_1])}. \tag{2.21}$$

Proof Suppose that x_k is a solution of problem (2.5), (2.6) with $\beta_k = 1$, $\varepsilon_{j,k} = \varepsilon_j$. Then, in view of Lemma 2.3, relations (2.9) hold. On the other hand, by Lemma 2.2 we get

$$\nu_n \int_{t_{0,k}}^{a_1} \left|x_k^{(m)}(s)\right|^2 ds \leq -w_n(x_k)(a_1) + \int_{t_{0,k}}^{a_1} (s-a)^{n-2m}\lambda(s)\left|x_k(s)\right| ds. \tag{2.22}$$

Now, on the basis of Lemma 2.1, Schwarz's and Young's inequalities, we get

$$\begin{aligned} &\left|\int_{t_{0,k}}^{a_1} (s-a)^{n-2m}\lambda(s)x_k(s)\, ds\right| \\ &\quad = \left|\int_{t_{0,k}}^{a_1} \left[(n-2m)x_k(s) + (s-a)^{n-2m}x_k'(s)\right]\left(\int_s^{a_1} \lambda(\xi)\, d\xi\right) ds\right| \end{aligned}$$

$$\leq \left[(n-2m)\left(\int_{t_{0,k}}^{a_1} \frac{x_k^2(s)}{(s-a)^{2m}}\,ds\right)^{1/2} + \left(\int_{t_{0,k}}^{a_1} \frac{x_k'^2(s)}{(s-a)^{2m-2}}\,ds\right)^{1/2}\right] \|\lambda\|_{\widetilde{L}_{2n-2m-2}(]a,a_1])}$$
$$\leq \frac{2^{m-1}(2m+1)}{(2m-1)!!}\left(\int_{t_{0k}}^{a_1} \left|x_k^{(m)}(s)\right|^2 ds\right)^{1/2} \|\lambda\|_{\widetilde{L}_{2n-2m-2}(]a,a_1])}$$
$$\leq \frac{1}{2}\int_{t_{0,k}}^{a_1} \left|x_k^{(m)}(s)\right|^2 ds + \frac{1}{2}\left(\frac{2^{m-1}(2m+1)}{(2m-1)!!}\right)^2 \|\lambda\|^2_{\widetilde{L}_{2n-2m-2}(]a,a_1])}.$$

Thus from (2.22) by the definition of numbers ν_n it immediately follows that the estimate

$$\int_{t_{0,k}}^{a_1} \left|x_k^{(m)}(s)\right| ds \leq 2\left|w_n(x_k)(a_1)\right| + \left(\frac{2^{m-1}(2m+1)}{(2m-1)!!}\right)^2 \|\lambda\|^2_{\widetilde{L}_{2n-2m-2}(]a,a_1])} \quad (k \in N).$$

By means of (2.9), from the last inequality, (2.20) and (2.21) follow. □

2.2 Lemmas on the Banach space $\widetilde{C}_1^{m-1}(]a,b])$

Definition 2.1 Let $\rho \in R^+$ and the function $\eta \in L_{\mathrm{loc}}(]a,b])$ be nonnegative. Then $S(\rho,\eta)$ is a set of such $y \in C_{\mathrm{loc}}^{n-1}(]a,b])$ that

$$\left|y^{(i-1)}\left(\frac{a+b}{2}\right)\right| \leq \rho \quad (i=1,\dots,n), \tag{2.23}$$

$$\left|y^{(n-1)}(t) - y^{(n-1)}(s)\right| \leq \int_s^t \eta(\xi)\,d\xi \quad \text{for } a < s \leq t \leq b, \tag{2.24}$$

and

$$y^{(i-1)}(a) = 0 \quad (i=1,\dots,m), \qquad y^{(j-1)}(b) = 0 \quad (j=m+1,\dots,n). \tag{2.25}$$

Lemma 2.5 *Let, for the function* $y \in \widetilde{C}^{n-1,m}(]a,b])$, *conditions* (2.25) *be satisfied. Then* $y \in \widetilde{C}_1^{m-1}(]a,b])$ *and the estimates*

$$\left|y^{(i-1)}(t)\right| \leq \frac{(t-a)^{m-i+1/2}}{(m-i)!(2m-2i+1)^{1/2}}\left(\int_a^t \left|y^{(m)}(s)\right|^2 ds\right)^{1/2}, \tag{2.26}$$

$a < t \leq b$, $i = 1,\dots,m$.

Proof First note that in view of inclusion $y \in \widetilde{C}^{n-1,m}(]a,b])$, the equality

$$y^{(i-1)}(t) = \sum_{j=i}^{l} \frac{(t-c)^{j-i}}{(j-i)!} y^{(j-1)}(c) + \frac{1}{(l-i)!}\int_c^t (t-s)^{l-i} y^{(l)}(s)\,ds \tag{2.27}$$

on $[a,b]$, for $i=1,\dots,l$, $l=1,\dots,n$, holds, where: (1) $c \in [a,b]$ if $l \leq m$; (2) $c \in]a,b]$ if $l > m$; and there exists $r > 0$ such that

$$\int_a^b \left|y^{(m)}(s)\right|^2 ds \leq r. \tag{2.28}$$

Equality (2.27), with $l = m$, $c = a$, by conditions (2.25), (2.28) and Schwarz's inequality yields (2.26). From (2.26) and (2.28) it is clear that $y \in \widetilde{C}_1^{m}(]a,b])$. □

Lemma 2.6 *Let $\rho \in R^+$, and let $\eta \in \widetilde{L}^2_{2n-2m-2}(]a,b])$ be a nonnegative function. Then $S(\rho,\eta)$ is a compact subset of the space $\widetilde{C}^{m-1}_1(]a,b])$.*

Proof Condition (2.24) yields the inequality $|y^{(n)}(t)| \leq \eta(t)$. Thus there exists such function $\eta_1 \in \widetilde{L}^2_{2n-2m-2}(]a,b])$ that

$$y^{(n)}(t) = \eta_1(t), \quad \text{for } a < t \leq b, \tag{2.29}$$

where

$$\big|\eta_1(t)\big| \leq \eta(t) \quad \text{for } a < t \leq b. \tag{2.30}$$

From Theorem 1.1 it follows that problem (2.29), (2.25) has the unique solution $y \in C^{n-1,m}(]a,b])$, *i.e.*, there exists $r > 0$ such that inequality (2.28) holds.

For any $y \in S(\rho,\eta)$, from equality (2.27) with $l = n$, by (2.23), (2.29) and (2.30), we get

$$\big|y^{(i-1)}(t)\big| \leq \gamma_i(t) \quad \text{for } a < t < b \ (i = 1,\dots,n), \tag{2.31}$$

where $\gamma_i(t) = \rho_i + \frac{1}{(n-i)!}|\int_c^t (t-s)^{n-i}\eta(s)\,ds|$ $(i = 1,\dots,n)$, and $\rho_i \in R^+$. Let now $y_k \in S(\rho,\eta)$ $(k \in N)$. By virtue of the Arzela-Ascoli lemma and conditions (2.24), (2.31), the sequence $\{y_k\}_{k=1}^{+\infty}$ contains a subsequence $\{y_{k_\ell}\}_{\ell=1}^{+\infty}$ such that $\{y_{k_\ell}^{(i-1)}\}_{\ell=1}^{+\infty}$ $(i = 1,\dots,n)$ are uniformly convergent on $]a,b]$. Thus, without loss of generality, we can assume that $\{y_k^{(i-1)}\}_{k=1}^{+\infty}$ $(i = 1,\dots,n)$ are uniformly convergent on $]a,b]$. Let $\lim_{k\to+\infty} y_k(t) = y_0(t)$, then $y_0 \in \widetilde{C}^{n-1}_{\mathrm{loc}}(]a,b])$ and

$$\lim_{k\to+\infty} y_k^{(i-1)}(t) = y_0^{(i-1)}(t) \quad (i = 1,\dots,n) \text{ uniformly on }]a,b]. \tag{2.32}$$

From (2.32), in view of the inclusions $y_k \in S(\rho,\eta)$, it immediately follows that

$$\left|y_0^{(i-1)}\left(\frac{a+b}{2}\right)\right| \leq \rho \quad (i = 1,\dots,n), \tag{2.33}$$

$$y_0^{(i-1)}(a) = 0 \quad (j = 1,\dots,m), \qquad y_0^{(j-1)}(b) = 0 \quad (j = m+1,\dots,n), \tag{2.34}$$

and

$$\big|y_0^{(n-1)}(t) - y_0^{(n-1)}(s)\big| \leq \int_s^t \eta(\xi)\,d\xi \quad \text{for } a < s \leq t \leq b. \tag{2.35}$$

From (2.33)-(2.35) it is clear that $y_0 \in S(\rho,\eta)$. To finish the proof, we must show that

$$\lim_{k\to+\infty} \big\|y_k(t) - y_0(t)\big\|_{\widetilde{C}^{m-1}_1} = 0 \tag{2.36}$$

and

$$S(\rho,\eta) \subset \widetilde{C}^{m-1}_1\big(]a,b]\big). \tag{2.37}$$

Let $x_k = y_0 - y_k$ and $a_1 \in]a,b]$. Then it is clear that $x_k \in S(\rho',\eta')$, where $\rho' = 2\rho$, $\eta' = 2\eta$. Thus, for any x_k, there exists $\eta_k \in \widetilde{L}^2_{2n-2m-2}(]a,b])$ such that

$$x_k^{(n)}(t) = \eta_k(t), \tag{2.38}$$

$$x_k^{(i-1)}(a) = 0 \quad (i = 1, \dots, m), \qquad x_k^{(j-1)}(b) = 0 \quad (j = m+1, \dots, n), \tag{2.39}$$

where

$$\left|\eta_k(t)\right| \le 2\eta(t) \quad \text{for } a < t \le b \ (k \in N). \tag{2.40}$$

On the other hand, from (2.26) with $y = x_k$, in view of (2.39) we get

$$\left|x_k^{(i-1)}(t)\right| \le \left(\int_a^t \left|x_k^{(m)}(s)\right|^2 ds\right)^{1/2} (t-a)^{m-i+1/2} \quad \text{for } a < t < a_1, \tag{2.41}$$

for $i = 1, \dots, m$.

Let now w_n be the operator defined in Lemma 2.2 and Θ_1 be a function defined by (2.21) with $\lambda = \eta_k$. Then conditions (2.32) yield

$$\lim_{k\to+\infty} w_n(x_k)(a_1) = 0 \quad (k \in N), \tag{2.42}$$

and from the definition of the norm $\|\cdot\|_{\widetilde{L}^2_\alpha}$, (2.40) and (2.42) it follows that for any $\varepsilon > 0$, we can choose $a_1 \in]a, \min\{a+1, b\}[$ and $k_0 \in N$ such that

$$\Theta_1(x_k, a_1, 2\eta) \le \frac{\varepsilon^2}{16m^2} \quad (k \ge k_0). \tag{2.43}$$

By using Lemma 2.4 for x_k, in view of (2.43) and (2.41), we get respectively

$$\int_a^{a_1} \left|x_k^{(m)}(s)\right|^2 ds \le \frac{\varepsilon^2}{16m^2} \quad (k \ge k_0) \tag{2.44}$$

and

$$\frac{|x_k^{(i-1)}(t)|}{(t-a)^{m-i-1/2}} \le \frac{\varepsilon}{4m} \quad \text{for } t \in]a, a_1] \ (1 \le i \le m, k \ge k_0). \tag{2.45}$$

Also, in view of (2.32), without loss of generality, we can assume that

$$\frac{|x_k^{(i-1)}(t)|}{(t-a)^{m-i-1/2}} \le \frac{\varepsilon}{4m} \quad \text{for } a_1 \le t \le b \ (1 \le i \le m, k \ge k_0), \tag{2.46}$$

and

$$\int_{a_1}^{b} \left|x_k^{(m)}(s)\right|^2 ds \le \frac{\varepsilon^2(4m^2-1)}{16m^2} \quad (k \ge k_0). \tag{2.47}$$

From (2.44)-(2.47), equality (2.36) immediately follows.

Let now $y \in S(\rho, \eta)$ and $y_k = \delta_k y$, where $\lim_{k\to+\infty} \delta_k = 0$. Then by (2.32) it is clear that $y_0 \equiv 0$, and then from (2.36) it follows that $y \in \widetilde{C}_1^{m-1}(]a, b])$, *i.e.*, the inclusion (2.37) holds. □

Now we introduce one lemma which is proved in [33].

Lemma 2.7 *Let $\tau \in M(]a,b])$, $\alpha, \beta, k_1 \geq 0$, and let there exist $\delta \in]0, b-a[$ such that*

$$\left|\tau(t) - t\right| \leq k_1 (t-a)^\beta \quad \text{for } a < t \leq a + \delta. \tag{2.48}$$

Then

$$\left|\int_t^{\tau(t)} (s-a)^\alpha \, ds\right| \leq \begin{cases} k_1[1 + k_1\delta^{\beta-1}]^\alpha (t-a)^{\alpha+\beta} & \text{for } \beta \geq 1, \\ k_1[\delta^{1-\beta} + k_1]^\alpha (t-a)^{\alpha\beta+\beta} & \text{for } 0 \leq \beta < 1, \end{cases}$$

for $a < t \leq a + \delta$.

2.3 Lemmas on the solutions of auxiliary problems

Throughout this section we assume that the operator $P : C_1^{m-1}(]a,b]) \times C_1^{m-1}(]a,b]) \to L_n(]a,b])$ is γ_0, γ consistent with boundary condition (1.2), and the operator $q : C_1^{m-1}(]a,b]) \to \widetilde{L}^2_{2n-2m-2}(]a,b])$ is continuous.

Consider, for any $x \in \widetilde{C}_1^{m-1}(]a,b]) \subset C_1^{m-1}(]a,b])$, the nonhomogeneous equation

$$y^{(n)}(t) = \sum_{i=1}^m p_i(x)(t) y^{(i-1)}\big(\tau_i(t)\big) + q(x)(t) \tag{2.49}$$

and the corresponding homogeneous equation

$$y^{(n)}(t) = \sum_{i=1}^m p_i(x)(t) y^{(i-1)}\big(\tau_i(t)\big), \tag{2.50}$$

and let E^n be a set of the solutions of problem (2.49), (2.25).

From inequality (1.17) of item (ii) of Definition 1.1, it follows that for any $x \in A_{\gamma_0}$ boundary problem (2.49), (2.25) has the unique solution y in the space $\widetilde{C}^{n-1,m}(]a,b])$ such that $y \in \widetilde{C}_1^{m-1}(]a,b])$. Thus $E^n \cap \widetilde{C}_1^{m-1}(]a,b]) \neq \emptyset$, and there exists the operator $U : A_{\gamma_0} \to E^n \cap \widetilde{C}_1^{m-1}(]a,b])$ defined by the equality

$$U(x)(t) = y(t).$$

Lemma 2.8 *$U : A_{\gamma_0} \to E^n \cap \widetilde{C}_1^{m-1}(]a,b])$ is a continuous operator.*

Proof Let $x_k \in A_{\gamma_0}$ and $y_k(t) = U(x_k)(t)$ $(k = 1,2)$, $y = y_2 - y_1$, and let the operator P be defined by (1.13). Then

$$y^{(n)}(t) = P(x_2, y)(t) + q_0(x_1, x_2)(t),$$

where $q_0(x_1,x_2)(t) = P(x_2,y_1)(t) - P(x_1,y_1)(t) + q(x_2)(t) - q(x_1)(t)$. Hence, by item (ii) of Definition 1.1 we have

$$\big\|U(x_2) - U(x_1)\big\|_{\widetilde{C}_1^{m-1}} \leq \gamma \big\|q_0(x_1,x_2)\big\|_{\widetilde{L}^2_{2n-2m-2}}.$$

Since the operators P and q are continuous, this estimate implies the continuity of the operator U. □

3 Proofs

Proof of Remark 1.2 Let x be a solution of problem (1.6), (1.2), then from inequalities (2.26) (with $y = x$), by the definition of the norm in the space $\widetilde{C}_1^{m-1}(]a,b])$ and estimate (1.11), estimate (1.12) immediately follows. □

Proof of Theorem 1.3 Let δ and λ be the functions and numbers appearing in Definition 1.1. We set

$$\eta(t) = \delta(t,\gamma_0)\gamma_0 + \widetilde{F}_p\bigl(t, \min\{2\rho_0,\gamma_0\}\bigr), \tag{3.1}$$

$$\chi(s) = \begin{cases} 1 & \text{for } 0 \le s \le \rho_0, \\ 2 - s/\rho_0 & \text{for } \rho_0 < s < 2\rho_0, \\ 0 & \text{for } s \ge 2\rho_0, \end{cases} \tag{3.2}$$

$$q(x)(t) = \chi\left(\|x\|_{\widetilde{C}_1^{m-1}}\right)F_p(x)(t). \tag{3.3}$$

From (1.18) it is clear that the nonnegative functions $\widetilde{F}_p$, η admit the inclusion

$$\widetilde{F}_p\bigl(\cdot, \min\{2\rho_0,\gamma_0\}\bigr), \eta \in \widetilde{L}^2_{2n-2m-2}\bigl(]a,b]\bigr), \tag{3.4}$$

and for every $x \in A_{\gamma_0} \subset \widetilde{C}_1^{m-1}(]a,b])$ and almost all $t \in]a,b]$, the inequality

$$\bigl|q(x)(t)\bigr| \le \widetilde{F}_p\bigl(t, \min\{2\rho_0,\gamma_0\}\bigr) \quad \text{for } a < t \le b \tag{3.5}$$

holds.

Let $U : A_{\gamma_0} \to E^n \cap \widetilde{C}_1^{m-1}(]a,b])$ be the operator in Lemma 2.8, from which it follows that U is a continuous operator. On the other hand, from items (i) and (ii) of Definition 1.1, (1.19) and (3.5), it is clear that for each $x \in A_{\gamma_0}$, the conditions

$$\|y\|_{\widetilde{C}_1^{m-1}} \le \gamma_0, \quad \bigl|y^{(n-1)}(t) - y^{(n-1)}(s)\bigr| \le \int_s^t \eta(\xi)\,d\xi \quad \text{for } a < t < b$$

hold. Thus, in view of Definition 2.1, the operator U maps the ball A_{γ_0} into its own subset $S(\rho_1,\eta)$. From Lemma 2.2 it follows that $S(\rho_1,\eta)$ is a compact subset of the ball $A_{\gamma_0} \subset \widetilde{C}_1^{m-1}(]a,b])$, *i.e.*, the operator u maps the ball A_{γ_0} into its own compact subset. Therefore, owing to Schauder's principle, there exists $x \in S(\rho_1,\eta) \subset A_{\gamma_0}$ such that

$$x(t) = U(x)(t) \quad \text{for } a < t \le b.$$

Thus by (2.49) and notation (3.3), the function x ($x \in A_{\gamma_0}$) is a solution of problem (1.20), (1.2), where

$$\lambda = \chi\left(\|x\|_{\widetilde{C}_1^{m-1}}\right). \tag{3.6}$$

If $\gamma_0 = \rho_0$, then in view of the condition $x \in A_{\gamma_0}$, by (3.2) we have that $\lambda = 1$, and then, in view of (2.49) and (3.3), the function x is a solution of problem (1.1), (1.2) which admits estimate (1.21).

Let us show now that x admits estimate (1.21) in the case when $\rho_0 < \gamma_0$. Assume the contrary. Then either

$$\rho_0 < \|x\|_{\widetilde{C}_1^{m-1}} < 2\rho_0, \tag{3.7}$$

or

$$\|x\|_{\widetilde{C}_1^{m-1}} \geq 2\rho_0. \tag{3.8}$$

If condition (3.7) holds, then by virtue of (3.2) and (3.6) we have that $\lambda \in]0,1[$, which by the conditions of our theorem guarantees the validity of estimate (1.21). But this contradicts (3.7).

Assume now that (3.8) is fulfilled. Then, by virtue of (3.2) and (3.6), we have that $\lambda = 0$. Therefore $x \in A_{\gamma_0}$ is a solution of problem (2.50), (1.2). Thus from item (ii) of Definition 1.1 it is obvious that $x \equiv 0$, because problem (2.50), (1.2) has only a trivial solution. But this contradicts condition (3.8), *i.e.*, estimate (1.21) is valid. From estimates (1.21) and (3.2) we have that $\lambda = 1$, and then in view of (2.49) and (3.3) the function x is a solution of problem (1.1), (1.2) which admits estimate (1.21). □

Proof of Corollary 1.2 First note that in view of condition (1.24) there exists such $\gamma_0 > 2\rho_0$ that condition (1.19) holds, and in view of Definition 1.2 the operator P is γ_0, γ consistent.

On the other hand, from (1.24) it follows the existence of the number ρ_0 such that

$$\gamma \left\|\eta(\cdot,\rho)\right\|_{\widetilde{L}^2_{2n-2m-2}} < \rho \quad \text{for } \rho > \rho_0. \tag{3.9}$$

Let x be a solution of problem (1.20), (1.2) for some $\lambda \in]0,1[$. Then $y = x$ is also a solution of problem (1.16), (1.2) where $q(t) = \lambda(F(x)(t) - P(x,x)(t))$. Let now $\rho = \|x\|_{\widetilde{C}_1^{m-1}}$ and assume that

$$\rho > \rho_0 \tag{3.10}$$

holds. Then in view of the γ-consistency of the operator p with boundary conditions (1.2), inequality (1.17) holds and thus by condition (1.22) we have

$$\rho = \|x\|_{\widetilde{C}_1^{m-1}} \leq \gamma \left\|q(x)\right\|_{\widetilde{L}^2_{2n-2m-2}} \leq \gamma \left\|\eta(\cdot,\rho)\right\|_{\widetilde{L}^2_{2n-2m-2}}.$$

But the last inequality contradicts (3.9). Thus assumption (3.10) is not valid and $\rho \leq \rho_0$. Therefore, for any $\lambda \in]0,1[$, an arbitrary solution of problem (1.20), (1.2) admits estimate (1.21). Therefore all the conditions of Theorem 1.3 are fulfilled, from which the solvability of problem (1.1), (1.2) follows. □

Proof of Theorem 1.4 Let r_n be the constant defined in Remark 1.2. First prove that the operator P is γ_0, r_n consistent with boundary conditions (1.2). From the conditions of our theorem it is obvious that item (i) of Definition 1.1 is satisfied. Let now x be an arbitrary fixed function from the set A_{γ_0}, and let $p_j(t) \equiv p_j(x)(t)$. Thus, in view of (1.27) and (1.9), all the assumptions of Theorem 1.1 are satisfied, and then for any $q \in \widetilde{L}^2_{2n-2m-2}(]a,b])$ problem

(1.16), (1.2) has the unique solution y. Also in view of Remark 1.2 there exists the constant $r_n > 0$ (which depends only on the numbers l_j, $\bar{l}_j$, γ_j ($j = 1,\dots,m$), and a, b, n) such that estimate (1.17) holds with $\gamma = r_n$, *i.e.*, the operator P is γ_0, r_n consistent with boundary conditions (1.2). Therefore all the assumptions of Corollary 1.1 are fulfilled, from which the solvability of problem (1.1), (1.2) follows. □

Proof of Theorem 1.5 Let r_n be the constant defined in Remark 1.2. First prove that the operator P is r_n consistent with boundary conditions (1.2). From the conditions of our theorem it is obvious that item (i) of Definition 1.1 is satisfied. Let now γ_0 be an arbitrary nonnegative number, x be an arbitrary fixed function from the space A_{γ_0}, and let $p_j(t) \equiv p_j(x)(t)$. Then in view of (1.29) and (1.9) all the assumptions of Theorem 1.1 are satisfied, and then for any $q \in \widetilde{L}^2_{2n-2m-2}(]a,b])$ problem (1.16), (1.2) has the unique solution y. Also in view of Remark 1.2 there exists the constant $r_n > 0$ (which depends only on the numbers l_j, $\bar{l}_j$, γ_j ($j = 1,\dots,m$), and a, b, n) such that estimate (1.17) holds with $\gamma = r_n$, *i.e.*, the operator P is γ_0, r_n consistent with boundary conditions (1.2) for arbitrary $\gamma_0 > 0$. Thus by Definition 1.1, the operator P is r_n consistent with boundary conditions (1.2). Therefore all the assumptions of Corollary 1.2 are fulfilled, from which the solvability of problem (1.1), (1.2) follows. □

Proof of Remark 1.3 By Schwarz's inequality, the definition of the norm $\|y\|_{\widetilde{C}_1^{m-1}}$ and inequalities (2.2) for any $x, y \in A_{\gamma_0}$ and $z = y - x$, we have

$$\begin{aligned}
&\left|p_j(y)(t)z^{(j-1)}\big(\tau_j(t)\big)\right| \\
&\quad = \left|p_j(y)(t)z^{(j-1)}(t)\right| + \left|p_j(y)(t)\right|\left|\int_t^{\tau_j(t)} z^{(j)}(\psi)\,d\psi\right| \\
&\quad \le \|z\|_{\widetilde{C}_1^{m-1}}\left|p_j(y)(t)\right|\alpha_j(t)\left(1 + \frac{2}{\alpha_j(t)}\left(\int_t^{\tau_j(t)} (\psi - a)^{2m-2j}\,d\psi\right)^{1/2}\right)
\end{aligned} \tag{3.11}$$

for $a < t \le b$. On the other hand, from conditions (1.31) by Lemma 2.7 it is clear that

$$\begin{aligned}
&\alpha_j^{-1}(s)\left(\int_s^{\tau_j(s)} (\xi - a)^{2m-2j}\,d\xi\right)^{1/2} \le \sqrt{\kappa}(1+\kappa)^m \quad \text{for } s \in]a, a+\varepsilon], \\
&\alpha_j^{-1}(s)\left(\int_s^{\tau_j(s)} (\xi - a)^{2m-2j}\,d\xi\right)^{1/2} \le \varepsilon^{-m+j-1/2}\left(\int_a^b (\xi - a)^{2m-2j}\,d\xi\right)^{1/2} \\
&\qquad = \frac{(b-a)^{m-j+1/2}}{\sqrt{2m-2j+1}\,\varepsilon^{m-j+1/2}} \quad \text{for } s \in]a+\varepsilon, b].
\end{aligned}$$

Then if we put

$$\kappa_0 = \max_{1 \le j \le m}\left\{\sqrt{\kappa}(1+\kappa)^m, \frac{(b-a)^{m-j+1/2}}{\sqrt{2m-2j+1}\,\varepsilon^{m-j+1/2}}\right\}, \tag{3.12}$$

from (3.11) by the last estimates and (1.30), we get the inequality

$$\begin{aligned}
\left|p_j(y)(t)z^{(j-1)}\big(\tau_j(t)\big)\right| &\le \|z\|_{\widetilde{C}_1^{m-1}}(1+\kappa_0)\left|p_j(y)(t)\right|\alpha_j(t) \\
&\le \|z\|_{\widetilde{C}_1^{m-1}}(1+\kappa_0)\delta_j\big(t, \|y\|_{\widetilde{C}_1^{m-1}}\big)
\end{aligned} \tag{3.13}$$

for $a < t \leq b$. Analogously, we get that

$$\left|\big(p_j(y)(t) - p_j(x)(t)\big)x^{(j-1)}\big(\tau_j(t)\big)\right| \leq \|x\|_{\widetilde{C}_1^{m-1}}(1+\kappa_0)\left|p_j(y)(t) - p_j(x)(t)\right|\alpha_j(t)$$

for $a < t \leq b$. From (3.13) and the last inequality it is obvious that the operator P defined by equality (1.13) continuously acts from A_{γ_0} to the space $L_n(]a,b])$, and item (ii) of Definition 1.1 holds with $\delta(t,\rho) = (1+\kappa_0)\sum_{j=1}^m \delta_j(t,\rho)$. □

Proof of Corollary 1.3 From conditions (1.33) and (1.31), by Remark 1.3, we obtain that the operator P defined by equality (1.13) with $p_j(x)(t) = p_j(t)$ continuously acts from A_{γ_0} to the space $L_n(]a,b])$ for any $\gamma_0 > 0$, *i.e.*, continuously acts from $\widetilde{C}_1^{m-1}(]a,b])$ to the space $L_n(]a,b])$.

Therefore it is clear that all the conditions of Theorem 1.5 would be satisfied with

$$F(x)(t) = f\big(t, x\big(\tau_1(t)\big), x'\big(\tau_2(t)\big), \dots, x^{(m-1)}\big(\tau_m(t)\big)\big), \qquad \delta(t,\rho) = (1+\kappa_0)\sum_{j=1}^m \left|p_j(t)\right|,$$

where the constant κ_0 is defined by equality (3.12). Thus problem (1.32), (1.2) is solvable. □

Proof of Corollary 1.4 Let the operators $F, p_1 : C^{m-1}(]a,b]) \to L_{\mathrm{loc}}(]a,b])$, and the function $\eta :]a,b] \times R^+ \to R^+$ be defined by the equalities

$$F(x)(t) = -\frac{\lambda|x(t)|^k}{(t-a)^{2+k/2}}x\big(\tau(t)\big) + q(x)(t), \qquad p_1(x)(t) = -\frac{\lambda|x(t)|^k}{(t-a)^{2+k/2}}$$

and

$$\begin{aligned} &\delta(t,\rho) = \lambda\frac{(\tau(t)-a)^{1/2}\rho^k}{(t-a)^2}, \qquad l_1 = \gamma_0^k\lambda, \qquad \bar{l}_1 = 4\gamma_0^k\lambda, \\ &r_2 = \frac{2}{1-4\lambda\gamma_0^k(1+[4(b-a)]^{1/4})}, \qquad B = 4\lambda\gamma_0^k\big(1+\big[4(b-a)\big]^{1/4}\big), \qquad \gamma_1 = \frac{1}{4}. \end{aligned} \tag{3.14}$$

Then it is easy to verify that, in view of (1.36)-(1.38), conditions (1.9), (1.15), (1.22), (1.27), (1.28) are satisfied and $\delta \in D_n(]a,b] \times R^+)$.

Thus all the condition of Theorem 1.4 are satisfied, from which the solvability of problem (1.34), (1.2) follows. □

Proof of Corollary 1.5 Let the operators $F, p_1 : C^{m-1}(]a,b]) \to L_{\mathrm{loc}}(]a,b])$ and the function $\eta :]a,b] \times R^+ \to R^+$ be defined by the equalities

$$F(x)(t) = -\frac{\lambda|\sin x^k(t)|}{(t-a)^2}x\big(\tau(t)\big) + q(x)(t), \qquad p_1(x)(t) = -\frac{\lambda|\sin x^k(t)|}{(t-a)^2}.$$

Then it is easy to verify that, in view of (1.24), (1.36) and (1.39), all the conditions of Theorem 1.5 follow, where δ, l_1, $\bar{l}_1$, r_2, B, γ_1 are defined by (3.14) with $\rho = 1$, $\gamma_0 = 1$, from which the solvability of problem (1.34), (1.2) follows. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed to each part of this work equally and read and approved the final manuscript.

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