OPTIMIZATION OF LINEAR DIFFERENTIAL SYSTEMS BY LYAPUNOV'S DIRECT METHOD

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Abstract: Two approaches to solving optimization problems of dynamic systems are well-known. The first approach needs to find a fixed control (program control) for which the system described by differential equations reaches a predetermined value and minimizes an integral quality criterion. Proposed by L.S. Pontryagin, this method was in essence a further development of general optimization methods for dynamical systems. The second method consists in finding a control function (in the form of a feedback) guaranteeing that, simultaneously, the zero solution is asymptotically stable and an integral quality criterion attains a minimum value. This method is based on what is called the second Lyapunov method and its founder is N.N. Krasovskii. In the paper, the latter method is applied to linear differential equations and systems with integral quality criteria.

Keywords: Lyapunov function, optimization problem, integral quality criterion, control function

1 INTRODUCTION

As it is well-known, there are two approaches to solving optimization problems of dynamic systems. The first approach is based on finding what is called a fixed control (program control) for which the system described by differential equations reaches a predetermined previously value and minimizes the integral quality criterion. This method was designed by L.S. Pontryagin and can be regarded as a further development of the general optimization methods for dynamical systems. The second method suggests finding a control function in the form of a feedback such that the zero solution will be asymptotically stable and, simultaneously, an integral quality criterion attains a minimum value. This method is based on what is called the second Lyapunov method, known from the theory of stability of differential equations and was proposed by N.N. Krasovskii. In the contribution, we apply the second method to linear differential equations and systems with integral quality criteria.

We describe the general scheme for the construction of an optimal control by the second Lyapunov method (see, e.g. [1, p. 476]).

The stabilization problems applied, together with the requirement of asymptotic stability of a given motion described by a system of differential equations

$$\frac{dx(t)}{dt} = f\left(t, x(t), u(t)\right), \; x \in R^n, \; u \in R^m, \; t \geq t_0, \tag{1}$$

require the best possible quality of the transition process. The best quality criterion is very often expressed as a condition of minimizing the integral

$$I\left[x(t), u(x(t))\right] = \int_{t_0}^{\infty} \omega(t, x(t), u(t))dt \tag{2}$$

where $\omega(t,x,u)$ is a positive function defined for $t \geq t_0$, $\|x\| < M$, $u \in R^m$, $\|\cdot\|$ is a norm, and M is a positive constant. Frequently, the integrand $\omega(t,x(t),u(t))$ is reduced to $\omega(x(t),u(t))$ where ω is assumed to have a quadratic form

$$\omega(x,u) = x^T Cx + u^T Du$$

with a positive semi-definite matrices C and D.

We will deal with the optimal stabilization problem (see [1, p. 479]). Let a quality criterion of a process $x(t)$ in the form (2) be fixed. It is necessary to find a control function $u = u_0(t)$ ensuring the asymptotic stability of non-perturbed motion $x(t) \equiv 0$ such that, for any other admissible control function $u = u^*(t)$, the inequality

$$\int_{t_0}^{\infty} \omega(t,x(t),u_0(t))dt \leq \int_{t_0}^{\infty} \omega(t,x(t),u^*(t))$$

holds. Then, the function $u = u_0(t)$ is called an optimal control function.

Define an auxiliary function

$$B[V,t,x,u] := \frac{\partial V(t,x)}{\partial t} + \text{grad}_x^T V(t,x) f(t,x,u)$$

where V is a Lyapunov function defined for $t \geq t_0$, $\|x\| < M$ and formulate the main theorem on optimal stabilization.

Theorem 1. [1, p. 485] Assume that, for the system of differential equations (1), there exist a Lyapunov function $V_0(t,x)$ having an infinitesimal upper bound and a function $u_0(t,x)$ such that

1. The function $\omega(x,t) = \omega(t,x,u_0(t,x))$ is positive definite for every $t \geq t_0$, $\|x\| < M$.
2. $B[V_0,t,x,u_0(t,x)] \equiv 0$.
3. $B[V_0,t,x,u(t,x)] \geq 0$ for any $u(t,x) \not\equiv u_0(t,x)$.

Then, the function $u_0(t,x)$ is a solution to the optimal control problem and

$$\int_{t_0}^{\infty} \omega(t,x(t),u_0(t,x))dt = \min_u \left[\int_{t_0}^{\infty} \omega(t,x(t),u(t,x))dt \right] = V_0(t_0,x(t_0)). \quad (3)$$

2 OPTIMIZATION OF LINEAR DIFFERENTIAL EQUATIONS AND SYSTEMS

Consider a scalar equation

$$\frac{dx(t)}{dt} = ax(t) + bu(x) \quad (4)$$

where a and b are real constants. We need to find a control function $u = u_0(x)$ for which the equation with $u = u_0(x)$ is asymptotically stable and a given integral quality criterion

$$I[x(t),u(x(t))] = \int_0^{\infty} \left(\alpha x^2(t) + \beta u^2(x(t))\right) dt, \ \alpha > 0, \ \beta > 0$$

attains a minimum value. Solving this problem, we look for a Lyapunov function in the form $V(x) = hx^2$. Its total derivative along the solutions of given equation is

$$\frac{d}{dt} V(x(t)) = 2hx(t)\dot{x}(t) = 2hx(t)[ax(t) + bu(x(t))].$$

Equating it with the integrand multiplied by -1 (this general recommendation applied here and below follows from (3)), we obtain

$$2hax^2(t) + 2hbx(t)u(x(t)) = -\alpha x^2(t) - \beta u^2(x(t)).$$

This equation will be satisfied if

$$2hax^2(t) = -\alpha x^2(t),\ \ 2hbx(t)u(x(t)) = -\beta u^2(x(t)).$$

We get $h = -\alpha/2a$ and $u\,(x(t)) = (\alpha b/\beta a)x(t)$. Hence, the optimal stabilization conditions are

$$h = -\frac{\alpha}{2a},\ \ u_0(x(t)) = \frac{\alpha b}{\beta a}x(t).$$

Thus, for the control function $u = u_0(x) = (\alpha b/\beta a)x(t)$ and Lyapunov function $V_0(x) = -(\alpha 2a)x^2$, $a < 0$, the equation (4) will be asymptotically stable and the integral criterion attains a minimum value.

As a next application, consider a linear system with scalar control function:

$$\frac{dx(t)}{dt} = Ax(t) + bu(x) \tag{5}$$

where $A \in R^{n\times n}$, $b \in R^n$, $x(t) \in R^n$, $u(t) \in R$. We need to find a control function $u = u_0(x)$ for which the system (5) is asymptotically stable and a given integral quality criterion

$$I\,[x(t), u(x(t))] = \int_0^\infty \left(x^T(t)Cx(t) + du^2(x(t))\right) dt \tag{6}$$

has a minimum value provided that C is a symmetric, positive definite matrix and $d > 0$. To solve this problem, we use Lyapunov function taken in the form $V(x) = x^T Hx$ where $H \in R^{n\times n}$ is a positive definite symmetric matrix. The total derivative of the Lyapunov function along the trajectories of (5) is

$$\frac{d}{dt}V(x(t)) = [Ax(t) + bu(x(t))]^T\, Hx(t) + x^T(t)H\,[Ax(t) + bu(x(t))]\,.$$

Equating it with the integrand (multiplied by -1), we obtain

$$[Ax(t) + bu(x(t))]^T\, Hx(t) + x^T(t)H\,[Ax(t) + bu(x(t))] = -x^T(t)Cx(t) - du^2(x(t)).$$

This equation will hold if

$$x^T(t)\left[A^T H + HA\right]x(t) = -x^T(t)Cx(t)$$

and

$$u(x(t))b^T Hx(t) + x^T(t)Hbu(x(t)) = -du^2(x(t)).$$

The first equation holds if H is the solution to the Lyapunov matrix equation

$$A^T H + HA = -C. \tag{7}$$

If the matrix A is asymptotically stable, then, for an arbitrary positive definite matrix C, the Lyapunov matrix equation has a unique solution - the positive definite matrix H (see, e.g. [2]). Consider the second expression. Since the control function $u(x)$ is a scalar, we derive

$$u(x(t)) = u_0(x) := -\frac{1}{d}\left[b^T Hx + x^T Hb\right]. \tag{8}$$

Thus, for the control function (8) and the Lyapunov function used, the system (5) is asymptotically stable and the quality criterion (6) has a minimum value.

As the last application, consider a system:

$$\frac{dx(t)}{dt} = Ax(t) + Bu(x) \tag{9}$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$, $x(t) \in R^n$, $u(t) \in R^m$. We need to find a control function $u = u_0(x)$ for which the system is asymptotically stable and an integral quality criterion

$$I[x(t),u(x(t))] = \int_0^\infty \left(x^T(t)Cx(t) + u^T(x(t))Du(x(t))\right)dt \tag{10}$$

takes a minimum value provided that $C \in R^{n\times n}$ is a symmetric, positive definite matrix and D is a diagonal control matrix, $D = \mathrm{diag}\{d_j\}$, $d_j > 0$, $j = 1,\ldots,m$. The total derivative of the Lyapunov function $V(x) = x^T Hx$ along the trajectories of (9) is

$$\frac{d}{dt}V(x(t)) = [Ax(t)+Bu(x(t))]^T Hx(t) + x^T(t)H[Ax(t)+Bu(x(t))].$$

Equating it with the integrand (multiplied by -1), we obtain

$$[Ax(t)+Bu(x(t))]^T Hx(t) + x^T(t)H[Ax(t)+Bu(x(t))] = -x^T(t)Cx(t) - u^T(x(t))Du(x(t)).$$

This equation will hold if

$$x^T(t)\left[A^T H + HA\right]x(t) = -x^T(t)Cx(t),$$

and

$$u^T(x(t))B^T Hx(t) + x^T(t)HBu(x(t)) = -u^T(x(t))Du(x(t)).$$

If the matrix H is a solution to the Lyapunov matrix equation (7), the first equation is satisfied. Consider the second equation. Set

$$b_i = \begin{pmatrix} b_{1i} \\ b_{2i} \\ \vdots \\ b_{ni} \end{pmatrix},\ i = 1,\ldots,m,\quad h_j = \begin{pmatrix} h_{1j} \\ h_{2j} \\ \vdots \\ h_{nj} \end{pmatrix},\ j = 1\ldots,n.$$

Then, the second equation can be rewritten as

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}^T \begin{pmatrix} b_1^T h_1 & b_1^T h_2 & \ldots & b_1^T h_n \\ b_2^T h_1 & b_2^T h_2 & \ldots & b_2^T h_n \\ \vdots & \vdots & \ddots & \vdots \\ b_m^T h_1 & b_m^T h_2 & \ldots & b_m^T h_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}^T \begin{pmatrix} h_1^T b_1 & h_1^T b_2 & \ldots & h_1^T b_m \\ h_2^T b_1 & h_2^T b_2 & \ldots & h_2^T b_m \\ \vdots & \vdots & \ddots & \vdots \\ h_n^T b_1 & h_n^T b_2 & \ldots & h_n^T b_m \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}$$

$$= -\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}^T \begin{pmatrix} d_1 & 0 & \ldots & 0 \\ 0 & d_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & d_m \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix},$$

or in the form

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}^T \begin{pmatrix} b_1^T(h_1x_1 + h_2x_2 + \ldots + h_nx_n) \\ b_2^T(h_1x_1 + h_2x_2 + \ldots + h_nx_n) \\ \vdots \\ b_m^T(h_1x_1 + h_2x_2 + \ldots + h_nx_n) \end{pmatrix}$$

$$+ \begin{pmatrix} \left(h_1^T x_1 + h_2^T x_2 + \ldots + h_n^T x_n\right) b_1 \\ \left(h_1^T x_1 + h_2^T x_2 + \ldots + h_n^T x_n\right) b_2 \\ \vdots \\ \left(h_1^T x_1 + h_2^T x_2 + \ldots + h_n^T x_n\right) b_m \end{pmatrix}^T \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} = -d_{11}u_1^2 - d_{22}u_2^2 - \cdots - d_{mm}u_m^2.$$

Hence, we obtain the system of equations

$$2\left(h_1^T x_1 + h_2^T x_2 + \ldots + h_n^T x_n\right) b_1 u_1 = -d_{11} u_1^2,$$

$$2\left(h_1^T x_1 + h_2^T x_2 + \ldots + h_n^T x_n\right) b_2 u_2 = -d_{22} u_2^2,$$

$$\cdots\cdots\cdots$$

$$2\left(h_1^T x_1 + h_2^T x_2 + \ldots + h_n^T x_n\right) b_m u_m = -d_{mm} u_m^2.$$

Thus, the optimal control has the form

$$u_1^0(x) = -\frac{2}{d_{11}}\left(h_1^T x_1 + h_2^T x_2 + \ldots + h_n^T x_n\right) b_1,$$

$$u_2^0(x) = -\frac{2}{d_{22}}\left(h_1^T x_1 + h_2^T x_2 + \ldots + h_n^T x_n\right) b_2,$$

$$\cdots\cdots\cdots$$

$$u_m^0(x) = -\frac{2}{d_{mm}}\left(h_1^T x_1 + h_2^T x_2 + \ldots + h_n^T x_n\right) b_m.$$

3 CONCLUSION

In the paper we applied a method developed by N.N. Krasovskii to solving optimal stabilization problems for several classes of differential equations and their systems. By this method, a control function can be found in the form of a feedback such that the zero solution of a given equation or system will be asymptotically stable and, simultaneously, an integral quality criterion attains a minimum value. Further investigation can be directed to the problem of optimal stabilization of system (9) if the matrix D in the integral quality criterion (10) is not diagonal.

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