

# Laguerre Polynomials' Scheme of Transient Analysis: Scale Factor and Number of Temporal Basis Functions

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**Abstract.** *The paper is focused on two important parameters of the scheme with weighted Laguerre polynomials: the scale factor of time axis and the number of temporal basis functions. In the first part of the paper, the approach for the determination of the number of temporal basis functions is proposed. Its number is determined during the run of the scheme.*

*In the second part of the paper, the influence of the choice of the scale factor on the efficiency of the scheme, with the possibility of its optimum choice, is investigated. The investigations show that a chosen scale factor strongly influences the efficiency of the scheme. If the scale factor is not chosen close to the optimum one, the scheme becomes time-consuming. However, a simple formula for its prediction can not be given. The choice of the scale factor seems to be the weak point of the scheme with Laguerre polynomials.*

## Keywords

Time domain analysis, method of moments, Laguerre polynomials.

## 1. Introduction

In recent years, the marching-on-in-degree (MOD) method [1]-[4] has been proposed for the solution of the time domain electric field integral equations (TD-EFIE). The method does not suffer from the late-time instability, in comparison with the classical marching-on-in-time (MOT) method [5]-[7].

At the MOD approach, weighted Laguerre polynomials are used as temporal basis and testing functions. There are five characteristic properties of the weighted Laguerre polynomials: they are (1) causal, (2) recursively computed, (3) orthogonal, (4) separating space and time variables, and (5) convergent. Thanks to those properties, the unconditional stable scheme is obtained. Therefore, a temporal response is approximated by a set of weighted Laguerre polynomials.

In order to use the Laguerre polynomial scheme, the scale factor of the time axis  $s$  and the number of temporal basis functions  $N_L$  needed for the sufficient approximation of the transient response have to be determined.

An experience with a choice of the scale factor for the Gaussian impulse as an excitation signal was described in [3]. However, a deeper investigation of the influence of the scale factor on the efficiency of the scheme has not been performed, and a potential choice of an optimum scale factor has not been examined.

In [1] and [2], the expression for the determination of the minimum number of temporal basis functions  $N_L$  was published. This one depends on the width of the frequency band of the excitation signal, and the length of the response. However, the length of the response is an unknown quantity in case we are interested in the steady state. In addition, the expression does not take into account the scale factor of the time axis. Thus, using the expression in [1], [2] is almost inapplicable.

In this paper, the approach for the determination of the number of temporal basis functions is proposed, and the influence of the choice of the scale factor (potentially the optimum one) on the efficiency of the MOD algorithm is investigated for a strip symmetric dipole.

The paper is organized as follows. Section 2 presents a formulation of a scheme with weighted Laguerre polynomials. Section 3 discusses and proposes an approach for the determination of the number of temporal basis functions. In Section 4, the influence of the choice of the scale factor on the efficiency of the scheme with weighted Laguerre polynomials is investigated, and the possibility of the optimum choice of the scale factor is discussed. Section 5 concludes the paper.

## 2. Formulation with Weighted Laguerre Polynomials

Let  $S$  denote the surface of a closed or open perfectly electric conducting (PEC) body illuminated by a transient electromagnetic wave. This incident wave induces a sur-

face current  $\mathbf{J}(\mathbf{r}, t)$  on  $S$ . For the scattered electric field  $\mathbf{E}^S(\mathbf{r}, t)$  computed from the surface current, we can write

$$\begin{aligned} & \frac{\mu}{4\pi} \int_S \frac{\partial J(r', t - R/c)}{\partial t} \frac{1}{R} dS' - \\ & - \frac{\nabla}{4\pi\epsilon} \iint \partial J(r', t - R/c) dt \frac{1}{R} dS' = -E^S(r, t) \end{aligned} \quad (1)$$

The permittivity and permeability of the surrounding medium are  $\mu$  and  $\epsilon$ , respectively,  $R=|\mathbf{r}-\mathbf{r}'|$  is the distance between an arbitrarily located observation point  $\mathbf{r}$  and source point  $\mathbf{r}'$  on  $S$ , and  $\tau = t-R/c$  is the retarded time. The velocity of propagation in the surrounding medium is  $c=(\mu\epsilon)^{-1/2}$ . Since the total tangential electric field is zero on the conducting surface for all times, we can write

$$[\mathbf{E}^i + \mathbf{E}^S]_{\text{tan}} = 0, \quad r \in S \quad (2)$$

where  $\mathbf{E}^i$  is the incident electric field on the scatterer and the subscript "tan" denotes a tangential component.

The equation (2) is solved by the method of moments (MoM) [2], [3]. The analyzed structure is approximated by planar triangular patches, and the RWG functions [8] are used to expand the spatial variation of the electric current density. The unknown current density  $\mathbf{J}(\mathbf{r}, t)$  is expanded with an unknown current coefficient  $I_{n,u}$ , a weighted Laguerre polynomial  $\varphi_u$  (of the order  $u$ ) and a spacial basis function  $\mathbf{f}_n$

$$\mathbf{J}(r, t) = \sum_{n=1}^N I_n(t) \mathbf{f}_n(r), \quad (3a)$$

$$I_n(t) = \sum_{u=0}^{\infty} I_{n,u} \varphi_u(ts) \quad (3b)$$

where  $N$  is the number of non-boundary edges [8] and  $s$  is the scale factor of the time axis. Substituting (2) and (3) to (1) and performing the testing procedure in the time and space, the unknown current coefficients can be computed from the following matrix equation

$$[\alpha_{mn}] [I_{n,v}] = [V_m^v] - [\beta_m^{v-1}], \quad v = 0, 1, 2, \dots, N_L \quad (4)$$

where

$$V_m^v = \int_0^{\infty} V_m(t) \varphi_v(st) s dt, \quad (5)$$

and  $[\alpha_{mn}]$  denotes a matrix which elements do not depend on the order of the testing function  $\varphi_v(st)$ . In (4),  $[\beta_m^{v-1}]$  is the matrix which contains computed terms of order from 0 to  $v-1$  and  $N_L$  is the number of temporal basis functions. The upper infinite limit in (5) can be replaced by a finite time interval  $T_t$ . This interval is chosen in such a way so that the waveforms of interest  $V_m(t)$  can practically decay to zero at that time. The matrix equation (4) is solved  $N_L$  times, but the inverse matrix of  $[\alpha_{mn}]$  is computed once only, since its elements do not depend on the order of the

testing function. Detailed steps of the derivation of the scheme and the comparison of the accuracy of the method with the solution in the frequency-domain are available in [2], [3].

### 3. Determination of the Number of Temporal Basis Functions

In [1], [2], the minimum number of temporal basis functions  $N_L$  needed for a sufficient approximation of the computed transient response is given by

$$N_L = 2BT_f + 1 \quad (6)$$

where  $B$  is the width of the frequency band of an excitation signal and  $T_f$  is the length of the transient response.

As mentioned in the introduction, the length of the response is an unknown quantity in case we are interested in the steady state. Next, (6) does not take into account the scale factor, which makes (6) practically inapplicable.

Due to the above reasons, we propose a procedure determining the number of temporal basis functions during the run of the analysis.

Let us analyze the symmetric strip dipole of the length  $L = 0.2$  m and the width 2 mm (Fig. 1). The body of the dipole is meshed by triangular elements.

The dipole is excited at its center by the harmonic signal modulated by Gaussian pulse

$$U(t) = U_0 \frac{4}{cT\sqrt{\pi}} e^{-\left[\frac{4}{T}(t-t_0)\right]^2} \cos(2\pi f_0 t) \quad (7)$$

where  $T$  is the width of the pulse,  $t_0$  is the time delay at which the pulse reaches its peak, and  $f_0$  is the frequency of the harmonic carrier (a center frequency). For the analysis, the parameters of the voltage impulse source are:  $U_0=120\pi$  V,  $T=2.53$  ns,  $t_0=4.66$  ns, and  $f_0=2.0$  GHz. The bandwidth of the signal is  $B=2.0$  GHz. Let us choose the scale factor  $s = 3.0 \cdot 10^{+10}$  and the number of temporal basis functions  $N_L = 300$ .

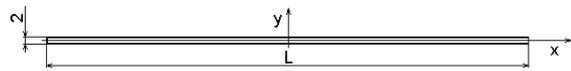


Fig.1. The symmetric strip dipole.

Applying the method described in Section 2, the current coefficients  $I_{n,u}$  (Fig. 2), and the transient response of the current at the center of dipole are obtained (Fig. 3).

One could think that the number of temporal basis functions needed for a sufficient approximation of the response can be determined from Fig. 2 so that the magnitude of coefficients can drop below a certain percentage value of their maximum value. However, the determination of that value is difficult. If the value is too high, the re-

sponse is approximated inaccurately. If the value is too low, the algorithm is time-consuming.

Thus, we propose a different approach which is based on the following idea: a transient response is approximated sufficiently if magnitudes of spectral components of the transient response are computed accurately enough at the lowest and the highest frequency of the important part of the spectrum of the excitation signal (considering the principle of the Laguerre polynomials' scheme [1]-[4]).

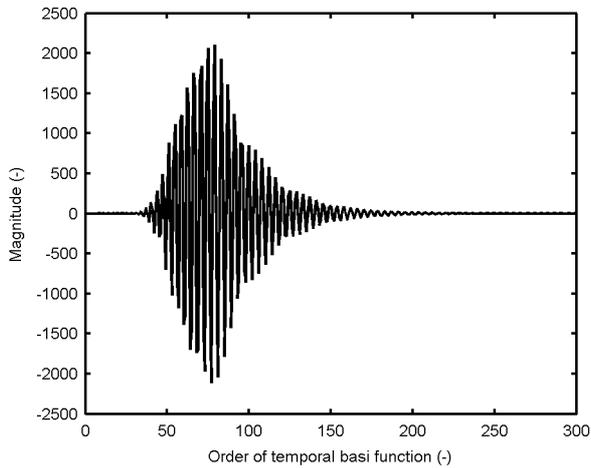


Fig. 2. Magnitude of approximated coefficients  $I_{n,u}$ .

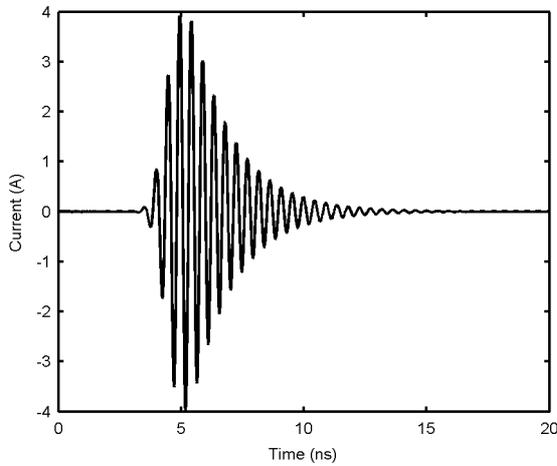


Fig. 3. The temporal response at the center of the symmetric strip dipole.

The way of evaluating the accuracy of the computation of spectral components is indicated in Fig. 4 showing the dependence of the real part of the dipole current at the frequency  $f = 3$  GHz (the solid black line) on the number of temporal basis functions  $N_L$ . This data was computed using [9]

$$I_n(f) = \sum_{u=0}^{N_L} I_{n,u} \frac{\left(-\frac{s}{2} + j2\pi f\right)^u}{\left(\frac{s}{2} + j2\pi f\right)^{u+1}}. \quad (8)$$

Obviously, the real part of the current oscillates. Increasing the number of temporal basis functions  $N_L$ , the real part of the current converges to the green dashed line. This line denotes the real part of current computed in the frequency domain [8]. Thus, the approximation error can be simply determined depending on the number of temporal basis functions  $N_L$ .

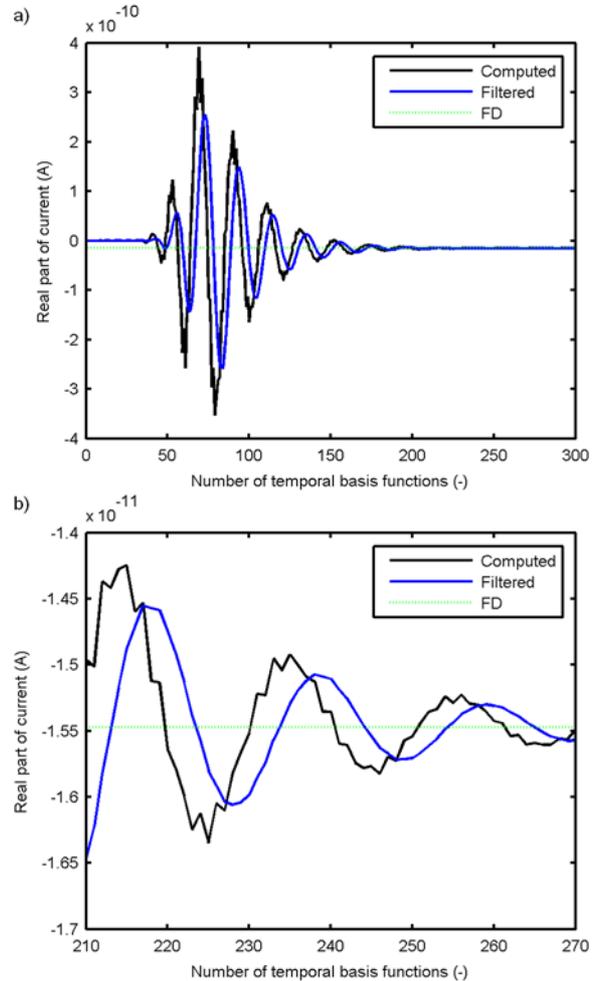


Fig. 4. Dependence of the real part of the current at  $f=3$  GHz on the number of temporal basis functions. a) the whole dependence for  $N_L \in \langle 0, 300 \rangle$ , b) the enlarged detail for  $N_L \in \langle 210, 270 \rangle$ .

Using rough data, the estimation of the error is quite difficult. The data is therefore smoothened using a digital filter (the average one, e.g.). The average is computed from  $N_A$  previous values. The filtration result is depicted in Fig. 4 ( $N_A=8$ : the blue solid line). The filtered signal is retarded with respect to the original one about  $N_A/2$  samples. The filtered signal converges to the accurate value a bit quicker than the original one. In order to find extremes easily, the filtered signal can be approximated piecewise by the first-order polynomial using the least square method. The extreme is at a point, where the sign of the direction is changed.

Of course, the imaginary part of the current should be checked also.

Although the number of temporal basis functions is determined when the scheme (4) runs, the proposed approach (8) of watching at two frequencies is not time-consuming. For each iteration of the scheme (4), just one term of (8) for a frequency is evaluated.

#### 4. Scale Factor

In this section, the influence of the scale factor on the efficiency of the scheme with weighted Laguerre polynomials is investigated, and the possibility of a choice of an optimum scale factor is discussed.

In this paper, the optimum scale factor denotes a scale factor, which makes the algorithm the most efficient (for a given structure and an excitation signal).

The investigation is performed considering the analysis of a symmetric strip dipole of the length  $L$  and the width 2 mm (Fig. 1). The dipole body is meshed by triangular elements. At its center, the dipole is excited by the voltage impulse source (7).

Since the computational complexity of the scheme with weighted Laguerre polynomials (4) is  $N^2 N_L(N_L+1)/2$ , the efficiency of the algorithm can be evaluated by the number of temporal basis functions. In order to determine the number of basis functions, the approach described in the previous section can be used. For the purposes of our investigation, this approach is modified to determine  $N_L$  more precisely. The following procedure is used: an analysis of a structure, for a given excitation signal, runs until the responses of the current computed according to (8) at the lowest and highest frequency are steady (the steady value is the correct value). After that, the responses are checked (the real and imaginary part separately) from the end to the beginning, and the error between the actual and steady current are computed. If the error is higher than 1%, the number of temporal basis functions is determined (from the end to the beginning of the response, the error grows, e.g. see Fig. 4). For the given scale factor, the resultant value corresponds to the maximum of four values – the real and imaginary part of the responses for two frequencies.

During the investigation, the length of the dipole is changed (different structures) and the parameters of the excitation signal stay constant, or the length of the dipole is constant, and the parameters of the excitation signal are changed (different excitations signals). Thus, a wide range of cases is obtained.

In the first case, the parameters of the voltage impulse source (7) are changed according to Tab. 1. The highest frequency of the spectrum of the excitation signal is  $f_{\max}=3$  GHz, but the bandwidth and the center frequency are different. In Tab. 1, the symbol  $TU_{0.1}$  denotes the time of the duration of the excitation signal for which the magnitude drops below 0.1% of its maximum value. The length of the dipole is  $L=0.2$  m.

B[GHz]	T[ns]	$f_0$ [GHz]	$TU_{0.1}$ [ns]
2.5	2.01	1.75	5.99
2.0	2.53	2.00	6.30
1.5	3.36	2.25	6.85
1.0	5.00	2.50	7.84

Tab. 1. Parameters of the voltage impulse source for the dependencies in Fig. 5.  $t_0=4.66$  ns.

For a different bandwidth of the voltage impulse source, the dependencies of the number of temporal basis functions on the scale factor are depicted in Fig. 5. Obviously, the minimum of all the responses is almost the same. Thus, also the optimum scale factor is almost the same. Moreover, the scheme (4) is more time-consuming for the scale factor smaller than the optimum one.

The detailed data of the responses from Fig. 5, including the length of the responses, is summarized in Tab. 2. Here,  $s_{\text{opt}}$  is the scale factor corresponding to the minimum of the given responses (the optimum one);  $s$  for 5% defines a region of the scale factor, where the number of temporal basis functions exceeds  $s_{\text{opt}}$  for 5% maximum; and  $Tl_{0.1}$  is the time duration of the current response where the magnitude drops below 0.1% of its maximum value.

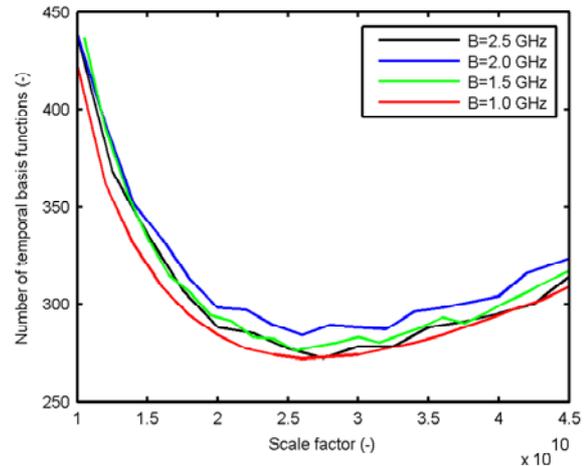


Fig. 5. Dependencies of the number of temporal basis functions on the scale factor for a different bandwidth of the voltage impulse source (7) defined by Tab. 1.

Hence, the following conclusion can be formulated: if the highest frequency of the spectrum of the excitation signal stays the same, then the width of the band of the excitation signal influences the position of the optimum scale factor for a given structure slightly.

B [GHz]	$s_{\text{opt}}$	$s$ for 5%	$Tl_{0.1}$ [ns]
2.5	$2.7 \cdot 10^{+10}$	$(2.2 \div 3.5) \cdot 10^{+10}$	19.38
2.0	$2.6 \cdot 10^{+10}$	$(2.0 \div 3.6) \cdot 10^{+10}$	17.50
1.5	$2.5 \cdot 10^{+10}$	$(2.1 \div 3.5) \cdot 10^{+10}$	17.66
1.0	$2.6 \cdot 10^{+10}$	$(2.0 \div 3.7) \cdot 10^{+10}$	16.84

Tab. 2. Detailed data for the first case of the investigation (Tab.1, Fig. 5).

In the second case, the voltage impulse source (7) is of the following parameters:  $T=5.0$  ns,  $t_0=4.66$  ns. The bandwidth of the signal is  $B=1.0$  GHz. Now, the center frequency  $f_0$  and the highest frequency  $f_{max}$  of the excitation signal are different. Their values are changed according to Tab. 3. The length of the dipole is again  $L=0.2$  m.

$f_0$ [GHz]	$f_{max}$ [GHz]	$TU_{0.1}$ [ns]
1.10	1.60	7.84
1.30	1.80	7.80
1.50	2.00	7.76
1.75	2.25	7.80
2.00	2.50	7.82
2.50	3.00	7.84

Tab. 3. Parameters of the voltage impulse source for the dependencies in Fig. 6.

The dependencies of the number of temporal basis functions on the scale factor for different center frequencies and different maximum frequencies of the excitation signal are depicted in Fig. 6. Obviously, different responses have a different minimum, so the optimum scale factor is different. Again, the scheme with weighted Laguerre polynomials (4) is more time-consuming for the scale factor smaller than the optimum one. The detailed data of these responses including the length of the responses is summarized in Tab. 4.

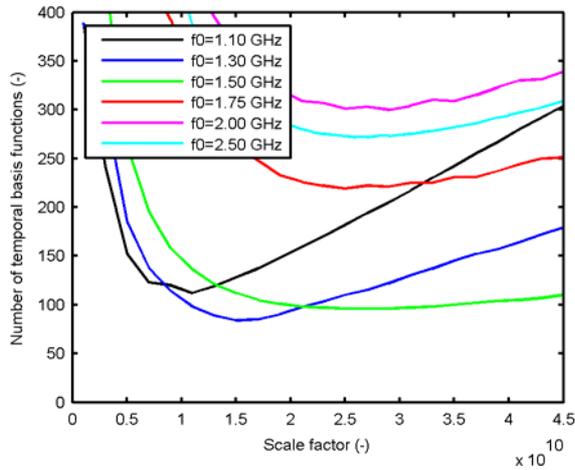


Fig. 6. Dependencies of the number of temporal basis functions on the scale factor for a different frequency  $f_0$  (the center frequency of the band), and the maximum frequency  $f_{max}$  of the voltage impulse source (7) defined by Tab. 3.

$f_0$ [GHz]	$s_{opt}$	$s$ for 5%	$Tl_{0.1}$ [ns]
1.10	$1.1 \cdot 10^{+10}$	$(1.0 \div 1.3) \cdot 10^{+10}$	23.56
1.30	$1.5 \cdot 10^{+10}$	$(1.3 \div 1.8) \cdot 10^{+10}$	15.78
1.50	$2.7 \cdot 10^{+10}$	$(1.9 \div 3.6) \cdot 10^{+10}$	8.46
1.75	$2.5 \cdot 10^{+10}$	$(2.0 \div 3.5) \cdot 10^{+10}$	14.48
2.00	$2.9 \cdot 10^{+10}$	$(2.0 \div 3.7) \cdot 10^{+10}$	17.54
2.50	$2.6 \cdot 10^{+10}$	$(2.0 \div 3.7) \cdot 10^{+10}$	16.84

Tab. 4. Detailed data for the second example of the investigation (Fig. 7).

Hence, the following conclusion can be formulated: if the width of the band of the excitation signal stays the same (however, the highest frequency of the spectrum of excitation signal grows), the position of the optimum scale factor grows.

In the last case, the voltage impulse source is of the following parameters:  $T=2.53$  ns,  $t_0=4.66$  ns, and  $f_0=2.0$  GHz. The time duration of the excitation signal is 6.29 ns, and the width of its band is  $B=2.0$  GHz. Now, the length of the dipole is changed:  $L=25$  mm,  $L=50$  mm,  $L=100$  mm,  $L=200$  mm,  $L=300$  mm, and  $L=400$  mm.

The dependencies of the number of temporal basis functions on the scale factor for different lengths of the symmetric strip dipole (Fig. 1) are depicted in Fig. 7. Again, different dependencies have a different minimum, so the optimum scale factor is different. The detailed data of these dependencies including the length of the responses is summarized in Tab. 5.

In this case, a "simple" conclusion can not be given. The position of the optimum scale factor depends on the length of the dipole.

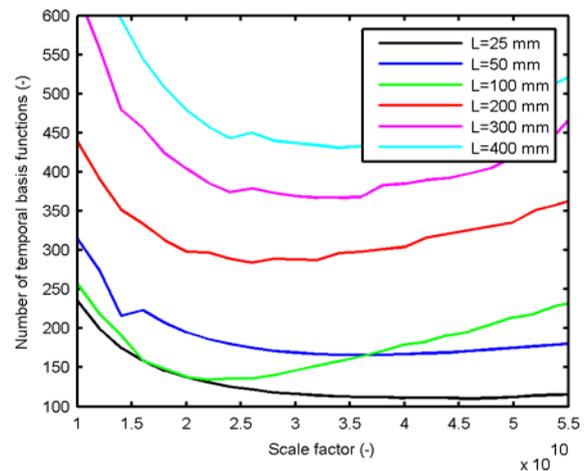


Fig. 7. Dependencies of the number of temporal basis functions on the scale factor for different lengths of the symmetric strip dipole (Fig. 1).

$L$ [mm]	$s_{min}$	$s$ for 5%	$Tl_{0.1}$ [ns]
25	$4.6 \cdot 10^{+10}$	$(3.0 \div 5.5) \cdot 10^{+10}$	6.34
50	$3.6 \cdot 10^{+10}$	$(2.6 \div 4.9) \cdot 10^{+10}$	8.50
100	$2.2 \cdot 10^{+10}$	$(1.9 \div 2.8) \cdot 10^{+10}$	13.71
200	$2.6 \cdot 10^{+10}$	$(2.0 \div 3.6) \cdot 10^{+10}$	17.50
300	$3.3 \cdot 10^{+10}$	$(2.2 \div 4.0) \cdot 10^{+10}$	22.87
400	$3.4 \cdot 10^{+10}$	$(2.3 \div 4.2) \cdot 10^{+10}$	27.51

Tab. 5. Detailed data for the third example of the investigation (Fig. 7).

From the investigation, the following conclusions can be given:

- The scale factor of the time axis significantly influences the efficiency of the scheme with weighted Laguerre polynomials (4). If the scale factor is not

chosen close to optimum for a given structure and a given excitation signal, the scheme becomes time-consuming (mainly for the scale factor smaller than the optimum one).

- Both the analyzed structure and the maximum frequency of the excitation signal significantly influence the position of the optimum scale factor. However, a simple expression for its determination can not be given.
- The choice of the scale factor (not an optimum one) could be described by the following expression:

$$s = k_s f_{\max} \quad (9)$$

where  $k_s$  is an coefficient describing the influence of the analyzed structure on the scale factor, and  $f_{\max}$  is the maximum frequency of the excitation signal. For the symmetric strip dipole, with regard to the column  $s$  for 5% in Tab. 2, 4 and 5, the value of  $k_s$  could be roughly determined about  $10 \text{ Hz}^{-1}$ . Although the choice of the scale factor according to (9) is possible, the value is quite far from the optimum one.

## 5. Conclusion

In this paper, the approach to the determination of the number of temporal basis functions has been proposed. Although the number of temporal basis function is determined when the scheme with Laguerre polynomials runs, the proposed approach is not time-consuming.

Further, the influence of the choice of the scale factor on the efficiency of the scheme, with the possibility of its optimum choice, has been investigated. The investigation results into the conclusion that a chosen scale factor strongly influences the efficiency of the scheme. If the scale factor is not chosen close to the optimum one, the scheme becomes time-consuming. However, a simple formula for the prediction of the optimum scale factor can not be given.

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## References

- [1] CHUNG, Y. S., SARKAR, T. K., JUNG, B. H., SALZAR-PALMA, M., JI, Z., JANG, S., KIM, K. Solution of time domain electric field integral equation using the Laguerre polynomials. *IEEE Transactions on Antennas and Propagation*, 2004, vol. 52, no. 9, p. 2319–2328.
- [2] JUNG, B. H., SARKAR, T. K., CHUNG, Y. S., SALZAR-PALMA, M., JI, Z., JANG, S., KIM, K. Transient electromagnetic scattering from dielectric objects using the electric field integral equation with Laguerre polynomials as temporal basis functions. *IEEE Transactions on Antennas and Propagation*, 2004, vol. 52, no. 9, p. 2329 – 2340.
- [3] LÁČÍK, J., RAIDA, Z. Modeling microwave structure in time domain using Laguerre polynomials. *Radioengineering*, 2006, vol. 16, no. 3, p. 1-9.
- [4] CHUNG, Y. S., SARKAR, T. K., JUNG, B. H., SALZAR-PALMA, M. Solving time domain electric field integral equation without the time variable. *IEEE Transactions on Antennas and Propagation*, 2006, vol. 54, no. 1, p. 258–262.
- [5] RAO, S. M. *Time Domain Electromagnetics*. London: Academic Press, 1999.
- [6] JUNG, B. H., SARKAR, T. K. Time-domain electric-field integral equation with central finite difference. *Microwave and Optical Technology Letters*. 2001, vol. 31, no. 6, p. 429-435.
- [7] WEILE, S. D., PISHARODY, G., CHEN, N., SHANKER, B., MICHIELSEN, E. A novel scheme for the solution on the time-domain integral equations of electromagnetics. *IEEE Transactions on Antennas and Propagation*, 2004, vol. 52, no. 1, p. 283 – 295.
- [8] RAO, S. M., WILTON, D. R., GLISSON, A. W. Electromagnetic scattering by surfaces of arbitrary shape. *IEEE Transactions on Antennas and Propagation*, 1982, vol. 30, no. 5, p. 409 – 418.
- [9] SARKAR, T. K., KOH, J., SALAZAR-PALMA, M. Generation of wideband electromagnetic response through a Laguerre expansion using early time and low frequency data. *IEEE MTT-S International Microwave Symposium Digest*. 2002, vol. 3, p. 1989-1992.

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